New gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole: Gyroscope precession

Donato Bini and Andrea Geralico
Istituto per le Applicazioni del Calcolo "M. Picone," CNR, I-00185 Rome, Italy

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We analytically compute the gravitational self-force correction to the gyroscope precession along slightly eccentric equatorial orbits in the Kerr spacetime, generalizing previous results for the Schwarzschild spacetime. Our results are accurate through the 9.5 post-Newtonian order and to second order in both eccentricity and rotation parameter. We also provide a post-Newtonian check of our results based on the currently known Hamiltonian for spinning binaries.

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I. INTRODUCTION

The last few years have witnessed the beginning of the era of gravitational-wave astronomy, after the discovery of the first signals by LIGO [1–6] associated with either binary black hole or neutron star mergers. The number of such events is expected to rapidly increase in the near future thanks to the improved sensitivity of Advanced LIGO [7] and to the contribution of the space-based interferometer LISA [8], which is designed to detect a wide range of low-frequency gravitational wave sources, including extreme mass ratio inspirals (EMRIs). The latter are binary systems in which one body is much more massive than the other, so that the dynamics is well described in the framework of gravitational self-force (GSF) theory by using standard perturbation methods (see, e.g., Ref. [9] for a recent review). Conservative effects are encoded in gauge-invariant quantities, whose calculation relies on subtle computational techniques to fully reconstruct the metric perturbation and suitably regularize it by removing its singular part. However, the final result allows for the extraction of physical information which can be used to perform comparisons with other approaches, like post-Newtonian (PN) theory and numerical relativity (NR) simulations, as well as to calibrate and enhance the effective-one-body (EOB) model [10–12].

Spin couplings are expected to significantly affect the two-body dynamics, thereby playing an important role in the gravitational wave detection and parameter estimation (see, e.g., Ref. [13] and references therein). Spin-orbital (i.e., linear-in-spin) and spin-spin (i.e., quadratic-in-spin) effects have been accounted for at the lowest PN levels by standard Hamiltonian methods [14–16] and effective field theory (EFT) techniques [17,18]. The first high-PN calculations within the GSF approach of the spin-orbit precession of a spinning compact body on a circular orbit around a Schwarzschild black hole have been done in Refs. [19–21]. These results have been extended to eccentric orbits in Refs. [22,23] by using the methodology introduced in Ref. [24] and soon after generalized to the Kerr case in Ref. [25].

We compute here the first-order GSF correction to the spin-precession invariant for slightly eccentric equatorial orbits in the Kerr spacetime through the 9.5PN order and to second order both in the eccentricity and spin parameter. The spin-dependent part mixing eccentricity and spin effects is completely new. We also improve to the 9.5PN level the current knowledge of the spin-precession invariant for eccentric orbits in the nonrotating case (9PN, Ref. [23]) and for circular orbits in the same Kerr case (8PN, Ref. [26]) up to the second order in the spin parameter. Furthermore, the circular orbit limit of the present result gives the self-force correction to the periastron advance around a Kerr black hole, which has been presented elsewhere [27]. Finally, as an independent check, we calculate the same invariant by using the current knowledge of the Arnowitt-Deser-Misner (ADM) Hamiltonian for two point masses with aligned spins [28].

We will denote by m_1 and m_2 and by S_1 and S_2 the masses and spins of the two bodies, respectively, with the convention that $m_1 \leq m_2$. We also define the total mass of the system $M = m_1 + m_2$, the mass ratios

$$q = \frac{m_1}{m_2}, \qquad \mu = \frac{m_1 m_2}{M}, \qquad \nu = \frac{\mu}{M} = \frac{q}{(1+q)^2}, \tag{1}$$

and the dimensionless mass difference

$$\frac{m_2 - m_1}{M} = \Delta = \sqrt{1 - 4\nu}, \tag{2}$$

as well as the dimensionless spin variables $\chi_{1,2} \equiv S_{1,2}/m_{1,2}^2$ associated with each body, as usual. GSF results are obtained in the limit of small mass ratio ($m_1 \ll m_2$, implying $q \sim \nu \ll 1$) and small spin $[|S_1|/(cGm_1^2) \ll 1]$ of the perturbing body. The metric signature is chosen to be $+2$ and units are such that $c = G = 1$ unless differently specified. Greek indices run from 0 to 3, whereas latin ones from 1 to 3.

II. GYROSCOPE PRECESSION IN THE BACKGROUND KERR SPACETIME

The background Kerr metric with parameters m_2 and $a_2 = a$ (with $\hat{a} = a/m_2$ dimensionless) written in Boyer-Lindquist coordinates (t, r, θ, ϕ) reads

$$\begin{aligned} d\bar{s}^2 &= \bar{g}_{\alpha\beta} dx^\alpha dx^\beta \\ &= -\left(1 - \frac{2m_2 r}{\Sigma}\right) dt^2 - \frac{4am_2 r \sin^2\theta}{\Sigma} dt d\phi \\ &\quad + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ &\quad + \left(r^2 + a^2 + \frac{2m_2 r a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2, \end{aligned} \tag{3}$$

where

$$\Delta = r^2 + a^2 - 2m_2 r, \qquad \Sigma = r^2 + a^2 \cos^2\theta. \tag{4}$$

A test gyroscope moving along an eccentric geodesic orbit on the equatorial plane ($\theta = \pi/2$) has 4-velocity

$$\begin{aligned} \bar{u} = \bar{u}^\alpha \partial_\alpha &= \frac{1}{r^2}\left(ax + \frac{r^2 + a^2}{\Delta} \bar{P}\right) \partial_t + \dot{r} \partial_r \\ &\quad + \frac{1}{r^2}\left(x + \frac{a}{\Delta} \bar{P}\right) \partial_\phi, \end{aligned} \tag{5}$$

where $\bar{P} = \bar{E} r^2 - ax$, with $x = \bar{L} - a\bar{E}$, and $\dot{r} \equiv \bar{u}^r$ is such that

$$\dot{r}^2 = \left(\frac{dr}{d\bar{\tau}}\right)^2 = \frac{1}{r^4}[\bar{P}^2 - \Delta(r^2 + x^2)]. \tag{6}$$

Here $\bar{E} = -\bar{u}_t$ and $\bar{L} = \bar{u}_\phi$ denote the conserved energy and angular momentum per unit mass of the particle, respectively, so that $\bar{E}$ and $\bar{L}/m_2$ are dimensionless together with their combination $\hat{x} = x/m_2$. The orbit can be parametrized either by the proper time $\bar{\tau}$ or by the relativistic anomaly $\chi \in [0, 2\pi]$, such that

$$r = \frac{m_2 p}{1 + e\cos\chi}, \tag{7}$$

which are related by

$$m_2 \frac{d\chi}{d\bar{\tau}} = u_p^{3/2}(1 + e\cos\chi)^2 [1 + u_p^2 \hat{x}^2 (e^2 - 2e\cos\chi - 3)]^{1/2}. \tag{8}$$

The (dimensionless) background orbital parameters, semi-latus rectum p (with reciprocal $u_p = 1/p$) and eccentricity e are defined by writing the minimum (pericenter, $r_{\rm peri}$) and maximum (apocenter, $r_{\rm apo}$) values of the radial coordinate along the orbit as

$$r_{\rm peri} = \frac{m_2 p}{1 + e}, \qquad r_{\rm apo} = \frac{m_2 p}{1 - e}. \tag{9}$$

The two conditions,

$$\left(\frac{dr}{d\bar{\tau}}\right)\bigg|_{r_{\rm peri}} = 0 = \left(\frac{dr}{d\bar{\tau}}\right)\bigg|_{r_{\rm apo}}, \tag{10}$$

can be imposed on Eq. (6) to solve them for $\bar{E} = \bar{E}(p, e)$ and $\bar{L} = \bar{L}(p, e)$. Their explicit expressions in terms of $(u_p, e, \hat{a})$ for prograde orbits are given by

$$\begin{aligned} \bar{E} &= \frac{1 - 2u_p + \hat{a} u_p^{3/2}}{\sqrt{1 - 3u_p + 2\hat{a} u_p^{3/2}}} \left\{1 - \left[\frac{1}{2} - \frac{2\hat{a} u_p^{5/2}}{(1 - 2u_p + \hat{a}^2 u_p^2)(1 - 2u_p)} + \frac{1 - 4u_p}{2(1 - 3u_p + 2\hat{a} u_p^{3/2})}\right.\right. \\ &\quad \left.\left. - \frac{1 - 4u_p + 2u_p^2}{(1 - 2\hat{a} u_p^{3/2} + \hat{a}^2 u_p^2)(1 - 2u_p)}\right] e^2\right\} + O(e^4), \\ \frac{\bar{L}}{m_2} &= \frac{1 - 2\hat{a} u_p^{3/2} + \hat{a}^2 u_p^2}{\sqrt{u_p(1 - 3u_p + 2\hat{a} u_p^{3/2})}} \left\{1 - \left[\frac{1}{2} + \frac{\hat{a} u_p^{1/2}(1 + u_p)}{1 - 2u_p + \hat{a}^2 u_p^2} + \frac{1 - 4u_p}{2(1 - 3u_p + 2\hat{a} u_p^{3/2})} - \frac{1 + \hat{a} u_p^{1/2}(1 - u_p)}{1 - 2\hat{a} u_p^{3/2} + \hat{a}^2 u_p^2}\right] e^2\right\} \\ &\quad + O(e^4), \end{aligned} \tag{11}$$

respectively, to the second order in eccentricity.

The motion is then governed by the following equations [29,30]:

$$\frac{dt}{d\chi}=\frac{m_2}{u_p^{3/2}}\frac{E+E\hat{a}^2u_p^2(1+e\cos\chi)^2-2\hat{a}u_p^3\hat{x}(1+e\cos\chi)^3}{(1+e\cos\chi)^2[1+u_p^2\hat{x}^2(e^2-2e\cos\chi-3)]^{1/2}[1-2u_p(1+e\cos\chi)+a^2u_p^2(1+e\cos\chi)^2]},$$
$$\frac{d\phi}{d\chi}=u_p^{1/2}\frac{\hat{x}+\hat{a}E-2u_p\hat{x}(1+e\cos\chi)}{[1+u_p^2\hat{x}^2(e^2-2e\cos\chi-3)]^{1/2}[1-2u_p(1+e\cos\chi)+a^2u_p^2(1+e\cos\chi)^2]}. \tag{12}$$

Integrating over a full radial orbit from periastron to periastron gives the coordinate time radial period $\bar{T}_r=\oint dt=\oint d\chi(dt/d\chi)$ and the accumulated azimuthal angle $\bar{\Phi}=\oint d\phi=\oint d\chi(d\phi/d\chi)$, with associated frequencies $\bar{\Omega}_r=2\pi/\bar{T}_r$ and $\bar{\Omega}_\phi=\bar{\Phi}/\bar{T}_r$.

A. Marck's "intermediate" frame and gyroscope precession

Using the Killing-Yano tensor Marck defined a parallelly propagated frame along a general geodesic in the Kerr spacetime [31]. Marck's geometric construction uses, as an "intermediate" frame, a convenient (degenerate) Frenet-Serret frame adapted to $\bar{u}$, which in the case of equatorial timelike geodesics reads

$$\bar{e}_1=\frac{r}{(r^2+x^2)^{1/2}}\left[\frac{\dot{r}(r^2+a^2)}{\Delta}\left(\partial_t+\frac{a}{r^2+a^2}\partial_\phi\right)+\frac{\bar{P}}{r^2}\partial_r\right],$$
$$\bar{e}_2=\frac{1}{r}\partial_\theta,$$
$$\bar{e}_3=\left(\frac{x(r^2+a^2)\bar{P}}{(r^2+x^2)^{1/2}\Delta r^2}+\frac{a(r^2+x^2)^{1/2}}{r^2}\right)\partial_t+\frac{x\dot{r}}{(r^2+x^2)^{1/2}}\partial_r+\left(\frac{ax\bar{P}}{(r^2+x^2)^{1/2}r^2\Delta}+\frac{(r^2+x^2)^{1/2}}{r^2}\right)\partial_\phi, \tag{13}$$

whose transport properties are

$$\nabla_{\bar{u}}\bar{e}_1=\bar{\omega}\bar{e}_3,\qquad \nabla_{\bar{u}}\bar{e}_3=-\bar{\omega}\bar{e}_1, \tag{14}$$

with

$$\bar{\omega}=\frac{\bar{E}x+a}{r^2+x^2}, \tag{15}$$

whereas $\nabla_{\bar{u}}\bar{e}_2=0$, since $\bar{e}_2$ is aligned with the θ-direction. The total spin-precession angle accumulated over a radial period is then

$$\bar{\Psi}=\int_0^{\bar{\mathcal{T}}_r}\bar{\omega}d\bar{\tau}=\int_0^{2\pi}\bar{\omega}\frac{d\bar{\tau}}{d\chi}d\chi, \tag{16}$$

$\bar{\mathcal{T}}_r=\oint d\bar{\tau}$ denoting the proper-time period. In order to remove the rotation of the Boyer-Lindquist spherical-like coordinate frame in the azimuthal direction, corresponding to comparing the spin direction with a "fixed" asymptotic Cartesian-like frame, one must subtract $\bar{\Phi}$ from $\bar{\Psi}$. The net precession angle of the test gyroscope dragged along $\bar{u}$ is then conveniently measured by the quantity

$$\bar{\psi}=1-\frac{\bar{\Psi}}{\bar{\Phi}}, \tag{17}$$

which reads

$$\bar{\psi}=1-\sqrt{1-3u_p+2\hat{a}u_p^{3/2}}+\frac{3u_p^2(1-\hat{a}u_p^{1/2})^2}{2(1-6u_p+8\hat{a}u_p^{3/2}-3\hat{a}^2u_p^2)^2\sqrt{1-3u_p+2\hat{a}u_p^{3/2}}(1-2u_p+\hat{a}^2u_p^2)}$$
$$\times[(1-6u_p)(1-4u_p)+2(5-22u_p)u_p^{3/2}\hat{a}+10\hat{a}^2u_p^3-2(1-15u_p)u_p^{5/2}\hat{a}^3-25\hat{a}^4u_p^4+6\hat{a}^5u_p^{9/2}]e^2+O(e^4), \tag{18}$$

to the second order in the eccentricity parameter.

III. SPIN PRECESSION IN THE PERTURBED SPACETIME

In this section we recall the basic theory underlying the derivation of the spin-precession invariant in the perturbed spacetime and its first-order self-force (SF) correction, following Refs. [24,25]. The gyroscope carrying a small mass m_1 and a small spin S_1 [so that $q=\frac{m_1}{m_2}\ll 1$ and $|S_1|/(cGm_1^2)\ll 1$] can be considered as following an eccentric geodesic orbit in a (regularized) perturbed spacetime $g^{\rm R}_{\alpha\beta}$, through order $O(q)$, while its associated spin vector is parallelly transported in $g^{\rm R}_{\alpha\beta}$, to linear order in spin. The regularized perturbed metric is decomposed as

$$g^{\rm R}_{\alpha\beta}=\bar{g}_{\alpha\beta}+h^{\rm R}_{\alpha\beta}+O(q^2), \tag{19}$$

where $\bar{g}_{\alpha\beta}$ is the background spacetime (3) and $h^{\rm R}_{\alpha\beta} = O(q)$ is the first-order SF metric perturbation. Henceforth, we shall omit the superscript R. The spin-precession invariant,

$$\psi(m_2\Omega_r, m_2\Omega_\phi; q) = 1 - \frac{\Psi}{\Phi}, \tag{20}$$

is assumed to be a function of the radial and (averaged) azimuthal angular frequencies $\Omega_r = 2\pi/T_r$ and $\Omega_\phi = \Phi/T_r$, for any value of the mass ratio. Furthermore, the geodesics in both background and perturbed spacetimes are assumed to have the same orbital parameters (p, e), so that any comparison between perturbed and unperturbed quantities is done at the same coordinate radius r (or the same anomaly χ), though not the same t and ϕ coordinates. Any such difference is not gauge invariant, in general. Gauge invariance is ensured by further assuming that the background and perturbed orbits both have the same orbital frequencies (or equivalently the same radial and azimuthal periods). The first-order SF correction to the spin-precession invariant is then defined as

$$\begin{aligned}\Delta\psi &= \frac{1}{q}[\psi(m_2\Omega_r, m_2\Omega_\phi; q) - \psi(m_2\Omega_r, m_2\Omega_\phi; 0)]\\ &= -\frac{\Delta\Psi}{\Phi},\end{aligned} \tag{21}$$

where

$$\Delta\Psi = \delta\Psi - \frac{\partial\bar{\Psi}}{\partial\bar{\Omega}_r}\delta\Omega_r - \frac{\partial\bar{\Psi}}{\partial\bar{\Omega}_\phi}\delta\Omega_\phi, \tag{22}$$

the operator δ denoting the $O(q)$ difference between a quantity on the perturbed geodesic and the same quantity on the background one with the same (p, e, χ), but which does not keep fixed the values of the two frequencies. After the computation of the function $\Delta\psi(\Omega_r, \Omega_\phi)$, one can reexpress it as a function of the inverse semilatus rectum u_p, and eccentricity e, of the unperturbed orbit.

A. Bound timelike geodesics

Bound timelike geodesics in the equatorial plane of the perturbed spacetime (19) have 4-velocity

$$u = u^\alpha\partial_\alpha = (\bar{u}^\alpha + \delta u^\alpha)\partial_\alpha, \tag{23}$$

with $\delta u^\alpha = O(h)$, and $u^\theta = 0 = \bar{u}^\theta$, so that $\delta u^\theta = 0$. Let us introduce the first-order quantities δE and δL such that the 4-velocity components u^α can be written exactly in the same form as those of the background (5) with the replacement $\bar{E} \to \bar{E} + \delta E$ and $\bar{L} \to \bar{L} + \delta L$, implying that

$$\begin{aligned}\delta u^t &= \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\right]\frac{\delta E}{r^2} + \left[1 - \frac{r^2+a^2}{\Delta}\right]\frac{a}{r^2}\delta L,\\ \delta u^\phi &= \left[\frac{r^2+a^2}{\Delta} - 1\right]\frac{a}{r^2}\delta E + \left[1 - \frac{a^2}{\Delta}\right]\frac{\delta L}{r^2},\end{aligned} \tag{24}$$

which can in turn be inverted to yield $\delta E = -\bar{g}_{t\alpha}\delta u^\alpha$ and $\delta L = \bar{g}_{\phi\alpha}\delta u^\alpha$. The correction δu^r to the radial component of the 4-velocity directly follows from the normalization condition of u $(u \cdot u = -1)$ with respect to the perturbed metric, which reads

$$\bar{g}_{rr}\bar{u}^r\delta u^r = \bar{u}^t\delta E - \bar{u}^\phi\delta L - \frac{1}{2}h_{00}, \tag{25}$$

where $h_{00} = h_{\alpha\beta}\bar{u}^\alpha\bar{u}^\beta$. Equivalently, one can normalize u with respect to the background metric as in the work of Barack and Sago [32] (a hat denoting the corresponding quantities), implying

$$\delta u^\alpha = \hat{\delta}u^\alpha + \frac{1}{2}h_{00}\bar{u}^\alpha, \tag{26}$$

leading to the relations

$$\begin{aligned}\hat{\delta}E &= \delta E - \frac{1}{2}\bar{E}h_{00},\\ \hat{\delta}u^r &= \delta u^r - \frac{1}{2}\bar{u}^r h_{00},\\ \hat{\delta}L &= \delta L - \frac{1}{2}\bar{L}h_{00},\end{aligned} \tag{27}$$

with

$$\bar{g}_{rr}\bar{u}^r\hat{\delta}u^r = \bar{u}^t\hat{\delta}E - \bar{u}^\phi\hat{\delta}L. \tag{28}$$

The geodesic equations

$$\frac{du_\alpha}{d\tau} - \frac{1}{2}(\bar{g}_{\lambda\mu,\alpha} + h_{\lambda\mu,\alpha})u^\lambda u^\mu = 0, \tag{29}$$

with

$$u_\alpha = \bar{u}_\alpha + h_{0\alpha} + \bar{g}_{\alpha\beta}\delta u^\beta, \tag{30}$$

determine the evolution of δu_t and δu_ϕ, or equivalently of the perturbations in energy $\hat{\delta}E$ and angular momentum $\hat{\delta}L$ by

$$\frac{d}{d\tau}\hat{\delta}E = -F_t, \qquad \frac{d}{d\tau}\hat{\delta}L = F_\phi, \tag{31}$$

where the functions F_t and F_ϕ are the covariant t and ϕ components of the self-force

$$F^\mu = -\frac{1}{2}(\bar{g}^{\mu\nu} + \bar{u}^\mu\bar{u}^\nu)\bar{u}^\lambda\bar{u}^\rho(2h_{\nu\lambda;\rho} - h_{\lambda\rho;\nu}) \equiv -\frac{1}{2}P(\bar{u})^{\mu\nu}\bar{u}^\lambda\bar{u}^\rho h_{\{\nu\lambda;\rho\}_-}, \tag{32}$$

the anticyclic permutation notation $A_{\{abc\}_-} = A_{abc} - A_{bca} + A_{cab}$ having been introduced. Here we are interested in conservative effects only; i.e., we assume that $F^\alpha = F^\alpha_{\rm cons}$ results in a periodic function of χ. Equations (31) can then be formally integrated as

$$\begin{aligned}\hat{\delta}E(\chi) &= -\int_0^\chi F_t^{\rm cons}(\chi)\frac{d\tau}{d\chi}d\chi + \hat{\delta}E(0) \\ &\equiv \mathcal{E}(\chi) + \hat{\delta}E(0), \\ \hat{\delta}L(\chi) &= \int_0^\chi F_\phi^{\rm cons}(\chi)\frac{d\tau}{d\chi}d\chi + \hat{\delta}L(0) \\ &\equiv \mathcal{L}(\chi) + \hat{\delta}L(0),\end{aligned} \tag{33}$$

where the conservative SF components are defined by $F_t^{\rm cons} = [F_t(\chi) - F_t(-\chi)]/2$ and $F_\phi^{\rm cons} = [F_\phi(\chi) - F_\phi(-\chi)]/2$. The integration constants $\hat{\delta}E(0)$ and $\hat{\delta}L(0)$ are computed by imposing the vanishing of $\hat{\delta}u^r$ both at the periastron ($\chi = 0$) and the apastron ($\chi = \pi$), i.e.,

$$\begin{aligned}0 &= \bar{u}^t(0)\hat{\delta}E(0) - \bar{u}^\phi(0)\hat{\delta}L(0), \\ 0 &= \bar{u}^t(\pi)\hat{\delta}E(\pi) - \bar{u}^\phi(\pi)\hat{\delta}L(\pi),\end{aligned} \tag{34}$$

from Eq. (28), leading to

$$\begin{aligned}\hat{\delta}E(0) &= -\bar{u}^\phi(0)\frac{[-\bar{u}^t(\pi)\mathcal{E}(\pi) + \bar{u}^\phi(\pi)\mathcal{L}(\pi)]}{S(0,\pi)} \\ \hat{\delta}L(0) &= -\bar{u}^t(0)\frac{[-\bar{u}^t(\pi)\mathcal{E}(\pi) + \bar{u}^\phi(\pi)\mathcal{L}(\pi)]}{S(0,\pi)},\end{aligned} \tag{35}$$

where $S(0,\pi) = \bar{u}^t(0)\bar{u}^\phi(\pi) - \bar{u}^t(\pi)\bar{u}^\phi(0)$.

B. GSF corrections to the spin-precession invariant

The spin precession has been calculated in Ref. [25] with respect to a suitably defined perturbed Marck-type frame $\{u, e_a\}$ adapted to u, with $e_a^\alpha = \bar{e}_a^\alpha + \delta e_a^\alpha$, with $\delta e_a^\alpha = O(h)$. The first-order SF correction $\Delta\psi$ to the spin-precession invariant (21) is expressed in terms of the corresponding correction $\Delta\Psi$ to the amount of precession angle accumulated by the spin vector over one radial period defined by Eq. (22), where

$$\delta\Psi = \int_0^{2\pi}\left(\frac{\delta\omega}{\bar{\omega}} - \frac{\delta u^r}{\bar{u}^r}\right)\bar{\omega}\frac{d\bar{\tau}}{d\chi}d\chi, \tag{36}$$

whereas the SF corrections to the frequencies are given by

$$\delta\Omega_r = -\bar{\Omega}_r\frac{\delta T_r}{\bar{T}_r}, \qquad \delta\Omega_\phi = -\bar{\Omega}_\phi\left(-\frac{\delta\Phi}{\bar{\Phi}} + \frac{\delta T_r}{\bar{T}_r}\right), \tag{37}$$

with

$$\begin{aligned}\delta T_r &= \int_0^{2\pi}\left(\frac{\delta u^t}{\bar{u}^t} - \frac{\delta u^r}{\bar{u}^r}\right)\bar{u}^t\frac{d\bar{\tau}}{d\chi}d\chi, \\ \delta\Phi &= \int_0^{2\pi}\left(\frac{\delta u^\phi}{\bar{u}^\phi} - \frac{\delta u^r}{\bar{u}^r}\right)\bar{u}^\phi\frac{d\bar{\tau}}{d\chi}d\chi.\end{aligned} \tag{38}$$

The quantity $\delta\omega$ is defined in Eq. (3.20) of Ref. [25]. It can be conveniently rewritten as

$$\hat{\delta}\omega = \delta\Gamma_{[31]0} + c_{01}\bar{\mathcal{R}}_{11,3} + c_{03}\bar{\mathcal{R}}_{13,3}, \tag{39}$$

where $\hat{\delta}\omega = \delta\omega - \frac{1}{2}\bar{\omega}h_{00}$,

$$\begin{aligned}c_{01} &= -\frac{\hat{\delta}u^r\bar{e}_3^\phi - \bar{e}_3^r\hat{\delta}u^\phi}{\bar{e}_1^\phi\bar{e}_3^r - \bar{e}_3^\phi\bar{e}_1^r}, \\ c_{03} &= \frac{-\hat{\delta}u^\phi\bar{e}_1^r + \bar{e}_1^\phi\hat{\delta}u^r}{\bar{e}_1^\phi\bar{e}_3^r - \bar{e}_3^\phi\bar{e}_1^r},\end{aligned} \tag{40}$$

and

$$\begin{aligned}\bar{\mathcal{R}}_{11,3} &= \frac{x\sqrt{r^2+x^2}}{r\bar{u}_r}\left(\frac{m_2}{r^3} - \bar{\omega}^2\right), \\ \bar{\mathcal{R}}_{13,3} &= \frac{E + \bar{\omega}x}{\sqrt{r^2+x^2}}\end{aligned} \tag{41}$$

are the Ricci rotation coefficients of the background frame

$$\bar{\mathcal{R}}_{\beta\alpha,\sigma} = \bar{e}_\sigma \cdot_{\bar{g}} \nabla_{\bar{e}_\alpha}\bar{e}_\beta. \tag{42}$$

Finally, the quantity $\delta\Gamma_{[31]0}$ is explicitly given in Appendix B of Ref. [25] in terms of the components of the metric perturbation and their first derivatives.

IV. SELF-FORCE CALCULATION

The method for obtaining the first-order metric perturbations of a Kerr spacetime by using the Teukolsky formalism in a radiation gauge is well established in the literature (see, e.g., Refs. [33,34]). The radiative part of the metric perturbation can be reconstructed from a scalar function, the Hertz potential, through the Chrzanowski-Cohen-Kegeles (CKK) procedure [35–37].

The nonradiative modes, instead, cannot be determined by the Teukolsky equation, since the spheroidal harmonics are not defined for $l < 2$, in contrast with the Schwarzschild case, where these lower multipoles can be expressed appropriately in terms of spherical harmonics using the Regge-Wheeler-Zerilli formalism [38,39]. The remaining part must be stationary and axially symmetric, simply corresponding to mass and angular momentum perturbations of the Kerr background in the vacuum region away from the particle's location, up to gauge modes [40]. Therefore, it must be computed separately. The issue of metric completion for bound orbits around a Kerr black hole has been addressed in Refs. [41,42], whereas the problem of gauge-smoothing of the perturbation across the particle position has been discussed in Ref. [43].

The procedure recalled above has been already applied to the computation of the corrections to the gyroscope precession along eccentric orbits in a Schwarzschild spacetime in Refs. [22,23] and for circular orbits in the same Kerr case in Ref. [26]. Therefore, we refer to these works for a detailed account of all the intermediate steps, including the subtleties concerning the regularization technique (see Sec. III E of Ref. [22] and Sec. III B of Ref. [26]) as well as the completion of the metric perturbation (see Sec. III C of Ref. [26]). We provide below only the relevant information about the nonradiative multipoles and the regularization parameter used in our analysis.

We first calculate the radiative part of the perturbation by using the CKK procedure, thereby constructing the quantities $\Delta\psi^{l,-}$ and $\Delta\psi^{l,+}$ (for $l \geq 2$) in the spacetime regions inside (left, $-$) and outside (right, $+$) the particle's location, respectively, with the property $\Delta\psi^{l,+} = \Delta\psi^{-l-1,-}$. We then add the contribution of the lowest modes $l = 0, 1$, which takes into account both the change in mass and angular momentum induced by the particle and the gauge-fixing of the completion piece, leading to

$$
\begin{aligned}
\Delta\psi^-_{l=0,1} = &-\frac{(-4+42u_p-121u_p^2+98u_p^3)}{(-1+3u_p)(4-39u_p+86u_p^2)}u_p \\
&+\frac{(24-558u_p+5281u_p^2-26410u_p^3+75061u_p^4-116396u_p^5+76996u_p^6)}{(-1+3u_p)^2(4-39u_p+86u_p^2)^2}u_p^{3/2}\hat{a} \\
&+\frac{u_p^2}{(-1+3u_p)^3(4-39u_p+86u_p^2)^3}(-32+1296u_p-22098u_p^2+208882u_p^3-1208315u_p^4+4451526u_p^5 \\
&-10533213u_p^6+15691174u_p^7-13712636u_p^8+5540680u_p^9)\hat{a}^2 \\
&+\left\{\frac{1}{2}\frac{(-40+1088u_p-12307u_p^2+75418u_p^3-273210u_p^4+594423u_p^5-732436u_p^6+400092u_p^7)}{(-1+3u_p)^2(4-39u_p+86u_p^2)^2(-1+2u_p)(-1+6u_p)}u_p^2\right. \\
&-\frac{1}{4}\frac{u_p^{5/2}}{(-1+3u_p)^3(4-39u_p+86u_p^2)^3(-1+2u_p)(-1+6u_p)^2}(-672+29032u_p-564560u_p^2 \\
&+6550123u_p^3-50644291u_p^4+275566031u_p^5-1082164705u_p^6+3075978930u_p^7 \\
&-6199296104u_p^8+8404397408u_p^9-6854125200u_p^{10}+2528579232u_p^{11})\hat{a} \\
&-\frac{1}{4}\frac{u_p^3}{(-1+3u_p)^4(4-39u_p+86u_p^2)^4(-1+2u_p)^2(-1+6u_p)^3}(-384-128u_p+703000u_p^2 \\
&-24934708u_p^3+457010141u_p^4-5436136756u_p^5+45820221810u_p^6-286235945992u_p^7 \\
&+1358692923261u_p^8-4964206286808u_p^9+14012232755836u_p^{10}-30394437681256u_p^{11} \\
&+49854730541696u_p^{12}-59906488552896u_p^{13}+49754497440960u_p^{14} \\
&\left.-25484985853056u_p^{15}+6047210836224u_p^{16})\hat{a}^2\right\}e^2+O(\hat{a}^3,e^4), \qquad (43)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\psi^{+}_{l=0,1} ={}& -\frac{(-4+36u_p-75u_p^2+14u_p^3)}{(-1+3u_p)(4-39u_p+86u_p^2)}u_p \\
&+\frac{(24-486u_p+3959u_p^2-16766u_p^3+40403u_p^4-55220u_p^5+34588u_p^6)}{(-1+3u_p)^2(4-39u_p+86u_p^2)^2}u_p^{3/2}\hat{a} \\
&+\frac{u_p^2}{(-1+3u_p)^3(4-39u_p+86u_p^2)^3}(-32+1584u_p-29674u_p^2+297266u_p^3-1801019u_p^4 \\
&+6933366u_p^5-17148037u_p^6+26630422u_p^7-23983868u_p^8+9746632u_p^9)\hat{a}^2 \\
&+\left\{-\frac{1}{2}\frac{(-8+256u_p-3533u_p^2+26468u_p^3-112802u_p^4+265855u_p^5-310004u_p^6+128604u_p^7)}{(-1+3u_p)^2(4-39u_p+86u_p^2)^2(-1+2u_p)(-1+6u_p)}u_p^2\right. \\
&+\frac{1}{4}\frac{u_p^{5/2}}{(-1+3u_p)^3(4-39u_p+86u_p^2)^3(-1+2u_p)(-1+6u_p)^2}(-480+22136u_p-463312u_p^2 \\
&+5780997u_p^3-47514985u_p^4+268657357u_p^5-1061333235u_p^6+2919149266u_p^7 \\
&-5467300616u_p^8+6641478880u_p^9-4724705136u_p^{10}+1503019296u_p^{11})\hat{a} \\
&-\frac{1}{4}\frac{u_p^3}{(-1+3u_p)^4(4-39u_p+86u_p^2)^4(-1+2u_p)^2(-1+6u_p)^3}(4224-272768u_p+8278968u_p^2 \\
&-156420964u_p^3+2054426153u_p^4-19850358252u_p^5+145756161682u_p^6-829041253480u_p^7 \\
&+3691145926841u_p^8-12912460620256u_p^9+35399969986324u_p^{10}-75321362601016u_p^{11} \\
&+122013289648128u_p^{12}-145497047785152u_p^{13}+120396791624256u_p^{14} \\
&\left.-61694700700032u_p^{15}+14720859668736u_p^{16})\hat{a}^2\right\}e^2+O(\hat{a}^3,e^4), \qquad (44)
\end{aligned}
$$

respectively.

The regularization procedure finally leads to

$$\Delta\psi=\sum_{\ell=0}^{\infty}(\langle\Delta\psi^l\rangle-B), \qquad (45)$$

where the subtraction term B denotes the PN-expanded large-l limit of the left-right average

$$\langle\Delta\psi^l\rangle=\frac{1}{2}(\Delta\psi^{l,+}+\Delta\psi^{l,-}). \qquad (46)$$

We find

$$B(u_p,e,\hat{a})=B_0(u_p,\hat{a})+e^2B_2(u_p,\hat{a})+O(e^4), \qquad (47)$$

with

$$
\begin{aligned}
B_0(u_p,\hat a) &= \frac{21}{16}u_p - \frac{201}{128}u_p^2 + \frac{529}{1024}u_p^3 + \frac{152197}{16384}u_p^4 + \frac{17145445}{262144}u_p^5 + \frac{886692225}{2097152}u_p^6 + \frac{45206277105}{16777216}u_p^7 \\
&+ \frac{9204713714385}{536870912}u_p^8 + \frac{1875482334818445}{17179869184}u_p^9 \\
&+ \left(-\frac{11}{16}u_p^{3/2} + \frac{19}{128}u_p^{5/2} - \frac{3187}{1024}u_p^{7/2} - \frac{897011}{16384}u_p^{9/2} - \frac{119529091}{262144}u_p^{11/2} - \frac{7322895475}{2097152}u_p^{13/2}\right. \\
&\left. - \frac{434363072475}{16777216}u_p^{15/2} - \frac{101048547627615}{536870912}u_p^{17/2} - \frac{23167337673070755}{17179869184}u_p^{19/2}\right)\hat a \\
&+ \left(\frac{1}{4}u_p^2 - \frac{117}{128}u_p^3 + \frac{707}{64}u_p^4 + \frac{2138193}{16384}u_p^5 + \frac{91049009}{65536}u_p^6 + \frac{27330703781}{2097152}u_p^7\right. \\
&\left. + \frac{15024987805}{131072}u_p^8 + \frac{517781575013205}{536870912}u_p^9\right)\hat a^2 + O(\hat a^3, u_p^{10}),
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
B_2(u_p,\hat a) &= -\frac{435}{512}u_p^2 - \frac{1155}{1024}u_p^3 - \frac{352849}{65536}u_p^4 - \frac{5100243}{131072}u_p^5 - \frac{2456459237}{8388608}u_p^6 - \frac{36003649389}{16777216}u_p^7 \\
&- \frac{32713771158557}{2147483648}u_p^8 - \frac{451723973383879}{4294967296}u_p^9 \\
&+ \left(\frac{605}{512}u_p^{5/2} + \frac{957}{256}u_p^{7/2} + \frac{558367}{65536}u_p^{9/2} + \frac{11521973}{65536}u_p^{11/2} + \frac{17120777051}{8388608}u_p^{13/2}\right. \\
&\left. + \frac{80347197891}{4194304}u_p^{15/2} + \frac{347929041937283}{2147483648}u_p^{17/2} + \frac{2760514246569789}{2147483648}u_p^{19/2}\right)\hat a \\
&+ \left(-\frac{523}{512}u_p^3 + \frac{843}{256}u_p^4 + \frac{892255}{65536}u_p^5 - \frac{5413651}{16384}u_p^6 - \frac{55709086485}{8388608}u_p^7\right. \\
&\left. - \frac{337847858229}{4194304}u_p^8 - \frac{1751794928899397}{2147483648}u_p^9\right)\hat a^2 + O(\hat a^3, u_p^{10}).
\end{aligned}
\tag{49}
$$

A. Results

Our final result for the spin-precession invariant $\Delta\psi(u_p, e, \hat a)$ reads

$$
\begin{aligned}
\Delta\psi(u_p,e,\hat a) &= \sum_{i,j=0}^{\infty} e^i \hat a^j \Delta\psi^{(e^i,a^j)}(u_p) \\
&= \Delta\psi^{(e^0,a^0)} + e^2\Delta\psi^{(e^2,a^0)} + \hat a\Delta\psi^{(e^0,a^1)} + \hat a^2\Delta\psi^{(e^0,a^2)} + \ldots + e^2\hat a\Delta\psi^{(e^2,a^1)} + e^2\hat a^2\Delta\psi^{(e^2,a^2)} + \ldots.
\end{aligned}
\tag{50}
$$

The spin-independent part has been computed in Refs. [22,23] up to the 9PN level, which we raise here to 9.5PN. The new terms are

$$
\Delta\psi^{(e^0,a^0)} = \Delta\psi^{(e^0,a^0)}|_{\text{Ref.[23]}} + \Delta\psi^{(e^0,a^0)}_{9.5\text{PN}}, \qquad \Delta\psi^{(e^2,a^0)} = \Delta\psi^{(e^2,a^0)}|_{\text{Ref.[23]}} + \Delta\psi^{(e^2,a^0)}_{9.5\text{PN}}, \tag{51}
$$

with

$$\Delta\psi^{(e^0,a^0)}_{9.5\mathrm{PN}} = \left(-\frac{313011924344499619464 7}{11453592870720000}-\frac{180728953}{11025}\pi^2+\frac{23055449891}{385875}\gamma+\frac{316521883}{15435}\ln(2)\right.$$
$$\left.+\frac{6854694417}{85750}\ln(3)+\frac{23055449891}{771750}\ln(u_p)\right)\pi u_p^{19/2},$$
$$\Delta\psi^{(e^2,a^0)}_{9.5\mathrm{PN}} = \left(-\frac{3527414571498810162815 57}{61085828643840000}-\frac{53999919103}{176400}\pi^2+\frac{19991310125293}{18522000}\gamma+\frac{264647617121}{35280}\ln(2)\right.$$
$$\left.-\frac{1983214856673}{1372000}\ln(3)-\frac{206298828125}{296352}\ln(5)+\frac{19991310125293}{37044000}\ln(u_p)\right)\pi u_p^{19/2}. \quad (52)$$

The zero-eccentricity spin-dependent terms are given by

$$\Delta\psi^{(e^0,a^1)} = C_{1.5}^{(e^0,a^1),\mathrm{c}}u_p^{3/2}+C_{2.5}^{(e^0,a^1),\mathrm{c}}u_p^{5/2}+C_{3.5}^{(e^0,a^1),\mathrm{c}}u_p^{7/2}+(C_{4.5}^{(e^0,a^1),\mathrm{c}}+C_{4.5}^{(e^0,a^1),\ln}\ln(u_p))u_p^{9/2}$$
$$+(C_{5.5}^{(e^0,a^1),\mathrm{c}}+C_{5.5}^{(e^0,a^1),\ln}\ln(u_p))u_p^{11/2}+C_{6}^{(e^0,a^1),\mathrm{c}}u_p^{6}+(C_{6.5}^{(e^0,a^1),\mathrm{c}}+C_{6.5}^{(e^0,a^1)\,\ln}\ln(u_p))u_p^{13/2}$$
$$+C_{7}^{(e^0,a^1),\mathrm{c}}u_p^{7}+(C_{7.5}^{(e^0,a^1),\mathrm{c}}+C_{7.5}^{(e^0,a^1),\ln}\ln(u_p)+C_{7.5}^{(e^0,a^1),\ln^2}\ln(u_p)^2)u_p^{15/2}+C_{8}^{(e^0,a^1),\mathrm{c}}u_p^{8}$$
$$+(C_{8.5}^{(e^0,a^1),\mathrm{c}}+C_{8.5}^{(e^0,a^1),\ln}\ln(u_p)+C_{8.5}^{(e^0,a^1),\ln^2}\ln(u_p)^2)u_p^{17/2}+(C_{9}^{(e^0,a^1),\mathrm{c}}+C_{9}^{(e^0,a^1),\ln}\ln(u_p))u_p^{9}$$
$$+(C_{9.5}^{(e^0,a^1),\mathrm{c}}+C_{9.5}^{(e^0,a^1),\ln}\ln(u_p)+C_{9.5}^{(e^0,a^1),\ln^2}\ln(u_p)^2)u_p^{19/2}+O_{\ln}(u_p^{10}), \quad (53)$$

with

$$C_{1.5}^{(e^0,a^1),\mathrm{c}}=-\frac{1}{2},\qquad C_{2.5}^{(e^0,a^1),\mathrm{c}}=-\frac{41}{8},\qquad C_{3.5}^{(e^0,a^1),\mathrm{c}}=\frac{237}{32}-\frac{123}{64}\pi^2,$$
$$C_{4.5}^{(e^0,a^1),\mathrm{c}}=-\frac{2580077}{5760}+\frac{52225}{6144}\pi^2+\frac{1256}{15}\gamma+\frac{296}{15}\ln(2)+\frac{729}{5}\ln(3),\qquad C_{4.5}^{(e^0,a^1),\ln}=\frac{628}{15},$$
$$C_{5.5}^{(e^0,a^1),\mathrm{c}}=-\frac{371061}{140}\ln(3)+\frac{16521221}{24576}\pi^2-\frac{653849867}{115200}+\frac{20186}{35}\ln(2)-\frac{131234}{105}\gamma,$$
$$C_{5.5}^{(e^0,a^1),\ln}=-\frac{65617}{105},\qquad C_{6}^{(e^0,a^1),\mathrm{c}}=\frac{49969}{315}\pi,$$
$$C_{6.5}^{(e^0,a^1),\mathrm{c}}=-\frac{34667196284353}{203212800}+\frac{43396897187}{2359296}\pi^2+\frac{4274383}{1890}\gamma-\frac{5127317}{378}\ln(2)+\frac{602397}{70}\ln(3)$$
$$+\frac{9765625}{9072}\ln(5)-\frac{7335303}{131072}\pi^4,$$
$$C_{6.5}^{(e^0,a^1),\ln}=\frac{4111087}{3780},\qquad C_{7}^{(e^0,a^1),\mathrm{c}}=-\frac{17884343}{6300}\pi,$$
$$C_{7.5}^{(e^0,a^1),\mathrm{c}}=-\frac{530755103526042557}{521579520000}+\frac{138120741638137}{1238630400}\pi^2+\frac{1796383502593}{43659000}\gamma+\frac{7478658446233}{43659000}\ln(2)$$
$$+\frac{60948732447}{8624000}\ln(3)-\frac{2216796875}{72576}\ln(5)-\frac{1951932086423}{1006632960}\pi^4+\frac{63488}{15}\zeta(3)-\frac{3396608}{1575}\gamma^2$$
$$-\frac{5149696}{1575}\gamma\ln(2)-\frac{936036}{175}\gamma\ln(3)-\frac{931328}{1575}\ln(2)^2-\frac{936036}{175}\ln(3)\ln(2)-\frac{468018}{175}\ln(3)^2,$$
$$C_{7.5}^{(e^0,a^1),\ln}=\frac{1781539442593}{87318000}-\frac{3396608}{1575}\gamma-\frac{2574848}{1575}\ln(2)-\frac{468018}{175}\ln(3),\qquad C_{7.5}^{(e^0,a^1),\ln^2}=-\frac{849152}{1575},$$
$$C_{8}^{(e^0,a^1),\mathrm{c}}=\frac{27936275503}{2910600}\pi,$$

$$
\begin{aligned}
C_{8.5}^{(e^0,a^1),\mathrm{c}} &= -\frac{6812666517192145626 49}{11277383749632000}+\frac{427384464568822843}{1109812838400}\pi^2-\frac{5947119623686361}{15891876000}\gamma \\
&\quad -\frac{5484803096524561}{15891876000}\ln(2)-\frac{2417456672510751}{3139136000}\ln(3)+\frac{72943791015625}{290594304}\ln(5) \\
&\quad +\frac{678223072849}{92664000}\ln(7)-\frac{466989768838667}{12884901888}\pi^4-\frac{6177152}{105}\zeta(3)+\frac{46649968}{1225}\gamma^2+\frac{30413792}{1225}\gamma\ln(2) \\
&\quad +\frac{137032317}{1225}\gamma\ln(3)-\frac{599506832}{11025}\ln(2)^2+\frac{137032317}{1225}\ln(3)\ln(2)+\frac{137032317}{2450}\ln(3)^2, \\
C_{8.5}^{(e^0,a^1),\ln} &= -\frac{5994748484165561}{31783752000}+\frac{46649968}{1225}\gamma+\frac{15206896}{1225}\ln(2)+\frac{137032317}{2450}\ln(3), \qquad C_{8.5}^{(e^0,a^1),\ln^2}=\frac{11662492}{1225}, \\
C_{9}^{(e^0,a^1),\mathrm{c}} &= \frac{116182866505170823}{2097727632000}\pi+\frac{10999172}{4725}\pi^3-\frac{1176911404}{165375}\pi\gamma-\frac{1001054764}{165375}\pi\ln(2)-\frac{50077926}{6125}\pi\ln(3), \\
C_{9}^{(e^0,a^1),\ln} &= -\frac{588455702}{165375}\pi, \\
C_{9.5}^{(e^0,a^1),\mathrm{c}} &= \frac{20916670104789977714594187495 1}{290279857715527680000}+\frac{458671024959899491}{775107379200}\pi^2+\frac{9661311739976843}{419545526400}\gamma \\
&\quad -\frac{50685948367874925289}{18879548688000}\ln(2)+\frac{69137888408251731}{27624396800}\ln(3)-\frac{4945799701746484375}{7249746696192}\ln(5) \\
&\quad -\frac{102411684000199}{370656000}\ln(7)-\frac{4917788235174981898 3}{6597069766656}\pi^4+\frac{1719608}{63}\zeta(3)-\frac{131422044664}{1091475}\gamma^2 \\
&\quad +\frac{1715126780656}{3274425}\gamma\ln(2)-\frac{7539716034}{13475}\gamma\ln(3)-\frac{76708984375}{1571724}\gamma\ln(5)+\frac{14049792489496}{9823275}\ln(2)^2 \\
&\quad -\frac{7539716034}{13475}\ln(3)\ln(2)-\frac{76708984375}{1571724}\ln(2)\ln(5)-\frac{3769858017}{13475}\ln(3)^2 \\
&\quad -\frac{76708984375}{3143448}\ln(5)^2+\frac{128148402261}{67108864}\pi^6, \\
C_{9.5}^{(e^0,a^1),\ln} &= -\frac{4238891698612841}{4195455264000}-\frac{131422044664}{1091475}\gamma+\frac{863777837336}{3274425}\ln(2)-\frac{3769858017}{13475}\ln(3)-\frac{76708984375}{3143448}\ln(5), \\
C_{9.5}^{(e^0,a^1),\ln^2} &= -\frac{31819769998}{1091475},
\end{aligned} \tag{54}
$$

and

$$
\begin{aligned}
\Delta\psi^{(e^0,a^2)} &= C_2^{(e^0,a^2),\mathrm{c}}u_p^2+C_3^{(e^0,a^2),\mathrm{c}}u_p^3+C_4^{(e^0,a^2),\mathrm{c}}u_p^4+(C_5^{(e^0,a^2),\mathrm{c}}+C_5^{(e^0,a^2),\ln}\ln(u_p))u_p^5+(C_6^{(e^0,a^2),\mathrm{c}}+C_6^{(e^0,a^2),\ln}\ln(u_p))u_p^6 \\
&\quad +C_{6.5}^{(e^0,a^2),\mathrm{c}}u_p^{13/2}+(C_7^{(e^0,a^2),\mathrm{c}}+C_7^{(e^0,a^2),\ln}\ln(u_p))u_p^7+C_{7.5}^{(e^0,a^2),\mathrm{c}}u_p^{15/2}+(C_8^{(e^0,a^2),\mathrm{c}}+C_8^{(e^0,a^2),\ln}\ln(u_p) \\
&\quad +C_8^{(e^0,a^2),\ln^2}\ln(u_p)^2)u_p^8+C_{8.5}^{(e^0,a^2),\mathrm{c}}u_p^{17/2}+(C_9^{(e^0,a^2),\mathrm{c}}+C_9^{(e^0,a^2),\ln}\ln(u_p)+C_9^{(e^0,a^2),\ln^2}\ln(u_p)^2)u_p^9 \\
&\quad +(C_{9.5}^{(e^0,a^2),\mathrm{c}}+C_{9.5}^{(e^0,a^2),\ln}\ln(u_p))u_p^{19/2}+O_{\ln}(u_p^{10}),
\end{aligned} \tag{55}
$$

with

$$
\begin{aligned}
C_2^{(e^0,a^2),\rm c} &= -1, \qquad C_3^{(e^0,a^2),\rm c} = \frac{15}{4}, \qquad C_4^{(e^0,a^2),\rm c} = \frac{843}{16} - \frac{123}{64}\pi^2,\\
C_5^{(e^0,a^2),\rm c} &= -\frac{41161}{2880} + \frac{5155}{1536}\pi^2 + \frac{1256}{15}\gamma + \frac{296}{15}\ln(2) + \frac{729}{5}\ln(3), \qquad C_5^{(e^0,a^2),\rm ln} = \frac{628}{15},\\
C_6^{(e^0,a^2),\rm c} &= -\frac{198163141}{57600} + \frac{14769449}{24576}\pi^2 - \frac{32484}{35}\gamma + \frac{53012}{105}\ln(2) - \frac{68526}{35}\ln(3), \qquad C_6^{(e^0,a^2),\rm ln} = -\frac{16242}{35},\\
C_{6.5}^{(e^0,a^2),\rm c} &= \frac{49969}{315}\pi,\\
C_7^{(e^0,a^2),\rm c} &= -\frac{31986710669261}{101606400} + \frac{79045202729}{2359296}\pi^2 + \frac{8403943}{1890}\gamma - \frac{35370913}{1890}\ln(2) + \frac{5276259}{280}\ln(3)\\
&\quad + \frac{9765625}{9072}\ln(5) - \frac{7335303}{131072}\pi^4, \qquad C_7^{(e^0,a^2),\rm ln} = \frac{8240647}{3780},\\
C_{7.5}^{(e^0,a^2),\rm c} &= -\frac{113991}{50}\pi,\\
C_8^{(e^0,a^2),\rm c} &= -\frac{1336810434105217691}{260789760000} + \frac{876841593090859}{1651507200}\pi^2 + \frac{24691487069}{606375}\gamma + \frac{241419814667}{1819125}\ln(2)\\
&\quad + \frac{367133665347}{8624000}\ln(3) - \frac{1689453125}{72576}\ln(5) - \frac{222475429201}{125829120}\pi^4 + \frac{89552}{15}\zeta(3) - \frac{3396608}{1575}\gamma^2\\
&\quad - \frac{5149696}{1575}\gamma\ln(2) - \frac{936036}{175}\gamma\ln(3) - \frac{931328}{1575}\ln(2)^2 - \frac{936036}{175}\ln(3)\ln(2) - \frac{468018}{175}\ln(3)^2,\\
C_8^{(e^0,a^2),\rm ln} &= \frac{26855194769}{1212750} - \frac{3396608}{1575}\gamma - \frac{2574848}{1575}\ln(2) - \frac{468018}{175}\ln(3), \qquad C_8^{(e^0,a^2),\rm ln^2} = -\frac{849152}{1575},\\
C_{8.5}^{(e^0,a^2),\rm c} &= \frac{2454668003}{138600}\pi,\\
C_9^{(e^0,a^2),\rm c} &= -\frac{5769163510952082588693 53}{28193459374080000} + \frac{1743178610265080953}{554906419200}\pi^2 - \frac{4986393924529319}{15891876000}\gamma\\
&\quad - \frac{766147036782487}{635675040}\ln(2) - \frac{208971977923053}{1569568000}\ln(3) + \frac{23691548828125}{72648576}\ln(5) + \frac{678223072849}{92664000}\ln(7)\\
&\quad - \frac{2174958156794893}{21474836480}\pi^4 - \frac{290740}{7}\zeta(3) + \frac{344032936}{11025}\gamma^2 + \frac{192844112}{11025}\gamma\ln(2) + \frac{112461372}{1225}\gamma\ln(3)\\
&\quad - \frac{569193304}{11025}\ln(2)^2 + \frac{112461372}{1225}\ln(3)\ln(2) + \frac{56230686}{1225}\ln(3)^2,\\
C_9^{(e^0,a^2),\rm ln} &= -\frac{4888595168274119}{31783752000} + \frac{344032936}{11025}\gamma + \frac{96422056}{11025}\ln(2) + \frac{56230686}{1225}\ln(3), \qquad C_9^{(e^0,a^2),\rm ln^2} = \frac{86008234}{11025},\\
C_{9.5}^{(e^0,a^2),\rm c} &= \frac{69193216974146623}{2097727632000}\pi + \frac{10999172}{4725}\pi^3 - \frac{1176911404}{165375}\pi\gamma - \frac{1001054764}{165375}\pi\ln(2) - \frac{50077926}{6125}\pi\ln(3),\\
C_{9.5}^{(e^0,a^2),\rm ln} &= -\frac{588455702}{165375}\pi.
\end{aligned} \tag{56}
$$

Finally, the spin-dependent part mixing eccentricity and spin effects is given by

$$
\begin{aligned}
\Delta\psi^{(e^2,a^1)} &= C_{2.5}^{(e^2,a^1),\rm c}u_p^{5/2} + C_{3.5}^{(e^2,a^1),\rm c}u_p^{7/2} + (C_{4.5}^{(e^2,a^1),\rm c} + C_{4.5}^{(e^2,a^1),\rm ln}\ln(u_p))u_p^{9/2}\\
&\quad + (C_{5.5}^{(e^2,a^1),\rm c} + C_{5.5}^{(e^2,a^1),\rm ln}\ln(u_p))u_p^{11/2} + C_6^{(e^2,a^1),\rm c}u_p^6 + (C_{6.5}^{(e^2,a^1),\rm c} + C_{6.5}^{(e^2,a^1),\rm ln}\ln(u_p))u_p^{13/2}\\
&\quad + C_7^{(e^2,a^1),\rm c}u_p^7 + (C_{7.5}^{(e^2,a^1),\rm c} + C_{7.5}^{(e^2,a^1),\rm ln}\ln(u_p) + C_{7.5}^{(e^2,a^1),\rm ln^2}\ln(u_p)^2)u_p^{15/2} + C_8^{(e^2,a^1),\rm c}u_p^8\\
&\quad + (C_{8.5}^{(e^2,a^1),\rm c} + C_{8.5}^{(e^2,a^1),\rm ln}\ln(u_p) + C_{8.5}^{(e^2,a^1),\rm ln^2}\ln(u_p)^2)u_p^{17/2} + (C_9^{(e^2,a^1),\rm c} + C_9^{(e^2,a^1),\rm ln}\ln(u_p))u_p^9\\
&\quad + (C_{9.5}^{(e^2,a^1),\rm c} + C_{9.5}^{(e^2,a^1),\rm ln}\ln(u_p) + C_{9.5}^{(e^2,a^1),\rm ln^2}\ln(u_p)^2)u_p^{19/2} + O_{\rm ln}(u_p^{10}),
\end{aligned} \tag{57}
$$

with

$$C_{2.5}^{(e^2,a^1),\mathrm{c}} = -\frac{1}{8}, \qquad C_{3.5}^{(e^2,a^1),\mathrm{c}} = -\frac{59}{16} - \frac{123}{256}\pi^2,$$

$$C_{4.5}^{(e^2,a^1),\mathrm{c}} = -\frac{274889}{640} - \frac{39529}{4096}\pi^2 + \frac{536}{5}\gamma + \frac{11720}{3}\ln(2) - \frac{10206}{5}\ln(3), \qquad C_{4.5}^{(e^2,a^1),\ln} = \frac{268}{5},$$

$$C_{5.5}^{(e^2,a^1),\mathrm{c}} = -\frac{47376713}{14400} + \frac{46450919}{49152}\pi^2 - \frac{38026}{15}\gamma - \frac{2049574}{21}\ln(2) + \frac{13574709}{320}\ln(3) + \frac{9765625}{1344}\ln(5),$$

$$C_{5.5}^{(e^2,a^1),\ln} = -\frac{19013}{15}, \qquad C_{6}^{(e^2,a^1),\mathrm{c}} = \frac{319609}{630}\pi,$$

$$C_{6.5}^{(e^2,a^1),\mathrm{c}} = -\frac{19506870722893}{29030400} + \frac{168336760679}{2359296}\pi^2 + \frac{187867}{27}\gamma + \frac{616924811}{945}\ln(2) - \frac{111860433}{1792}\ln(3)$$
$$- \frac{4701171875}{20736}\ln(5) - \frac{146026515}{1048576}\pi^4, \qquad C_{6.5}^{(e^2,a^1),\ln} = \frac{179443}{54},$$

$$C_{7}^{(e^2,a^1),\mathrm{c}} = -\frac{629699771}{47040}\pi,$$

$$C_{7.5}^{(e^2,a^1),\mathrm{c}} = -\frac{6966671370033684457}{782369280000} + \frac{6083404435612271}{6606028800}\pi^2 + \frac{4282750559249}{14553000}\gamma - \frac{23584521073621}{8731800}\ln(2)$$
$$- \frac{271718217011673}{275968000}\ln(3) + \frac{56509677734375}{25546752}\ln(5) + \frac{678223072849}{6082560}\ln(7) - \frac{3390769890109}{335544320}\pi^4$$
$$+ \frac{134944}{5}\zeta(3) - \frac{7219504}{525}\gamma^2 - \frac{79652512}{315}\gamma\ln(2) + \frac{15912612}{175}\gamma\ln(3) - \frac{80263696}{175}\ln(2)^2$$
$$+ \frac{15912612}{175}\ln(3)\ln(2) + \frac{7956306}{175}\ln(3)^2,$$

$$C_{7.5}^{(e^2,a^1),\ln} = \frac{4253184684449}{29106000} - \frac{7219504}{525}\gamma - \frac{39826256}{315}\ln(2) + \frac{7956306}{175}\ln(3), \qquad C_{7.5}^{(e^2,a^1),\ln^2} = -\frac{1804876}{525},$$

$$C_{8}^{(e^2,a^1),\mathrm{c}} = \frac{21791194144427}{279417600}\pi,$$

$$C_{8.5}^{(e^2,a^1),\mathrm{c}} = -\frac{3353144684464231686478 97}{11277383749632000} + \frac{1017319845627088081 3}{2219625676800}\pi^2 - \frac{27456615049027529}{7945938000}\gamma$$
$$- \frac{170592889938737863}{7945938000}\ln(2) + \frac{530521027102134009}{20090470400}\ln(3) - \frac{558244786663203125}{65093124096}\ln(5)$$
$$- \frac{106301099544967201}{23721984000}\ln(7) - \frac{31433138995123013}{257698037760}\pi^4 - \frac{49939936}{105}\zeta(3) + \frac{1202995352}{3675}\gamma^2$$
$$+ \frac{15075112912}{2205}\gamma\ln(2) - \frac{41368207977}{19600}\gamma\ln(3) - \frac{3173828125}{7056}\gamma\ln(5) + \frac{138535911176}{11025}\ln(2)^2$$
$$- \frac{41368207977}{19600}\ln(3)\ln(2) - \frac{3173828125}{7056}\ln(2)\ln(5) - \frac{41368207977}{39200}\ln(3)^2 - \frac{3173828125}{14112}\ln(5)^2,$$

$$C_{8.5}^{(e^2,a^1),\ln} = -\frac{27648022365728129}{15891876000} + \frac{1202995352}{3675}\gamma + \frac{7537556456}{2205}\ln(2) - \frac{41368207977}{39200}\ln(3) - \frac{3173828125}{14112}\ln(5),$$

$$C_{8.5}^{(e^2,a^1),\ln^2} = \frac{300748838}{3675},$$

$$C_{9}^{(e^2,a^1),\mathrm{c}} = \frac{21196624539283901299}{33563642112000}\pi + \frac{123567238}{4725}\pi^3 - \frac{13221694466}{165375}\pi\gamma - \frac{166628746}{315}\pi\ln(2) + \frac{926441631}{6125}\pi\ln(3),$$

$$C_{9}^{(e^2,a^1),\ln} = -\frac{6610847233}{165375}\pi,$$

$$
\begin{aligned}
C_{9.5}^{(e^2,a^1),\mathrm{c}} &= \frac{397392906234932463640834034 21}{6597269493534720000} + \frac{73642835756807424659}{8878502707200}\pi^2 + \frac{151919323439718707}{27243216000}\gamma \\
&+ \frac{82978352154780707393}{699242544000}\ln(2) - \frac{23067310031610488793}{339992576000}\ln(3) - \frac{16664548214107826421875}{231991894278144}\ln(5) \\
&+ \frac{16814208792873724897}{284663808000}\ln(7) - \frac{41216057496684356495 21}{65970697666560}\pi^4 + \frac{27893212}{45}\zeta(3) - \frac{25587350981}{14175}\gamma^2 \\
&- \frac{63046858231586}{1091475}\gamma\ln(2) + \frac{1269422541711}{172480}\gamma\ln(3) + \frac{374473583984375}{25147584}\gamma\ln(5) \\
&- \frac{1074807834943193}{9823275}\ln(2)^2 + \frac{931671919503}{172480}\ln(3)\ln(2) + \frac{374473583984375}{25147584}\ln(2)\ln(5) \\
&+ \frac{1269422541711}{344960}\ln(3)^2 + \frac{374473583984375}{50295168}\ln(5)^2 + \frac{962681186487}{268435456}\pi^6, \\
C_{9.5}^{(e^2,a^1),\ln} &= \frac{145588636751786243}{54486432000} - \frac{25587350981}{14175}\gamma - \frac{1259924060281}{43659}\ln(2) + \frac{1269422541711}{344960}\ln(3) \\
&+ \frac{374473583984375}{50295168}\ln(5), \qquad C_{9.5}^{(e^2,a^1),\ln^2} = -\frac{3561355859}{8100}, \qquad (58)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\psi^{(e^2,a^2)} &= C_3^{(e^2,a^2),\mathrm{c}}u_p^3 + C_4^{(e^2,a^2),\mathrm{c}}u_p^4 + (C_5^{(e^2,a^2),\mathrm{c}} + C_5^{(e^2,a^2),\ln}\ln(u_p))u_p^5 \\
&+ (C_6^{(e^2,a^2),\mathrm{c}} + C_6^{(e^2,a^2),\ln}\ln(u_p))u_p^6 + C_{6.5}^{(e^2,a^2),\mathrm{c}}u_p^{13/2} \\
&+ (C_7^{(e^2,a^2),\mathrm{c}} + C_7^{(e^2,a^2),\ln}\ln(u_p))u_p^7 + C_{7.5}^{(e^2,a^2),\mathrm{c}}u_p^{15/2} \\
&+ (C_8^{(e^2,a^2),\mathrm{c}} + C_8^{(e^2,a^2),\ln}\ln(u_p) + C_8^{(e^2,a^2),\ln^2}\ln(u_p)^2)u_p^8 + C_{8.5}^{(e^2,a^2),\mathrm{c}}u_p^{17/2} \\
&+ (C_9^{(e^2,a^2),\mathrm{c}} + C_9^{(e^2,a^2),\ln}\ln(u_p) + C_9^{(e^2,a^2),\ln^2}\ln(u_p)^2)u_p^9 \\
&+ (C_{9.5}^{(e^2,a^2),\mathrm{c}} + C_{9.5}^{(e^2,a^2),\ln}\ln(u_p))u_p^{19/2} + O_{\ln}(u_p^{10}), \qquad (59)
\end{aligned}
$$

with

$$
\begin{aligned}
C_3^{(e^2,a^2),\mathrm{c}} &= -2, \qquad C_4^{(e^2,a^2),\mathrm{c}} = -\frac{13}{4} - \frac{123}{256}\pi^2, \\
C_5^{(e^2,a^2),\mathrm{c}} &= -\frac{65091}{160} - \frac{22037}{2048}\pi^2 + \frac{536}{5}\gamma + \frac{11720}{3}\ln(2) - \frac{10206}{5}\ln(3), \qquad C_5^{(e^2,a^2),\ln} = \frac{268}{5}, \\
C_6^{(e^2,a^2),\mathrm{c}} &= -\frac{9371747}{3600} + \frac{33970805}{49152}\pi^2 - \frac{31018}{15}\gamma - \frac{8107718}{105}\ln(2) + \frac{10023021}{320}\ln(3) + \frac{9765625}{1344}\ln(5), \\
C_6^{(e^2,a^2),\ln} &= -\frac{15509}{15}, \qquad C_{6.5}^{(e^2,a^2),\mathrm{c}} = \frac{319609}{630}\pi, \\
C_7^{(e^2,a^2),\mathrm{c}} &= -\frac{9278011192573}{7257600} + \frac{308714314565}{2359296}\pi^2 + \frac{18377071}{1890}\gamma + \frac{1961637691}{1890}\ln(2) - \frac{39782259}{112}\ln(3) \\
&- \frac{3302734375}{18144}\ln(5) - \frac{146026515}{1048576}\pi^4, \qquad C_7^{(e^2,a^2),\ln} = \frac{17787391}{3780}, \qquad C_{7.5}^{(e^2,a^2),\mathrm{c}} = -\frac{916708909}{78400}\pi,
\end{aligned}
$$

$$
\begin{aligned}
C_8^{(e^2,a^2),\mathrm{c}} &= -\frac{7515099422720578033}{195592320000}+\frac{2867592560250501}{734003200}\pi^2+\frac{3469177106249}{14553000}\gamma-\frac{19693190989441}{8731800}\ln(2)\\
&\quad-\frac{625767867535473}{275968000}\ln(3)+\frac{3414326171875}{1216512}\ln(5)+\frac{678223072849}{6082560}\ln(7)-\frac{365600352653}{41943040}\pi^4\\
&\quad+\frac{196096}{5}\zeta(3)-\frac{7219504}{525}\gamma^2-\frac{79652512}{315}\gamma\ln(2)+\frac{15912612}{175}\gamma\ln(3)-\frac{80263696}{175}\ln(2)^2\\
&\quad+\frac{15912612}{175}\ln(2)\ln(3)+\frac{7956306}{175}\ln(3)^2,\\
C_8^{(e^2,a^2),\ln} &= \frac{3819871419449}{29106000}-\frac{7219504}{525}\gamma-\frac{39826256}{315}\ln(2)+\frac{7956306}{175}\ln(3), \qquad C_8^{(e^2,a^2),\ln^2}=-\frac{1804876}{525},\\
C_{8.5}^{(e^2,a^2),\mathrm{c}} &= \frac{16093843572391}{139708800}\pi,\\
C_9^{(e^2,a^2),\mathrm{c}} &= -\frac{9376361039961996929927831}{2819345937408000}+\frac{5639207064852757987}{138726604800}\pi^2-\frac{57750252464296303}{15891876000}\gamma\\
&\quad+\frac{217123592856512149}{15891876000}\ln(2)+\frac{127161994962779433}{12556544000}\ln(3)-\frac{116148004073984375}{8136640512}\ln(5)\\
&\quad-\frac{478922378441801}{134784000}\ln(7)-\frac{167245560394225319}{257698037760}\pi^4-\frac{39051856}{105}\zeta(3)+\frac{358048294}{1225}\gamma^2\\
&\quad+\frac{13108799164}{2205}\gamma\ln(2)-\frac{1359614889}{784}\gamma\ln(3)-\frac{3173828125}{7056}\gamma\ln(5)+\frac{24087679582}{2205}\ln(2)^2\\
&\quad-\frac{1359614889}{784}\ln(2)\ln(3)-\frac{3173828125}{7056}\ln(2)\ln(5)-\frac{1359614889}{1568}\ln(3)^2-\frac{3173828125}{14112}\ln(5)^2,\\
C_9^{(e^2,a^2),\ln} &= -\frac{57076219203195103}{31783752000}+\frac{358048294}{1225}\gamma+\frac{6554399582}{2205}\ln(2)-\frac{1359614889}{1568}\ln(3)-\frac{3173828125}{14112}\ln(5),\\
C_9^{(e^2,a^2),\ln^2} &= \frac{179024147}{2450},\\
C_{9.5}^{(e^2,a^2),\mathrm{c}} &= \frac{85875146281603555479}{33563642112000}\pi+\frac{123567238}{4725}\pi^3-\frac{13221694466}{165375}\pi\gamma-\frac{166628746}{315}\pi\ln(2)+\frac{926441631}{6125}\pi\ln(3),\\
C_{9.5}^{(e^2,a^2),\ln} &= -\frac{6610847233}{165375}\pi. \qquad (60)
\end{aligned}
$$

The structure of the first PN terms shows an interesting resummation property, which has been discussed in Ref. [27] [see Eq. (6) there].

B. Circular orbit limit

Let us consider now the zero-eccentricity limit of the above expressions. In the nonspinning case Akcay *et al.* [24] showed that the difference between the limit for vanishing eccentricity of $\Delta\psi$, i.e., $\lim_{e\to 0}\Delta\psi$, and the corresponding quantity $\Delta\psi^{\rm circ}$ calculated for circular orbits is proportional to the SF correction to the fractional periastron advance, which is fully known up to the 9.5PN order in terms of the EOB function ρ [44,45]. The same functional relation has been argued to hold in the Kerr case [25], even if the gauge-invariant SF correction to the periastron advance for circular equatorial orbits in a Kerr spacetime is not explicitly known, i.e.,

$$
\lim_{e\to 0}\Delta\psi-\Delta\psi^{\rm circ}=\bar G_\psi \Delta k, \qquad (61)
$$

where

$$
2\pi\Delta k=\Delta\Phi|_{e\to 0}=\delta\Phi|_{e\to 0}-\frac{\partial\bar\Phi^{\rm circ}}{\partial\bar\Omega_\phi^{\rm circ}}\delta\Omega_\phi|_{e\to 0}, \qquad (62)
$$

and

$$
\bar G_\psi=-\frac{2\pi}{\bar g_1}\frac{\partial\bar\psi}{\partial\bar\Omega_r}, \qquad \bar g_1=-\frac{1}{2\pi}\bar T_r\bar\Phi|_{e\to 0}, \qquad (63)
$$

which turns out to be

$$\bar{G}_\psi = -\frac{2(1-6u_p)^{5/2}(1-3u_p)^{1/2}}{(86u_p^2-39u_p+4)} + \frac{2u_p^{1/2}(1-6u_p)^{3/2}}{(86u_p^2-39u_p+4)^2(1-3u_p)^{1/2}}(744u_p^4-384u_p^3-28u_p^2+37u_p-4)\hat{a}$$
$$-\frac{u_p(1-6u_p)^{1/2}}{(86u_p^2-39u_p+4)^3(1-3u_p)^{3/2}}(794376u_p^8-2135148u_p^7+2333418u_p^6-1376961u_p^5$$
$$+484745u_p^4-105147u_p^3+13836u_p^2-1016u_p+32)\hat{a}^2+O(\hat{a}^3), \tag{64}$$

to the second order in the rotation parameter. Therefore, one needs to compute also the GSF corrections (38) to the periods and (37) to the associated frequencies.

The correction $\Delta\psi^{\rm circ}$ to the spin-precession invariant for circular orbits has been calculated in Ref. [26] through the 8PN order and to all orders in spin [see Eqs. (4.1)–(4.2) there]. We have checked that Eq. (61) reproduces such results up to the second order in spin. As a by-product, we can improve it to the 9.5PN order with the addition of the following new terms:

$$\Delta\psi^{{\rm circ},a^1} = \Delta\psi^{{\rm circ},a^1}|_{\rm Ref.[26]}$$
$$+\left(-\frac{163659814070959}{382016250}+\frac{13576618358917}{309657600}\pi^2+\frac{23552516744}{5457375}\gamma+\frac{1137772376}{218295}\ln(2)+\frac{3826683}{3520}\ln(3)\right.$$
$$+\frac{9765625}{19008}\ln(5)-\frac{3418003793}{67108864}\pi^4+\frac{4064}{5}\zeta(3)-\frac{217424}{525}\gamma^2-\frac{58208}{35}\gamma\ln(2)-\frac{124976}{75}\ln(2)^2$$
$$\left.+\frac{11828649172}{5457375}\ln(y)-\frac{217424}{525}\gamma\ln(y)-\frac{29104}{35}\ln(2)\ln(y)-\frac{54356}{525}\ln(y)^2\right)y^{17/2}-\frac{2201017711}{6548850}\pi y^9$$
$$+\left(\frac{16542752726965594}{1251485235}+\frac{109676435084511079}{1664719257600}\pi^2+\frac{2310004910264}{1489863375}\gamma-\frac{65918048552}{30405375}\ln(2)\right.$$
$$+\frac{6150898410939}{392392000}\ln(3)-\frac{10900390625}{2223936}\ln(5)-\frac{1835842082140957}{12884901888}\pi^4+\frac{30656}{105}\zeta(3)+\frac{492928}{1225}\gamma^2$$
$$+\frac{18697984}{4725}\gamma\ln(2)-\frac{113724}{49}\gamma\ln(3)+\frac{11314048}{2205}\ln(2)^2-\frac{113724}{49}\ln(3)\ln(2)-\frac{56862}{49}\ln(3)^2$$
$$+\frac{1183607831932}{1489863375}\ln(y)+\frac{492928}{1225}\gamma\ln(y)+\frac{9348992}{4725}\ln(2)\ln(y)-\frac{56862}{49}\ln(y)\ln(3)$$
$$\left.+\frac{123232}{1225}\ln(y)^2\right)y^{19/2}+O_{\ln}(y^{10}) \tag{65}$$

(linear in the dimensionless spin parameter $\hat{a}$) and

$$\Delta\psi^{{\rm circ},a^2} = \Delta\psi^{{\rm circ},a^2}|_{\rm Ref.[26]}-\frac{188848}{1575}\pi y^{17/2}+\left(-\frac{3255185322968}{893025}+\frac{231004545858251}{619315200}\pi^2+\frac{1085768}{945}\gamma+\frac{103352}{945}\ln(2)\right.$$
$$\left.+\frac{75087}{70}\ln(3)-\frac{27914012553}{67108864}\pi^4+\frac{576}{5}\zeta(3)+\frac{648724}{945}\ln(y)\right)y^9-\frac{12389548}{33075}\pi y^{19/2}+O_{\ln}(y^{10}) \tag{66}$$

(quadratic in $\hat{a}$), where we have used the dimensionless frequency variable y related to u_p by $u_p = y/(1-\hat{a}y^{3/2})^{2/3}$.

Furthermore, the GSF correction to the periastron advance for circular equatorial orbits in a Kerr spacetime in terms of the variable y turns out to be

$$\Delta k^{\rm circ} = \Delta k^{{\rm circ},a^0} + \hat{a}\Delta k^{{\rm circ},a^1} + \hat{a}^2\Delta k^{{\rm circ},a^2}, \tag{67}$$

which has been already presented in Ref. [27]. We will discuss below the corresponding PN expectation for completeness.

V. PN RESULTS

In this section we will check the first PN terms of our results by using the center-of-mass Hamiltonian description of a two-body system with spin. Let us start by defining the spin-precession frequency of the body 1 when spin couplings higher than the spin-orbit one are taken into account. The Hamiltonian can then be formally written as

$$\begin{aligned} H(\mathbf{q},\mathbf{p},\mathbf{S}_1,\mathbf{S}_2) &= H_{\rm orb}(\mathbf{q},\mathbf{p}) + \mathbf{\Omega}_1(\mathbf{q},\mathbf{p})\cdot\mathbf{S}_1 + \mathbf{\Omega}_2(\mathbf{q},\mathbf{p})\cdot\mathbf{S}_2 \\ &\quad + Q^{11}{}_{jk}(\mathbf{q},\mathbf{p})\mathbf{S}_1^j\mathbf{S}_1^k + 2Q^{12}{}_{jk}(\mathbf{q},\mathbf{p})\mathbf{S}_1^j\mathbf{S}_2^k \\ &\quad + Q^{22}{}_{jk}(\mathbf{q},\mathbf{p})\mathbf{S}_2^j\mathbf{S}_2^k + O^{111}{}_{ijk}(\mathbf{q},\mathbf{p})\mathbf{S}_1^i\mathbf{S}_1^j\mathbf{S}_1^k \\ &\quad + 2O^{112}{}_{ijk}(\mathbf{q},\mathbf{p})\mathbf{S}_1^i\mathbf{S}_1^j\mathbf{S}_2^k + 2O^{122}{}_{ijk}(\mathbf{q},\mathbf{p})\mathbf{S}_1^i\mathbf{S}_2^j\mathbf{S}_2^k \\ &\quad + O^{222}{}_{ijk}(\mathbf{q},\mathbf{p})\mathbf{S}_2^i\mathbf{S}_2^j\mathbf{S}_2^k + O(\text{spin}^4), \end{aligned} \tag{68}$$

where quadrupolar and octupolar interaction terms have been included. Here $(\mathbf{q},\mathbf{p})$ are phase-space variables and $(\mathbf{S}_1,\mathbf{S}_2)$ the spins of the two bodies. Omitting the explicit dependence on the variables to ease notation, we see that the spin-precession frequency follows from the spin evolution equations [see Eqs. (3.1)–(3.4) of Ref. [14]]

$$\frac{dS_1^r}{dt} = \{S_1^r, H\}, \qquad \{S_1^i, S_1^j\} = \epsilon^{ijk}S_1^k \equiv S_1^{ij}, \quad \text{etc.} \tag{69}$$

We find

$$\begin{aligned} \frac{dS_1^r}{dt} &= \Omega_{1k}\{S_1^r, S_1^k\} + Q^{11}{}_{jk}\{S_1^r, S_1^jS_1^k\} + 2Q^{12}{}_{jk}\{S_1^r, S_1^j\}S_2^k \\ &\quad + O^{111}{}_{ijk}\{S_1^r, S_1^iS_1^jS_1^k\} + 2O^{112}{}_{ijk}\{S_1^r, S_1^iS_1^j\}S_2^k \\ &\quad + 2O^{122}{}_{ijk}\{S_1^r, S_1^i\}S_2^jS_2^k \\ &= [\Omega_1\times S_1]^r + Q^{11}{}_{jk}[\epsilon^{rjm}S_{1m}S_1^k + S_1^j\epsilon^{rkm}S_{1m}] \\ &\quad + 2Q^{12}{}_{jk}\epsilon^{rji}S_{1i}S_2^k + O^{111}{}_{ijk}\{S_1^r, S_1^iS_1^jS_1^k\} \\ &\quad + 2O^{112}{}_{ijk}\{S_1^r, S_1^iS_1^j\}S_2^k + 2O^{122}{}_{ijk}\epsilon^{rim}S_{1m}S_2^jS_2^k, \end{aligned} \tag{70}$$

which can be cast in the form

$$\frac{dS_1^r}{dt} = S_1^{rj}\Omega_{S_{1j}}, \tag{71}$$

with

$$\begin{aligned} \Omega_{S_{1j}} &= \Omega_{1j} + 2Q^{11}{}_{jk}S_1^k + 2Q^{12}{}_{jk}S_2^k + 3O^{111}{}_{ijk}S_1^iS_1^k \\ &\quad + 4O^{112}{}_{ijk}S_1^iS_2^k + 2O^{122}{}_{ijk}S_2^iS_2^k. \end{aligned} \tag{72}$$

If both spins are aligned with the orbital angular momentum $\mathbf{L} = Le_z$, i.e., $\mathbf{S}_1 = S_1e_z$ and $\mathbf{S}_2 = S_2e_z$, and in addition have constant magnitudes, then $\Omega_{S_{1j}}$ can only be directed along the z-axis too, i.e., $\Omega_{S_{1j}} = \Omega_{S_1}\delta_j^z$, implying

$$\Omega_{S_1} = \frac{\partial H}{\partial S_1}. \tag{73}$$

We will compute the so-defined spin-precession frequency by using the center-of-mass ADM Hamiltonian, $H = H^{\rm ADM}$, with

$$H^{\rm ADM} = m_1 + m_2 + \mu\hat{H}^{\rm ADM}, \tag{74}$$

and

$$\hat{H}^{\rm ADM} = \hat{H}^{\rm ADM}_{\rm orb} + \hat{H}^{\rm ADM}_{\rm SO} + \hat{H}^{\rm ADM}_{\rm SS} + \hat{H}^{\rm ADM}_{\rm SSS}, \tag{75}$$

including linear, quadratic and cubic spin terms up to the present knowledge, namely next-to-next-to-leading order (NNLO) for the linear-in-spin terms, next-to-leading order (NLO) for the quadratic-in-spin terms and leading order (LO) for the cubic-in-spin terms (see Ref. [28] for a recent review). We will limit ourselves to the case of two point masses with aligned spins, orthogonal to the orbital motion. We refer to Ref. [46] for the explicit expressions of the ADM Hamiltonian terms up to the spin squared. Here we include also the LO cubic-in-spin term

$$\begin{aligned} \hat{H}^{\rm ADM,LO}_{\rm SSS} &= \left(-\frac{3}{4}\nu^2 + \frac{1}{4}\Delta\nu + \frac{1}{8} + \frac{1}{8}\Delta\right)\frac{L}{r^5}S_1^3 \\ &\quad + \left(\frac{3}{4}\Delta\nu + \frac{3}{4}\nu + \frac{3}{4}\nu^2\right)\frac{L}{r^5}S_2S_1^2 + 1\leftrightarrow 2, \end{aligned} \tag{76}$$

where the symbol $1\leftrightarrow 2$ stands for all the spin-dependent terms with the particle labels 1 and 2 exchanged ($S_1\leftrightarrow S_2$ and $\Delta\leftrightarrow-\Delta$).

A. Computing the gyroscope precession invariant

With the ADM Hamiltonian written above and physical dimensions restored, we will compute the (averaged) spin frequency of the body 1,

$$\langle\Omega_{S_1}\rangle_t = \frac{1}{T_r}\oint\frac{\partial H}{\partial S_1}dt, \tag{77}$$

where all phase-space variables (except to S_1) are kept as constant. The analogous quantity to the spin-precession invariant (20) is then defined by

$$\psi = \frac{\langle\Omega_{S_1}\rangle_t}{\Omega_\phi}. \tag{78}$$

The periods of the radial and azimuthal motion as well as the associated frequencies follow from the definition

$$T_r = \oint dt = \oint \left(\frac{\partial H}{\partial p_r}\right)^{-1} dr = 2\int_0^\pi \left(\frac{\partial H}{\partial p_r}\right)^{-1} \frac{dr}{d\chi} d\chi,$$
$$\Phi = \oint d\phi = \oint \frac{\partial H}{\partial L} dt = 2\int_0^\pi \frac{\partial H}{\partial L} \left(\frac{\partial H}{\partial p_r}\right)^{-1} \frac{dr}{d\chi} d\chi, \tag{79}$$

and

$$\Omega_r = \frac{2\pi}{T_r}, \qquad \Omega_\phi = \frac{\Phi}{T_r}, \tag{80}$$

where we have introduced the new radial variable parametrization for eccentric (equatorial) orbits

$$r = \frac{1}{u(1+e\cos(\chi))}, \tag{81}$$

with u denoting the reciprocal of the semilatus rectum and e the eccentricity, which are now ADM variables. Both such quantities are coordinate dependent and then gauge dependent. The latter should then be reexpressed in terms of a (convenient) pair of gauge invariant variables. A convenient choice is

$$\hat{k} = \frac{k}{3}, \qquad \iota = \frac{x}{\hat{k}}, \tag{82}$$

which are simply related to the (fractional) periastron advance per radial period $k = \frac{\Phi}{2\pi} - 1$ and the dimensionless azimuthal frequency $x = (M\Omega_\phi)^{2/3}$. Computing these two quantities allows one to express u and e in terms of $\hat{k}$ and ι, or equivalently ι and x (see Ref. [46] for details).

The spin-precession invariant (78) as a function of ι and x then turns out to be

$$\psi(\iota,x) = \psi_{S^0}(\iota,x) + \psi_{S^1}(\iota,x) + \psi_{S^2}(\iota,x), \tag{83}$$

with

$$\begin{aligned}\psi_{S^0}(\iota,x) = {}& \left(\frac{3}{4}\Delta + \frac{1}{2}\nu + \frac{3}{4}\right)\frac{x}{\iota} + \left\{\left[\left(-\frac{9}{16}+\frac{3}{8}\nu\right)\Delta - 2\nu + \frac{1}{4}\nu^2 - \frac{9}{16}\right]\frac{1}{\iota} + \left[\left(\frac{3}{4}\nu - \frac{9}{4}\right)\Delta + \frac{7}{8}\nu^2 - \frac{9}{4} + \frac{11}{4}\nu\right]\frac{1}{\iota^2}\right\}x^2\\
&+ \left\{\left[\left(\frac{5}{32}\nu^2 - \frac{75}{64} + \frac{3}{8}\nu\right)\Delta - \frac{93}{32}\nu - \frac{29}{96}\nu^2 - \frac{75}{64} + \frac{5}{48}\nu^3\right]\frac{1}{\iota}\right.\\
&+ \left[\left(\frac{3}{4}\nu^2 - \frac{103}{16}\nu + \frac{75}{64} + \frac{123}{512}\nu\pi^2\right)\Delta + \frac{75}{64} - \frac{141}{32}\nu + \frac{123}{512}\nu\pi^2 - \frac{233}{48}\nu^2 + \frac{7}{8}\nu^3 + \frac{41}{256}\nu^2\pi^2\right]\frac{1}{\iota^2}\\
&+ \left[\left(\frac{43}{2}\nu - \frac{615}{512}\nu\pi^2 + \frac{27}{4} + \frac{15}{8}\nu^2\right)\Delta\right.\\
&\left.\left. + \frac{37}{4}\nu - \frac{615}{512}\nu\pi^2 - \frac{205}{256}\nu^2\pi^2 + \frac{79}{3}\nu^2 + \frac{5}{2}\nu^3 + \frac{27}{4}\right]\frac{1}{\iota^3}\right\}x^3 + O(x^4),\end{aligned} \tag{84}$$

$$\begin{aligned}\psi_{S^1}(\iota,x) = {}& \left\{\left[-\frac{7}{12}\chi_1\nu + \left(\frac{1}{2}+\frac{1}{12}\nu\right)\chi_2\right]\Delta + \left(\frac{13}{12}\nu - \frac{1}{6}\nu^2\right)\chi_1 + \left(\frac{1}{2} - \frac{1}{6}\nu^2 - \frac{11}{12}\nu\right)\chi_2\right\}\frac{x^{3/2}}{\iota^{3/2}}\\
&+ \left\{\left\{\left[\left(-\frac{7}{16}\nu^2 - \frac{7}{96}\nu\right)\chi_1 + \left(\frac{15}{16} + \frac{13}{96}\nu + \frac{1}{16}\nu^2\right)\chi_2\right]\Delta\right.\right.\\
&\left. + \left(\frac{31}{24}\nu^2 + \frac{133}{96}\nu - \frac{1}{8}\nu^3\right)\chi_1 + \left(-\frac{1}{8}\nu^3 - \frac{5}{24}\nu^2 - \frac{167}{96}\nu + \frac{15}{16}\right)\chi_2\right\}\frac{1}{\iota^{3/2}}\\
&+ \left\{\left[\left(-\frac{125}{48}\nu^2 - \frac{37}{96}\nu\right)\chi_1 + \left(-\frac{45}{16} + \frac{283}{96}\nu + \frac{23}{48}\nu^2\right)\chi_2\right]\Delta + \left(-\frac{23}{24}\nu^3 + \frac{31}{12}\nu^2 - \frac{377}{96}\nu\right)\chi_1\right.\\
&\left.\left. + \left(-\frac{45}{16} - \frac{23}{24}\nu^3 + \frac{823}{96}\nu - \frac{65}{12}\nu^2\right)\chi_2\right\}\frac{1}{\iota^{5/2}}\right\}x^{5/2} + O(x^{7/2}),\end{aligned} \tag{85}$$

and

$$\begin{aligned}\psi_{S^2}(\iota,x)=&\left\{\left[\left(-\frac{7}{6}\nu+\frac{11}{24}\nu^2\right)\chi_1^2+\left(\frac{1}{2}\nu+\frac{1}{4}\nu^2\right)\chi_2\chi_1+\left(-\frac{5}{6}\nu-\frac{5}{24}\nu^2+\frac{5}{8}\right)\chi_2^2\right]\Delta+\left(\frac{7}{6}\nu-\frac{67}{24}\nu^2+\frac{1}{12}\nu^3\right)\chi_1^2\right.\\&\left.+\left(-\frac{25}{12}\nu+\frac{5}{24}\nu^2+\frac{5}{8}+\frac{1}{12}\nu^3\right)\chi_2^2+\left(\frac{1}{2}\nu+\frac{1}{6}\nu^3+\frac{7}{4}\nu^2\right)\chi_2\chi_1\right\}\frac{x^2}{\iota^2}\\&+\left\{\left\{\left[\left(-\frac{17}{48}\nu-\frac{7}{16}\nu^2+\frac{15}{32}-\frac{5}{24}\nu^3\right)\chi_2^2+\left(-\frac{5}{24}\nu+\frac{11}{24}\nu^3-\frac{27}{16}\nu^2\right)\chi_1^2+\left(\frac{3}{8}\nu^2+\frac{3}{8}\nu+\frac{1}{4}\nu^3\right)\chi_2\chi_1\right]\Delta\right.\right.\\&+\left(\frac{1}{12}\nu^4-\frac{31}{24}\nu-\frac{2}{3}\nu^2-\frac{1}{8}\nu^3+\frac{15}{32}\right)\chi_2^2+\left(\frac{5}{24}\nu+\frac{1}{12}\nu^4-\frac{25}{8}\nu^3+\frac{17}{24}\nu^2\right)\chi_1^2\\&\left.+\left(\frac{13}{12}\nu^3+\frac{3}{8}\nu+\frac{3}{2}\nu^2+\frac{1}{6}\nu^4\right)\chi_2\chi_1\right\}\frac{1}{\iota^2}\\&+\left\{\left[\left(\frac{73}{18}\nu-\frac{445}{72}\nu^2+\frac{149}{36}\nu^3\right)\chi_1^2+\left(\frac{13}{6}\nu^3+\frac{19}{4}\nu^2-6\nu\right)\chi_2\chi_1+\left(-\frac{71}{36}\nu^3-\frac{587}{72}\nu^2+\frac{929}{72}\nu-\frac{79}{16}\right)\chi_2^2\right]\Delta\right.\\&+\left(\frac{343}{24}\nu^2-\frac{73}{18}\nu-\frac{367}{18}\nu^3+\frac{8}{9}\nu^4\right)\chi_1^2+\left(-\frac{289}{12}\nu^2+\frac{8}{9}\nu^4+\frac{205}{9}\nu-\frac{79}{16}+\frac{41}{18}\nu^3\right)\chi_2^2\\&\left.\left.+\left(\frac{16}{9}\nu^4-3\nu^2-6\nu+\frac{151}{9}\nu^3\right)\chi_2\chi_1\right\}\frac{1}{\iota^3}\right\}x^3+O(x^4),\end{aligned}\tag{86}$$

where we have used the spin variables χ_1 and χ_2 instead of S_1 and S_2.

The GSF contribution can be extracted by substituting the new variables $y=(m_2\Omega_\phi)^{2/3}$ and $\lambda=y/\hat{k}$, which are related to x and ι by $x=y(1+q)^{2/3}$ and $\iota=\lambda(1+q)^{2/3}$, into the previous expressions, expanding them in power series of the mass ratio q and selecting the first-order terms. One then gets the first-order SF (1SF) part:

$$\begin{aligned}\psi_{1SF,S^0}(y,\lambda)&=-\frac{y}{\lambda}+\left(-\frac{5}{4\lambda}+\frac{8}{\lambda^2}\right)y^2+\left[-\frac{53}{16\lambda}+\left(\frac{123}{256}\pi^2-\frac{93}{8}\right)\frac{1}{\lambda^2}+\left(\frac{69}{4}-\frac{615}{256}\pi^2\right)\frac{1}{\lambda^3}\right]y^3+O(y^4),\\\psi_{1SF,S^1}(y,\lambda)&=\left(-\frac{11}{6}\chi_2+\frac{1}{2}\chi_1\right)\frac{y^{3/2}}{\lambda^{3/2}}+\left[\left(-\frac{107}{48}\chi_2+\frac{21}{16}\chi_1\right)\frac{1}{\lambda^{3/2}}+\left(\frac{823}{48}\chi_2-\frac{69}{16}\chi_1\right)\frac{1}{\lambda^{5/2}}\right]y^{5/2}+O(y^{7/2}),\\\psi_{1SF,S^2}(y,\lambda)&=\left(-\frac{25}{6}\chi_2^2+\chi_2\chi_1\right)\frac{y^2}{\lambda^2}+\left[\left(-\frac{47}{24}\chi_2^2+\frac{3}{4}\chi_2\chi_1\right)\frac{1}{\lambda^2}+\left(\frac{410}{9}\chi_2^2-12\chi_2\chi_1\right)\frac{1}{\lambda^3}\right]y^3+O(y^4).\end{aligned}\tag{87}$$

The last step consists in computing the Kerr background values for y and λ, both functions of u_p and e_p (say, to distinguish them from the corresponding ADM quantities u and e), and substituting them into the previous 1SF expressions. Setting $\chi_2=\hat{a}$ we find

$$\begin{aligned}\psi_{1SF,S^0}(u_p,e_p)&=-u_p+\left(\frac{9}{4}+e_p^2\right)u_p^2+\left[\frac{739}{16}-\frac{123}{64}\pi^2+\left(\frac{341}{16}-\frac{123}{256}\pi^2\right)e_p^2-\frac{1}{2}e_p^4\right]u_p^3+O(u_p^4),\\\psi_{1SF,S^1}(u_p,e_p)&=\left(-\frac{1}{2}\hat{a}+\frac{1}{2}\chi_1\right)u_p^{3/2}+\left[\left(-\frac{9}{8}\chi_1-\frac{1}{8}\hat{a}\right)e_p^2-\frac{41}{8}\hat{a}+\frac{3}{8}\chi_1\right]u_p^{5/2}+O(u_p^{7/2}),\\\psi_{1SF,S^2}(u_p,e_p)&=-\hat{a}^2u_p^2+\left[\left(-2\hat{a}^2+\frac{9}{4}\hat{a}\chi_1\right)e_p^2+\frac{15}{4}\hat{a}^2-\frac{7}{4}\hat{a}\chi_1\right]u_p^3+O(u_p^4),\end{aligned}\tag{88}$$

which coincide with the GSF results for $\Delta\psi$ of the previous section for $\chi_1=0$.

B. Circular limit

Let us discuss the circular limit of previous results. The variables ι and x are not independent in this limit. Recalling the definition (82), in order to express ι as a function of x it is enough to use the relation $k_{\rm circ}(x)$ for the fractional periastron advance [see Eqs. (9a)–(9h) in Ref. [47]]

$$\begin{aligned}k_{\rm circ}(x)&=k_{\rm circ,O}(x)+k_{\rm circ,S}(x)+k_{\rm circ,SS}(x),\\k_{\rm circ,SS}(x)&=k_{\rm circ,S_1S_2}(x)+k_{{\rm circ},S_{1,2}^2}(x),\end{aligned}\tag{89}$$

where

$$
\begin{aligned}
k_{\mathrm{circ,O}}(x) &= 3x + \left(\frac{27}{2} - 7\nu\right)x^2 + \left(7\nu^2 - \frac{649}{4}\nu + \frac{135}{2} + \frac{123}{32}\nu\pi^2\right)x^3 + O(x^4),\\
k_{\mathrm{circ,S}}(x) &= \left[(-2 + 2\Delta + \nu)x^{3/2} + \left(-\frac{17}{4}\Delta\nu - 17 - \nu^2 + 17\Delta + \frac{81}{4}\nu\right)x^{5/2}\right.\\
&\quad \left. + \left(-\frac{733}{12}\nu^2 + \frac{1}{3}\nu^3 + \frac{11581}{48}\nu + \frac{11}{3}\Delta\nu^2 - 126 - \frac{5317}{48}\Delta\nu + 126\Delta\right)x^{7/2} + O(x^{9/2})\right]\chi_1 + 1 \leftrightarrow 2,\\
k_{\mathrm{circ,S_1S_2}}(x) &= [3\nu x^2 + (2\nu^2 + 45\nu)x^3 + O(x^4)]\chi_2\chi_1,\\
k_{\mathrm{circ,S_{1,2}^2}}(x) &= \left[\left(\frac{3}{4} - \frac{3}{2}\nu - \frac{3}{4}\Delta\right)x^2 + \left(6\nu^2 - \frac{189}{4}\nu + \frac{67}{4} - \frac{67}{4}\Delta + \frac{55}{4}\Delta\nu\right)x^3 + O(x^4)\right]\chi_1^2 + 1 \leftrightarrow 2,
\end{aligned} \tag{90}
$$

so that

$$
\iota_{\mathrm{circ}}(x) = \frac{3x}{k_{\mathrm{circ}}(x)}. \tag{91}
$$

We then find

$$
\psi_{\mathrm{circ}}(x) = \psi_{\mathrm{circ,S^0}}(x) + \psi_{\mathrm{circ,S^1}}(x) + \psi_{\mathrm{circ,S^2}}(x), \tag{92}
$$

where

$$
\begin{aligned}
\psi_{\mathrm{circ,S^0}}(x) &= \left(\frac{3}{4}\Delta + \frac{1}{2}\nu + \frac{3}{4}\right)x + \left[\left(-\frac{5}{8}\nu + \frac{9}{16}\right)\Delta + \frac{9}{16} + \frac{5}{4}\nu - \frac{1}{24}\nu^2\right]x^2\\
&\quad + \left[\left(\frac{27}{32} - \frac{39}{8}\nu + \frac{5}{32}\nu^2\right)\Delta + \frac{3}{16}\nu - \frac{1}{48}\nu^3 + \frac{27}{32} - \frac{105}{32}\nu^2\right]x^3 + O(x^4),\\
\psi_{\mathrm{circ,S^1}}(x) &= \left[-\frac{1}{2}\Delta\chi_2 - \chi_1\nu + \left(\nu - \frac{1}{2}\right)\chi_2\right]x^{3/2}\\
&\quad + \left\{\left[\frac{11}{6}\chi_1\nu + \left(-\frac{1}{4} - \frac{1}{12}\nu\right)\chi_2\right]\Delta + \left(-\frac{1}{3}\nu + \frac{7}{6}\nu^2\right)\chi_1 + \left(-\frac{1}{4} + \frac{1}{6}\nu^2 + \frac{5}{12}\nu\right)\chi_2\right\}x^{5/2} + O(x^{7/2}),\\
\psi_{\mathrm{circ,S^2}}(x) &= \left[\left(-\chi_2\chi_1\nu - \frac{1}{4}\chi_2^2\right)\Delta - \frac{3}{2}\chi_1^2\nu^2 + (\nu^2 - \nu)\chi_2\chi_1 + \left(\frac{1}{2}\nu - \frac{1}{4} + \frac{1}{2}\nu^2\right)\chi_2^2\right]x^3 + O(x^4).
\end{aligned} \tag{93}
$$

The spin-orbit term $\psi_{\mathrm{circ,S^0}}$ is given by Eq. (9) of Ref. [19]. The other terms agree with those computed in Ref. [26] by using the EFT results of Ref. [48], which allow for the inclusion of the following further term:

$$
\begin{aligned}
\psi_{\mathrm{circ,S^1NNLO}}(x) &= \left\{\left[\left(-\frac{137}{36}\nu^2 + \frac{19}{4}\nu\right)\chi_1 + \left(\frac{143}{48}\nu - \frac{15}{16} + \frac{53}{144}\nu^2\right)\chi_2\right]\Delta + \left(-\frac{29}{8}\nu + \frac{931}{72}\nu^2 - \frac{59}{72}\nu^3\right)\chi_1\right.\\
&\quad \left. + \left(-\frac{805}{144}\nu^2 + \frac{233}{48}\nu - \frac{15}{16} - \frac{53}{72}\nu^3\right)\chi_2\right\}x^{7/2}.
\end{aligned} \tag{94}
$$

The corresponding 1SF expansion then reads

$$
\psi_{\mathrm{circ,1SF}}(y) = y^2 - 3y^3 + (\chi_2 - \chi_1)y^{3/2} + \frac{3}{2}y^{5/2}\chi_1 + \left(\frac{16}{3}\chi_2 + \frac{9}{8}\chi_1\right)y^{7/2} - 2\chi_1\chi_2 y^3 + O(y^4), \tag{95}
$$

which agrees with Eq. (4.8) of Ref. [26] for $\chi_1 = 0$.

Finally, the 1SF expansion of the fractional periastron advance is

$$
\begin{aligned}
k_{\mathrm{circ,1SF,O}}(y) &= 2y + 11y^2 + \left(\frac{123}{32}\pi^2 - \frac{109}{4}\right)y^3 + O(y^4),\\
k_{\mathrm{circ,1SF,S^1}}(y) &= (\chi_2 - 3\chi_1)y^{3/2} + \left(\frac{11}{6}\chi_2 - 18\chi_1\right)y^{5/2} + \left(-\frac{243}{2}\chi_1 + \frac{385}{24}\chi_2\right)y^{7/2} + O(y^{9/2}),\\
k_{\mathrm{circ,1SF,S^2}}(y) &= (-\chi_2^2 + 3\chi_2\chi_1)y^2 + \left(-\frac{55}{2}\chi_2^2 + 45\chi_2\chi_1\right)y^3 + O(y^4),
\end{aligned}
\tag{96}
$$

which agrees with previous results [27] for $\chi_1 = 0$ and $\chi_2 = \hat{a}$.

VI. CONCLUDING REMARKS

We have analytically computed the gravitational self-force correction to the gyroscope precession along slightly eccentric equatorial orbits in the Kerr spacetime, generalizing known expressions in the Schwarzschild case. Our results are accurate through the 9.5PN order and to second order in both eccentricity and rotation parameter. We have also improved to the 9.5PN level the current knowledge of the spin-precession invariant for eccentric orbits in the nonrotating case and for circular orbits in the same Kerr case. As an independent check, we have calculated the same invariant by using the current knowledge of the ADM Hamiltonian for two point masses with aligned spins. The full transcription of such a high-PN analytical result within other approaches, like the EOB model, will be considered elsewhere.

[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. **116**, 241103 (2016).

[3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, Phys. Rev. Lett. **119**, 141101 (2017).

[4] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[5] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. **118**, 221101 (2017).

[6] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170608: Observation of a 19-solar-mass binary black hole coalescence, Astrophys. J. **851**, L35 (2017).

[7] See the aLIGO website, https://www.advancedligo.mit.edu/.

[8] See the eLISA website, https://www.elisascience.org/.

[9] L. Barack and A. Pound, Self-force and radiation reaction in general relativity, Rep. Prog. Phys. **82**, 016904 (2019).

[10] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[11] A. Buonanno and T. Damour, Transition from inspiral to plunge in binary black hole coalescences, Phys. Rev. D **62**, 064015 (2000).

[12] T. Damour, Coalescence of two spinning black holes: An effective one-body approach, Phys. Rev. D **64**, 124013 (2001).

[13] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity **17**, 2 (2014).

[14] T. Damour, P. Jaranowski, and G. Schaefer, Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling, Phys. Rev. D **77**, 064032 (2008).

[15] E. Barausse, E. Racine, and A. Buonanno, Hamiltonian of a spinning test-particle in curved spacetime, Phys. Rev. D **80**, 104025 (2009); **85**, 069904(E) (2012).

[16] J. Vines, D. Kunst, J. Steinhoff, and T. Hinderer, Canonical Hamiltonian for an extended test body in curved spacetime: To quadratic order in spin, Phys. Rev. D **93**, 103008 (2016).

[17] M. Levi and J. Steinhoff, Spinning gravitating objects in the effective field theory in the post-Newtonian scheme, J. High Energy Phys. 09 (2015) 219.

[18] M. Levi and J. Steinhoff, Complete conservative dynamics for inspiralling compact binaries with spins at fourth post-Newtonian order, arXiv:1607.04252.

[19] S. R. Dolan, N. Warburton, A. I. Harte, A. Le Tiec, B. Wardell, and L. Barack, Gravitational self-torque and spin precession in compact binaries, Phys. Rev. D **89**, 064011 (2014).

[20] D. Bini and T. Damour, Two-body gravitational spin-orbit interaction at linear order in the mass ratio, Phys. Rev. D **90**, 024039 (2014).

[21] D. Bini and T. Damour, Analytic determination of high-order post-Newtonian self-force contributions to gravitational spin precession, Phys. Rev. D **91**, 064064 (2015).

[22] C. Kavanagh, D. Bini, T. Damour, S. Hopper, A. C. Ottewill, and B. Wardell, Spin-orbit precession along eccentric orbits for extreme mass ratio black hole binaries and its effective-one-body transcription, Phys. Rev. D **96**, 064012 (2017).

[23] D. Bini, T. Damour, and A. Geralico, Spin-orbit precession along eccentric orbits: Improving the knowledge of self-force corrections and of their effective-one-body counterparts, Phys. Rev. D **97**, 104046 (2018).

[24] S. Akcay, D. Dempsey, and S. R. Dolan, Spin-orbit precession for eccentric black hole binaries at first order in the mass ratio, Classical Quantum Gravity **34**, 084001 (2017).

[25] S. Akcay, Self-force correction to geodetic spin precession in Kerr spacetime, Phys. Rev. D **96**, 044024 (2017).

[26] D. Bini, T. Damour, A. Geralico, C. Kavanagh, and M. van de Meent, Gravitational self-force corrections to gyroscope precession along circular orbits in the Kerr spacetime, Phys. Rev. D **98**, 104062 (2018).

[27] D. Bini and A. Geralico, Analytical determination of the periastron advance in spinning binaries from self-force computations (to be published).

[28] G. Schfer and P. Jaranowski, Hamiltonian formulation of general relativity and post-Newtonian dynamics of compact binaries, Living Rev. Relativity **21**, 7 (2018).

[29] K. Glampedakis and D. Kennefick, Zoom and whirl: Eccentric equatorial orbits around spinning black holes and their evolution under gravitational radiation reaction, Phys. Rev. D **66**, 044002 (2002).

[30] D. Bini, A. Geralico, and R. T. Jantzen, Gyroscope precession along bound equatorial plane orbits around a Kerr black hole, Phys. Rev. D **94**, 064066 (2016).

[31] J.-A. Marck, Solution to the equations of parallel transport in Kerr geometry; tidal tensor, Proc. R. Soc. A **385**, 431 (1983).

[32] L. Barack and N. Sago, Beyond the geodesic approximation: Conservative effects of the gravitational self-force in eccentric orbits around a Schwarzschild black hole, Phys. Rev. D **83**, 084023 (2011).

[33] T. S. Keidl, A. G. Shah, J. L. Friedman, D. H. Kim, and L. R. Price, Gravitational self-force in a radiation gauge, Phys. Rev. D **82**, 124012 (2010); **90**, 109902(E) (2014).

[34] A. G. Shah, J. L. Friedman, and T. S. Keidl, EMRI corrections to the angular velocity and redshift factor of a mass in circular orbit about a Kerr black hole, Phys. Rev. D **86**, 084059 (2012).

[35] J. M. Cohen and L. S. Kegeles, Electromagnetic fields in curved spaces: A constructive procedure, Phys. Rev. D **10**, 1070 (1974).

[36] P. L. Chrzanowski, Vector potential and metric perturbations of a rotating black hole, Phys. Rev. D **11**, 2042 (1975).

[37] L. S. Kegeles and J. M. Cohen, Constructive procedure for perturbations of space-times, Phys. Rev. D **19**, 1641 (1979).

[38] T. Regge and J. A. Wheeler, Stability of a Schwarzschild singularity, Phys. Rev. **108**, 1063 (1957).

[39] F. J. Zerilli, Gravitational field of a particle falling in a Schwarzschild geometry analyzed in tensor harmonics, Phys. Rev. D **2**, 2141 (1970).

[40] R. M. Wald, Construction of Solutions of Gravitational and Electromagnetic, or Other Perturbation Equations from Solutions of Decoupled Equations, Phys. Rev. Lett. **41**, 203 (1978).

[41] C. Merlin, A. Ori, L. Barack, A. Pound, and M. van de Meent, Completion of metric reconstruction for a particle orbiting a Kerr black hole, Phys. Rev. D **94**, 104066 (2016).

[42] M. van De Meent, The mass and angular momentum of reconstructed metric perturbations, Classical Quantum Gravity **34**, 124003 (2017).

[43] D. Bini and A. Geralico, Gauge-fixing for the completion problem of reconstructed metric perturbations of a Kerr spacetime, arXiv:1908.03191.

[44] T. Damour, Gravitational self-force in a Schwarzschild background and the effective one body formalism, Phys. Rev. D **81**, 024017 (2010).

[45] D. Bini, T. Damour, and A. Geralico, New gravitational self-force analytical results for eccentric orbits around a Schwarzschild black hole, Phys. Rev. D **93**, 104017 (2016).

[46] D. Bini and A. Geralico, preceding paper, New gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole: Redshift invariant, Phys. Rev. D **100**, 104002 (2019)

[47] A. Le Tiec *et al.*, Periastron advance in spinning black hole binaries: Gravitational self-force from numerical relativity, Phys. Rev. D **88**, 124027 (2013).

[48] M. Levi and J. Steinhoff, Equivalence of ADM Hamiltonian and effective field theory approaches at next-to-next-to-leading order spin1-spin2 coupling of binary inspirals, J. Cosmol. Astropart. Phys. 12 (2014) 003.