Superstatistics: Consequences on gravitation and cosmology

Kamel Ourabah,[1,*] Edésio M. Barboza, Jr.,[2,†] Everton M. C. Abreu,[3,4,5,‡] and Jorge Ananias Neto[4,§]

[1]*Theoretical Physics Laboratory, Faculty of Physics, University of Bab-Ezzouar, USTHB, Boite Postale 32, El Alia, Algiers 16111, Algeria*

[2]*Departamento de Física, Universidade do Estado do Rio Grande do Norte, 59610-210 Mossoró, Rio Grande do Norte, Brazil*

[3]*Grupo de Física Teórica e Matemática Física, Departamento de Física, Universidade Federal Rural do Rio de Janeiro, 23890-971 Seropédica, Rio de Janeiro, Brazil*

[4]*Departamento de Física, Universidade Federal de Juiz de Fora, 36036-330, Juiz de Fora, Minas Gerais, Brazil*

[5]*Programa de Pós-Graduação Interdisciplinar em Física Aplicada, Instituto de Física, Universidade Federal do Rio de Janeiro, 21941-972 Rio de Janeiro, Rio de Janeiro, Brazil*

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Fluctuations can be incorporated into an equilibrium statistical mechanics setting through a superposition of different statistics, i.e., superstatistics. Herein, we have combined equipartition theorems arising out of superstatistics together with the holographic principle, to address the main consequences of fluctuations on gravitation and cosmology. We explored the effect of the three universality classes of superstatistics, namely, χ^2, inverse χ^2, and log-normal superstatistics, on the Jeans criterion and the Friedmann equations. We used some of the most recent cosmological data to constrain the universality parameter q arising from these superstatistics. The dataset employed in our analysis provides $q > 1$ at 1σ for the three superstatistics classes, indicating a deviation of the standard Maxwell-Boltzmann equilibrium statistics, $q = 1$.

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I. INTRODUCTION

Connections between thermodynamics and gravitation can be traced back to the earlier works of Bekenstein and Hawking on black hole physics [1,2]. Since then, it has become clear that a connection between gravitation and thermodynamics can bring new insight into our understanding of the evolution of the Universe, and much effort has been given in building such a bridge. For instance, Jacobson [3] interpreted Einstein field equations as a thermodynamic identity, and Padmanabhan [4,5] gave an interpretation of gravity as an equipartition theorem. However, the most robust connection between gravitation and thermodynamics is probably due to Verlinde [6], who brought a heuristic derivation of gravity from thermodynamic considerations. In this formalism, gravitation is understood as an entropic force originated from perturbations in the information "manifold" caused by the motion of a massive body when it moves away from the holographic screen—*a storage device of information*.

Most of the effort in this direction has been done, so far, in the context of equilibrium thermodynamics and equilibrium statistical mechanics [7–12]. However, the Universe is full of complex systems that are either far away from equilibrium or exhibit only local equilibrium. Such systems, usually, exhibit fluctuations, and one may inquire about the consequences of fluctuations on entropic gravity and cosmology.

In this respect, a great deal of interest has been devoted recently to understanding both gravitation and cosmology from the perspective of non-Gaussian statistics [13–22]. Especially, much effort has been devoted to revisiting cosmology from the point of view of the Tsallis nonextensive statistical mechanics [23]. The Tsallis formalism is based upon a generalization of both the entropy and the distributions arising from it, and it brought new insight into cosmology. Such an approach, however, has a couple of drawbacks, one of which is the parameter "q," which underpins Tsallis entropy and the non-Gaussian distributions arising from it, the physical meaning of which remains quite obscure [24]. Another shortcoming of *phenomenologically* introducing these distributions is that there is a critical value of the Tsallis parameter, namely, $q = 5/3$, that leads to singularities [17,18].

In this paper, we have taken a different path, and we based our approach on the so-called *superstatistics* [25]. The latter is by now a standard method in nonequilibrium statistical mechanics, providing a simple approach to systems in a nonequilibrium steady state exhibiting fluctuations.

*kam.ourabah@gmail.com
†edesiobarboza@uern.br
‡evertonabreu@ufrrj.br
§jorge@fisica.ufjf.br

The central idea of superstatistics consists of decomposing the system's dynamics in different scales, so that its statistical properties are given by a superposition of statistics. So far, the formalism of superstatistics has been successfully applied to many systems exhibiting fluctuations in different areas, such as space and plasma physics [26–28], high-energy physics [29,30], and quantum information [31,32], to mention only a few. The motivation behind our approach is twofold. First, distributions arising within the nonextensive statistical mechanics emerge, as a particular case, from superstatistics, together with a transparent definition of the parameter q [33,34]. It is related to the strength of fluctuations, opening therefore a perspective for an empirical validation. Second, superstatistics generates other distributions, beyond those emerging within nonextensive statistical mechanics and for which experimental evidence can be found in the literature [27,28,35], that do not suffer from singularities.

The paper is organized as follows. In Sec. II, we obtain the equipartition theorems for the three universality classes of superstatistics. In Sec. III, we discuss their implication on the entropic gravity. In Sec. IV, we examine the effect of fluctuation on the Jeans criterion. In Sec. V, we obtain the modified Friedmann equations that emerge from superstatistics and the q modified ΛCDM models associated with it. In Sec. VI, we obeservationally constrain the parameters of q modified ΛCDM models. We present our conclusions in the last section.

II. SUPERSTATISTICS AND EQUIPARTITION THEOREMS

Tsallis thermostatistics [23] relies strongly upon a free parameter that has different values for each system. A validity test is to recover Boltzmann-Gibbs (BG's) statistics always when $q \to 1$. Another very interesting entropy formulation is the one that depends only on the probability [36]. It differs from BG's entropy when applied to large probabilities, namely, when the number of microstates of the systems is not big enough [37–39]. Tsallis entropy [23] and Obregón entropy [40,41] are, respectively,

$$S_q = \frac{1}{q-1}\left(1 - \sum_{i=1}^{\Omega} p_i^q\right) \tag{1}$$

and

$$S = \sum_{i=1}^{\Omega}(1 - p_i^{p_i}), \tag{2}$$

where, from Tsallis' expression in Eq. (1), the BG entropy is straightforwardly recovered when $q \to 1$. For the Obregón entropy, in Eq. (2), the BG entropy appears as the first term in the expansion of Eq. (2) [36]. In that sense, the Obregón entropy provides corrections for the BG's entropy, the implications on gravitation and cosmology of which have been examined in Refs. [36–39,41].

Herein, we consider an alternative approach, which does not rely on a generalization of the entropy but on a superposition of statistics, in short, superstatistics. For that purpose, let us consider a system of noninteracting particles at equilibrium temperature T. It is characterized by a Maxwell-Boltzmann (MB) velocity distribution,

$$f_{\mathrm{MB}}(v) = \sqrt{\frac{\beta m}{2\pi}} \exp\left[-\frac{\beta m v^2}{2}\right], \tag{3}$$

where $\beta \equiv 1/k_B T$ is the inverse temperature in energy units (henceforth, for simplicity, we set $k_B \equiv 1$). It is implicitly assumed in Eq. (3) that the temperature is well defined, well known, and constant. In many systems, however, the temperature can present spatiotemporal fluctuations. Such systems cannot be characterized by a canonical distribution in Eq. (3) with a well-defined temperature and, in principle, should be treated through nonequilibrium statistical mechanics. The obtention of the steady state of a nonequilibrium system is a highly nontrivial task. But the task becomes relatively easy to handle in the case of slow fluctuations operating on a long timescale. In this case, fluctuations can be incorporated in an equilibrium statistical mechanics, which is defined by considering a superposition of distributions [25,33,34]. The nonequilibrium system in a steady state can be divided up into cells—small regions characterized by a sharp value of the inverse temperature β—such that in each cell the MB distribution in Eq. (3) holds. As the temperature (or equivalently its inverse β) is varying from cell to cell, this variation among the different cells can be modeled by a distribution, say, $f(\beta)$. In the long run, the system, made up of many smaller cells that are temporarily in local equilibrium, follows a velocity distribution that arises out from the MB distribution, which holds in each cell, averaged over $f(\beta)$, which characterizes the variation of the temperature among the different cells, i.e.,

$$B(v) = \int_0^\infty d\beta f(\beta) \sqrt{\frac{\beta m}{2\pi}} \exp\left[-\frac{\beta m v^2}{2}\right]. \tag{4}$$

Velocity distributions in the form of Eq. (4) are typically non-Gaussian distributions exhibiting fat tails, which make them suitable to describe many space systems. The approach leading to distributions in Eq. (4) is known as superstatistics. It generalizes distributions arising within the so-called nonextensive statistical mechanics, and it provides the foundations for the occurrence of non-Gaussian statistics in many systems based on temperature fluctuations [33,34].

The distribution $f(\beta)$ is, of course, a central choice since it defines the velocity distribution in Eq. (4). In principle, $f(\beta)$ could be any normalized probability density, but it is known that all relevant superstatistics fall into three fundamental universality classes [42]:

(1) χ^2 *superstatistics.*—This case holds if there are many (nearly) independent microscopic random variables acting on β in an *additive* way. For a large number of variables, due to the *central limit theorem*, their rescaled sum approaches a Gaussian one. The inverse temperature β, which is positive, is obtained by squaring these Gaussian random variables, which results in a χ^2 distribution, that is,

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{n}{2\beta_0}\right)^{n/2}\beta^{n/2-1}e^{-\frac{n\beta}{2\beta_0}}, \tag{5}$$

where β_0 is the average of β, i.e., $\beta_0 \equiv \int_0^\infty \beta f(\beta)d\beta$, and n is a real parameter. In this case, the emergent velocity distribution in Eq. (4) deviates from the MB distribution and corresponds to the so-called q-Gaussian [23] distribution, which, apart from a normalization factor, reads as

$$f^{(q)}(v) \propto \left(1+(q-1)\frac{\beta_0 m v^2}{2}\right)^{1/(1-q)}, \tag{6}$$

with $q \equiv 1 + 2/n$. Distributions in the form of Eq. (6) have been observed in many situations [43–47], and their consequences on gravitation and cosmology have been addressed in a list of papers [13–22].

(2) *Inverse χ^2 superstatistics.*—In this case, instead of β, the temperature (β^{-1}) itself arises from a large number of independent or nearly independent variables, which results in an inverse χ^2 distribution,

$$f(\beta) = \frac{\beta_0}{\Gamma(\frac{n}{2})}\left(\frac{n\beta_0}{2}\right)^{n/2}\beta^{-n/2-2}e^{-\frac{n\beta_0}{2\beta}}. \tag{7}$$

(3) *Log-normal superstatistics.*—Instead of being a sum of random variables, one may have many (nearly) independent microscopic random variables acting on β in a *multiplicative* way. In view of the central limit theorem, for a large number of microscopic variables, the rescaled sum of their logarithms follows a Gaussian distribution, which results therefore in a log-normal distribution of β,

$$f(\beta) = \frac{1}{\sqrt{2\pi}s\beta}\exp\left\{\frac{-(\ln\frac{\beta}{\mu})^2}{2s^2}\right\}, \tag{8}$$

where μ and s are real parameters.

In opposition to χ^2 superstatistics, from which the velocity distribution, Eq. (6), is explicitly known, the velocity distribution emerging from the last two superstatistics, Eqs. (7) and (8), is not accessible in closed form. However, the main ingredient in our approach, together with the holographic principle, is the theorem of equipartition of energy that can be obtained in closed form, as long as the moments of $f(\beta)$ are known. To this end, we have to evaluate $\langle v^2\rangle$, which, apart from a constant factor, corresponds to the mean kinetic energy.

In the case of χ^2 superstatistics, the velocity distribution is explicitly known, and it corresponds to the q-Gaussian distribution in Eq. (6) from which it follows that

$$\langle v^2\rangle_{\chi^2} = \frac{2}{5-3q}\langle v^2\rangle_{\rm MB}, \quad (q < 5/3), \tag{9}$$

where $\langle v^2\rangle_{\rm MB} = T/m$, where $T \equiv 1/\beta_0$ is the average temperature. The same result is obtained in the formalism of nonextensive statistical mechanics (see, for instance, Refs. [48,49]). For the two other superstatistics, although the velocity distribution $B(v)$ does not have a closed form, $\langle v^2\rangle$ can be obtained by combining the moments of the inverse χ^2 distribution in Eq. (7) and the log-normal distribution in Eq. (8),

$$\langle\beta^l\rangle_{{\rm inv}.\chi^2} = \frac{\Gamma(\frac{n}{2}+1-l)}{\Gamma(\frac{n}{2})}\left(\frac{n}{2}\right)^{l-1}\beta_0^l,$$
$$\langle\beta^l\rangle_{LN} = \mu^l e^{\frac{1}{2}l^2 s^2}, \tag{10}$$

where the moments of the MB distribution are given by

$$\langle v^l\rangle_{\rm MB} = \frac{(l-1)!!}{(\beta m)^{l/2}}(l\,\text{even}). \tag{11}$$

By proceeding in this way, we obtain that

$$\langle v^2\rangle_{{\rm inv}.\chi^2} = \frac{n+2}{n}\langle v^2\rangle_{\rm MB},$$
$$\langle v^2\rangle_{LN} = e^{s^2}\langle v^2\rangle_{\rm MB}. \tag{12}$$

To compare the effects of different superstatistics, it is convenient to define one universal parameter extending the parameter that underpins the q-Gaussian distribution [50]. One may link this parameter to the strength of fluctuations as $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2$, preserving therefore the distribution in Eq. (6), and it is still valid for the other superstatistics. For the three distributions, Eqs. (5), (7), and (8), it reads, respectively, as

$$q \equiv \frac{\langle\beta^2\rangle_{\chi^2}}{\langle\beta\rangle^2_{\chi^2}} = \frac{n+2}{n}, \tag{13}$$

$$q \equiv \frac{\langle\beta^2\rangle_{{\rm inv}.\chi^2}}{\langle\beta\rangle^2_{{\rm inv}.\chi^2}} = \frac{n}{n-2}, \quad \text{and} \tag{14}$$

$$q \equiv \frac{\langle\beta^2\rangle_{LN}}{\langle\beta\rangle^2_{LN}} = e^{s^2}, \tag{15}$$

from which $\langle v^2\rangle$ can be written in a unified manner as

$$\langle v^2\rangle_i = \phi_i(q)\langle v^2\rangle_{\rm MB} \quad (i = 1, 2, 3), \tag{16}$$

where

$$\phi_1(q) \equiv \frac{2}{5-3q} \quad (0 < q < 5/3) \tag{17}$$

$$\phi_2(q) \equiv \frac{2q-1}{q} \quad (q > 1/2) \tag{18}$$

$$\phi_3(q) \equiv q \quad (q > 0), \tag{19}$$

and the indices 1, 2, and 3 correspond, respectively, to χ^2 superstatistics, inverse χ^2 superstatistics, and log-normal superstatistics, the positiveness of $\langle v^2\rangle_i$ defining the domain of $\phi_i(q)$ ($i = 1$, 2, 3). Note that, in this approach, one has *by construction* $q \geq 1$. However, to observationally constrain the parameters of our model, we examine both cases $q \geq 1$ and $q < 1$, inasmuch as other mechanisms, not addressed here, may generate non-Gaussian distributions with $q < 1$ (see, for instance, Refs. [51–54]). Figure 1 shows the relative deviation,

$$\delta\langle v^2\rangle_i \equiv \frac{\langle v^2\rangle_i - \langle v^2\rangle_{\rm MB}}{\langle v^2\rangle_{\rm MB}}, \tag{20}$$

in the neighborhood of $q = 1$. Note that the inverse χ^2 and the log-normal superstatistics are almost indistinguishable from each other inside the range $0.96 < q < 1.04$. Moreover, considering the three superstatistics, the χ^2 is the one that presents the higher deviation of the standard MB result as q moves away from 1.

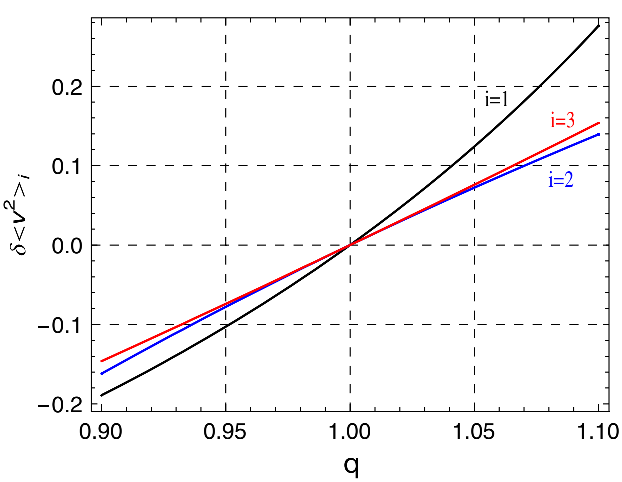

FIG. 1. Plot of $\delta\langle v^2\rangle_i \equiv (\langle v^2\rangle_i - \langle v^2\rangle_{\rm MB})/\langle v^2\rangle_{\rm MB}$ for different superstatistics: $i = 1, \chi^2$ (black line); $i = 2$, inverse χ^2 (blue line); and $i = 3$, log-normal (red line) superstatistics, against $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2$ in the neighborhood of $q = 1$.

Notice that for all superstatistics $\langle v^2\rangle_i$ reduces to the MB case in the limit $q \to 1$. This is because this limit corresponds to a vanishing variance of the distribution $f(\beta)$, which approaches a Dirac delta centered at β_0, in which case the velocity distribution in Eq. (4) reduces to the MB velocity distribution in Eq. (3), with inverse temperature β_0. From Eq. (16), the equipartition theorem for the three different superstatistics is given by

$$E_i = \phi_i(q)\frac{N}{2}T \quad (i = 1, 2, 3), \tag{21}$$

where ϕ_i is given in Eqs. (17)–(19) and N will be defined as the number of bits, which will be shown in Eq. (22).

III. VERLINDE'S FORMALISM

The main ingredients of Verlinde's formalism [4,6] are the holographic principle on the one hand and the theorem of equipartition of energy on the other. The model considers a spherical surface as being the holographic screen, with a particle of mass M positioned in its center. The holographic screen can be thought of as a storage device for information. The number of bits—the smallest units of information—in the holographic screen is assumed to be proportional to the holographic screen area A, i.e.,

$$N = \frac{A}{\ell_P^2}, \tag{22}$$

where $A = 4\pi r^2$ and $\ell_P \equiv \sqrt{G\hbar/c^3}$ is the Planck length. The theorem of equipartition of energy counts the bits' total energy on the screen. The energy of the particle inside the holographic screen is assumed to be equally divided among the bits in such a way that we have

$$Mc^2 = \frac{1}{2}NT. \tag{23}$$

From the number of bits in Eq. (22) and the Unruh temperature [55],

$$T = \frac{1}{2\pi}\frac{\hbar a}{c}, \tag{24}$$

one arrives at the gravitational acceleration as

$$\begin{aligned} a &= \frac{l_P^2 c^3}{\hbar}\frac{M}{r^2} \\ &= G\frac{M}{r^2}, \end{aligned} \tag{25}$$

in which the gravitational constant G appears in terms of the fundamental constants as $G \equiv l_P^2 c^3/\hbar$. So far, much efforts has been devoted to understanding the implications of a modification of the equipartition theorem on Verlinde's formalism. One important direction is to investigate the effects of q-equipartition theorem that emerges from the nonextensive statistical mechanics [16–20]. In this respect, the modified acceleration is given by

$$a = G_q \frac{M}{r^2}, \tag{26}$$

where the (effective) gravitational constant in a nonextensive scenario reads as

$$G_q = \frac{5-3q}{2} G. \tag{27}$$

Combining the equipartition theorems in Eq. (21) arising from the superstatistics, together with Eqs. (22), (23), and (24), one may deduce the effective gravitational constants due to the three different superstatistics as

$$G_i = \frac{G}{\phi_i(q)} \quad (i = 1, 2, 3). \tag{28}$$

Figure 2 shows the relative deviation of the gravitational constant,

$$\delta G_i \equiv \frac{G_i - G}{G}, \tag{29}$$

in the neighborhood of $q = 1$. Note that the strength of the gravitational field decreases as q increases for all superstatistics. Armed with Eq. (28), we have all the ingredients to analyze the implications of superstatistics on gravitation and cosmology.

Note that fluctuations of the gravitational constant G, given by Eq. (28), and the numerical results plotted in Fig. 2 can be corroborated by the results obtained in Refs. [56,57], which point out that fluctuations of the gravitational constant may be slightly correlated with the variations in Earth's rotation velocity within a period of 5.9 years together with other geophysical factors [56,57].

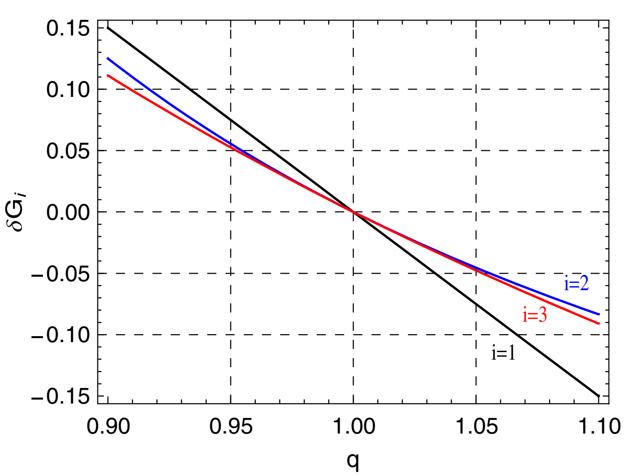

FIG. 2. Plot of $\delta G_i \equiv (G_i - G)/G$ for different superstatistics: $i = 1, \chi^2$ (black line); $i = 2$, inverse χ^2 (blue line); and $i = 3$, log-normal (red line) superstatistics, against $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2$.

Hence, the documented variations of the gravitational constant shall be considered an intrinsic feature that cannot be neglected but is taken into account in the modification of G from 1.35 to 2.2, via the q parameter. In other words, it seems that the parameter q can be adjusted to fit an Earth phenomenon already known, which does not offer a new interpretation of the fundamental problems in cosmology. On the other hand, it would be desirable to modify the dynamical and kinematical structure of Friedmann equations in order to contrast with the usual models. That will be addressed in Sec. V.

IV. GRAVITATIONAL COLLAPSE

One direct application of Eq. (28) is the Jeans criterion of gravitational instability that causes the collapse of interstellar gas clouds and subsequent star formation. The so-called Jeans length is given by [58]

$$\lambda_J = \sqrt{\frac{\pi T}{\mu m_H G \rho_0}}, \tag{30}$$

where μ, m_H, and ρ_0 stand, respectively, for the mean molecular weight, the atomic mass of hydrogen, and the equilibrium mass density. G is the gravitational constant. The Jeans criterion states that if the wavelength λ of density fluctuations is greater than λ_J the density grows exponentially with time, and the system becomes gravitationally unstable. Using the effective gravitational constants in Eq. (28), the Jeans length becomes, for the three superstatistics,

$$\lambda_i = \sqrt{\phi_i(q)}\lambda_J \quad (i = 1, 2, 3). \tag{31}$$

The Jeans length emerging from χ^2 superstatistics corresponds to the Jeans length established by Jiulin in the nonextensive scenario [59]. The three expressions in Eq. (31) reduce to λ_J in Eq. (30) in the limit $q \to 1$. In Fig. 3, $\delta\lambda_i \equiv (\lambda_i - \lambda_J)/\lambda_J$ plotted against q for different superstatistics. As we can see, for the three superstatistics, the deviation of λ_i is lower than 10% when q deviates 10% from 1. For $q > 1$, it shows that all superstatistics tend to increase the critical length, above which the system becomes unstable. This can be easily understood in light of the equipartition theorem in superstatistics in Eq. (21): as q becomes greater than 1, the mean kinetic energy increases, which allows the system to remain stable for larger oscillation wavelengths. Another way to convince ourselves of that is to compute the pressure corresponding to the three superstatistics. Namely,

$$P = \frac{nm}{3} \int\int\int B(v) v^2 d^3 v = nm\langle v^2\rangle, \tag{32}$$

where $\langle v^2\rangle$ is given in Eq. (16). Clearly, the pressure increases as q becomes greater than 1, preventing therefore the gravitational collapse from occurring for larger oscillations. If $q < 1$, the opposite of the situation described above

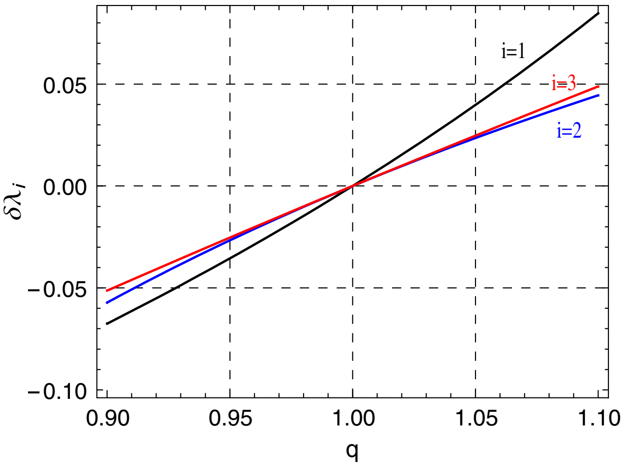

FIG. 3. Plot of $\delta\lambda_i \equiv (\lambda_i - \lambda_J)/\lambda_J$ for different superstatistics: $i = 1, \chi^2$ (black line); $i = 2$, inverse χ^2 (blue line); and $i = 3$, log-normal (red line) superstatistics, against $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2$.

occurs, and smaller wavelengths are required to make the system gravitationally unstable.

The Jeans wavelength, i.e., Eq. (30), defines the threshold wavelength, characterizing density fluctuations, beyond which gravitational collapse and subsequent star formation may occur. Similarly, one may consider the case in which the density is nearly homogeneous and seek for the mass of a gas cloud beyond which it undergoes gravitational collapse and subsequent fragmentation; this is the so-called Jeans mass.

It is worth it to address here the effect of the three universality classes of superstatistics on this quantity. One motivation behind this task is that the Jeans mass can be easily worked out in interstellar clouds of dust and gas. In some cases, as happens for the Bok globules discussed in Refs. [60,61], the predicted Jeans mass is not capable of explaining the structure formation at different scales; while some Bok globules experience the star formation process, their mass is lower than their corresponding Jeans mass, which requires the introduction of new physics [61,62]. For instance, the CB 188 Bok globule contains a protostar [60], while its mass is smaller than its corresponding Jeans mass; this demands new mechanisms that may account for a reduction of the Jeans mass.

The starting point in computing the Jeans mass is to consider the gravitational potential energy of a cloud of mass M and radius R, formed due to gravitational potential $V(r)$. Namely,

$$U = \int_0^M V(r)dM. \tag{33}$$

Considering Newtonian gravity, one has that $V(r) = -\frac{GM}{r}$, and the gravitational potential energy reads as

$$U = -\int_0^R \frac{GM}{r} 4\pi\rho(r)r^2 dr = -\frac{3GM^2}{5R}. \tag{34}$$

The occurrence of a gravitational collapse is entirely determined by the interplay between the gravitational potential energy U and the kinetic energy $\langle K\rangle$ of the cloud, approximated by an ideal gas composed of N identical noninteracting particles of mass μ. The critical mass, beyond which the cloud undergoes gravitational collapse, can be determined following a simple argument that relies upon the virial theorem; the cloud is in virial equilibrium if $\langle K\rangle = -\frac{1}{2}U$, whereas gravitational collapse may occur if $\langle K\rangle < -\frac{1}{2}U$, i.e., for

$$NT < \frac{GM^2}{5R}. \tag{35}$$

The Jeans mass is obtained by saturating the above inequality, which, upon observing that the radius R is related to the density ρ through $R = (\frac{3M}{4\pi\rho_0})^{\frac{1}{3}}$, gives

$$M_J \equiv \left(\frac{5T}{G\mu}\right)^{\frac{3}{2}} \left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}. \tag{36}$$

The cloud is stable (unstable) provided that $M \leq M_J$ ($M > M_J$). The effect of superstatistics on the Jeans mass can be straightforwardly deduced by relying on the equipartition theorems established in Sec. II. In this case, the kinetic energy (considering 3 degrees of freedom per particle) reads as

$$\langle K\rangle_q = \phi_i(q)\frac{3N}{2}T, \tag{37}$$

where $\phi_i(q)$ ($i = 1, 2, 3$) are given in Eq. (17) for the three superstatistics. The modified Jeans mass can be obtained, by repeating the same argument, to give

$$M_J^{(q)} = \phi_i(q)^{3/2} M_J. \tag{38}$$

As an example, consider the Bok globule CB 188 that has a mass $M_{CB188} \approx 7.19\ M_\odot$ and a Jeans mass $M_J \approx 7.7\ M_\odot$ [60] ($M_\odot$ being the solar mass) and is known to contain a protostar [60–62]. In this case, assuming $M_{CB188} = M_J^{(q)}$, as the upper bound of the Jeans mass that is capable of accounting for the presence of a protostar, it yields

$$\phi_i(q) = 0.955342, \tag{39}$$

from which the values of q, for the three superstatistics, follow as

$$\begin{aligned} q_1 &= 0.680182 \\ q_2 &= 0.755173 \\ q_3 &= 0.955342, \end{aligned} \tag{40}$$

where $i = 1, 2, 3$ correspond, respectively, to χ^2 superstatistics, inverse χ^2 superstatistics, and log-normal superstatistics.

TABLE I. Values of the universal parameter q for the three superstatistics ($i = 1$: χ^2; $i = 2$: inverse χ^2; and $i = 3$: log normal), in some of the Bok globules provided in Refs. [60,61].

Bok globule	$M/M_\odot$	$M_J/M_\odot$	$\phi_i(q)$	q_i
CB 87	2.73 ± 0.24	9.6	0.432442	q_1: 0.125034 q_2: 0.637935 q_3: 0.432442
CB110	7.21 ± 1.64	8.5	0.896075	q_1: 0.922681 q_2: 0.905859 q_3: 0.896075
CB131	7.83 ± 2.35	8.1	0.977652	q_1: 0.984761 q_2: 0.978141 q_3: 0.977652
CB161	2.79 ± 0.72	5.4	0.643882	q_1: 0.631280 q_2: 0.737399 q_3: 0.643882
CB184	4.70 ± 1.76	11.4	0.554725	q_1: 0.464870 q_2: 0.691910 q_3: 0.554725
FeSt 1-457	1.12 ± 0.23	1.4	0.861774	q_1: 0.893069 q_2: 0.878560 q_3: 0.861774
Lynds 495	2.95 ± 0.77	6.6	0.584591	q_1: 0.526268 q_2: 0.706510 q_3: 0.584591

For any q smaller than the values given in Eq. (40), one always has $M_{CB188} > M_J^{(q)}$, consistent with the presence of protostars in this Bok globule. The same discussion can be extended to other Bok globules, which are known to experience the star formation process [60]. The results are presented in Table I.

A word of caution is needed before closing this section. Notice that the values of q allowing for a sufficient variation of the Jeans mass, to account for the structure formation, deviate significantly from the standard MB case, i.e., $q = 1$. Thus, although the superstatistical picture allows an effective description of the gravitational collapse, capable of accounting for the star formation process, it is more likely that other mechanisms come into play. For instance, modified gravity models, such as $f(R)$ gravity [61], may introduce a notable deviation of the Jeans mass. Similarly, quantum gravitational effects, accounted for through the generalized uncertainty principle or the extended uncertainty principle, can modify the limit of collapse [62].

V. FRIEDMANN EQUATIONS FROM SUPERSTATISTICS EQUIPARTITION THEOREMS

The dynamical evolution of the Friedmann-Robertson-Walker (FRW) Universe is governed by the Friedmann equation, which can be obtained through the holographic principle together with the equipartition law of energy [6,63]. Here, we will analyze the implications of the superstatistical equipartition theorems in Eq. (21) on Friedman equations. Our starting point is the Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{41}$$

where $a(t)$ is the scale factor of the Universe and $k = -1, 0, 1$ is the curvature constant. Following Ref. [6], let us consider a compact spatial region $\mathcal{S}$ with a compact boundary $\partial\mathcal{S}$—*a sphere with physical radius* $\tilde{r} = ar$—that plays the role of the holographic screen. Let M be the mass that would emerge in the compact spatial region $\mathcal{S}$, and assume that the Universe is filled by a perfect fluid that is characterized by the stress-energy tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{42}$$

where ρ and p are, respectively, the total density and the total pressure of the fluid. Changes in the mass are due to the work given by the pressure, i.e., $dM = pdV$. Consequently, the continuity equation reads as

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{43}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. The total mass inside the spatial region Λ is given by

$$M = \int_\Lambda dV T_{\mu\nu} u^\mu u^\nu, \tag{44}$$

where $T_{\mu\nu} u^\mu u^\nu$ is the energy density measured by a comoving observer. A comoving observer at position r of the screen measures an acceleration, caused by the matter in the spatial region enclosed by the boundary $\partial\mathcal{S}$, as

$$a_r = -\frac{d^2\tilde{r}}{dt^2} = -\ddot{a}r, \tag{45}$$

from which the Unruh temperature in Eq. (24) can be rewritten as

$$T = \frac{1}{2\pi c}\hbar\ddot{a}r. \tag{46}$$

Let us define the active gravitational mass $\mathcal{M}$, different from the total mass in the spatial region $\mathcal{S}$. It is the well-known Tolman-Komar mass [63]

$$\begin{aligned}\mathcal{M} &= 2\int_\Lambda dV\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)u^\mu u^\nu \\ &= \frac{4\pi}{3}a^3 r^3\left(\rho + \frac{3p}{c^2}\right).\end{aligned} \tag{47}$$

Using the number of bits in Eq. (22) and the equipartition theorems in Eq. (21), where the mass is considered to be the active gravitational mass in Eq. (47), one has, for the three universality classes of superstatistics, the acceleration equation for the dynamical evolution of the FRW Universe as

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G_i(\rho + 3p) \quad (i = 1, 2, 3). \tag{48}$$

By integrating Eq. (48) and using the continuity equation in Eq. (43), we can write

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_i}{3}\rho \quad (i = 1, 2, 3), \tag{49}$$

which are the Friedmann equations for the three universality classes of superstatistics. In the above equations, the integration constant k is the curvature constant, which appears in the Robertson-Walker metric (41). The Friedmann equation corresponding to χ^2 superstatistics is the one obtained in nonextensive statistical mechanics and discussed in a series of papers [17,18]. The usual Friedmann equation is recovered in the absence of fluctuations, i.e., $q = 1$.

A. q modified ΛCDM model

Consider a FRW universe containing nonrelativistic matter (baryonic and dark matter), radiation, and a cosmological term. By taking into account the superstatistics modifications of the Friedmann equation in Eq. (49), the Universe expansion rate becomes

$$H = \frac{H_0}{\sqrt{\phi_i(q)}} \sqrt{\frac{\Omega_{\gamma,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2} + \Omega_{\Lambda,0}}, \tag{50}$$

where $i = 1, 2, 3$ and

$$\Omega_{j,0} = \frac{8\pi G \rho_{i,0}}{(3H_0^2)},$$

is the density parameter of the jth component; $j \equiv \gamma$, m, and Λ for radiation, matter (baryonic plus dark), and quantum vacuum, respectively. The normalization conditions for each case read as

$$\Omega_{\gamma,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{x,0} = \phi_i(q) \quad (i = 1, 2, 3). \tag{51}$$

In what follows, we will examine the observational constraints on the parameters $(h, \Omega_{b,0}h^2, \Omega_{m,0}, q)$ of this model.

TABLE II. Measurements of d and $\mathcal{A}$ used in this paper.

z	d	$\mathcal{A}$	Ref.
0.106	0.336 ± 0.015	…	[75]
0.57	0.0732 ± 0.0012	…	[76]
0.2	0.1905 ± 0.0061	…	[77]
0.35	0.1097 ± 0.0036	…	[77]
0.44	…	0.474 ± 0.034	[78]
0.6	…	0.441 ± 0.020	[78]
0.73	…	0.424 ± 0.021	[78]

VI. OBSERVATIONAL CONSTRAINTS

To constrain the model parameters, we use distance measurements of type Ia supernovae from the binned Joint Light-Curve Analysis (JLA) dataset [64]; the cosmic microwave background radiation (CMB) distance priors $\mathcal{R}$ and l_A along with the baryon density $\Omega_b h^2$ derived from Planck 2018 TT, TE, EE + lowE [65] in Ref. [66]; the 31 measurements of $H(z)$ compiled from Refs. [67–73]; the local measurement of the Hubble constant H_0 obtained in Ref. [74]; and measurements of the distilled parameters d_z and $\mathcal{A}$ from baryonic acoustic oscillations (BAOs) listed in Table II. Here, we assume that $\Omega_{k,0} = 0$. Our results are summarized in Table III and Fig. 4.

Table III contains the best fit parameters for the $q\Lambda$CDM models derived in this paper. The errors correspond to a 1σ ($\Delta\chi^2 = 1$) confidence interval for each parameter. For the sake of comparison, Table III also displays the best fit for the ΛCDM model. The last two columns of Table III list the corrected Akaike Information Criterion ($\mathrm{AIC_C}$) [79] and the Bayesian Information Criterion (BIC) [80] values for the ΛCDM and $q\Lambda$CDM models. These information criteria, which provide a fair way to compare models with different numbers of parameters, are defined, respectively, by

TABLE III. Observational constraints on the models free parameters for the ΛCDM and the $q-\Lambda$CDM models. The errors correspond to a 1σ ($\Delta\chi^2 = 1$) statistical uncertainty for each parameter.

Model	$\Omega_{\mathrm{m},0}$	q	h	$\Omega_{b,0}h^2 \times 10^2$	χ^2_{min}	$\mathrm{AIC_C}$	BIC
χ^2 superstatistics	$0.295^{+0.0091}_{-0.0098}$	$1.009^{+0.0065}_{-0.0052}$	$0.687^{+0.006}_{-0.005}$	$2.2427^{+0.0143}_{0.0143}$	66.291	74.85	83.67
Inverse χ^2 superstatistics	$0.295^{+0.0090}_{-0.0096}$	$1.015^{+0.0102}_{-0.0102}$	$0.687^{+0.006}_{-0.005}$	$2.2416^{+0.0153}_{-0.0136}$	66.290	74.85	83.67
Log-normal superstatistics	$0.294^{+0.0096}_{-0.0090}$	$1.015^{+0.0096}_{-0.0102}$	$0.687^{+0.006}_{-0.006}$	$2.2408^{+0.0162}_{-0.0128}$	66.291	74.85	83.67
ΛCDM	$0.305^{+0.0064}_{-0.0064}$	1.000	$0.683^{+0.0048}_{-0.0048}$	$2.2509^{+0.0136}_{-0.0136}$	68.444	74.77	81.47

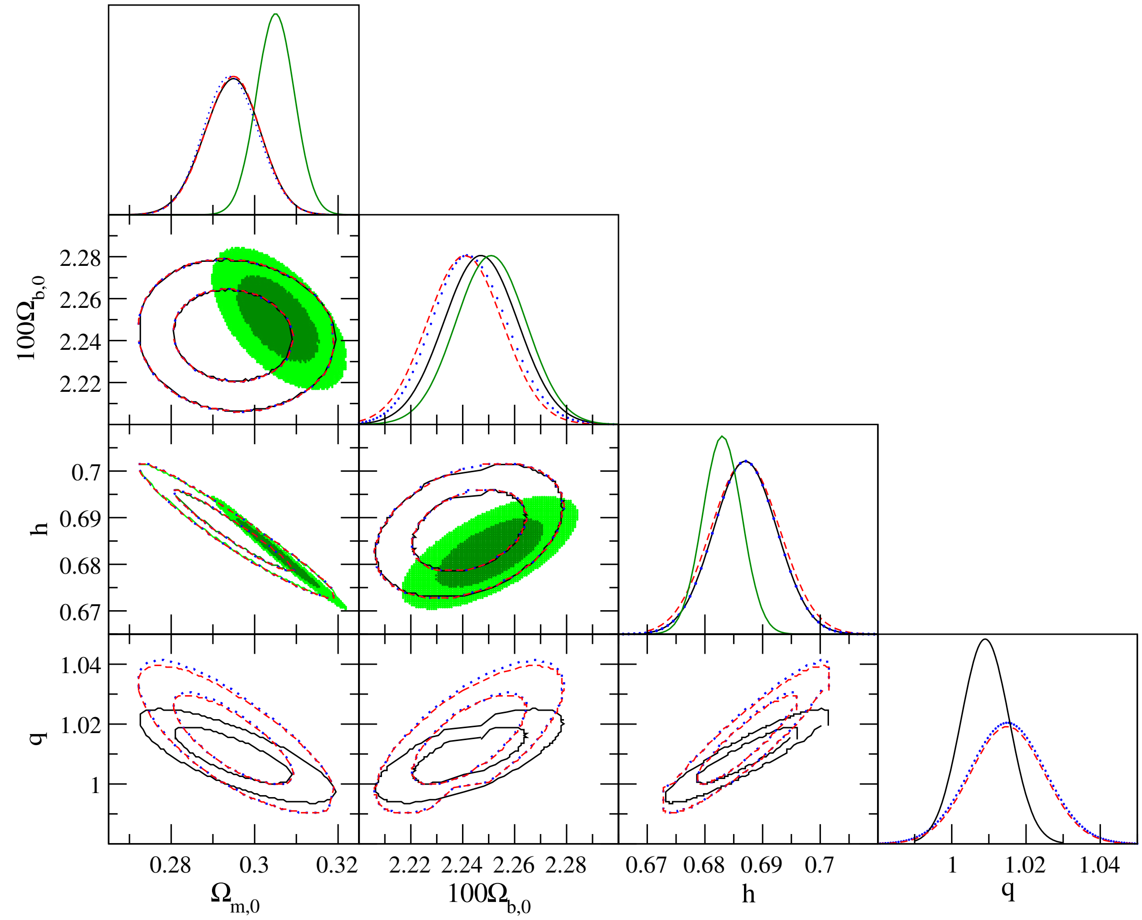

FIG. 4. Probability distribution functions and marginalized confidence regions at 1σ and 2σ for the free parameters of the model. The black continuous line, red dashed line, and blue dotted line denote, respectively, the χ^2, inverse χ^2, and log-normal superstatistics. The continuous green line represents the ΛCDM model.

$$\mathrm{AIC_C} \equiv \chi^2_{\mathrm{min}} + \frac{2kN}{N-k-1} \tag{52}$$

and

$$\mathrm{BIC} \equiv \chi^2_{\mathrm{min}} + k \ln N, \tag{53}$$

where k is the number of parameters of a given model and N is the number of data points. In the AIC$_\mathrm{C}$ language, the $q\Lambda$CDM models have substantial evidence support ($\Delta\mathrm{AIC_C} < 2$), while in the BIC terminology, the evidence against these models is positive ($2 < \Delta\mathrm{BIC} < 6$).

Figure 4 shows the probability distribution functions and the marginalized confidence regions at 1σ and 2σ. The black continuous line, red dashed line, and blue dotted line denotes, respectively, the χ^2, inverse χ^2, and log-normal superstatistics. The continuous green line represents the ΛCDM model. At 1σ, the dataset employed in our analysis constrains $q > 1$, favoring the interpretation of non-Gaussian distributions emerging from fluctuations. The χ^2 superstatistics provides tighter constraints on $q = \langle\beta^2\rangle/\langle\beta\rangle^2$ than the inverse χ^2 and the log-normal superstatistics, which are almost indistinguishable from one another. This result is expected since in the χ^2 superstatistics frame small deviations of $q = 1$ lead to deviations of the standard MB greater than in the inverse χ^2 and the log-normal superstatistics frames. The value of the Hubble constant H_0 is slightly higher in the $q\Lambda$CDM models than in the standard ΛCDM model, as we should expect theoretically since for $q > 1$ the gravitational field is weaker. However, this small difference is not enough to alleviate the tension between the local and the global estimates of H_0.

VII. CONCLUSION

We have studied the effect of non-Gaussian statistics on gravitation and cosmology by adopting the viewpoint of superstatistics, in which non-Gaussian statistics are seen as due to fluctuations. Combining the equipartition theorems for the three universality classes of superstatistics (χ^2, inverse χ^2, and log normal) with the holographic principle, we have addressed the main consequences on gravitation and cosmology, generalizing therefore the results previously addressed within nonextensive statistical mechanics [16–18,20–22]. In our formulation, the latter emerge as a special case, produced by χ^2 fluctuations. We have explored the effect of fluctuations on the Jeans criterion, Friedmann equations, and the ΛCDM model. In the case of Jeans instability, fluctuations appear to enhance the critical Jeans length ($q > 1$), a feature that can be attributed to the "fat tails" exhibited by non-Gaussian distributions, leading therefore to an increase in the mean energy. For $q < 1$, smaller wavelengths λ of density fluctuation cause the gravitational collapse. We have investigated the effects of superstatistics on the cosmic expansion. To observationally constrain the model parameters, we have used the distance measurements of type Ia Supernovae, BAO, CMB, cosmic chronometers, and the local H_0 measurement. Although our results are compatible with the ΛCDM model, we have obtained, at 1σ confidence level, that the universality parameter q is greater than 1 for all classes of superstatistics, favoring therefore the interpretation of non-Gaussian distributions as emerging from fluctuations.

Our approach can be understood as a first step toward dealing with entropic gravitation and cosmology in non-equilibrium statistical mechanics. It opens some perspectives for further investigations. Possible application areas of research include cosmos expansion and black hole physics. Note also that, although we have analyzed herein the three universality classes of fluctuations that have a transparent statistical origin, it is not excluded that other distributions, not identified yet, can be used to model a variety of cosmological problems and fit observational data.

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