

Thermodynamical hairs of the four-dimensional Taub-Newman-Unti-Tamburino spacetimes

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It is demonstrated that the generic four-dimensional Taub-Newman-Unti-Tamburino (Taub-NUT) spacetimes can be perfectly described in terms of three or four different kinds of thermodynamic hairs: the Komar mass ($M = m$), the “angular momentum” ($J_n = mn$), the gravitomagnetic charge ($N = n$), and/or the dual (magnetic) mass ($\tilde{M} = n$). In other words, the NUT charge is a thermodynamic multihair which means that it simultaneously has both rotation-like and electromagnetic charge-like characteristics; this is in sharp contrast with the previous knowledge that it has only one physical feature, or that it is purely a single solution parameter. To arrive at this novel result, we put forward a simple, systematic way to investigate the consistent thermodynamic first law and Bekenstein-Smarr mass formulas of all four-dimensional spacetimes that contain a nonzero NUT charge, facilitated by first deriving a meaningful Christodoulou-Ruffini-type squared-mass formula. In this way, not only can the elegant Bekenstein-Hawking one-quarter area-entropy relation be naturally restored in the Lorentzian and Euclidian sectors of generic Taub-NUT-type spacetimes without imposing any constraint condition, but also the physical meaning of the NUT parameter as a poly-facet can be completely clarified in the thermodynamic sense for the first time.

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I. INTRODUCTION

Ever since the seminal work of Bekenstein [1] and Hawking [2], it has been well known that the (Bekenstein-Hawking) entropy of a black hole is proportional to the area of the horizon and its Hawking temperature to the surface gravity at the horizon. In terms of the natural unit system, the relations between them are simply given by

$$S = A/4, \quad T = \kappa/(2\pi). \quad (1)$$

The four laws of black hole thermodynamics were also found for asymptotically flat and anti-de Sitter (AdS) black holes. In particular, the differential first law and the integral Bekenstein-Smarr mass formula in the $D = 4$ asymptotically flat case read [3,4]

$$\begin{aligned} dM &= TdS + \Omega dJ + \Phi dQ + \dots, \\ M &= 2TS + 2\Omega J + \Phi Q + \dots. \end{aligned} \quad (2)$$

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It is remarkable that these mass formulas provide the more elaborate relationship between the global conserved charges (M, J, Q , etc.) measured at infinity and the horizon temperature, entropy, and other quantities (Ω, Φ) which are evaluated at the horizon but relative to infinity. When a negative cosmological constant is included for the anti-de Sitter case, the above mass formulas (2) should include a modified term ($+VdP, -2VP$), respectively, where V is the thermodynamic volume conjugate to the pressure $P = 3g^2/(8\pi)$ with g being the inverse of the cosmological radius.

What is more, a Christodoulou-Ruffini-type squared-mass formula was found to be [5,6]

$$M^2 = \frac{\pi}{4S} \left(\frac{S}{\pi} + Q^2 \right)^2 + \frac{\pi J^2}{S} \quad (3)$$

for the Kerr-Newman black hole, and was later generalized to the Kerr-Newman-AdS₄ case [7]. It should be mentioned that the first law in Eq. (2) can be simply deduced [8] via differentiating the squared-mass formula (3) with respect to all of its thermodynamic variables, and then the Bekenstein-Smarr mass formula can be easily verified.

Recently, the thermodynamics of accelerating (charged and rotating) AdS₄ black holes [9] was discussed in Refs. [10–13] where all of the above formulas (1)–(3)

were extended to this kind of spacetimes which belong to the class of double-black-hole solutions with one black hole's event horizon becoming the Rindler horizon.

However, it still seems to be a great exception to asymptotically locally flat spacetimes that are endowed with a nonzero NUT charge in general relativity. Until now, no consistent thermodynamic formula similar to the above-mentioned relations (1)–(3) has been fully and satisfactorily found for this class of spacetimes in both the Lorentzian and Euclidian sectors, even for the simplest Taub-NUT spacetime [14,15]. As far as the first law (2) is concerned, to the authors' best knowledge, the differential mass formulas for the NUT-charged spacetimes that appeared in Refs. [16–24] are either inconsistent (here “inconsistent” means that the thermodynamic quantities cannot constitute the ordinary conjugate pairs) or even are false. (Nevertheless, it should be noted that consistent mass formulas for the Demianski-Newman “black hole” were already conjectured in Ref. [25] without any “derivation” almost two decades ago!)

In some recent attempts [26–30], the so-called “consistent thermodynamical first law” was pursued for the Lorentzian Taub-NUT-type spacetimes. However, these formulas could not really represent the actual first law from our viewpoint, since the imported $\bar{\psi}\text{-}\mathcal{N}$ pair (which was later called the “Misner gravitational charge”) does not possess the conventional characteristics of global charges that are measured at infinity; rather, it combines the contributions of the Misner strings at the horizon, contrary to common wisdom. Recently, this thermodynamic pair were alternatively explained in Ref. [31] as the angular velocity and angular momentum of the Misner strings, rather than being interpreted as the temperature and entropy as in Ref. [27], since this will seriously challenge the zeroth law of black hole thermodynamics. According to common sense, it is hardly believable that for a static axisymmetric Taub-NUT-type spacetime, the temperatures (or surface gravities) at the north and south poles are different from those at the remaining part of the event horizon. If such an interpretation [27] is true, then the horizon must be a nonequilibrium system for a stationary black hole, thus violating the well-established zeroth law.

In this paper, based upon a previously unpublished talk [32], we put forward a simple, systematic routine to investigate the thermodynamics of the four-dimensional spacetimes with a nonzero NUT charge by first deriving a meaningful Christodoulou-Ruffini-type squared-mass formula, where a new “angular momentum” $J_n = mn$ is additionally introduced as an extra conserved charge. Starting from this squared-mass formula, we derive a consistent first law and Bekenstein-Smarr mass formula for the NUT-charged spacetimes just like the usual black holes, without assuming that the famous one-quarter area-entropy relation should hold true in order to get a consistent thermodynamic first law. In this way, we show that the

NUT charge is a thermodynamic multihair, which means that it simultaneously has both rotation-like and electromagnetic charge-like characteristics. The novelty of this new viewpoint is that it can plausibly explain many of the peculiar properties of the NUT-charged spacetime, such as why the NUT parameter has so many different names and why there are different interpretations of the physical source of Taub-NUT-type spacetimes.

II. THE LORENTZIAN TAUB-NUT GEOMETRY

To begin with, let us first recapitulate some known basic facts of the four-dimensional Taub-NUT metric in the Lorentzian sector [15]. We adopt the following line element in which the Misner strings [33] are symmetrically distributed along the polar axis:

$$ds^2 = -\frac{f(r)}{r^2 + n^2} (dt + 2n \cos \theta d\phi)^2 + \frac{r^2 + n^2}{f(r)} dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

where $f(r) = r^2 - 2mr - n^2$.

The spacetime (4) has many peculiar properties that are mainly due to the presence of wire/line singularities at the polar axis ($\theta = 0, \pi$), which are often called Misner strings. Misner [33] proposed removing these singularities (so as to ensure the metric's regularity) by introducing a time-periodical identification condition: $\beta = 8\pi n$. Then, the unavoidable presence of closed timelike curves led him [34] to declare that the NUT parameter is unphysical and the Taub-NUT spacetime is “a counter example to almost anything” in general relativity. The evil consequence is that the dominant community does not usually consider Taub-NUT-type spacetimes to be black holes (although sometimes they were called “black holes” in some low-level articles).

In the following, we will derive various mass formulas of four-dimensional Taub-NUT-type spacetimes without imposing the time-periodicity condition, as was done in Refs. [26,28,30,31,35–38]. We will also keep the Misner strings symmetrically present at the polar axis and only care about the conical singularities where $f(r) = 0$, corresponding to the outer and inner horizons located at $r_h = r_{\pm} = m \pm \sqrt{m^2 + n^2}$. Below, we will focus on the (exterior) event horizon; however, the discussions are also true for the interior (Cauchy) horizon.

The area and surface gravity at the horizon are easily computed via the standard method as

$$A_h = 4\pi(r_h^2 + n^2), \quad \kappa = \frac{f'(r_h)}{2(r_h^2 + n^2)} = \frac{r_h - m}{r_h^2 + n^2} = \frac{1}{2r_h}. \quad (5)$$

As for the global conserved charges, the well-known Komar mass at infinity related to the timelike Killing

vector ∂_t and the NUT charge or gravitational magnetic (gravitomagnetic) charge [39] can be computed, respectively, as $M = m$ and $N = n$ [40,41]. Its horizon mass [41,42] reads $M_h = r_h - m$. On the other hand, one can also determine the dual or magnetic-type mass [43–46] as $\tilde{M} = n \equiv N$. It is clear that one cannot distinguish the dual or magnetic-type mass from the gravitomagnetic charge in the present case; however, they will be significantly different from each other in the case where a nonzero cosmological constant is included.

III. NEW CHARGE $J_n = mn$ AND SQUARED-MASS FORMULA

In order to derive a reasonable first law, we follow the method used in Ref. [8] to deduce a meaningful Christodoulou-Ruffini-type squared-mass formula which is the starting point of our work. Introducing the “reduced horizon area” $\mathcal{A}_h = A_h/(4\pi)$ as in Ref. [8] just for the sake of simplicity,

$$\mathcal{A}_h = r_h^2 + n^2 = 2mr_h + 2n^2, \quad (6)$$

and shifting the $2n^2$ term to the left-hand side and squaring the obtained formula, we get the identity

$$(\mathcal{A}_h - 2n^2)^2 = 4m^2r_h^2 = 4m^2\mathcal{A}_h - 4m^2n^2. \quad (7)$$

Using only $M = m$ and $N = n$ as the conserved charges would lead to the squared-mass formula that appeared in Ref. [47], $4M^2 = (\mathcal{A}_h - 2N^2)^2/(\mathcal{A}_h - N^2)$, which will give rise to inconsistent versions [16–24] of the first law and integral mass formula. Then, nothing new would take place and the story would end.

On the contrary, suppose we introduce a new quantity that is closely analogous to the angular momentum $J = ma$ of the Kerr(-Newman) black hole and is given by

$$J_n = mn. \quad (8)$$

Then, we obtain a new squared-mass formula that is almost completely analogous to that of a Kerr-type black hole presented in Ref. [6],

$$M^2 = \frac{1}{4\mathcal{A}_h}(\mathcal{A}_h - 2N^2)^2 + \frac{J_n^2}{\mathcal{A}_h} = \frac{\mathcal{A}_h}{4} + \frac{J_n^2 + N^4}{\mathcal{A}_h} - N^2, \quad (9)$$

which forms the basis of our work.

As shown in Ref. [48], the fact that $J_n = mn \equiv M_5$ corresponds to the mass of a five-dimensional gravitational magnetic monopole means that it is very natural to consider it as a conserved charge, at least from the viewpoint of five dimensions. There are also a lot of reasons to support such an idea. For example, it explains the gyromagnetic ratio

[49,50] of Kerr-Taub-NUT-type spacetimes. Furthermore, not only is it related to the gravitational action [51,52] of the four-dimensional Taub-NUT-type spacetimes, but it also possesses the general feature of angular momentum [37,38,53]. Recently, it was included in Ref. [30] to get a consistent first law for the Lorentzian Taub-NUT spacetimes.

It should be pointed out that our identification of $J_n = mn$ as a new “conserved charge” is the only input of our procedure. In the following, the complete set of conserved charges for the Taub-NUT spacetime that we will work with is $M = m$, $N = n$, and $J_n = mn$.

IV. DERIVATION OF DIFFERENTIAL AND INTEGRAL MASS FORMULAS

Now, as in Ref. [8], we can view the mass as an implicit function $M = M(\mathcal{A}_h, J_n, N)$, and after differentiating the squared-mass formula (9) (multiplied by $4\mathcal{A}_h$) with respect to its variables we get a new reasonable differential mass formula,

$$dM = (\kappa/2)d\mathcal{A}_h + \omega_h dJ_n + \psi_h dN, \quad (10)$$

where

$$\begin{aligned} \kappa &= 2 \left. \frac{\partial M}{\partial \mathcal{A}_h} \right|_{J_n, N} = \frac{\mathcal{A}_h - 2N^2 - 2M^2}{2M\mathcal{A}_h} = \frac{r_h - m}{r_h^2 + n^2}, \\ \omega_h &= \left. \frac{\partial M}{\partial J_n} \right|_{\mathcal{A}_h, N} = \frac{J_n}{M\mathcal{A}_h} = \frac{n}{r_h^2 + n^2}, \\ \psi_h &= \left. \frac{\partial M}{\partial N} \right|_{\mathcal{A}_h, J_n} = \frac{-N(\mathcal{A}_h - 2N^2)}{M\mathcal{A}_h} = \frac{-2nr_h}{r_h^2 + n^2}. \end{aligned}$$

Then, it is easy to verify directly that the Bekenstein-Smarr integral mass formula is completely satisfied,

$$M = \kappa\mathcal{A}_h + 2\omega_h J_n + \psi_h N, \quad (11)$$

after using the horizon equation $f(r_h) \equiv r_h^2 - 2mr_h - n^2 = 0$.

Comparing our new mass formulas presented in Eqs. (9)–(11) with the standard ones (2)–(3), it makes sense to make the familiar identifications

$$T = \kappa/(2\pi), \quad S = \pi\mathcal{A}_h = A/4, \quad (12)$$

which restore the famous Bekenstein-Hawking one-quarter area-entropy relation in a very simple manner. It is quite remarkable that one should assign a geometric entropy to the Taub-NUT spacetime, which is just one quarter of its horizon area. In the above “derivation” we did not require that the relations (12) should hold true ahead to get a reasonable first law, rather it is a very natural result via the above thermodynamic deduction.

It is remarkable that, unlike some recent attempts [26–31], our first law (10) and the Bekenstein-Smarr mass formula (11) attain their traditional forms which relate the global conserved charges (M, N, J_n) measured at infinity to those quantities (T, S, ψ_h, ω_h) evaluated at the horizon. At this step, it is quite reasonable to infer that all four laws of the usual black hole thermodynamics are applicable to the Taub-NUT spacetime, which should no longer bear the bad reputation of being “a counter example to almost anything.” It is now time to formally call it a genuine black hole, at least as far as its thermodynamics are considered.

V. IMPACT OF MISNER STRINGS

Without attempting to endow each of the Misner strings with an entropy (and thus also a temperature), we can see that each string attached at the south and north poles carries the same amount of rotation-like and electromagnetic-like energies. The total contribution of both strings to the differential and integral mass formulas (10)–(11) is a cumulative effect that consists of two terms that can be rewritten, respectively, as follows:

$$\begin{aligned} \omega_h dJ_n + \psi_h dN &= \frac{n^2 dm + n(m - 2r_h) dn}{r_h^2 + n^2} \\ &\equiv -\frac{1}{2n} d\left(\frac{n^3}{r_h}\right) \equiv \frac{1}{n} dJ_n - \frac{1}{2n} d(nr_h), \end{aligned} \quad (13)$$

$$\begin{aligned} 2\omega_h J_n + \psi_h N &= \frac{2n^2(m - r_h)}{r_h^2 + n^2} = \frac{-n^2}{r_h} = 2m - r_h \\ &\equiv -2\frac{1}{2n} \left(\frac{n^3}{r_h}\right) \equiv 2\frac{J_n}{n} - 2\frac{nr_h}{2n}. \end{aligned} \quad (14)$$

This explains why recent efforts to formulate a “consistent thermodynamic first law” that is mathematically consistent were only partially successful [26,30]. Clearly, neither the potential $\bar{\psi} = 1/(4n)$ introduced in Ref. [26] nor the angular velocity $(1/n$ or $1/(2n))$ proposed in Refs. [30,31] has a well-defined limit when the NUT charge vanishes. Furthermore, the charges n^3/r_h and nr_h in Refs. [26,30] do not really have the general characteristics of a globally conserved charge; rather, they must be some quantities related to the horizon. On the contrary, no such fatal defect exists in the present work, which smoothly reduces to the Schwarzschild case.

VI. ANALYTICAL CONTINUATION TO THE EUCLIDEAN SECTOR

The Euclidean sector [40,54,55] is obtained via the Wick rotation $t = i\tau$ and $n = iN$. Now $f(r) = r^2 - 2Mr + N^2$, with $M = m$. We will work with the generic values of the solution parameters (M, N) , neither introducing the time-periodic identification condition $\beta = 8\pi N$ nor imposing any constraint condition on the solution parameters.

The horizons are located at $r_H = M \pm \sqrt{M^2 - N^2}$, which are determined by $f(r_H) = 0$. The event horizon area is $A_H = 4\pi\mathcal{A}_H$, where $\mathcal{A}_H = r_H^2 - N^2 = 2Mr_H - 2N^2$, and its surface gravity is easily evaluated as

$$\kappa = \frac{f'(r_h)}{2\mathcal{A}_H} = \frac{r_H - M}{r_H^2 - N^2} = \frac{1}{2r_H}. \quad (15)$$

After introducing a conserved quantity $J_N = MN$ as before, we arrive at a squared-mass formula

$$M^2 = \frac{(\mathcal{A}_H + 2N^2)^2 - 4J_N^2}{4\mathcal{A}_H}. \quad (16)$$

Similar to the Lorentzian case, one can deduce the following differential and integral mass formulas:

$$\begin{aligned} dM &= (\kappa/2)d\mathcal{A}_H + \omega_H dJ_N + \psi_H dN, \\ M &= \kappa\mathcal{A}_H + 2\omega_H J_N + \psi_H N, \end{aligned} \quad (17)$$

where $\kappa = (r_H - M)/\mathcal{A}_H$, $\psi_H = 2Nr_H/\mathcal{A}_H$, and $\omega_H = -N/\mathcal{A}_H$.

It is natural to suggest that one should identify a geometric entropy with $S = A_H/4 = \pi\mathcal{A}_H$ and the Hawking temperature via $T = \kappa/(2\pi)$, so that a fairly satisfactory relation also holds true for the thermodynamics of the Euclidean Taub-NUT solution with the generic parameters (M, N) , which can be roughly interpreted as a spinning Misner string with angular momentum $J_N = MN$ and gravitomagnetic charge N .

Now, let us impose the periodic condition $\beta = 8\pi N = 2\pi/\kappa$ and discuss the nut and bolt cases, separately. A self-dual Taub-NUT solution is the special case when $M = N$, with a nut at $r_H = N$ with zero horizon area ($\mathcal{A}_H = 0$) and surface gravity $\kappa = \partial_r[(r - N)/(r + N)]|_{r=N}/2 = 1/(4N)$. In this case, both ω_H and ψ_H become infinite; however, the compositions

$$\begin{aligned} \omega_H dJ_N + \psi_H dN &= dN \equiv dM, \\ 2\omega_H J_N + \psi_H N &= N \equiv M \end{aligned} \quad (18)$$

remain finite. [For example, one can let $r_H = N + \varepsilon$ and then take the $\varepsilon \rightarrow 0$ limit. Note that when this limit is applied to Eq. (15), an additional factor of 1/2 should be multiplied to get the correct value $\kappa = 1/(4N)$ for the surface gravity.] The contribution of the Misner strings to the nut solution is $N^2/r_H = N$.

In contrast, the regular bolt solution is obtained when $M = 5N/4$. In this case, the horizon is located at $r_H = 2N$, and the other quantities are $\kappa = 1/(4N)$, $\mathcal{A}_H = 3N^2$, $\omega_H = -1/(3N)$, $J_N = 5N^2/4$, and $\psi_H = 4/3$, so it is easy to show that

$$\begin{aligned}\omega_H dJ_N + \psi_H dN &= dN/2 \equiv M - (\kappa/2)d\mathcal{A}_H, \\ 2\omega_H J_N + \psi_H N &= N/2 \equiv M - \kappa\mathcal{A}_H,\end{aligned}\quad (19)$$

which shows that the contribution of the Misner strings to the bolt solution is $N^2/r_H = N/2$.

Since the bolt and nut solutions are matched with the identical surface gravity $\kappa = 1/(4N)$, their mass difference is $\Delta M = 5N/4 - N = N/4$, while their reduced area difference is $\Delta\mathcal{A}_H = 3\pi N^2$. Thus, the contribution of the Misner strings is $\Delta M - \kappa\Delta\mathcal{A}_H = N/4 - 3N/4 = -N/2$, which coincides with the total contribution of the Misner strings $N/2 - N = -N/2$.

One can understand that the entropy obtained by Hawking *et al.* [40,54,55] and Mann [56] is a (generalized) relative entropy, which is computed for the bolt solution with respect to the reference background, namely, the self-dual nut solution.

VII. ADDING A NONZERO NEGATIVE COSMOLOGICAL CONSTANT AND ELECTRIC CHARGE

Now we can extend the above work to the Lorentzian Reissner-Nordström-Taub-NUT-AdS₄ spacetime with a nonzero cosmological constant and a pure electric charge. The metric is still given by Eq. (4) with the parameters (m, n) replaced by (M, N) , and now $f(r) = r^2 - 2Mr - N^2 + Q^2 + g^2(r^4 + 6N^2r^2 - 3N^4)$. In addition, the electromagnetic gauge potential one-form is

$$\mathbf{A} = \frac{Qr}{r^2 + N^2}(dt + 2N \cos\theta d\phi). \quad (20)$$

It is easy to compute the electric charge Q and gravito-magnetic charge N of the spacetime, while its electric mass M and dual (magnetic) mass [57] $\tilde{M} = N(1 + 4g^2N^2)$, which is different from the NUT charge now, can be calculated using the conformal completion method.

The solution admits Killing horizons which are determined by $f(r_h) = 0$, where the electrostatic potential is $\Phi = Qr_h/(r_h^2 + N^2)$. The horizon area is $A_h = 4\pi\mathcal{A}_h$ and the surface gravity is given by

$$\kappa = \frac{f'(r_h)}{2\mathcal{A}_h} = \frac{r_h - M + 2g^2(r_h^2 + 3N^2)r_h}{\mathcal{A}_h}, \quad (21)$$

where $\mathcal{A}_h = r_h^2 + N^2 = 2Mr_h + 2N^2 - Q^2 - g^2(r_h^4 + 6N^2r_h^2 - 3N^4) = 2Mr_h + 2N^2 - Q^2 - g^2(\mathcal{A}_h^2 + 4N^2\mathcal{A}_h - 8N^4)$.

Now, after introducing $J_N = MN$, squaring the identity $2Mr_h = (1 + 4g^2N^2)(\mathcal{A}_h - 2N^2) + Q^2 + g^2\mathcal{A}_h^2$, and adding a term $4M^2N^2$ to it, we get

$$M^2\mathcal{A}_h = J_N^2 + \frac{1}{4}[(1 + 4g^2N^2)(\mathcal{A}_h - 2N^2) + g^2\mathcal{A}_h^2 + Q^2]^2,$$

which is nothing but the squared-mass formula

$$M^2 = \frac{1}{\mathcal{A}_h} \left[\left(1 + \frac{32\pi}{3}PN^2 \right) (\mathcal{A}_h - 2N^2) + Q^2 + \frac{8\pi}{3}P\mathcal{A}_h^2 \right]^2 + \frac{J_N^2}{\mathcal{A}_h}, \quad (22)$$

where $P = 3g^2/(8\pi)$ is the generalized pressure.

The differentiation of the mass formula (22) leads to the first law

$$dM = (\kappa/2)d\mathcal{A}_h + \omega_h dJ_N + \psi_h dN + \Phi dQ + V dP, \quad (23)$$

where

$$\begin{aligned}\kappa &= \frac{r_h - M + 2g^2(r_h + 3N^2)r_h}{\mathcal{A}_h}, & \Phi &= \frac{Qr_h}{\mathcal{A}_h}, \\ \psi_h &= 2Nr_h \frac{-1 + 2g^2(r_h^2 - 3N^2)}{\mathcal{A}_h}, & \omega_h &= \frac{N}{\mathcal{A}_h}, \\ V &= \frac{4\pi(r_h^4 + 6N^2r_h^2 - 3N^4)r_h}{3\mathcal{A}_h}.\end{aligned}$$

Then, we can directly verify that the Bekenstein-Smarr mass formula

$$\kappa\mathcal{A}_h + 2\omega_h J_n + \psi_h N + \Phi Q - 2VP = M \quad (24)$$

is completely satisfied. It is natural to recognize $S = A_h/4 = \pi\mathcal{A}_h$ and $T = \kappa/(2\pi)$, so that the solution behaves like a genuine black hole without violating the beautiful one-quarter area/entropy law. In sharp contrast with Refs. [26–30], here we neither insist that this law be obeyed nor require that the first law and the integral mass formula be consistent. This is a very natural product of the pure thermodynamic deduction.

In the above, the derived conjugate thermodynamic volume V is not equal to $\tilde{V} = 4\pi(r_h^2 + 3N^2)/3$ as given in Refs. [26–29]. If one prefers to use such a thermodynamic volume, then the dual (magnetic) mass $\tilde{M} = N(1 + 4g^2N^2)$ can be further introduced as an additional conserved charge into the differential and integral mass formulas:

$$\begin{aligned}dM &= (\kappa/2)d\mathcal{A}_h + \omega_h dJ_N + \tilde{\psi}_h dN + \zeta d\tilde{M} \\ &\quad + \Phi dQ + \tilde{V} dP, \\ M &= \kappa\mathcal{A}_h + 2\omega_h J_n + \tilde{\psi}_h N + \zeta\tilde{M} + \Phi Q - 2\tilde{V}P,\end{aligned}$$

where

$$\tilde{\psi}_h = -\frac{2Nr_h}{\mathcal{A}_h} - (1 - 4g^2N^2)\zeta, \quad \zeta = \frac{r_h(r_h^2 - 3N^2)}{4N\mathcal{A}_h},$$

suggesting that the NUT charge is a thermodynamic trihair rather than a bihair. We will not attempt to discuss the thermodynamics of its Euclidean sector further [58,59].

VIII. EXTENSION TO THE KERR-NEWMAN-TAUB-NUT SPACETIME

Finally, let us discuss the general case including a nonzero rotation parameter but without a cosmological constant. The line element of the Kerr-Newman-Taub-NUT [60] or Demianski-Newman [45] spacetime with the Misner strings symmetrically distributed along the rotation axis and the electromagnetic one-form are

$$ds^2 = -\frac{\Delta(r)}{\Sigma} [dt + (2N \cos \theta - a \sin^2 \theta) d\phi]^2 + \frac{\Sigma}{\Delta(r)} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [adt - (r^2 + a^2 + N^2) d\phi]^2, \quad (25)$$

$$\mathbf{A} = \frac{Qr}{\Sigma} [dt + (2N \cos \theta - a \sin^2 \theta) d\phi], \quad (26)$$

where $\Sigma = r^2 + (N + a \cos \theta)^2$ and $\Delta(r) = r^2 + a^2 - 2Mr - N^2 + Q^2$.

The global conserved charges for this spacetime are the Komar mass M , angular momentum $J = Ma$, electric charge Q , and gravitomagnetic charge or dual (magnetic) mass (both of which are identical to the NUT charge N).

The horizons are determined by $\Delta(r_h) = 0$, which gives $r_h = M \pm \sqrt{M^2 + N^2 - Q^2 - a^2}$. The event horizon area is $A_h = 4\pi \mathcal{A}_h$, where $\mathcal{A}_h = r_h^2 + a^2 + N^2 = 2Mr_h + 2N^2 - Q^2$. At the horizon, the surface gravity, angular velocity, and electrostatic potential can be evaluated via the standard method as

$$\kappa = \frac{\Delta'(r_h)}{2\mathcal{A}_h} = \frac{r_h - M}{\mathcal{A}_h}, \quad \Omega = \frac{a}{\mathcal{A}_h}, \quad \Phi = \frac{Qr_h}{\mathcal{A}_h}. \quad (27)$$

Following the above procedure, we square the identity $2Mr_h = \mathcal{A}_h - 2N^2 + Q^2$, and then after adding $4M^2(a^2 + N^2)$ to it, we can obtain the useful identity

$$4M^2 \mathcal{A}_h = 4J^2 + 4M^2 N^2 + (\mathcal{A}_h - 2N^2 + Q^2)^2,$$

which is exactly our squared-mass formula

$$M^2 = \frac{(\mathcal{A}_h - 2N^2 + Q^2)^2 + 4J_N^2 + 4J^2}{4\mathcal{A}_h} \quad (28)$$

if we introduce $J_N = MN$ as a new conserved charge, as before.

Differentiation of the above squared-mass formula (28) yields the first law

$$dM = (\kappa/2)d\mathcal{A}_h + \Omega dJ + \omega_h dJ_N + \psi_h dN + \Phi dQ, \quad (29)$$

where (κ, Ω, Φ) are given by Eq. (27) and

$$\omega_h = \frac{N}{\mathcal{A}_h}, \quad \psi_h = -\frac{2Nr_h}{\mathcal{A}_h}. \quad (30)$$

One can verify that the integral mass formula

$$\kappa \mathcal{A}_h + 2\Omega J + 2\omega_h J_N + \psi_h N + \Phi Q = M \quad (31)$$

is completely satisfied. This completes the simple algebraic derivation of the mass formulas conjectured in Ref. [25] for the Demianski-Newman ‘‘black hole.’’

The consistency of the above thermodynamic formalism suggests that one should restore the well-known Bekenstein-Hawking area/entropy relation $S = A_h/4 = \pi \mathcal{A}_h$ and Hawking temperature $T = \kappa/(2\pi)$, which means that the whole class of NUT-charged spacetimes should be viewed as generic black holes.

IX. CONCLUDING REMARKS

In this work we have presented a simple, systematic way to naturally derive the thermodynamical first law and Bekenstein-Smarr mass formula of four-dimensional Taub-NUT-type spacetimes. This might be the most appropriate candidate framework to address the longstanding problem of the thermodynamics of both Lorentzian and Euclidean Taub-NUT-type spacetimes with the generic parameters. Not only can the beautiful Bekenstein-Hawking one-quarter area-entropy relation be naturally restored, but also all four laws of the usual black hole thermodynamics are shown to be completely applicable to the Taub-NUT-type spacetimes, without imposing any constraint condition. Furthermore, the physical meaning of the NUT parameter as a multihair has been clarified via its thermodynamics, namely, it can explain for the first time why the NUT parameter has so many different names and why there are different interpretations of the physical source of Taub-NUT-type spacetimes.

It would be suitable to think that our mass formulas properly describe a thermodynamic system made up of the horizon and two turning points (soliton and antisoliton pair sitting at two south and north poles) rather than all of the Misner strings, whose impacts can be coherently decomposed into two parts: rotation-like and electromagnetic charge-like effects.

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