Weak-U(1) × strong-U(1) effective gauge field theories and electron-monopole scattering

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We' present a gauge and Lorentz invariant model for the scattering of matter off magnetic poles, which justifies the presence of velocity-dependent magnetic charges as an effective description of either the behavior of monopoles in scattering with matter or their production from matter particles at colliders. Hence, in such an approach, perturbativity of the magnetic charge is ensured for relative low velocities of monopoles with respect to matter particles. The model employs a $U(1)_{weak} \times U(1)_{strong}$ effective gauge field theory under which electrons and monopoles (assumed to be fermions) are appropriately charged. The nonperturbative quantum effects of the strongly coupled sector of the theory lead to dressed effective couplings of the monopole/dyon with the electromagnetic photon, due to nontrivial wave-function renormalization effects. For slowly moving monopole/dyons, such effects lead to weak coupling, thus turning the bare nonperturbative magnetic charge, which is large due to the Dirac/Schwinger quantization rule, into a perturbative effective, velocity-("β")-dependent magnetic coupling. Our work thus offers formal support to previous conjectural studies, employing effective U(1)-electromagnetic gauge field theories for the description of monopole production from standard model matter, which are used in contemporary collider searches of such objects. This work necessarily pertains to composite monopoles, as seems to be the case of all known monopoles so far, that are solutions of specific particle physics models. This is a consequence of the fact that the wave-function renormalization of the (slowly moving) monopole fermion turns out to violate unitarity bounds that would characterize asymptotic elementary particle states.

I. INTRODUCTION AND MOTIVATION

The quantum theory of structureless magnetic poles as introduced by Dirac [1] was characterized by nonlocal hidden degrees of freedom (d.o.f.), the "Dirac string," whose invisibility in physical processes lead to the charge quantization condition. The presence of the string leads also to Lorentz noninvariance of the configuration and the associated field theory treatments, which manifested itself in various contexts, such as the local formulation of the magnetic charges by Zwanziger [2], which necessitated presence of a fixed four vector in the associated effective Lagrangian, containing two gauge potentials, associated with electric and magnetic current sources, or the famous Weinberg's paradox [3], according to which the leading perturbative term in the scattering amplitude between en electric charge and a magnetic pole was not Lorentz invariant. Schwinger's formulation of the dyon [4], which generalized the magnetic monopole to an object carrying

both electric q_e and magnetic q_m charge, restored Lorentz invariance, but at the cost of introducing a nonlocal Hamiltonian formulation of the dyon field, provided Schwinger generalization of Dirac's quantization condition is valid, for the scattering of dyon configurations with charges q_e^n, g_m^ℓ, n, ℓ positive integers,

$$(q_e^n g_m^\ell - q_e^\ell g_m^n)/4\pi = \mathcal{Z}_{n\ell} \in \mathsf{Z}, \qquad (1)$$

where Z denotes the set of integers. In the absence of electric charges, this condition leads to the Dirac quantization (but the fundamental charge unit is twice that of Dirac [1]). It should be remarked that, upon the imposition of (1), any Lorentz noninvariant effect in the effective two-gauge-potential Lagrangian of Zwanziger [2] disappears, and this is also associated with an integrability condition of the representation of the Poincaré Lie algebra that stems from Poincaré invariance into a representation of the finite Poincaré group [2].

Although the initial concept of the monopole as envisaged by Dirac was structureless, subsequently, 't Hooft and Polyakov [5] proposed composite monopoles, which were topological soliton solutions of phenomenologically realistic gauge and Lorentz invariant field theories, involving

spontaneous (Higgs-like) symmetry breaking. Unlike the Dirac monopole, such solutions were smooth, with no Dirac-string singularities, being characterized by finite energy.[1] The solution is localized around the origin, where the gauge group is unbroken, but very far from it the gauge group G is broken to a subgroup H, and the 't Hooft-Polyakov monopole reduces to the Dirac one. 't Hooft considered the Georgi-Glashow model [6] involving a Higgs triplet that spontaneously breaks the SU(2) group. The importance of this type of simply connected gauge group is that it is characterized by a nontrivial homotopy, e.g., for the SU(2) gauge group $\pi_2(\mathrm{SU}(2)) = \mathsf{Z}$, with Z being the set of integers. This defines how many times the spatial three-sphere, where the Higgs field lives, wraps around the internal (gauge) space sphere spanned by the Higgs triplet of the Georgi-Glashow model, which leads to the magnetic charge quantization condition [(1) for the magnetic monopole, i.e., $q_e^\ell = 0$]. Extension of the monopole and dyon solutions to phenomenologically realistic grand unified theories (GUT) with gauge group SU(5), with similar conclusions, was made in [7]. In such models, the monopoles have masses of order of the GUT scale 10^{14}–10^{16} GeV, which implies that inflation (whose scale is believed to be closed to this scale) would have inflated them away, which probably explains why such large GUT scale monopoles have not been discovered in cosmic searches so far [8]. In superstring theories, with appropriate grand unifying gauge groups, broken by stringy methods (Wilson lines), structures with magnetic charges have been first considered in [9]. Moreover, in D-brane-inspired GUT models [10], the unification scale (and hence the mass of the appropriate magnetic monopoles/dyons) can be lowered significantly, down to 10^4–10^6 GeV, relevant for future collider or cosmic-ray searches.

Unfortunately, unlike the SU(2) or SU(5) or other GUT-like-group monopoles, the gauge group of the standard model (SM) $\mathrm{SU}(2) \times U_Y(1)$ does not have this simple structure allowing for topological charge quantization, because of the hypercharge $U_Y(1)$ factor. As a result, after Higgs breaking, the quotient group $\mathrm{SU}(2) \times U_Y(1)/U_{\mathrm{em}}(1)$ is not characterized by a nontrivial second homotopy; thus monopoles were not expected to exist in the standard model. Phenomenologically, such a conclusion would imply that current collider (LHC) monopole searches might be futile.

However, in [11], it was argued that one can look for nontrivial topology in the Higgs field structure. Indeed, in the presence of the $U_Y(1)$ hypercharge group factor, the Weinberg Salam model is viewed as a gauge CP^1 model with the (normalized) Higgs doublet field playing the role

of the corresponding CP^1 field, characterized by a non-trivial homotopy $\pi_2(CP_1) = \mathsf{Z}$ thus allowing, in principle, for a topological quantization *à la* 't Hooft-Polyakov. This monopole is not characterized by a Dirac string, given that the latter had been argued to be a gauge artifact, provided the $U_Y(1)$ hypercharge group is endowed with a nontrivial bundle structure [11]. Unfortunately, however, the resulting monopole or dyon solutions of [11] were characterized by infinite energy.

Recently, though, it was observed that finite energy monopoles of the type proposed in [11] can characterize extensions of the standard model, with either appropriate nonminimally coupled Higgs and hypercharge sectors [12], or higher-derivative extensions of the hypercharge sector, for instance, in a (string-inspired) Born-Infeld configuration [13]. Such monopole/dyon solutions could have masses accessible to the scales of current or future colliders. Other finite-energy structured monopole/dyon solutions with potentially low mass (characterized by a Dirac string, though) can be found in string-inspired models with axionlike structures [14].

In view of the above theoretical advances in the field of relatively light monopoles/dyons, the development of effective field theory methods that could allow study of their production at colliders or scattering off SM matter has gained renewed interest. So far, lacking a fundamental theory for the description of the interaction of magnetic poles with standard model matter, like leptons, quarks, and photons, only *ad hoc* phenomenological effective U(1) gauge field theory models are used for collider searches of monopoles [15].

These models are essentially dual extensions of electrically charged particles of various spins interacting with photons, in which the electric charge in the coupling with the photon is replaced by an effective magnetic charge. Such effective descriptions may contain some truth in them, provided one considers the above-mentioned cases of the composite monopole solutions in (extensions of) the SM. Indeed, as discussed in [16], one may think of the magnetic monopole charge in such cases as a collective coupling to photons of (electrically) charged constituent d.o.f., such as charged W-bosons and Higgs fields, which the monopole is composed of. Modeling these constituent fields as quantum harmonic oscillators, the authors of [16] argued that the monopole might be viewed as a coherent superposition of $\sim \frac{1}{\alpha}$ such quantum states, with the result that the collective coupling to photons is $\frac{1}{\alpha}e$, consistent with the lowest nontrivial sector of the charge quantization condition (1).

In this work we attempt to construct (strongly coupled) effective gauge field theories for the above type of composite monopoles, by extending nontrivially the ideas of Zwanziger [2] that were developed for structureless monopoles. When we consider our effective theory sufficiently far away from the monopole centre, the 't-Hooft-Polyakov-type monopoles mentioned above resemble the

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[1]The fact that such solutions were obtained from gauge and Lorentz invariant field theories made Weinberg's observation on the non-Lorentz invariant nature of the (perturbative) scattering of electric charges off (generic) magnetic poles a true paradox.

structureless Dirac ones, to a good approximation. Nonetheless, as we explain below, the role of compositeness turns out to be crucial for us, since, as we see, when we consider the quantization of our nonperturbative effective field theory, the resulting wave-function renormalization for the monopole field does not respect the appropriate unitarity bounds for an elementary field, thus making our effective description suitable only for composite fields [17]. We are quite generic though in our considerations, and we do not specify the type of the composite monopole/dyon. For us, it might be one of the known types, mentioned above, or an as yet unknown solution of some beyond the SM theory with or without Dirac-string singularities.

Let us commence our study by first reviewing Zwanziger's approach [2], which we make partial use of in this work. The approach employs two related gauge fields, which allows for the construction of a local Lagrangian for the description of dyons, albeit non-Lorentz invariant. If one considers electric and magnetic currents, J_e^μ and J_m^μ, respectively, then, as shown in [2], the corresponding Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = J_e^\nu, \qquad \partial_\mu {}^\star F^{\mu\nu} = J_m^\nu, \qquad (2)$$

where $F^{\mu\nu}$ denotes the electromagnetic field-strength tensor, with ${}^\star F^{\mu\nu} \equiv \frac{1}{2}\epsilon_{\rho\sigma}^{\mu\nu} F^{\rho\sigma}$ being the dual tensor, and $\epsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric Levi-Civita symbol, with $\epsilon^{0123} = +1$, etc. We work throughout this work in a flat Minkowski space-time with metric $\eta^{\mu\nu} = (1, -1, -1, -1)$. For future purposes, we note the *axial*-vector (pseudovector) nature of the magnetic current in (2).

The general solution of the two Maxwell's equations can be expressed in terms of two (related, as we see) potentials $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ and a fixed four vector η^μ, as follows [2]:

First equation with electric current: $F = -{}^\star(\partial \wedge \mathcal{B}) + (\eta \cdot \partial)^{-1}(\eta \wedge J_e), \qquad {}^\star F = \partial \wedge \mathcal{B} + (\eta \cdot \partial)^{-1}{}^\star(\eta \wedge J_e), \quad (3)$

Second equation with magnetic current: ${}^\star F = {}^\star(\partial \wedge A) + (\eta \cdot \partial)^{-1}(\eta \wedge J_m), \qquad F = \partial \wedge A - (\eta \cdot \partial)^{-1}{}^\star(\eta \wedge J_m), \quad (4)$

where we used a form notation for brevity, with $\wedge$ ($\cdot$) denoting exterior (interior-dot) product as usual, such that for four vectors $(a \wedge b)^{\mu\nu} \equiv a^\mu b^\nu - a^\nu b^\mu$, $a \cdot b \equiv a^\mu b_\mu$. With these conventions we have for any antisymmetric second-rank tensor $\mathcal{F}_{\mu\nu} = \mathcal{F}_{\nu\mu}$: ${}^{\star\star}\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\mu\nu}$.

As discussed in [2], one can eliminate the currents from (3) and (4), and express these equations solely in terms of the potentials $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$. The following representation of the kernel $(\eta \cdot \partial)^{-1}(x)$ [satisfying $\eta \cdot \partial(\eta \cdot \partial)^{-1}(x) = \delta^{(4)}(x)$] is used:

$$(\eta \cdot \partial)^{-1}(x) = c_1 \int_0^\infty \delta^{(4)}(x - \eta s)ds$$
$$- (1 - c_1) \int_0^\infty \delta^{(4)}(x + \eta s)ds, \quad (5)$$

with c_1 being a real constant, appropriately defined in order to obtain the correct form of the Lorentz force in the classical relativistic particle limit of the dyon field [2]. The form (5) implies that, in the point-particle case, the support of $(\eta \cdot \partial)^{-1}(x_i - x_f)$ is reduced to $x_i^\mu(\tau_i) - x_i^\mu(\tau_f) = \eta^\mu s$, for $-\infty < \tau_i, \tau_f, s < +\infty$, with τ being the proper time.

The gauge potentials $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ depend on η^μ and on the gauge choice. For convenience, in the approach of Ref. [2], the fixed four vector η^μ was chosen to be spacelike $\eta^\mu \eta_\mu < 0$. The gauge potentials are not independent, as they are associated with a single field strength F, since the dual ${}^\star F$ is expressed in terms of F. Indeed, from (3), equating the expressions for the field strength F between

the first equation in (3) and the second equation in (4) yields

$$\partial \wedge \mathcal{A} + {}^\star(\partial \wedge \mathcal{B}) = (\eta \cdot \partial)^{-1}[\eta \wedge J_e + {}^\star(\eta \wedge J_m)], \quad (6)$$

so that only two photon d.o.f. propagate on shell in this local theory, despite the fact that two potentials are needed to ensure a local formulation of the dyon. The constraint (6) would imply that $\mathcal{B}$ is an axial vector, since this is the case with the magnetic current as well, and in this way one obtains consistent transformations under the (improper) Lorentz group, involving reflexions. This is essential for our purposes in this work, when we formulate in the next section the effective quantum gauge field theory for the monopole.

We also remark at this point that, upon using the following identity for an antisymmetric second-rank tensor $\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\nu\mu}$,

$$\mathrm{tr}(\mathcal{F} \cdot \mathcal{F}) \equiv \mathcal{F}_{\mu\nu}\mathcal{F}^{\nu\mu} = \frac{2}{\eta^2}[-(\eta \cdot \mathcal{F})^2 + (\eta \cdot {}^\star\mathcal{F})^2], \quad (7)$$

with the notation $\eta^2 = \eta_\mu \eta^\mu$, $(\eta \cdot \mathcal{F})^\nu = \eta_\mu \mathcal{F}^{\mu\nu}$, etc., it is possible express the electromagnetic tensor F and its dual ${}^\star F$, (3), (4), in terms of the potentials $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ alone [2],

$$F = \frac{1}{\eta^2}(\eta \wedge (\eta \cdot [\partial \wedge \mathcal{A}]) - {}^\star\{\eta \wedge (\eta \cdot [\partial \wedge \mathcal{B}])\}),$$

$${}^\star F = \frac{1}{\eta^2}({}^\star\{\eta \wedge (\eta \cdot [\partial \wedge \mathcal{A}])\} + \{\eta \wedge (\eta \cdot [\partial \wedge \mathcal{B}])\}), \quad (8)$$

in differential form notation.

Taking into account the identity (7), the Lagrangian yielding the field equations (2), in agreement with the solutions (3) and (4), reads [2]

$$\mathcal{L} = \frac{1}{8}\text{tr}[(\partial \wedge \mathcal{A}) \cdot (\partial \wedge \mathcal{A})] + \frac{1}{8}\text{tr}[(\partial \wedge \mathcal{B}) \cdot (\partial \wedge \mathcal{B})]$$
$$- J_e \cdot \mathcal{A} - J_m \cdot \mathcal{B} - \frac{1}{4\eta^2}(\eta \cdot [(\partial \wedge \mathcal{A}) +^\star (\partial \wedge \mathcal{B})])^2$$
$$- \frac{1}{4\eta^2}(\eta \cdot [(\partial \wedge \mathcal{B}) -^\star (\partial \wedge \mathcal{A})])^2. \qquad (9)$$

The expression (9) has the advantage of separating, formally at least, the Lorentz-violating effects of the fixed four vector η from the conventional Lorentz-invariant form of the quantum electrodynamics (QED) Lagrangian in terms of the photon field. Indeed, the first line of (9) would correspond to the conventional (Lorentz invariant, isotropic) QED terms, independent of the four vector η^μ, while the second line would correspond to the Lorentz-violating effects of the Dirac string in the dyon/monopole case, under the assumption that the Dirac string direction is aligned with that of the fixed spacelike four vector η^μ. The presence of the monopole, and its topologically nontrivial nature (the solutions belong to field-theory sectors of monopole number $n = 1, 2, \ldots$) is reflected precisely in the impossibility to deform continuously the vector so as $\eta^\mu \to 0^\mu \equiv (0, 0, 0, 0)^T$, with T denoting matrix transposition. However, by using the form (9), such a limit can be formally taken. Indeed, Eq. (6) implies in such a case

$$\partial \wedge \mathcal{A} +^\star (\partial \wedge \mathcal{B}) \overset{\eta^\mu \to 0^\mu}{=} 0, \qquad (10)$$

since, in view of (5), we can formally set the η-dependent right-hand side to 0 [given that it approaches 0 faster than $\eta^2 \to 0$, since the operator $(\eta \cdot \partial)^{-1}(x_i - x_f)$ has zero support in the limit $\eta^\mu \to 0^\mu$]. The Lagrangian (9) then does not contain any Lorentz-symmetry violating term, and involves two gauge fields related by the constraint (10). Our starting point is a similar model, as we explain in the next section.

In [18], it was argued, by means of topological considerations, that any Lorentz-violating effects of a magnetic pole (due to the Dirac string), can be resummed in a nonperturbative way in such a way that the scattering amplitude of an electric charge off a magnetic charge contains all such Lorentz-violating effects in a phase, which thus drops out of physical quantities such as cross sections. Moreover, this phase turns out to be a multiple of 2π, provided the quantization condition (1), and thus in such a case the amplitude is Lorentz invariant. Such a conclusion was reached by means of studying a toy model employing perturbative magnetic charges in a dark sector, using the two potential formalism (9) appropriately in both the visible and dark sectors, and assuming a perturbative small

mixing of ordinary photons with dark photons, that leads to perturbative couplings of magnetic charges to ordinary photons in the visible sector. This allows for a perturbative resummation of the dark monopole effects, making use of appropriate soft emissions of both gauge fields A^μ and B^μ in the pertinent Feynman diagrams, describing the scattering of electrons off monopoles, which in turn leads to the aforementioned decoupling of the Lorentz-violating (Dirac-stringlike) effects of the vector η^μ from the relevant cross sections.

Before proceeding further, we mention another peculiarity of the dyon, discussed in [19], which turns out to be of pivotal importance for our purposes here. When considering dyon-dyon scattering within a nonrelativistic quantum-mechanical framework, as appropriate for small relative velocities of the scattered dyons, Schwinger *et al.* arrived at the following differential cross section for the particular case of an electric charge e scattered off a magnetic one g (in units with the speed of light *in vacuo* is $c = 1$):

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{eg}{2\mu v}\right)^2 \frac{1}{(\theta/2)^4}, \qquad (11)$$

where μ is the reduced mass, θ is the (small) scattering angle of the nonrelativistic scattering, and $v = |\vec{v}|$ is the magnitude of relative velocity $\vec{v}$ of the magnetic charge.

If the magnetic pole is a dyon, carrying also an electric charge e_d, then (11) is extended classically to [19],

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{1}{2\mu v}\right)^2 \left[(eg)^2 + \frac{(ee_d)^2}{v^2}\right] \frac{1}{(\theta/2)^4}. \qquad (12)$$

However, when consider quantum scattering, of an electron off a dyon, the small-angle formula for the differential cross section contains [19] only the first term inside the brackets of (12) and thus coincides with (11). This turns out to be important for our work, when we define the magnetic charge in Sec. IV using scattering arguments.

One observes that the cross section (11) of electron-magnetic-monopole scattering can be obtained from the Rutherford differential cross section

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{Ruth}} = \left(\frac{e^2}{2\mu v^2}\right)^2 \frac{1}{(\theta/2)^4} \qquad (13)$$

upon the replacement $e^2 \to eg_{\text{eff}}$, where the effective monopole charge is (we reinstate the units of c in this formula for notational clarity, so that the reader connects easily to the existing literature [19])

$$g_{\text{eff}} \equiv g\frac{v}{c} \equiv g\beta. \qquad (14)$$

Upon invoking electric-magnetic duality, one therefore might expect that g_{eff} defines an effective "velocity-dependent"

magnetic charge that describes the behavior of a magnetic monopole in matter (or equivalently its production from the collision of matter standard model particles, such as quarks or charged leptons, at colliders). The important part of having a magnetic coupling (14) is its perturbative nature for small v, which can be used in monopole searches at colliders to place monopole mass bounds [20]. In fact, in the case of monopole-antimonopole pair production mechanisms at colliders, one may use such effective magnetic charges but in a Lorentz-invariant manner, using the center-of-mass velocity [15]

$$\beta = \sqrt{1 - \frac{4M^2}{s}}, \qquad (15)$$

where M is the monopole mass, and $s = (p_1 + p_2)^2$ is the Mandelstam variable, with p_i, $i = 1, 2$ being the momenta of the colliding particles (quarks of photons, in Drell-Yan or photon fusion processes for monopole production, respectively).

It is the purpose of this work to justify the use of an effective magnetic charge (14) in scattering processes involving magnetic poles by constructing a Lorentz and gauge invariant effective field theory of monopoles within a toy model, which we describe below. Our model is motivated by the model of Zwanziger [2] (9), combined with the findings/arguments of [18] that any Lorentz-violating effects of the monopole will not be present in cross sections [or scattering amplitudes, if the quantization condition (1) is valid]. Our results are based on a nonperturbative dressing of the coupling of the monopole to the real photon. At present, it is not technically possible to work directly with the Lagrangian (9), due to complications arising from the presence of the four vector η^μ, and the associated constraint (6). However, as our main purpose here is to demonstrate the emergence of nonperturbative dressing of the magnetic coupling, we deviate from the letter of the Zwanziger approach [2], keeping as much as possible of its spirit.

To this end, we consider a model Lagrangian, involving two independent U(1) gauge potentials A_μ, B_μ, with A_μ denoting the ordinary photon of the weakly coupled quantum electrodynamics [denoted by U(1)$_{\text{weak}}$]. The gauge field B_μ, referred to as a dual photon, belongs to the Lie algebra of a strongly coupled U(1)$_{\text{strong}}$, which is independent of that of electromagnetism. We also ignore any Lorentz-violating effects, anticipating the results of [18], based on soft gauge-field resummation, which we discuss briefly at the end of our article. The ordinary standard matter is represented as a spin-1/2 fermion, for simplicity, carrying electric charge only. For concreteness, in this work we assume the matter fermion to be an electron of charge e, but extension to any other electrically charged fermion in the standard model sector is straightforward. The dyon, on the other hand, is represented as a dual of the ordinary electron, under the aforementioned electric-magnetic duality, which leads to the symmetry of Maxwell's equations if magnetic poles are present. Hence it

is a fermion field itself (spin 1/2), which however carries both electric and magnetic charge, and thus couples to both A_μ and B_μ fields. Our approach consists in studying coupled Schwinger-Dyson (SD) equations, so as to study the effective vertex of the coupling of the dyon to the real photon, which, as we demonstrate, can be identified with an effective magnetic charge of the form (14), consistently with electric-magnetic duality.

The structure of the article is the following: in the Sec. II, we present the U(1)$_{\text{weak}}$ × U(1)$_{\text{strong}}$ model, discuss its properties, and present the relevant set of coupled SD equations. In Sec. III we solve these equations in the nonperturbative regime for the coupling of the dual photon, but in the weak-coupling limit of the ordinary QED, and obtain expressions for the dressed dual photon propagator and the photon-monopole vertices. Of particular interest to our study is the ordinary-photon-monopole dressed vertex, which turns out to be proportional to the monopole-field wave-function renormalization. The latter is computed in a self-consistent way in the strong-coupling limit of the dual U(1) gauge theory, under the assumption that it is approximately momentum independent. The gauge independence of the associated physical observables is also discussed. In the following Sec. IV, we discuss electron (matter-) monopole scattering, compute the relevant cross section, and derive the effective velocity-dependent magnetic charge (14) as a dressed coupling of the monopole-real-photon vertex derived previously. An important role in this is played by the appropriate physical interpretation of the dimensional-transmutation scale than enters the expression for the wave-function renormalization within our dimensional-regularization treatment. For slowly moving monopoles, the resulting wave-function renormalization turns out to violate unitarity bounds for elementary particle states. Thus, our effective field theory can only describe composite monopoles. Finally, Sec. V contains our conclusions and outlook. Technical aspects of our analysis are given in several Appendixes.

II. THE MODEL

We consider a gauge field theory with two independent sectors: (i) the standard model, with electric charges only and a corresponding U(1)$_{\text{weak}}$ gauge field A_μ, and (ii) the monopole/dyon sector, electrically charged under A_μ (with a bare charge e_A) and magnetically coupled to an independent U(1)$_{\text{strong}}$ axial gauge field B_μ (with the corresponding bare coupling e_B). Before proceeding, we should make some important comments regarding the characterization of the monopole/dyon as electrically charged under U(1)$_{\text{weak}}$. From a naive point of view it seems that we are dealing here with a dyon in the sense of Schwinger [4] or Zwanziger [2], whose bare electric charge is e_A. However, as we discuss in this work, the nonperturbative dressing of the corresponding monopole-electromagnetic-photon

vertex leads to a nonconventional dressed electric charge proportional to the (square of the) magnetic charge e_B [cf. (70) and (72) in Sec. IV, below], and thus vanishing when $e_B \to 0$. In this sense, this dressed coupling is the induced magnetic charge coupling of the monopole/dyon to the real photons, assumed *ad hoc* in the phenomenological searches of magnetic monopoles using effective field theories [15]. Our approach provides a microscopic derivation of this effect.

For our purposes it is enough to consider a simplified matter sector, consisting of an electron ψ of (bare) mass m, coupled to the standard model photon A_μ with a perturbative electric charge e. We also consider a spin-$\frac{1}{2}$ monopole χ, of (bare) mass M, coupled electrically to A_μ and magnetically to the axial dual photon B_μ. Unlike the Zwanziger approach [2], here A_μ and B_μ are independent U(1) gauge d.o.f., which suffice for our main purpose, which is a demonstration of the emergence of an effective velocity-dependent coupling of the monopole to the ordinary photon, after nonperturbative quantum dressing. As already mentioned, the restriction to a fermion monopole makes direct contact with the work of [2] and is motivated by the electric-magnetic duality, under which the dual of an electron is such a fermion monopole, although it must be said that consistent formulations of field theories of monopoles with spin one or zero might also exist.

A. Lagrangian

The Lagrangian of our model is given by

$$L = -\frac{1}{4} F_A^{\mu\nu} F_{\mu\nu}^A - \frac{1}{4} F_B^{\mu\nu} F_{\mu\nu}^B + \overline{\psi}(i\gamma^\mu D_\mu^A - m)\psi + \bar{\chi}(i\gamma^\mu D_\mu^{A+B} - M)\chi, \tag{16}$$

where A_μ is a vector and B_μ is an axial vector (pseudo-vector). The corresponding field-strength tensors are given by

$$F_A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad F_B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu, \tag{17}$$

and the covariant derivatives are

$$D_\mu^A = \partial_\mu - ieA_\mu \quad \text{and} \quad D_\mu^{A+B} = \partial_\mu - ie_A A_\mu - ie_B B_\mu, \tag{18}$$

where e is the coupling the electron to the gauge field A_μ and e_A and e_B are the couplings of the monopole to A_μ and B_μ, respectively. These covariant derivatives ensure that the Lagrangian is invariant under the gauge transformation

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \theta_A \\ B_\mu &\to B_\mu + \partial_\mu \theta_B \\ \psi &\to \exp(ie\theta_A)\psi \\ \chi &\to \exp(ie_A \theta_A + ie_B \theta_B)\chi. \end{aligned} \tag{19}$$

A few remarks are in order at this point.

(i) The axial nature of the gauge field B_μ is required for consistency of the field equations [see Eq. (23) below], under improper Lorentz transformations, including spatial reflections and reversal in time. As a consequence, the Lagrangian (16) breaks parity, P, and time reversal symmetry, T, but preserves CPT, where C denotes charge conjugation. It should be noted that such an explicit parity and time reversal symmetry breaking, but CPT conservation, is a generic feature of theories with magnetic charges [21].

(ii) The (bare) electric charges e and e_A are both assumed to be perturbative: $e \ll 1$ and $e_A \ll 1$; as already mentioned, we assume here, for concreteness, that the electrically charged matter fermion ψ is an electron of charge e. Our analysis can of course be extended trivially to incorporate any other charged matter, in which case the coupling e is replaced by the corresponding electric charge q_e.

(iii) The coupling e_B of the spin-$\frac{1}{2}$ monopole to the dual photon should not be identified immediately with the magnetic charge of the monopole/dyon. The latter is defined appropriately later on, in Sec. IV, via studying the scattering process of dyons with matter fermions in our effective theory, and identifying the relevant cross sections in the nonrelativistic limit with the corresponding ones in the quantum-mechanical approach of [19]. It is this charge that satisfies the quantization condition (1). As we discuss in this work, this quantization condition implies strong coupling for the dual photons $e_B \gg e_A$, and it is the dressing of the electric vertex of the monopole with the real photon by non-perturbative quantum corrections of the strongly coupled dual photon that leads in general to non-perturbative electromagnetic couplings of the monopole/dyons to the real photon. This coupling is identified with the "magnetic charge coupling g" of the monopole to photons, appearing in phenomenological effective field theories used in monopole/matter scattering or production of monopoles at colliders [15]. As we discuss in Sec. IV, this magnetic charge g coincides with the one defined by Dirac [1] in the expression for the (singular) magnetic field of the monopole. In our context, it turns out to be a product of e_A with the monopole/dyon wave-function renormalization Z, $g = Ze_A$, where the factor Z is due to the (strongly coupled, nonperturbative) quantum effects associated with the dual photon of the gauge group factor $U_{\text{strong}}(1)$ of our $U(1)_{\text{weak}} \otimes U_{\text{strong}}(1)$ effective field theory. This wave-function renormalization factor is itself a nontrivial function of both couplings e_B and e_A, and under the requirement of the validity of the charge quantization (1), turns out to be proportional

to e_B [cf. (70) and (72)], thus vanishing in the absence of a magnetic coupling e_B. Moreover, and most importantly, as our effective field theory approach also indicates, such magnetic charge dressed couplings can become perturbative for slowly moving monopole/dyons, in agreement with the electric-magnetic-duality-inspired conjecture of [19] on the emergence of an effective magnetic charge (14) in scattering processes of monopole/dyons with matter. We stress once again that for us, the presence of a bare electric coupling e_A of the monopole associated with the weak $U(1)_{\text{weak}}$ electromagnetic gauge group is essential.

After the above necessary remarks, we are in position to commence our study. Minimizing the action with respect to A_μ and B_μ respectively leads to

$$\partial_\mu F_A^{\mu\nu} = e j_\psi^\nu + e_A j_\chi^\nu \quad \text{and} \quad \partial_\mu F_B^{\mu\nu} = e_B j_\chi^\nu, \quad (20)$$

where the currents are defined here without the charges

$$j_\psi^\nu = \bar\psi \gamma^\nu \psi \quad \text{and} \quad j_\chi^\nu = \bar\chi \gamma^\nu \chi. \quad (21)$$

Contracting Eqs. (20) with the divergence ∂_ν, one can see that both vector currents are conserved individually,

$$\partial_\nu j_\psi^\nu = \partial_\nu j_\chi^\nu = 0, \quad (22)$$

which is expected, since the gauge functions θ_A and θ_B are independent. To make the connection with Zwanziger's approach [2] and the (electromagnetic) field-strength tensor F, which appears in the field equations (2), we require that the gauge fields A_μ and B_μ satisfy the on-shell constraints

$$F_A^{\mu\nu} = F^{\mu\nu}, \qquad F_B^{\mu\nu} = {}^\star F^{\mu\nu}. \quad (23)$$

This leads to a condition identical to Eq. (10), since $^\star({}^\star F) = -F$. The fields A_μ and B_μ in our model are treated as independent variables in the quantum theory, though, and the constraints (23) then characterize external photon lines with the relevant fields put on shell. This is implied in what follows.

Our main point in doing this is to demonstrate the existence of effective nonperturbatively dressed couplings of the photons A^μ and B^μ with the monopole, which depend on a wave-function renormalization factor [cf. (33) below] that vanishes for vanishing monopole velocities [cf. (63) and (68) below], thus becoming perturbative for slowly moving monopoles. This allows for the soft-photon resummation arguments of [18] to go through, implying that any η^μ-dependent term in (9) can be ignored as it will contribute only to the phase of the electron-monopole scattering amplitude; upon the quantization condition (1), such η^μ-dependent terms in the phase would vanish, thus leaving the amplitude itself Lorentz and gauge invariant.

With the above in mind we proceed to study nonperturbatively the model described by (16).

B. Properties of the quantum theory

The notations used here are defined in Appendix A, where the properties of the one-particle-irreducible graph generating functional Γ are given.

1. Current conservation and Furry's theorem

Because of the two independent conserved currents, there is no mixing between ψ and χ, and the following inverse propagators vanish:

$$\left.\frac{\delta^2\Gamma}{\delta\bar\psi\delta\chi}\right|_0 = 0 = \left.\frac{\delta^2\Gamma}{\delta\bar\chi\delta\psi}\right|_0. \quad (24)$$

Also, Furry's theorem [17], based on charge conjugation of magnetic currents, states that any graph involving a monopole loop vanishes if it has an odd number of dual photon insertions. This result is also valid for an axial vector B_μ, since it relies on the charge-conjugation properties of the current only. As a consequence, the gauge field propagator is diagonal in gauge fields space and

$$\left.\frac{\delta^2\Gamma}{\delta A_\mu \delta B_\nu}\right|_0 = 0, \quad (25)$$

since any corresponding graph necessarily contains an odd number of monopole/dual-photon vertices. For the same reason, the dressed electron/dual-photon vertex also vanishes, $\Lambda_\mu^{B\psi} = 0$, which implies that the electron does not couple to a single dual photon. The electron can couple to two dual photons though, through a two-loop process that involves a monopole loop, and it is therefore possible to detect the electromagnetic field generated by B_μ via the electron.

2. Ward identities

It is shown in Appendix B that the usual Ward identities hold for fermions: in the limit of vanishing momentum for the gauge fields, one obtains

$$\frac{1}{e}\Lambda_\mu^{A\psi}(p, 0) = \frac{\partial G_\psi^{-1}}{\partial p^\mu},$$
$$\frac{1}{e_B}\Lambda_\mu^{B\chi}(p, 0) = \frac{\partial G_\chi^{-1}}{\partial p^\mu}. \quad (26)$$

We also find

$$q^\mu \Lambda_\mu^{B\psi}(p, q) = 0, \quad (27)$$

with q denoting the gauge-field momentum, which is consistent with the fact that the electron does not couple

to the dual photon. An additional property one finds is that [cf. (B9)]

$$e_B q^\mu \Lambda_\mu^{A\chi}(p, q) = e_A q^\mu \Lambda_\mu^{B\chi}(p, q), \qquad (28)$$

showing that the interactions between the monopole and the gauge fields are dressed with the same quantum corrections. The condition (28) also implies that the U(1)$_{strong}$ nonperturbative quantum corrections due to the dual photons induce a nonperturbative coupling of the monopole/dyon to the electromagnetic photon, which is an important feature for the matter-monopole/dyon scattering or production of monopole/dyons in our framework, as we discuss later in Sec. IV.

3. Schwinger-Dyson equations

Following the usual derivation of SD equations and taking into account the property (24), we show in Appendix C that the fermion self-energies satisfy

$$G_\psi^{-1} - S_\psi^{-1} = ie\gamma^\mu G_\psi \Lambda_{A\psi}^\nu \Delta_{\mu\nu}^A,$$
$$G_\chi^{-1} - S_\chi^{-1} = ie_A\gamma^\mu G_\chi \Lambda_{A\chi}^\nu \Delta_{\mu\nu}^A + ie_B\gamma^\mu G_\chi \Lambda_{B\chi}^\nu \Delta_{\mu\nu}^B, \quad (29)$$

where integration over repeated space-time coordinates is understood. Because of the property (25), the gauge propagator is diagonal, and the gauge-field self-energies satisfy

$$(\Delta_{\mu\nu}^A)^{-1} - D_{\mu\nu}^{-1} = i\text{tr}\{e\,\gamma_\mu G_\psi \Lambda_\nu^{A\psi} G_\psi + e_A\,\gamma_\mu G_\chi \Lambda_\nu^{A\chi} G_\chi\},$$
$$(\Delta_{\mu\nu}^B)^{-1} - D_{\mu\nu}^{-1} = ie_B\text{tr}\{\gamma_\mu G_\chi \Lambda_\nu^{B\chi} G_\chi\}. \qquad (30)$$

In the next section we apply these nonperturbative equations to study a specific limit, where the monopole does not propagate.

III. SOLUTION OF SCHWINGER-DYSON EQUATIONS IN THE NONPERTURBATIVE REGIME

We consider a regime that is perturbative in the electric charge $e_A \ll 1$, but where the coupling e_B is large, $e_B \gg 1$. Thus, quantum corrections to the electronic sector are neglected, and the SD equations are used to calculate quantum corrections in the monopole sector only. In what follows, we neglect the momentum dependence of quantum corrections for the calculation of loop integrals. Because of regularization, though, these corrections acquire a scale dependence, which is interpreted as an external momentum dependence, in graphs describing the relevant scattering process we are interested in.

A. Quantum corrections

We derive here the monopole self-energy, as well as the polarization tensor for the dual photon. In a perturbative

approach, these two quantities would be determined independently but, in the nonperturbative framework of SD equations, as appropriate for strong e_B couplings, the corresponding quantum corrections are coupled and depend on one another.

The bare fermion propagators are

$$S_\psi = i\frac{\not p + m}{p^2 - m^2}, \qquad S_\chi = i\frac{\not p + M}{p^2 - M^2}, \qquad (31)$$

and we consider the approximation where the dressed fermion propagators are

$$G_\psi \simeq S_\psi, \qquad G_\chi = i\frac{Z\not p + \tilde M}{Z^2 p^2 - \tilde M^2}, \qquad (32)$$

where Z is the monopole wave-function renormalization and $\tilde M$ is the dressed monopole mass. If we assume that Z and the dressed vertices $\Lambda_\mu^{B\chi}$ and $\Lambda_\mu^{A\chi}$ are momentum independent, the Ward identities given in Sec. II B 2 imply

$$\Lambda_\mu^{B\chi} = e_B Z\gamma_\mu \quad \text{and} \quad \Lambda_\mu^{A\chi} = e_A Z\gamma_\mu. \qquad (33)$$

In a generic covariant gauge parametrized by $\lambda > 0$, the gauge propagators assume the form

$$\Delta_{\mu\nu}^A \simeq D_{\mu\nu} = \frac{-i}{q^2}\left(\eta_{\mu\nu} + \frac{1-\lambda}{\lambda}\frac{q_\mu q_\nu}{q^2}\right)$$
$$\Delta_{\mu\nu}^B = \frac{-i}{(1+\omega)q^2}\left(\eta_{\mu\nu} + \frac{1+\omega-\lambda}{\lambda}\frac{q_\mu q_\nu}{q^2}\right), \quad (34)$$

where ω is the quantum correction responsible for the dual photon transverse polarization tensor, defined by

$$\Delta_{\mu\nu}^{B-1} - D_{\mu\nu}^{-1} = i\omega(q^2\eta_{\mu\nu} - q_\mu q_\nu), \qquad (35)$$

and is also assumed to be momentum independent.

We show in Appendix D that the SD equation for the monopole self-energy, given in Sec. II B 3, leads to

$$\frac{\delta M}{\tilde M} = \frac{1}{8\pi^2\lambda Z}\left(e_A^2(1+3\lambda) + e_B^2\frac{1+3\lambda+\omega}{1+\omega}\right)\frac{1}{\epsilon}\left(\frac{Zk}{\tilde M}\right)^\epsilon$$
$$+ \text{finite}, \qquad (36)$$

and

$$Z = 1 + \frac{e_A^2 + e_B^2}{8\pi^2\lambda}\frac{1}{\epsilon}\left(\frac{Zk}{\tilde M}\right)^\epsilon + \text{finite}, \qquad (37)$$

where

$$\delta M \equiv \tilde M - M, \qquad (38)$$

and dimensional regularization is used. The latter introduces the arbitrary (transmutation) mass scale k in dimensions

$d = 4 - \epsilon$. It is interesting to note that the expression for Z does not involve ω, although it does correspond to a nontrivial resummation of higher-order loops.

The SD equation for the dual photon self-energy gives the same integral as in the perturbative case, except for a factor $1/Z$,

$$\omega = \frac{e_B^2}{6\pi^2 Z} \frac{1}{\epsilon} \left(\frac{Zk}{\tilde{M}}\right)^\epsilon + \text{finite as } \epsilon \to 0^+, \quad (39)$$

which is a nonperturbative effect.

B. Strong-coupling limit

The SD equations relevant to our model lead us to the quantum corrections given by Eqs. (36), (37), and (39). In a perturbative context, these corrections vanish in the limit where the coupling constants go to 0, but we are interested here in a nontrivial solution of the SD equations, which does not necessarily reproduce the perturbative results for small couplings. In order to achieve this, we first derive the coupled differential equations obtained from Eqs. (36), (37), and (39), which describe the evolution of $\tilde{M}, Z, \omega$ with the scale k. Instead of taking boundary conditions set by the perturbative regime, we explore here the possibility of having $Z = 0$ as part of the boundary conditions. We stress that it is only in the framework of effective theories that such a nonperturbative regime is relevant, and the renormalization flows we derive in this section are not related to the usual perturbative ones.

Before proceeding, we remark that the usual perturbative renormalization procedure, which consists of appropriately absorbing the $1/\epsilon$ divergence in (37), introduces the rescaling $\chi_B \to \sqrt{Z_B}\chi_B$, where χ_B is the bare fermion field. One chooses then $Z_B Z = 1$ such that perturbatively,

$$Z_B = 1 - \frac{e_A^2 + e_B^2}{8\pi^2 \lambda} \frac{1}{\epsilon} \left(\frac{Zk}{\tilde{M}}\right)^\epsilon + \cdots, \quad (40)$$

where the dots represent higher orders in e_A, e_B. As expected from a consistent unitary description of asymptotic states [17], given the interpretation of the wave-function renormalization as the probability for finding such states, the unitarity bound $0 \leq Z_B < 1$ is indeed satisfied. In principle, this implies

$$Z > 1, \quad (41)$$

whereas we are interested in the regime where $Z \ll 1$. The latter regime violates the unitarity bound (41), which is possible in the case of composite monopoles. As mentioned in the introduction, these are the only type of known solutions of field theories of phenomenological interest so far [5,7,12,13]. Thus, the effective field theory developed here can only be associated with composite monopoles/dyons, not necessarily restricted to the aforementioned types, but also encompassing new composite monopole/dyon solutions that may exist in beyond the standard model physics.

Equation (37) implies

$$\partial_k Z = \frac{e_A^2 + e_B^2}{8\pi^2 \lambda} \frac{\tilde{M}}{Zk} \partial_k \left(\frac{Zk}{\tilde{M}}\right), \quad (42)$$

with solution

$$Z = C_1 + \frac{e_A^2 + e_B^2}{8\pi^2 \lambda} \ln\left(\frac{Zk}{\tilde{M}}\right), \quad (43)$$

where C_1 is a constant. We then introduce the "running mass" $M_r(k)$ via the definition

$$M_r(k) \equiv \frac{\tilde{M}}{Z}, \quad (44)$$

from which the scale k_0, at which $Z \to 0$, can be defined as the solution of the self-consistent equation[2]

$$k_0 = M_0 \exp\left(-\frac{8\pi^2 \lambda C_1}{e_A^2 + e_B^2}\right) \quad \text{where } M_0 \equiv M_r(k_0). \quad (45)$$

This allows one to express Z as

$$Z = \frac{e_A^2 + e_B^2}{8\pi^2 \lambda} \ln\left(\frac{kM_0}{k_0 M_r(k)}\right). \quad (46)$$

For the polarization tensor, we obtain from Eq. (39)

$$\partial_k (Z\omega) = \frac{e_B^2}{6\pi^2} \frac{\tilde{M}}{Zk} \partial_k \left(\frac{Zk}{\tilde{M}}\right)$$
$$\Rightarrow Z\omega = C_2 + \frac{e_B^2}{6\pi^2} \ln\left(\frac{k}{M_r(k)}\right), \quad (47)$$

where C_2 is a constant. On requiring a finite limit $\omega \to \omega_0$ when $Z \to 0$, we can easily determine C_2, implying, on account of (46), that ω is actually independent of k,

$$\omega = \frac{e_B^2}{6\pi^2 Z} \ln\left(\frac{kM_0}{k_0 M_r(k)}\right) = \frac{4\lambda e_B^2}{3(e_A^2 + e_B^2)} = \omega_0. \quad (48)$$

Notice that $\omega_0 \geq 0$, since the gauge parameter λ is positive.

For the mass correction, we obtain from (36)

$$\partial_k \left(\frac{\delta M}{M_r}\right) = \frac{\kappa^2}{8\pi^2 \lambda} \frac{M_r}{k} \partial_k \left(\frac{k}{M_r}\right), \quad (49)$$

---

[2]The reader should bear in mind that the scale k_0 is gauge independent, which can be guaranteed by requiring an appropriate dependence of the constant C_1 and the bare coupling e_B on the gauge parameter λ.

where the effective coupling κ is given by

$$\kappa^2 \equiv e_A^2 + e_B^2 + 3\lambda\left(e_A^2 + \frac{e_B^2}{1+\omega_0}\right). \qquad (50)$$

As a consequence,

$$\frac{\delta M}{M_r} = C_3 + \frac{\kappa^2}{8\pi^2\lambda}\ln\left(\frac{k}{M_r}\right), \qquad (51)$$

where C_3 is a constant. Taking into account the definition $\delta M = \tilde{M} - M$ and the expression (46) for Z, we obtain the self-consistent equation that must be satisfied by the function $M_r(k)$,

$$\frac{M}{M_r(k)} = \frac{e_A^2 + e_B^2 - \kappa^2}{8\pi^2\lambda}\ln\left(\frac{k}{M_r(k)}\right)$$
$$+ \frac{e_A^2 + e_B^2}{8\pi^2\lambda}\ln\left(\frac{M_0}{k_0}\right) - C_3. \qquad (52)$$

One can check that there is indeed an appropriate solution for M_0: the latter equation for $k = k_0$ is of the form

$$X = -a\ln(X) + b, \qquad (53)$$

where $X = k_0/M_0$, a is a positive constant, and b is a constant of integration. It can be easily seen graphically that this equation has always a finite and positive solution, which ensures the consistency of the present derivation.

We note that M_r increases with k since, from Eq. (52),

$$k\partial_k M_r = \frac{(\kappa^2 - e_A^2 - e_B^2)M_r}{\kappa^2 - e_A^2 - e_B^2 + 8\pi^2\lambda M/M_r}$$
$$= \frac{3[e_A^2 + e_B^2/(1+\omega_0)]M_r}{3[e_A^2 + e_B^2/(1+\omega_0)] + 8\pi^2 M/M_r} > 0. \qquad (54)$$

Also, one can see that for $M > 0$ (and $\omega_0 > 0$)

$$k\partial_k M_r < M_r, \qquad (55)$$

implying that Z increases with k,

$$k\partial_k Z = \frac{e_A^2 + e_B^2}{8\pi^2\lambda}\left(1 - \frac{k\partial_k M_r}{M_r}\right) > 0. \qquad (56)$$

The above analysis has shown that it was possible to find a consistent solution of the SD equations, which allowed for the nonperturbative limit $Z \to 0$ to be taken. Such a solution is parametrized by two scales with dimensions of mass, k_0 and M_0. We also note that as $k \to k_0$, the renormalized mass $\tilde{M}(k) \to \tilde{M}(k_0) = 0$, since $Z \to 0$ and $M_r = \tilde{M}/Z$ goes to the finite value M_0. As a consequence, the corrections to the mass of the monopole/dyon are negative when $k \to k_0$, since [cf. (38)] $\delta M(k_0) = -M < 0$.

This feature is specific to the nonperturbative regime we consider here, and we emphasize again that it does not reduce to the usual perturbative solution when the coupling constant becomes small: it represents instead a disconnected configuration of quantum fluctuations in the model.

C. Gauge dependence

The polarization ω_0 given in the expression (48) depends on λ, which is allowed by the nonperturbative resummation provided by the SD equations. Unfortunately this has an effect on the gauge dependence of physical observables. Indeed, let us consider the scalar potential V seen by a dual charge e_B, and generated by a pointlike dual charge at rest, assumed to correspond to a magnetic current $j_\chi^\nu(r) = (e_B\delta(\vec{r}), 0)$, which in Fourier components reads $j_\chi^\nu(q) = (e_B, 0)$. We have

$$V = e_B B_0 = e_B\Delta_{0\nu}^B j_\chi^\nu, \qquad (57)$$

where $\Delta_{\mu\nu}^B$ is the dual photon propagator (34). For the static modes $q^\mu = (0, \vec{q})$, this leads to the Coulomb potential

$$V(\vec{r}) = e_B\int\frac{d^3q}{(2\pi)^3}\Delta_{00}^B(q)j_\chi^0(q)e^{i\vec{q}\cdot\vec{r}}$$
$$= \frac{e_B^2}{1+\omega_0}\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{r}}}{(\vec{q})^2} = \frac{e_B^2}{4\pi|\vec{r}|(1+\omega_0)}, \qquad (58)$$

which depends on the gauge through ω_0.

A similar situation characterizes dynamically generated masses in nonperturbative gauge theories [22]. Nevertheless, it is possible that, through appropriate resummation of special classes of appropriate Feynman graphs ("pinched technique" [23]), one recovers the gauge-independent value of physical quantities, such as masses, or in our case dressed charges, and such values seem to correspond to the value one would obtain in the truncated SD treatment in Feynman gauge $\lambda = 1$. In what follows, we therefore make this gauge choice, and we identify the effective dual coupling

$$\tilde{e}_B^2 \equiv \frac{e_B^2}{1+\omega_0(\lambda=1)} = \frac{3e_B^2(e_A^2 + e_B^2)}{3e_A^2 + 7e_B^2}, \qquad (59)$$

from which the flow for the running mass given in Eq. (54) reads

$$k\partial_k M_r = \frac{(e_A^2 + \tilde{e}_B^2)M_r}{e_A^2 + \tilde{e}_B^2 + (8\pi^2/3)M/M_r}, \qquad (60)$$

and is used in the discussion below. We finally note that the pinched technique is based on the description of physical processes, like a scattering process, for which it is shown that the longitudinal component of the gauge propagator cancels out in the calculation of physical observables. Such

processes are exactly those that motivate our present study, which therefore singles out the Feynman gauge as the physical one.

IV. ELECTRON-MONOPOLE SCATTERING AND AN EFFECTIVE VELOCITY-DEPENDENT MAGNETIC CHARGE FOR (COMPOSITE) MONOPOLES

As we have discussed in the introduction, one may obtain the differential cross section for nonrelativistic electron-monopole scattering at small angles using quantum-mechanical treatment, leading to expression (11), which scales with the relative momentum $p = \mu v$ of the incident matter particle (in the frame where the heavy monopole is initially at rest) as $1/p^2$ instead of the traditional $1/p^4$ of the conventional Ruhterford scattering differential cross section (13) between charged matter particles. As already mentioned previously, upon invoking electric-magnetic duality, one may obtain the Rutherford scattering formula from (11), by replacing the monopole magnetic coupling with an effective velocity (β)-dependent magnetic coupling, (14) and (15). In what follows, we propose an origin of such a coupling based on the nonperturbative results derived in the previous section.

A. Perturbative effective magnetic charge for slowly moving (composite) monopoles

To this end, we consider slowly moving monopoles, relative to the incident matter particle, assumed for concreteness to be an electron of charge e. The one-photon exchange scattering Feynman graph of Fig. 1, between the fermions ψ (electron) and χ (monopole fermion), proves
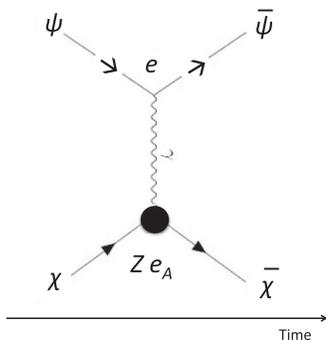
FIG. 1. Typical scattering graph between an electron (ψ) and a fermion monopole (χ). Wavy lines denote electrodynamical photons, γ. Straight lines with arrows denote the scattered fermions (dashed arrow lines denote electrons and continuous arrow lines denote the monopole). The dark blob denotes the dressed coupling Ze_A due to quantum corrections induced by the strongly coupled dual photon, which only the monopole couples to, characterized by the presence of the Z factor. By rotating the graph by 90° counterclockwise, one obtains a Drell-Yan diagram for monopole-antimonopole pair production from matter fermions.

sufficient for this purpose, as a result of the perturbative nature of both couplings, the electron charge $e \ll 1$, but also the coupling of the monopole χ to the real photon. The latter is determined by the appropriate dressed vertex $\Lambda_\mu^{A\chi} = e_A Z \gamma_\mu$, (33), with the wave function Z to be determined from (46), and as we see below it can become perturbative, when the scale k approaches k_0, for which $Z \ll 1$, with $Z(k_0) = 0$. We are also able to express the magnetic charge g of the monopole, which is subjected to the quantization condition (1), in terms of e.

We first notice that, from the differential equation (60), one can expand M_r to first order in $k - k_0$ to find

$$\frac{M_r(k)}{M} \simeq \frac{M_0}{M} + a\,\frac{k - k_0}{k_0},\qquad(61)$$

where

$$a \equiv \frac{(e_A^2 + \tilde{e}_B^2)(M_0/M)^2}{8\pi^2/3 + (e_A^2 + \tilde{e}_B^2)M_0/M}.\qquad(62)$$

From the expression (46) for Z we obtain then, to first order in $k - k_0$,

$$Z \simeq \frac{Z_0}{k_0}(k - k_0),\qquad(63)$$

where

$$Z_0 = \frac{e_A^2 + e_B^2}{8\pi^2 + 3(e_A^2 + \tilde{e}_B^2)M_0/M},\qquad(64)$$

with $\tilde{e}_B$ given by (59).

The Feynman diagram of Fig. 1 makes sense if the dressed coupling of the monopole to the electromagnetic photon γ is small, which necessitates $Z < 1$ in (63), and can be achieved for $k \to k_0$. Then the scattering process resembles that of two ordinary charged particles in (perturbative) electrodynamics with charges e and Ze_A, both small. The result for the corresponding differential cross section can be borrowed from standard quantum electrodynamics then. Since our interest is to compare this result with the differential cross section (11) of the nonrelativistic scattering of an electron off a monopole with magnetic charge g, discussed in [19], we consider here the appropriate limit, of a low relative velocity between the electron and the monopole. The result from the differential cross section at small angles θ, in a frame where the monopole is initially at rest, reads then (in appropriate units in which the fine structure constant $\alpha = e^2$, as per the conventions of [19]) [24]

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{LAB}} \simeq \left(\frac{Ze_A e}{2\mu|\vec{v}|^2 \sin^2\frac{\theta}{2}}\right)^2$$
$$\underset{\theta \ll 1}{\simeq} \left(\frac{Z_0 \frac{k-k_0}{k_0} e_A e}{2\mu|\vec{v}|^2}\right)^2 \frac{1}{(\theta/2)^4},\qquad(65)$$

where μ is the reduced mass of the electron-monopole system.

Comparing (65) with (11), which as we discussed in Sec. I describes quantum scattering of an electron off a dyon at small angles, we then observe that one can define an effective magnetic charge, describing the coupling of the monopole to the photon, by [cf. (63)]

$$g_{\text{eff}} = Ze_A \simeq Z_0 \frac{k - k_0}{k_0} e_A, \qquad (66)$$

with the scale $|k - k_0|$ being identified with a proper (Lorentz invariant) center-of-mass momentum scale, and k_0 is identified with a mass scale through the self-consistent relation (45), where we choose the constant of integration C_1 such that

$$k_0 = 2M_0. \qquad (67)$$

Consistently with the effective field theory approach for monopole-antimonopole pair production [15], we may take the mass scale M_0 to represent the monopole (rest) mass, and in this way one can reproduce the formula (11) of [19], and arrive at an effective velocity-dependent magnetic charge (14)[3]

$$\frac{|k - k_0|}{k_0} \to \frac{\sqrt{E^2 - 4M_0^2}}{2M_0} = \frac{E}{2M_0}\sqrt{1 - \frac{4M_0^2}{s}} \simeq \beta, \quad (68)$$

with $E^2 = s = (p_1 + p_2)^2$ being the relevant Mandelstam variable [with p_1 (p_2) being the incoming (outgoing) four momenta in the scattering process of Fig. 1, or the corresponding fused momenta in the case of production]. In such an identification, E denotes the exchanged photon momentum scale [see Fig. 1]. In the last approximate equality of (68), we considered slowly moving monopoles, for which $E \simeq 2M_0$ (since their kinetic energy is small), and this case corresponds to $k \to k_0$ where perturbativity applies, and

$$g_{\text{eff}} \simeq Z_0 \beta e_A, \qquad (69)$$

with Z_0 given by (64). In the duality-compatible approach of [19], $g_{\text{eff}} = g\beta$, with g being the magnetic charge that in the monopole case satisfies the appropriate quantization rule (1). Hence, it is natural to identify in the context of our effective field theory

$$g = Z_0 e_A. \qquad (70)$$

---

[3]The reader should bear in mind the Lorentz-invariant nature of the scale (68), characteristic of the effective field theory [15], which was not evident in the initial nonrelativistic quantum-mechanical approach to dyon-dyon scattering of [19].

Schwinger's quantization condition for the monopole (1) would then require

$$Z_0 e_A = \frac{n}{\alpha} e, \qquad n \in \mathsf{Z}, \qquad (71)$$

with Z being the set of integers, and α the fine structure constant of electromagnetism.

The value of Z_0 depends on several parameters, and the quantization condition (71) is not compatible with all of them. For bare couplings $e_B^2 \gg e_A^2$, we note that we also have $\tilde{e}_B^2 \simeq 3e_B^2/7 \gg e_A^2$ and we can see, for instance, that the case $M_0 \gg M$, would be inconsistent with (71). On the other hand, for $M_0 \ll M$, we have from (64),

$$Z_0 \simeq \frac{7\tilde{e}_B^2}{24\pi^2}, \qquad (M_0 \ll M), \qquad (72)$$

which, together with Eq. (71), implies a quantization condition for the (gauge-independent) dressed dual photon coupling $\tilde{e}_B$ (59). In this sense, we understand why the dual photon coupling is proportional to the magnetic charge of the monopole, and how our effective theory (9) describes the scattering of the latter with charged matter or its production from charged matter [19].

At this point we make some important remarks, which relate to the fact that the above effective field theory approach can only describe composite monopoles, like the ones that exist currently as topological soliton solutions of several field theories of phenomenological relevance [5,7,12,13], as discussed in the introduction of the article. This becomes clear by noticing that, because of the presence of the transmutation scale k, the wave-function renormalization Z (63) for the monopole fermion can, and in fact does for $k \to k_0$, become smaller than unity. This violates the unitarity bound [17] that requires the wave-function renormalization to be larger than one [cf. (41)], should the monopole be an elementary particle asymptotic state. However, such a bound can be evaded for composite states, either of the type discussed in the literature so far, or new, yet-unknown structured solutions to be discovered in theories beyond the standard model.

We stress that in our approach, the arguments of [16] on the strong suppression of the cross section for the collider production of such (composite) monopole-antimonopole pairs by form factors of order $\mathcal{O}(e^{-\frac{4}{\alpha}}) \simeq 10^{-238}$, are not valid for slowly moving monopoles; effectively, such factors would be replaced by terms of order $\mathcal{O}(e^{-\frac{4k-k_0}{\alpha\,k_0}})$ in our case, which would be of $\mathcal{O}(1)$ for slowly moving heavy monopoles ($k \to k_0$), with masses much larger than the SM quarks or leptons, whose collisions can lead to the monopole production. At present, such remarks of course remain speculations, as we do not have a complete microscopic understanding on the strong $U(1)$ dual interactions involved in our effective model.

B. Soft gauge-field-emission resummation and disappearance of Dirac-String effects from cross sections: a brief review of the arguments of [18]

Before closing the discussion, we review briefly the arguments of [18] on the decoupling of any Lorentz-violating Dirac-string effects from the physical cross sections [or even the scattering amplitudes if the quantization condition (1) is valid]. These arguments were based on a perturbative resummation of soft photons in the toy model for the monopole used in that work, where the magnetic charge was perturbative. This was achieved in [18] by assuming dark sector monopoles, and weak coupling of the monopole sector with the visible sector. In our case, as discussed above, both the magnetic and electric charge couplings of the monopole to photon and dual photon are perturbative for slowly moving monopoles, due to the wave-function renormalization screening effects on the effective magnetic charge (33) and (69) [see also (70)–(72)].

For such perturbative couplings, the leading soft gauge field (photons and dual photons) emissions that affect generic scattering processes of electrons off fermion monopoles in our effective $U(1)_{\mathrm{weak}} \times U(1)_{\mathrm{strong}}$ gauge theory are depicted in Fig. 2. The analysis has been performed in [18] and in previous relevant literature

referenced there, and we do not repeat it here. We only point out a trivial modification in our case as compared to those works, namely, the addition of the soft-B^μ (dual-photon-) emission graph, since, in contrast to the standard Zwanziger approach [2], here the fields A^μ and B^μ are treated as independent gauge fields, not related through (6), except when one considers classical solutions (10). Such solutions are the on-shell solutions represented by the external gauge field wavy lines in the graphs, and combine the A and B photon emission into one; this is indicated with the pertinent + sign in Fig. 2.

The presence of a classical solution, satisfying the constraint (10), implies a classical Lagrangian (9) that has Chern-Simons-like topological terms mixing the $\mathcal{A}$ and $\mathcal{B}$ fields, which depend on the Lorentz-violating Dirac-string effects ($\propto \eta^\mu$ fixed four vector). However, upon resumming soft photon and dual photon emissions, such effects exponentiate in the pertinent scattering amplitude, in such a way that they appear only in the phases of the amplitude, as demonstrated in [18], whose arguments apply here as a consequence of our perturbative magnetic charge couplings. Upon the quantization condition (1), which should be respected by the nonperturbative dressing, as discussed above, the effects disappear also from the phase of the
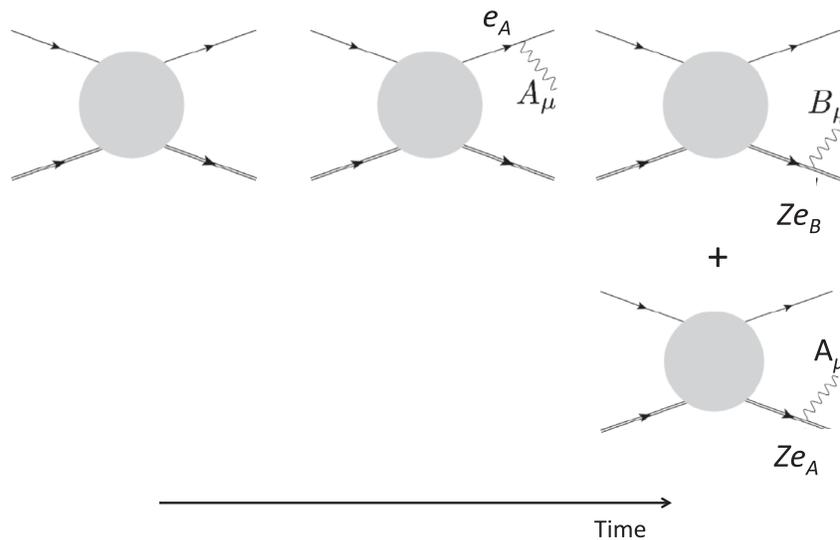
FIG. 2. Typical scattering graphs between an electron (representing generically charged matter) and a fermion monopole. Wavy lines denote real (on-shell) soft electrodynamical photons A_μ, or gauge bosons B^μ of the strongly coupled $U(1)_{\mathrm{strong}}$, the latter associated only with the monopole/dyon. Thin straight lines with arrows denote the scattered electrons, while double arrowed straight lines denote the monopole/dyon. The gray blobs denote generic scattering processes, involving all fields. The dressed couplings of the monopole to the A^μ (photon) and B^μ (dual-photon) fields are given by Ze_A and Ze_B, respectively, where Z is the wave-function renormalization factor, computed nonperturbatively due to the quantum corrections induced by the strongly coupled $U(1)_{\mathrm{strong}}$ gauge interactions. Notice that in our approach, in contrast to that of Zwanziger [2], which was adopted in [18], the photon (A^μ) and the dual photon (B^μ) are independent fields, which explains the presence of the last two graphs on the right-hand side of the figure (connected with a + sign). When however considering the monopole, one should impose on the classical on-shell gauge fields the constraint (10), which brings back the Dirac-string effects. For slowly moving monopoles, $Z \ll 1$ and this ensures perturbativity of all couplings; hence the depicted graphs denote the leading corrections in both A^μ and B^μ sectors. In such a case, resummation of the soft on-shell photons exponentiates such string effects into phases of the pertinent scattering amplitudes [18].

scattering amplitude. This completes our discussion on the self-consistency of our perturbative effective gauge field theory approach to the (slow) monopole-matter scattering or (slow) monopole-antimonopole-pair-production-from-matter processes, which have been conjectured in the previous literature and used in monopole searches at colliders [15,20].

V. CONCLUSIONS AND OUTLOOK

In this work we have considered a rather toy model of magnetically charged spin-$\frac{1}{2}$ dyons interacting with ordinary electrically charged matter fermions (electrons for concreteness). We have considered an effective gauge field theory $U(1)_{\text{weak}} \times U(1)_{\text{strong}}$, where the weak $U(1)$ represents electromagnetism, while the strong $U(1)$ effectively describes interactions that would lead to the presence of magnetic poles in then theory. The presence of two potentials was adopted by Zwanziger [2] in an attempt to describe the effective theory of Dirac monopoles in a local fashion, avoiding the use of infinite strings. However, the theory lacked Lorentz invariance. In a recent work [18], employing a toy model of perturbative magnetic charge, it was argued that all such string effects can be resummed to a physically irrelevant phase of the scattering amplitude between monopoles and matter, and in case the charge quantization condition applies, such phases vanish, so the amplitude itself was Lorentz (and gauge) invariant. In realistic monopole models, perturbative magnetic charges have been argued by Schwinger, Milton, and collaborators [19] to characterize the interaction of monopoles with matter as a result of electric magnetic duality. However, such arguments were purely qualitative, not supported by any detailed modeling. They were based on quantum-mechanical scattering of electrons off magnetic poles, whose cross sections had a form that, in order to be obtained from Rutherford cross section of normal charged particles by means of imposing electromagnetic duality, one should impose that the magnetic charge was effectively proportional to the monopole velocity, hence perturbative for slowly moving monopoles.

Motivated by these results, we have discussed in this work a toy model for describing the interaction of magnetic monopoles/dyons with matter, where quantum fluctuation effects of the magnetic charge are represented by the presence of a second independent strong U(1) gauge potential. However, we have ignored any Lorentz (or gauge) violation effects, anticipating the results of [18]. Our analysis has led to a consistent result, namely, the emergence of an effective magnetic coupling of the monopole/dyon to the real photons, depending on the monopole velocity, in the spirit conjectured in [19]. This result was obtained as a mathematically self-consistent solution of the appropriate Schwinger-Dyson equations. The role of the velocity-dependent factor that appeared in

the work of [19] is played here by the transmutation mass scale that appears in the Schwinger-Dyson solution for the monopole wave-function renormalization, which enters the expression of the monopole-photon vertex. The role of the nonperturbative corrections of the dual $U(1)_{\text{strong}}$ is crucial to this effect. However, we stress that in the case of slowly moving monopoles/dyons, the wave-function renormalization violates unitarity bounds set for elementary particle asymptotic states (41), thus necessitating the application of our effective $U(1)_{\text{weak}} \times U(1)_{\text{strong}}$ gauge field theory only to composite monopoles/dyons, of generic type though, not necessarily restricted to the known solutions existing in the literature.

Once such a perturbative monopole-photon coupling is established [for slowly moving, heavy (compared to the matter-fermion mass) monopoles], the arguments of [18] are in operation, and one may assume that any Lorentz noninvariant string effects that might appear in the theory of [2] affect *only* the phases of the respective scattering amplitudes, and they actually disappear once the charge quantization condition (1) is imposed. In our approach we have explained how such a quantization arises naturally for the dual photon coupling. Although in the theory of [2] such string effects implied that the dual photon is not independent of the real photon, and that on shell there are only two propagating d.o.f., in our fully quantum theory, the dual photon quantum fluctuations have been assumed independent of electromagnetism. Nonetheless, one may impose the monopole solution of Zwanziger as a consistent Lorentz noninvariant solution to the respective equations of motion, thus making contact with the original Dirac monopole case.

We should stress though that the emergence of the strong U(1) interactions from microscopic considerations in concrete composite monopole models existing in the current literature [5,7,12,13] is still not understood. It might be possible that the very existence of a magnetic pole implies automatically such interactions, as an effective way of describing the monopole quantum fluctuations.[4] Nonetheless, our effective field theory description of monopole-matter interactions developed in the current work offers support to the relativistic effective gauge field theory approach of [15], used so far in collider searches of magnetically charged particles [20]. Formulating our theory on the lattice might then be a way forward for obtaining results in the nonperturbative regime of the effective magnetic couplings, when the monopoles are not slowly moving, or appear as virtual particles in quantum loops of, say, light-by-light scattering processes,

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[4]In this latter respect, we mention that an interesting research direction would be to attempt and find connections, if any, of the current approach with the manifestly dual quantum field theory proposal for electric and magnetic charges of [25], in which the Dirac string (and thus the associated fixed vector η^μ in the Zwanziger approach [2]) becomes dynamical.

of interest to collider searches for monopoles heavier than the respective collider production threshold. In the latter case, an enhanced light-by-light scattering cross section, as compared to the standard-model case, has been claimed in Ref. [26] as an indirect way of detecting the existence of monopoles. In that work, analytical estimates for the corresponding cross sections have been made on assuming *ad hoc* photon-energy dependent monopole-photon couplings of the form (66) and (68), which are weak for sufficiently low photon energies compared to the monopole mass. Our effective gauge field theory not only provides justification for such assumptions, as we discussed above, but, by being placed on the lattice, can also lead, in principle, to an extension of the considerations of [26] to higher photon energies and lower monopole masses, thus enhancing the appropriate parameter space region in future collider searches. Such issues constitute interesting research material for future work.

Finally, before closing, we mention that another interesting open issue is the existence of nonperturbative effects associated with the strongly coupled $U(1)_{\text{strong}}$ sector. One way such effects could be manifested is through the monopole-antimonopole pair creation from the vacuum in the presence of external magnetic fields [27] and/or high temperatures [28]. In this approach, which generalizes Schwinger electron-positron pair production in external electric fields [29] to the magnetic charge case, one calculates the monopole production rate Γ_{mono} per unit volume, in the presence of a (weak) magnetic field of intensity B. The rate Γ_{mono} is calculated from the imaginary part of the total energy of the configuration $\Gamma_{\text{mono}} = \frac{2}{\hbar}\text{Im}E_0$, which implies $|\langle 0|\exp(-i\mathcal{H}t/\hbar)|0\rangle|^2 \exp(-\Gamma_{\text{mono}}t)$, where $\mathcal{H}$ is the Hamiltonian. Following the calculation in [27] one obtains for weak magnetic fields B and heavy monopoles of mass M

$$
\begin{aligned}
\Gamma_{\text{mono}} &= \exp(-E_0 T) \sim \int_{\text{T}\to\infty} D\phi \exp(-I(\phi)) \\
&= \mathcal{C}[1 - i\mathcal{K}\exp(-I_{\text{cl}}(\phi))VT] \\
&\simeq \frac{g^2 B^2}{8\pi^3}\exp\left(-\frac{\pi M^2}{gB} + \frac{g^2}{4}\right)\left[1 + \mathcal{O}\left(\frac{g^3 B}{M^2}\right) + \cdots\right],
\end{aligned}
\tag{73}
$$

where $I_{\text{cl}}(\phi)$ is the classical instanton action and $\mathcal{K}$ a one-loop factor [the dilute instanton gas approximation was adopted in [27], with $\exp(-I_{\text{cl}})$ the instanton density]. In (73), g denotes the monopole magnetic charge, while the … denote terms which in the specific (scalar) monopole model of Georgi-Glashow SO(3) type considered in [27] denote terms of order $\mathcal{O}(e^2)$ where e is the gauge SO(3) coupling constant related to g via $g = 4\pi/e$, where in the presence of weak magnetic fields B then, one can calculate I_{cl} and the one-loop factor $\mathcal{K}$, which yields the result (73). In the computation of [27] no charge renormalization effects have been taken into account.

In this respect, in our case, should the result (73) be valid, one would expect to consider the full monopole propagators obtained in the context of our Schwinger-Dyson treatment to circulate in the loop. According to our analysis in this work, the electromagnetic coupling of our fermion monopole to photons, and thus to external magnetic fields, is given by g_{eff} (69), which contains the nontrivial wave-function renormalization and the monopole velocity β dependence. For slowly moving monopoles, $\beta \to 0$ and hence the pair production would be suppressed according to (73). However, the vacuum polarization produces monopoles with all βs and the production rate of monopole-antimonopole pairs with $\beta \sim 1$ is unsuppressed. The total rate, should one accept formula (73), should then be given by replacing g in (73) by $g_{\text{eff}} = Z_0 e_A \beta \equiv g\beta$ [cf. (68) and (69)] and integrating over all possible velocities $\beta \in [0, 1]$,

$$
\begin{aligned}
\Gamma_{\text{mono}} &= \int_0^1 d\beta \frac{g^2\beta^2 B^2}{8\pi^3}\exp\left(-\frac{\pi M^2}{g\beta B} + \frac{g^2\beta^2}{4}\right)\left[1 + \mathcal{O}\left(\frac{g^3\beta^3 B}{M^2}\right) + \cdots\right] \\
&\simeq \frac{g^2 B^2}{48\pi^3}\exp\left(-\frac{\pi M^2}{gB}\right)\left(2 + \left(-1 + \frac{\pi M^2}{gB}\right)\frac{\pi M^2}{gB}\right. \\
&\quad + \left(\frac{\pi M^2}{gB}\right)^3 \exp\left(\frac{\pi M^2}{gB}\right)\left[\text{CoshIntegral}\left(\frac{\pi M^2}{gB}\right) - \text{SinhIntegral}\left(\frac{\pi M^2}{gB}\right)\right]\bigg) \\
&\simeq \frac{g^2 B^2}{24\pi^3}\exp\left(-\frac{\pi M^2}{gB}\right) \ll 1, \quad \text{for} \quad \frac{\pi M^2}{gB} \gg 1,
\end{aligned}
\tag{74}
$$

which is the case of relatively heavy monopoles and weak magnetic fields considered in [27]. We thus observe that the rate (74) is 1/3 of the rate (73) (they are both very strongly suppressed of course). To arrive at the result (74) we took into account that $\text{CoshIntegral}(x) = \gamma + \ln|x| + \int_0^x dt \frac{\cosh(t)-1}{t}$, $\text{SinhIntegral}(x) = \int_0^x dt \frac{\sinh(t)}{t}$, with $\gamma = 0.577216\ldots$ the

Euler-Mascheroni constant, and we used the asymptotic expression of these integrals for large $x \gg 1$ [30]: $x^3(\text{CoshIntegral}(x) - \text{SinhIntegral}(x)) \simeq x(-x+1)\exp(-x)$.

The above remarks should be considered as speculative, though, given that the precise computation of such vacuum nonperturbative effects for fermion monopoles that we

discussed in our approach is still pending [the computation of [27] pertained to scalar (spinless) monopoles of a specific microscopic model].

Another kind of nonperturbative effect can shortly be discussed, which is the possibility to dynamically generate the monopole mass. From Eq. (54), we can see that a vanishing monopole bare mass $M = 0$ leads to the consistent solution $M_r = c_1 k$ for the dressed mass, where $c_1 > 0$ is a numerical (dimensionless) integration constant (taken to be positive to ensure positivity of the mass, since $k > 0$). This dynamical mass is independent of the coupling constants, and hence purely a nonperturbative phenomenon. k is a transmutational scale, and therefore is not fixed by the model but instead should be considered as a physical parameter of the theory. The latter can be defined from the nonvanishing bare mass of the standard-model fermion, which is a natural scale in our description. Therefore, a monopole dynamical mass is in principle consistent, within the nonperturbative regime we describe here. We also note that dynamical mass generation is related to the existence of bound states, which, as we already discussed in this article, is a necessary interpretation to avoid the unitarity bound $Z > 1$. The formation of bound states between monopoles and antimonopoles in our effective theory (termed "monopolia"), as a result of the strongly coupled $U_{\mathrm{strong}}(1)$ sector, is a very interesting open issue, which requires a separate study, in order the establish the existence of a potentially confining phase. Such a study might prove useful for the discussion of monopolium production at colliders [31].

The above nonperturbative effects, as well a phase diagram of our effective theory, including potential confinement, deserve a full study, which is left for future work.

ACKNOWLEDGMENTS

APPENDIX A: ONE-PARTICLE-IRREDUCIBLE (1PI) EFFECTIVE ACTION

(i) The partition function is

$$\mathcal{Z} = \int \mathcal{D}[\phi] \exp\left(iI + iI_s\right), \qquad (A1)$$

where ϕ denotes collectively the different d.o.f. A_μ, B_μ, $\psi\bar{\psi}, \chi, \bar{\chi}$ with sources $j_A^\mu, j_B^\mu, \bar{\eta}, \eta, \bar{\xi}, \xi$, respectively, such that the source term is

$$I_s = \int_x j_A^\mu A_\mu + j_B^\mu B_\mu + \bar{\eta}\psi + \bar{\psi}\eta + \bar{\xi}\chi + \bar{\chi}\xi. \quad (A2)$$

The connected-graphs generating functional is

$$W = -i \ln \mathcal{Z}, \qquad (A3)$$

whose functional derivatives define the background fields

$$\psi_b = \langle\psi\rangle = \frac{\delta W}{\delta\bar{\eta}}, \qquad \bar{\psi}_b = \langle\bar{\psi}\rangle = \frac{\delta W}{\delta\eta}$$

$$\chi_b = \langle\chi\rangle = \frac{\delta W}{\delta\bar{\xi}}, \qquad \bar{\chi}_b = \langle\bar{\chi}\rangle = \frac{\delta W}{\delta\xi}$$

$$A_b^\mu = \langle A^\mu\rangle = \frac{\delta W}{\delta j_\mu^A}, \qquad B_b^\mu = \langle B^\mu\rangle = \frac{\delta W}{\delta j_\mu^B}, \qquad (A4)$$

with

$$\langle\cdots\rangle \equiv \frac{1}{\mathcal{Z}} \int \mathcal{D}[\phi](\cdots) \exp\left(iI + iI_s\right). \qquad (A5)$$

(ii) The 1PI-graphs generating functional, the effective action Γ, is a functional of the background fields and is obtained via a Legendre transform of W, after inverting the relations (A4) and treating the sources as functionals of the background fields

$$\Gamma = W - \int_x j_A^\mu A_\mu^b + j_B^\mu B_\mu^b + \bar{\eta}\psi_b + \bar{\psi}_b\eta$$
$$+ \bar{\xi}\chi_b + \bar{\chi}_b\xi. \qquad (A6)$$

Γ has the following functional derivatives:

$$\frac{\delta\Gamma}{\delta\psi_b} = -\bar{\eta}, \qquad \frac{\delta\Gamma}{\delta\bar{\psi}_b} = -\eta$$

$$\frac{\delta\Gamma}{\delta\chi_b} = -\bar{\xi}, \qquad \frac{\delta\Gamma}{\delta\bar{\chi}_b} = -\xi$$

$$\frac{\delta\Gamma}{\delta A_b^\mu} = -j_\mu^A, \qquad \frac{\delta\Gamma}{\delta B_b^\mu} = -j_\mu^B. \qquad (A7)$$

(iii) The bare propagators are defined by

$$S_\psi = \left(\frac{\delta^2 I}{\delta\psi\delta\bar{\psi}}\right)_0^{-1}, \qquad S_\chi = \left(\frac{\delta^2 I}{\delta\chi\delta\bar{\chi}}\right)_0^{-1}$$

$$D_{\mu\nu} = \left(\frac{\delta^2 I}{\delta A^\nu \delta A^\mu}\right)_0^{-1} = \left(\frac{\delta^2 I}{\delta B^\nu \delta B^\mu}\right)_0^{-1}, \qquad (A8)$$

where the subscript 0 denotes vanishing fields. As explained in Sec. II B 1, the dressed propagators are diagonal, both for fermions and gauge fields, and are defined by

$$G_\psi = \left(\frac{\delta^2 \Gamma}{\delta \psi_b \delta \bar\psi_b} \right)^{-1}, \quad G_\chi = \left(\frac{\delta^2 \Gamma}{\delta \chi_b \delta \bar\chi_b} \right)^{-1}$$

$$\Delta^A_{\mu\nu} = \left(\frac{\delta^2 \Gamma}{\delta A^\nu_b \delta A^\mu_b} \right)^{-1}, \quad \Delta^B_{\mu\nu} = \left(\frac{\delta^2 \Gamma}{\delta B^\nu_b \delta B^\mu_b} \right)^{-1}. \quad \text{(A9)}$$

The dressed vertices are

$$\Lambda^{A\psi}_\mu = \left. \frac{\delta^3 \Gamma}{\delta \psi_b \delta A^\mu_b \delta \bar\psi_b} \right|_0, \quad \Lambda^{A\chi}_\mu = \left. \frac{\delta^3 \Gamma}{\delta \chi_b \delta A^\mu_b \delta \bar\chi_b} \right|_0$$

$$\Lambda^{B\psi}_\mu = \left. \frac{\delta^3 \Gamma}{\delta \psi_b \delta B^\mu_b \delta \bar\psi_b} \right|_0, \quad \Lambda^{B\chi}_\mu = \left. \frac{\delta^3 \Gamma}{\delta \chi_b \delta B^\mu_b \delta \bar\chi_b} \right|_0. \quad \text{(A10)}$$

(iv) From the first functional derivatives (A7), one can obtain the following identities, relating the second functional derivatives of W and Γ,

$$\frac{\delta^2 \Gamma}{\delta \psi_b \delta \bar\psi_b} = -(\delta^2 W)^{-1}_{\eta,\bar\eta}, \quad \frac{\delta^2 \Gamma}{\delta \chi_b \delta \bar\chi_b} = -(\delta^2 W)^{-1}_{\xi\bar\xi}$$

$$\frac{\delta^2 \Gamma}{\delta A^\mu_b \delta A^\nu_b} = -(\delta^2 W)^{-1}_{j^A_\mu j^A_\nu}, \quad \frac{\delta^2 \Gamma}{\delta B^\mu_b \delta B^\nu_b} = -(\delta^2 W)^{-1}_{j^B_\mu j^B_\nu}, \quad \text{(A11)}$$

where the inverse has to be read as the element of the inverse matrix.

APPENDIX B: WARD IDENTITIES

(i) The gauge transformation (19) leaves the action S invariant and modifies the source terms, such that the partition function (A1) is modified as

$$\mathcal{Z} \to \mathcal{Z} + i \int \mathcal{D}[\phi] \exp(iI + iI_s)$$
$$\times \int_x (j^\mu_A \partial_\mu \theta_A + j^\mu_B \partial_\mu \theta_B + ie\theta_A(\bar\eta\psi - \bar\psi\eta)$$
$$+ i(e_A \theta_A + e_B \theta_B)(\bar\xi\chi - \bar\chi\xi)) + \mathcal{O}(\theta^{A,B})^2. \quad \text{(B1)}$$

Integration by parts lead to

$$\mathcal{Z} \to \mathcal{Z} + i \int_x \theta_A \int \mathcal{D}[\phi] \exp(iI + iI_s)$$
$$\times (-\partial_\mu j^\mu_A + ie(\bar\eta\psi - \bar\psi\eta) + ie_A(\bar\xi\chi - \bar\chi\xi))$$
$$+ i \int_x \theta_B \int \mathcal{D}[\phi] \exp(iI + iI_s)$$
$$\times (-\partial_\mu j^\mu_B + ie_B(\bar\xi\chi - \bar\chi\xi)) + \mathcal{O}(\theta^{A,B})^2, \quad \text{(B2)}$$

and gauge invariance for independent functions θ_A and θ_B leads to

$$0 = \int \mathcal{D}[\phi] \exp(iI + iI_s)(-\partial_\mu j^\mu_A + ie(\bar\eta\psi - \bar\psi\eta)$$
$$+ ie_A(\bar\xi\chi - \bar\chi\xi)),$$

$$0 = \int \mathcal{D}[\phi] \exp(iI + iI_s)(-\partial_\mu j^\mu_B + ie_B(\bar\xi\chi - \bar\chi\xi)). \quad \text{(B3)}$$

Together with the identities given in Appendix A, we obtain then

$$0 = \partial_\mu \left(\frac{\delta\Gamma}{\delta A_\mu} \right) + ie \frac{\delta\Gamma}{\delta\psi_b}\psi_b - ie\bar\psi_b \frac{\delta\Gamma}{\delta\bar\psi_b}$$
$$+ ie_A \frac{\delta\Gamma}{\delta\chi_b}\chi_b - ie_A\bar\chi_b \frac{\delta\Gamma}{\delta\bar\chi_b}$$

$$0 = \partial_\mu \left(\frac{\delta\Gamma}{\delta B_\mu} \right) + ie_B \frac{\delta\Gamma}{\delta\chi_b}\chi - ie_B\bar\chi \frac{\delta\Gamma}{\delta\bar\chi_b}. \quad \text{(B4)}$$

(ii) On account of the property (24), taking the functional derivatives of Eqs. (B4) with respect to ψ_b and $\bar\psi_b$, followed by setting the background fields to 0, leads to

$$\frac{1}{e}\frac{\partial\Lambda^{A\psi}_\mu}{\partial z_\mu}(x, y, z) = iG^{-1}_\psi(x, z)\delta(x - y)$$
$$- iG^{-1}_\psi(y, z)\delta(x - y)$$

$$\frac{\partial\Lambda^{B\psi}_\mu}{\partial z_\mu}(x, y, z) = 0. \quad \text{(B5)}$$

(iii) For the same reason, on taking the functional derivatives of Eqs. (B4) with respect to χ_b and $\bar\chi_b$, and then setting the background fields to 0, leads to

$$\frac{1}{e_A}\frac{\partial\Lambda^{A\chi}_\mu}{\partial z_\mu}(x, y, z) = iG^{-1}_\chi(x, z)\delta(x - y)$$
$$- iG^{-1}_\chi(y, z)\delta(x - y),$$

$$\frac{1}{e_B}\frac{\partial\Lambda^{B\chi}_\mu}{\partial z_\mu}(x, y, z) = iG^{-1}_\chi(x, z)\delta(x - y)$$
$$- iG^{-1}_\chi(y, z)\delta(x - y). \quad \text{(B6)}$$

In Fourier components, we have

$$iG^{-1}(x, z) - iG^{-1}(y, z) \to G^{-1}(p + q) - G^{-1}(p)$$
$$= q^\mu \left. \frac{\partial G^{-1}}{\partial p^\mu} \right|_p + \mathcal{O}(q^2), \quad \text{(B7)}$$

where p is an incoming fermion momentum and q is the gauge-field momentum. In the limit $q \to 0$, we then obtain the Ward identities

$$\frac{1}{e}\Lambda_\mu^{A\psi}(p,0) = \frac{\partial G_\psi^{-1}}{\partial p^\mu}, \qquad \frac{1}{e_B}\Lambda_\mu^{B\chi}(p,0) = \frac{\partial G_\chi^{-1}}{\partial p^\mu},$$

$$\text{(B8)}$$

and the additional properties

$$q^\mu \Lambda_\mu^{B\psi}(p,q) = 0,$$

$$e_B q^\mu \Lambda_\mu^{A\chi}(p,q) = e_A q^\mu \Lambda_\mu^{B\chi}(p,q). \qquad \text{(B9)}$$

APPENDIX C: SD EQUATIONS

(i) The SD equation for the ψ self-energy is obtained by noting that the integral of the following functional derivative vanishes,

$$\int \mathcal{D}[\phi]\frac{\delta}{\delta\bar\psi}\exp(iI + iI_s) = 0, \qquad \text{(C1)}$$

such that

$$\int \mathcal{D}[\phi](i\gamma^\mu D_\mu^A \psi - m\psi + \eta)\exp(iI + iI_s) = 0. \quad \text{(C2)}$$

Using the identities in Appendix A, the latter equation can also be written,

$$0 = (i\gamma^\mu \partial_\mu - m)\psi_b - \frac{\delta\Gamma}{\delta\bar\psi_b} - e\gamma^\mu \langle A_\mu \psi \rangle, \qquad \text{(C3)}$$

and a functional derivative with respect to ψ_b, followed by setting the fields to 0, leads to

$$G_\psi^{-1} - S_\psi^{-1} = -e\gamma^\mu \frac{\delta}{\delta\psi_b}\langle A_\mu \psi \rangle_0. \qquad \text{(C4)}$$

We have

$$\langle A_\mu \psi \rangle = A_\mu^b \psi_b - i\frac{\delta^2 W}{\delta j_A^\mu \delta\bar\eta}, \qquad \text{(C5)}$$

and

$$\left.\frac{\delta}{\delta\psi_b}\frac{\delta^2 W}{\delta j_A^\mu \delta\bar\eta}\right|_0 = -\left.\frac{\delta^3 W}{\delta\eta\delta j_A^\mu \delta\bar\eta}\right|_0 G_\psi^{-1}, \qquad \text{(C6)}$$

where the missing terms vanish when fields are set to 0, and the integral over space-time coordinates is understood. The next step consists in noting that

$$\left.\frac{\delta^3 W}{\delta\eta\delta j_A^\mu \delta\bar\eta}\right|_0$$

$$= -\left.\frac{\delta}{\delta j_A^\mu}\left(\frac{\delta^2 \Gamma}{\delta\psi_b \delta\bar\psi_b}\right)^{-1}\right|_0$$

$$= G_\psi\left(\frac{\delta^3 \Gamma}{\delta\psi_b \delta A_\nu \delta\bar\psi_b}\frac{\delta A_\nu}{\delta j_A^\mu} + \frac{\delta^3 \Gamma}{\delta\psi_b \delta B_\nu \delta\bar\psi_b}\frac{\delta B_\nu}{\delta j_A^\mu}\right)_0 G_\psi$$

$$\text{(C7)}$$

$$= G_\psi \Lambda_{A\psi}^\nu \Delta_{\mu\nu}^A G_\psi, \qquad \text{(C8)}$$

where $\delta B_\nu/\delta j_\mu^A|_0 = 0$ since the gauge propagator is diagonal. We finally obtain

$$G_\psi^{-1} - S_\psi^{-1} = ie\gamma^\mu G_\psi \Lambda_{A\psi}^\nu \Delta_{\mu\nu}^A, \qquad \text{(C9)}$$

where the integration over space-time coordinates is understood.

(ii) The SD equation for the χ-fermion self-energy is obtained by noting that

$$\int \mathcal{D}[\phi]\frac{\delta}{\delta\bar\chi}\exp(iI + iI_s) = 0, \qquad \text{(C10)}$$

such that

$$\int \mathcal{D}[\phi](i\gamma^\mu D_\mu^{A+B}\chi - M\chi + \xi)\exp(iI + iI_s) = 0,$$

$$\text{(C11)}$$

and the same steps as the ones described above lead to

$$G_\chi^{-1} - S_\chi^{-1} = ie_A\gamma^\mu G_\chi \Lambda_{A\chi}^\nu \Delta_{\mu\nu}^A + ie_B\gamma^\mu G_\chi \Lambda_{B\chi}^\nu \Delta_{\mu\nu}^B.$$

$$\text{(C12)}$$

(iii) The SD equation for the A_μ self-energy is obtained by noting that

$$\int \mathcal{D}[\phi]\frac{\delta}{\delta A_\mu}\exp(iI + iI_s) = 0, \qquad \text{(C13)}$$

such that

$$D_{\mu\nu}^{-1}A_b^\nu - e\langle\bar\psi\gamma_\mu\psi\rangle - e_A\langle\bar\chi\gamma_\mu\chi\rangle + j_\mu^A = 0. \qquad \text{(C14)}$$

A functional derivative with respect to A_b^ν leads to

$$(\Delta_{\mu\nu}^A)^{-1} - D_{\mu\nu}^{-1} = ie\,\text{tr}\left\{\gamma_\mu\frac{\delta}{\delta A_b^\nu}\frac{\delta^2 W}{\delta\eta\delta\bar\eta}\right\}$$

$$+ ie_A\,\text{tr}\left\{\gamma_\mu\frac{\delta}{\delta A_b^\nu}\frac{\delta^2 W}{\delta\xi\delta\bar\xi}\right\} \qquad \text{(C15)}$$

and for vanishing fields we have

$$\frac{\delta}{\delta A_b^\nu}\frac{\delta^2 W}{\delta\eta\delta\bar\eta}\bigg|_0 = \frac{\delta}{\delta A_b^\nu}\left(\frac{\delta^2\Gamma}{\delta\psi_b\delta\bar\psi_b}\right)^{-1}_0 = -G_\psi\Lambda_\nu^{A\psi}G_\psi,$$

(C16)

$$\frac{\delta}{\delta A_b^\nu}\frac{\delta^2 W}{\delta\xi\delta\bar{\bar\xi}}\bigg|_0 = \frac{\delta}{\delta A_b^\nu}\left(\frac{\delta^2\Gamma}{\delta\psi_b\delta\bar\psi_b}\right)^{-1}_0 = -G_\chi\Lambda_\nu^{A\chi}G_\chi.$$

(C17)

Finally we obtain

$$(\Delta_{\mu\nu}^A)^{-1} - D_{\mu\nu}^{-1} = i\,\mathrm{tr}\{e\gamma_\mu G_\psi\Lambda_\nu^{A\psi}G_\psi + e_A\gamma_\mu G_\chi\Lambda_\nu^{A\chi}G_\chi\}.$$

(C18)

(iv) The SD equation for the B_μ self-energy is obtained by noting that

$$\int\mathcal{D}[\phi]\frac{\delta}{\delta B_\mu}\exp(iI + iI_s) = 0,$$

(C19)

such that

$$D_{\mu\nu}^{-1}B_b^\nu - e_B\langle\bar\chi\gamma_\mu\chi\rangle + j_\mu^B = 0,$$

(C20)

and similar steps as the ones described above lead to

$$(\Delta_{\mu\nu}^{-1})^B - D_{\mu\nu}^{-1} = ie_B\,\mathrm{tr}\{\gamma_\mu G_\chi\Lambda_\nu^{B\chi}G_\chi\}.$$

(C21)

APPENDIX D: CALCULATION OF QUANTUM CORRECTIONS

(i) Within the approximations described in Sec. III, the SD equation for the monopole self-energy is

$$(Z-1)\slashed{p} - \delta M$$
$$= ie_A^2\int\frac{d^4q}{(2\pi)^4}\gamma^\mu G_\chi(p+q)Z\gamma^\nu D_{\mu\nu}(q)$$
$$+ ie_B^2\int\frac{d^4q}{(2\pi)^4}\gamma^\mu G_\chi(p+q)Z\gamma^\nu\Delta_{\mu\nu}^B(q),$$

(D1)

and the SD equation for the dual photon polarization tensor is

$$\omega(q^2\eta_{\mu\nu} - q_\mu q_\nu)$$
$$= ie_B^2\,\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\gamma_\mu G_\chi(p)Z\gamma_\nu G_\chi(p+q),$$

(D2)

where the propagators and vertices are given in Sec. III.

(ii) For the monopole self-energy (D1), the mass corrections are obtained by setting the external momentum to 0,

$$\delta M = ie_A^2 k^\epsilon\int\frac{d^d q}{(2\pi)^d}\gamma^\mu\frac{\tilde M}{Z^2 q^2 - \tilde M^2}\gamma^\nu\left(\frac{\eta_{\mu\nu}}{q^2} + \frac{1-\lambda}{\lambda}\frac{q_\mu q_\nu}{q^4}\right)$$
$$+ ie_B^2 k^\epsilon\int\frac{d^d q}{(2\pi)^d}\gamma^\mu\frac{\tilde M}{Z^2 q^2 - \tilde M^2}\frac{Z\gamma^\nu}{1+\omega}$$
$$\times\left(\frac{\eta_{\mu\nu}}{q^2} + \frac{1+\omega-\lambda}{\lambda}\frac{q_\mu q_\nu}{q^4}\right).$$

(D3)

and the wave-function renormalization is obtained by taking a derivative with respect to the external momentum component p^σ, which is then set to 0,

$$(Z-1)\gamma_\sigma$$
$$= ie_A^2 k^\epsilon\int\frac{d^d q}{(2\pi)^d}\gamma^\mu\left(\frac{Z\gamma_\sigma}{Z^2 q^2 - \tilde M^2} - \frac{2Z^2(Z\slashed{q}-\tilde M)q_\sigma}{(Z^2 q^2 - \tilde M^2)^2}\right)$$
$$\times\gamma^\nu\left(\frac{\eta_{\mu\nu}}{q^2} + \frac{1-\lambda}{\lambda}\frac{q_\mu q_\nu}{q^4}\right)$$
$$+ ie_B^2 k^\epsilon\int\frac{d^d q}{(2\pi)^d}\gamma^\mu\left(\frac{Z\gamma_\sigma}{Z^2 q^2 - \tilde M^2} - \frac{2Z^2(Z\slashed{q}-\tilde M)q_\sigma}{(Z^2 q^2 - \tilde M^2)^2}\right)$$
$$\times\frac{Z\gamma^\nu}{1+\omega}\left(\frac{\eta_{\mu\nu}}{q^2} + \frac{1+\omega-\lambda}{\lambda}\frac{q_\mu q_\nu}{q^4}\right).$$

(D4)

In the latter expressions dimensional regularization is used, in dimension $d = 4 - \epsilon$ and with the arbitrary mass scale k. In both cases the simplification of the integrand is straightforward, and we obtain for the mass corrections

$$\frac{\delta M}{\tilde M} = \frac{i}{\lambda}\left(e_A^2(1+3\lambda) + e_B^2\frac{1+3\lambda+\omega}{1+\omega}\right)$$
$$\times\frac{k^\epsilon}{Z^{d-3}}\int\frac{d^d q}{(2\pi)^d}\frac{1}{q^2(q^2 - \tilde M^2)} + \mathrm{finite}$$
$$= \frac{1}{8\pi^2\lambda Z}\left(e_A^2(1+3\lambda) + e_B^2\frac{1+3\lambda+\omega}{1+\omega}\right)\frac{1}{\epsilon}\left(\frac{Zk}{\tilde M}\right)^\epsilon$$
$$+ \mathrm{finite}.$$

(D5)

(iii) For the monopole wave-function renormalization, we observe that both Z and ω cancel in the integrand (both in the diverging and finite parts), such that the corrections arising from A_μ are identical to those arising from B_μ, and we obtain

$$Z = 1 - i\frac{e_A^2 + e_B^2}{\lambda}\frac{k^\epsilon}{Z^{d-4}}\int\frac{d^d q}{(2\pi)^d}\frac{1}{(q^2 - \tilde M^2)^2} + \mathrm{finite}$$
$$= 1 + \frac{e_A^2 + e_B^2}{8\pi^2\lambda}\frac{1}{\epsilon}\left(\frac{Zk}{\tilde M}\right)^\epsilon + \mathrm{finite}.$$

(D6)

(iv) For the dual photon polarization, the integral in Eq. (D2) is the same as in the usual perturbative case, except for an overall factor $1/Z$, which is a nonperturbative feature,

$$\omega = \frac{e_B^2}{6\pi^2 Z} \frac{1}{\epsilon} \left(\frac{Zk}{\tilde{M}}\right)^\epsilon + \text{finite}.$$

$$(D7)$$

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