Mass shifts of ^3P_J heavy quarkonia due to the effect of two-gluon annihilation

Hui-Yun Cao and Hai-Qing Zhou[*]

School of Physics, Southeast University, NanJing 211189, China

(Received 20 May 2019; published 8 November 2019)

In this work, we calculate the nonrelativistic asymptotic behavior of the amplitudes of $q\bar{q} \to 2g \to q\bar{q}$ in the leading order of α_s with $q\bar{q}$ in the ^3P_J channels. In the practical calculation we take the momenta of quarks and antiquarks on shell and expand the amplitudes on the three-momentum of the quarks and antiquarks to order 6 and get three nonzero terms. The imaginary parts of the results are comparable with the known nonrelativistic QCD results. The real parts of the results have IR divergence. When applying the results to the heavy quarkonia, the corresponding amplitude of $q\bar{q} \to 1g \to q\bar{q}$ with $q\bar{q}$ in the color octet $^3S_1$ channel is considered to absorb the IR divergence in a unitary way in the leading order of v. The final results can be used to estimate the mass shifts of the ^3P_J heavy quarkonia due to the effect of two-gluon annihilation. The numerical estimation shows that the contributions to the mass shifts of $\chi_{c0,c1,c2}$ are about $1.23 \sim 1.58$ MeV, $1.57 \sim 1.86$ MeV, and $5.92 \sim 5.45$ MeV when taking $\alpha_s \approx 0.25 \sim 0.35$.

DOI: 10.1103/PhysRevD.100.094004

I. INTRODUCTION

The energy spectrum of an elemental system is a basic question after the breakthrough of quantum mechanism. Currently, the energy spectrum of hadrons is still an unsolved problem in QCD due to the complex nonperturbative property. Many phenomenological models have been used to study the energy spectrum of hadrons in the quark level such as the quark model [1], QCD sum rules [2], Dyson-Schwinger equation and Bethe-Salpeter equation [3], etc. In these calculations, the annihilation effect whose imaginary and real parts correspond to the decay width and the mass shift is usually neglected. For heavy quarkonia, their inclusive decays can be well described by the effective theory nonrelativistic QCD (NRQCD) [4]. In NRQCD, the imaginary part of the coefficients of four fermions interactions are matched from the imaginary parts of the on-shell scattering amplitudes $q\bar{q} \to 1g, 2g, 3g \to q\bar{q}$ or the decay widths of $q\bar{q} \to 1g, 2g, 3g$ in perturbative QCD order by order. In a previous paper [5], we calculated the real parts of these coefficients in the leading order of α_s (LO-α_s) in the $^1S_0$ channel with the momenta of quarks and antiquarks off shell and find the results are gauge invariant, while the similar calculation cannot be directly extended to the ^3P_J channels due to the gauge invariance. In principle, the corresponding four-point Green's function of $q\bar{q} \to q\bar{q}$ with quarks and antiquarks off shell provides the interaction kernel of the Bethe-Salpeter and then determines the mass shifts. Usually the Green's function with the off-shell quarks and antiquarks is not gauge invariant. To avoid this problem, the most simple way is to do an approximation by taking the quarks and antiquarks on shell. After using such an approximation, the results are gauge invariant and the calculation is similar with what was done in NRQCD. In this work, we calculate the amplitudes of $q\bar{q} \to 2g \to q\bar{q}$ in the ^3P_J channels and $q\bar{q} \to 1g \to q\bar{q}$ in the $^3S_1$ channel with $q\bar{q}$ in color single and color octet states, respectively.

We organize the paper as follows. In Sec. II we give an introduction on the basic formula, in Sec. III we describe our calculation and present the analytic results for the coefficients to order 6 after the nonrelativistic expansion, in Sec. IV we compare the above results with the corresponding NRQCD coefficients and from this comparison the real parts of the corresponding NRQCD coefficients can be read directly, in Sec. V we apply the above results to discuss the mass shift of ^3P_J states, and in Sec. VI we estimate the effects to the mass shifts numerically and discuss the interesting properties of the results.

II. BASIC FORMULA

Following the idea of NRQCD, for a heavy quarkonium $H(^3P_J)$ in the J^{++} state there are two contributions in the amplitudes of $H(^3P_J) \to H(^3P_J)$ in the leading order of v (LO-v) which can be expressed as

[*]zhouhq@seu.edu.cn

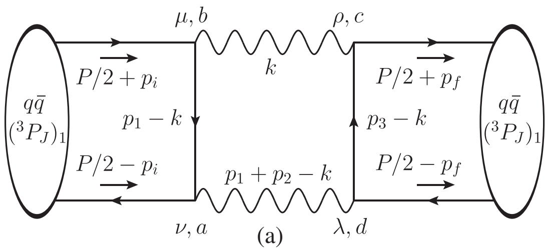

FIG. 1. The Feynman diagrams for $q\bar{q} \to 2g \to q\bar{q}$ in the ^3P_J channels in the leading order of α_s.

$$\mathcal{M}(H(^3P_J) \to H(^3P_J)) = \mathcal{M}(q\bar{q}(^3P_J)_1 \to q\bar{q}(^3P_J)_1)H_1 + \mathcal{M}(q\bar{q}(^3S_1)_8 \to q\bar{q}(^3S_1)_8)H_8, \quad (1)$$

where H_1 and H_8 are some nonperturbative matrix elements, $\mathcal{M}(q\bar{q}(^3P_J)_1 \to q\bar{q}(^3P_J)_1)$ and $\mathcal{M}(q\bar{q}(^3S_1)_8 \to q\bar{q}(^3S_1)_8)$ are the amplitudes with the momenta of the quarks and antiquarks on shell, and the indexes 1 and 8 refer to the color signal and color octet states, respectively. The amplitudes at quark level can be calculated perturbatively. In the perturbation theory, the corresponding Feynman diagrams for the amplitudes of $q\bar{q}(^3P_J)_1 \to 2g \to q\bar{q}(^3P_J)_1$ are shown in Fig. 1, and the transition $q\bar{q}(^3S_1)_8 \to g \to q\bar{q}(^3S_1)_8$ is shown in Fig. 2.

In the center of mass frame, we choose the momenta as follows:

$$p_1 \triangleq \frac{1}{2}P + p_i, \qquad p_2 \triangleq \frac{1}{2}P - p_i,$$
$$p_3 \triangleq \frac{1}{2}P + p_f, \qquad p_4 \triangleq \frac{1}{2}P - p_f, \quad (2)$$

with $p_1^2 = p_2^2 = p_3^2 = p_4^2 = m^2$ and m the mass of the heavy quark. We can define $P \triangleq (E, 0, 0, 0)$, $p_i \triangleq (0, \mathbf{p}_i)$, and $p_f \triangleq (0, \mathbf{p}_f)$. For the heavy quark and antiquark pairs we can take $|\mathbf{p}_{i,f}|/m$ as small variables, and then can expand the expressions on these small variables. In the on-shell case, we have the relations $|\mathbf{p}_i| = |\mathbf{p}_f| \triangleq p$ and $E = \sqrt{m^2 + p^2}$. This relation means we cannot distinguish $|\mathbf{p}_i|$ from $|\mathbf{p}_f|$ and a nonuniqueness may happen when applying the final expressions to the bound states. There is no such nonuniqueness in the direct calculation of the imaginary parts since the momenta p_i and p_f appear independently after cutting the gluon lines. Fortunately, we can see the first two nonzero orders can be gotten uniquely due to the symmetry which will be discussed in the following.

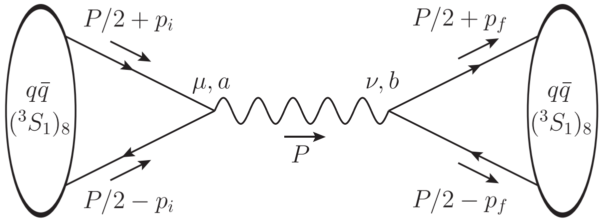

FIG. 2. The Feynman diagrams for $q\bar{q} \to 1g \to q\bar{q}$ in the $^3S_1$ channel in the leading order of α_s.

To project the quark and antiquark pairs to the ^3P_J state and the $^3S_1$ state, we use the project matrix in the on-shell case [6,7] and have the following:

$$\sum \bar{v}(p_2, s_2)Tu(p_1, s_1)\left\langle \frac{1}{2}s_1; \frac{1}{2}s_2 | 1s_i \right\rangle \triangleq \mathrm{Tr}[T.\Pi_i(s_i)],$$
$$\sum \bar{u}(p_3, s_3)Tv(p_4, s_4)\left\langle \frac{1}{2}s_3; \frac{1}{2}s_4 | 1s_f \right\rangle \triangleq \mathrm{Tr}[T.\Pi_f(s_f)], \quad (3)$$

where the Clebsch-Gordan coefficients are the standard ones as in Ref. [7] and the Dirac spinors are normalized as $u^+u = v^+v = 1$, whose expressions are written as

$$u(p_1, s_1) \triangleq \frac{\not{p}_1 + m}{\sqrt{E_1(E_1 + m)}} \begin{pmatrix} \xi^{s_1} \\ 0 \end{pmatrix},$$
$$v(p_2, s_2) \triangleq \frac{-\not{p}_2 + m}{\sqrt{E_2(E_2 + m)}} \begin{pmatrix} 0 \\ \eta^{s_2} \end{pmatrix}, \quad (4)$$

with $E_{1,2} \triangleq \sqrt{\mathbf{p}_{1,2}^2 + m^2} = E$, $\xi^{1/2} = (1, 0)^T$, $\xi^{-1/2} = (0, 1)^T$, $\eta^{1/2} = (0, 1)^T$, and $\eta^{-1/2} = (-1, 0)^T$. The combination of the above expressions results in the following:

$$\Pi_i(s_i) = -\frac{1}{8\sqrt{2}E_i^2(E_i + m)}(\not{p}_1 + m)(2E_i + \not{P}) \times \not{\epsilon}(s_i)(-\not{p}_2 + m),$$
$$\Pi_f(s_f) = -\frac{1}{8\sqrt{2}E_f^2(E_f + m)}(-\not{p}_4 + m) \times \not{\epsilon}^*(s_f)(2E_f + \not{P})(\not{p}_3 + m), \quad (5)$$

where $E_i = E_f \triangleq \sqrt{\mathbf{p}_{i,f}^2 + m^2}$ and

$$\epsilon^\mu(0) \triangleq (0,0,0,1),$$
$$\epsilon^\mu(\pm 1) \triangleq (0,\mp 1,-i,0)/\sqrt{2}. \tag{6}$$

Here the relative sign of $\Pi_i(s_i)$ and $\Pi_f(s_i)$ is positive which is different from the $^1S_0$ case.

To discuss the mass shifts of ^3P_J heavy quarkonia due to the effect of two-gluon annihilation, in principle, one can go to calculate the full four-point Green's function with quarks and antiquarks of shell and then take the four-points Green's function as the interaction kernel of the Bethe-Salper equation to discuss the mass shifts. In our calculation, similarly with NRQCD we take the on-shell approximation to the four-point Green's functions. Since we take the approximated four-point Green's function as the interaction kernel of the Bethe-Salpeter equation, we only calculate it in the perturbative QCD and do not go to match them with the corresponding amplitudes in NRQCD. In the practical calculation, we directly include the structure of the heavy quarkonia and assume the following structure for the $H(^3P_J)$:

$$|H(^3P_J)\rangle \sim \phi_1(|\mathbf{p}|)\frac{\delta_{ij}}{\sqrt{N_c}}|(q^i\bar{q}^j)_1(^3P_J)\rangle + \phi_0(|\mathbf{p}|)\frac{T_a^{ij}}{\sqrt{(N_c^2-1)/2}}|(q^i\bar{q}^j)_8(^3S_1)g^a\rangle \tag{7}$$

where the color factors $1/\sqrt{N_c}$ and $1/\sqrt{(N_c^2-1)/2}$ are used to normalized the color parts to 1, and $\phi_{0,1}(|\mathbf{p}|)$ refer to the wave functions of the $H(^3P_J)$ in the momentum space in the color single and color octet states, respectively. The relations between the wave functions $\phi_{0,1}$ with the wave functions in the coordinate space are defined as

$$\phi_l(|\mathbf{p}|)Y_{lm}(\Omega_\mathbf{p}) \triangleq \int d^3\mathbf{r}\frac{1}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{r}}R_l(|\mathbf{r}|)Y_{lm}(\Omega_\mathbf{r}). \tag{8}$$

Using the structure of the $H(^3P_J)$ and the above project matrices, the amplitudes can be expressed as follows:

$$\begin{aligned}
\mathcal{M}(^3P_J) &\triangleq \mathcal{M}(q\bar{q}(^3P_J)_1 \to q\bar{q}(^3P_J)_1)H_1 \\
&= \int d|\mathbf{p}_i|d|\mathbf{p}_f||\mathbf{p}_i|^2|\mathbf{p}_f|^2\phi_1(|\mathbf{p}_f|)\phi_1^*(|\mathbf{p}_i|)\bar{G}^{(a+b)}(^3P_J), \\
\mathcal{M}(^3S_1) &\triangleq \mathcal{M}(q\bar{q}(^3S_1)_8 \to q\bar{q}(^3S_1)_8)H_8 \\
&= \int d|\mathbf{p}_i|d|\mathbf{p}_f||\mathbf{p}_i|^2|\mathbf{p}_f|^2\phi_0(|\mathbf{p}_f|)\phi_0^*(|\mathbf{p}_i|)\bar{G}^{(c)}(^3S_1),
\end{aligned} \tag{9}$$

where $\bar{G}^{(a,b)}(^3P_J)$ and $\bar{G}^{(c)}(^3S_1)$ are expressed as

$$\begin{aligned}
\bar{G}^{(a,b)}(^3P_J) &= \sum_{s_i,s_f}\langle JJ_z|1s_f;1m_f\rangle\langle JJ_z|1s_i;1m_i\rangle\int d\Omega_{\mathbf{p}_i}d\Omega_{\mathbf{p}_f}Y_{1m_i}(\Omega_i)Y_{1m_f}^*(\Omega_f)G^{(a,b)}(s_i,s_f), \\
\bar{G}^{(c)}(^3S_1) &= \langle 1J_z1s_f|\rangle\langle 1s_i|1J_z\rangle\int d\Omega_{\mathbf{p}_i}d\Omega_{\mathbf{p}_f}Y_{00}(\Omega_i)Y_{00}^*(\Omega_f)G^{(c)}(s_i,s_f),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
G^{(a)}(s_i,s_f) &= -ic_f^{(2g)}\int\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}[T_1\Pi_i(s_i)]\mathrm{Tr}[T_2\Pi_f(s_f)]D_{\mu\rho}(k)D_{\nu\lambda}(p_1+p_2-k), \\
G^{(b)}(s_i,s_f) &= -ic_f^{(2g)}\int\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}[T_1\Pi_i(s_i)]\mathrm{Tr}[T_3\Pi_f(s_f)]D_{\mu\lambda}(k)D_{\nu\rho}(p_1+p_2-k), \\
G^{(c)}(s_i,s_f) &= -ic_f^{(1g)}\mathrm{Tr}[(-ig_s\gamma^\mu\Pi_i(s_i)]\mathrm{Tr}[(-ig_s\gamma^\nu\Pi_f(s_f)]D_{\mu\nu}(p_1+p_2),
\end{aligned} \tag{11}$$

with $D=4-2\epsilon$ and the color factor $c_f^{(2g)}$

$$c_f^{(2g)} = \left(\frac{\delta_{ij}}{\sqrt{N_c}} T_a^{jm} T_b^{mi}\right)\left(\frac{\delta_{i'j'}}{\sqrt{N_c}} T_c^{j'm'} T_d^{m'i'}\right)\delta_{ad}\delta_{bc} = \frac{C_A C_F}{2N_c} = \frac{N_c^2-1}{4N_c} = \frac{2}{3},$$

$$c_f^{(1g)} = \left(\frac{T_a^{ij}}{\sqrt{(N_c^2-1)/2}} T_b^{ij}\right)\left(\frac{T_c^{i'j'}}{\sqrt{(N_c^2-1)/2}} T_d^{i'j'}\right)\delta_{bc} = \frac{1}{2}, \tag{12}$$

the hard kernel T_i

$$\begin{aligned} T_1 &= (-ig_s\gamma^\nu)\cdot S_F(p_1-k)\cdot(-ig_s\gamma^\mu), \\ T_2 &= (-ig_s\gamma^\rho)\cdot S_F(p_3-k)\cdot(-ig_s\gamma^\lambda), \\ T_3 &= (-ig_s\gamma^\rho)\cdot S_F(k-p_4)\cdot(-ig_s\gamma^\lambda), \end{aligned} \tag{13}$$

and

$$\begin{aligned} S_F(q) &= \frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}, \\ D_{\mu\rho}(q) &= \frac{-i}{q^2+i\epsilon}\left(g_{\mu\rho} - \xi\frac{q_\mu q_\rho}{q^2}\right). \end{aligned} \tag{14}$$

In the real bound states, the values of $|\mathbf{p}_i|$ and $|\mathbf{p}_f|$ are independent which is different from the on-shell case, so we label $|\mathbf{p}_i|$ and $|\mathbf{p}_f|$ independently in the above original expressions.

To calculate $G^{(a,b,c)}(s_i, s_f)$, we use the package Feyncalc [8] to do the trace of Dirac matrices in the D dimension and then expand the expressions on the variable p to a special order. After the expansion, we use the tensor decomposition to reexpress the loop integrations and finally use the package FIESTA [9] to do the sector decomposition and then use *Mathematica* to do the analytic integration.

After the loop integrations, the form of $G^{(a+b,c)}(s_i, s_f)$ can be expressed as follows:

$$\begin{aligned} G^{(a+b,c)}(s_i,s_f) = {} & C_1^{(a+b,c)}\epsilon(s_i)\cdot\epsilon^*(s_f) \\ & + C_2^{(a+b,c)}\epsilon(s_i)\cdot p_i\epsilon^*(s_f)\cdot p_f \\ & + C_3^{(a+b,c)}\epsilon(s_i)\cdot p_f\epsilon^*(s_f)\cdot p_i \\ & + C_4^{(a+b,c)}\epsilon(s_i)\cdot p_i\epsilon^*(s_f)\cdot p_i \\ & + C_5^{(a+b,c)}\epsilon(s_i)\cdot p_f\epsilon^*(s_f)\cdot p_f, \end{aligned} \tag{15}$$

with

$$C_i^{(a+b,c)} = \sum_{n=0}^{3} C_{\rm in}^{(a+b,c)}(p_i^2, p_f^2)(p_i\cdot p_f)^n. \tag{16}$$

After getting the coefficients $C_{\rm in}^{(a+b,c)}$, usually the properties of the integrations of the angle and the sums of the spins are independently used to simplify the expressions as in Ref. [6]. In our calculation, for simplification we directly calculate the sums of the spins and the integrations of angles together. We define

$$\begin{aligned} P(J,X,n) &\triangleq \sum_{s_i,s_f}\langle JJ_z|1s_f;1m_f\rangle\langle JJ_z|1s_i;1m_i\rangle\int d\Omega_{\mathbf{p}_i}d\Omega_{\mathbf{p}_f}Y_{1m_i}(\Omega_i)Y^*_{1m_f}(\Omega_f)(\hat{p}_i\cdot\hat{p}_f)^nX, \\ Q(X,n) &\triangleq \langle 1J_z|1s_f\rangle\langle 1s_i|1J_z\rangle\int d\Omega_{\mathbf{p}_i}d\Omega_{\mathbf{p}_f}Y_{00}(\Omega_i)Y^*_{00}(\Omega_f)(\hat{p}_i\cdot\hat{p}_f)^nX, \end{aligned} \tag{17}$$

where X are some functions dependent on $\hat{p}_i$, $\hat{p}_f$, $\epsilon(s_i)$, and $\epsilon^*(s_f)$ with $\hat{p}_{i,f}\triangleq p_{i,f}/|\mathbf{p}_{i,f}|$, $n=0$, 1, 2, and 3, $J=0$, 1, and 2, $P(X,n)$ and $Q(J,X,n)$ are not dependent on J_z whose manifest forms are directly listed in the Appendix. Using the expressions of $P(J,X,n)$ and $Q(X,n)$, $\mathcal{M}(^3P_J)$ and $\mathcal{M}(^3S_1)$ can be calculated easily.

III. THE ANALYTIC RESULTS FOR THE ASYMPTOTIC BEHAVIOR

In the practical calculation, we expand the expressions on p to order 6. Since the calculation is taken with the momenta on shell, the gauge invariance is manifest. The final result can be expressed as

$$\begin{aligned} \bar{G}^{(a+b)}(^3P_J)|_p &= c_f^{(2g)}\alpha_s^2\pi\left[\frac{p^2}{m^4}c_{J,2} + \frac{p^4}{m^6}c_{J,4} + \frac{p^6}{m^8}c_{J,6} + \text{higher order}\right], \\ \bar{G}^{(c)}(^3S_1)|_p &= c_f^{(1g)}\alpha_s\pi\left[\frac{1}{m^2}d_0 + \frac{p^2}{m^4}d_2 + \frac{p^4}{m^6}d_4 + \text{higher order}\right], \end{aligned} \tag{18}$$

where the subindexes p mean to expand the expressions on p. For the real bound states, the terms p^2 and p^4 only receive contributions from the terms $|\mathbf{p}_i||\mathbf{p}_f|$ and $\frac{1}{2}(|\mathbf{p}_i||\mathbf{p}_f|^3+|\mathbf{p}_i|^3|\mathbf{p}_f|)$, respectively, since only they are nonzero. The term p^6 receives the contributions both from the terms $\frac{1}{2}(|\mathbf{p}_i||\mathbf{p}_f|^5+|\mathbf{p}_i|^5|\mathbf{p}_f|)$ and $|\mathbf{p}_i|^3|\mathbf{p}_f|^3$, which results in that one cannot distinguish them in a unitary from.

The imaginary parts of $c_{J,i}$ and d_i are expressed as follows:

$$\begin{aligned}
&\mathrm{Im}[c_{0,2}]=8\pi, \quad \mathrm{Im}[c_{0,4}]=-\frac{56}{3}\pi,\\
&\mathrm{Im}[c_{0,6}]=\frac{1384}{45}\pi, \quad \mathrm{Im}[c_{1,i}]=0,\\
&\mathrm{Im}[c_{2,2}]=\frac{32}{15}\pi, \qquad \mathrm{Im}[c_{2,4}]=-\frac{64}{15}\pi,\\
&\mathrm{Im}[c_{2,6}]=\frac{51088}{7875}\pi, \qquad \mathrm{Im}[d_i]=0.
\end{aligned} \tag{19}$$

The real parts of $c_{J,i}$ and d_i are expressed as follows:

$$\begin{aligned}
\mathrm{Re}[c_{0,2}]&=-\frac{8}{3}-16\log 2-\frac{64}{9}C_{\mathrm{IR}},\\
\mathrm{Re}[c_{0,4}]&=\frac{32}{45}+\frac{112}{3}\log 2+\frac{832}{45}C_{\mathrm{IR}},\\
\mathrm{Re}[c_{0,6}]&=\frac{9452}{1575}-\frac{2786}{45}\log 2-\frac{17216}{525}C_{\mathrm{IR}},
\end{aligned} \tag{20}$$

$$\begin{aligned}
\mathrm{Re}[c_{1,2}]&=-\frac{16}{9}-\frac{64}{9}C_{\mathrm{IR}},\\
\mathrm{Re}[c_{1,4}]&=\frac{16}{45}+\frac{512}{45}C_{\mathrm{IR}},\\
\mathrm{Re}[c_{1,6}]&=\frac{464}{315}-\frac{22528}{1575}C_{\mathrm{IR}},
\end{aligned} \tag{21}$$

$$\begin{aligned}
\mathrm{Re}[c_{2,2}]&=\frac{32}{15}-\frac{64}{15}\log 2-\frac{64}{9}C_{\mathrm{IR}},\\
\mathrm{Re}[c_{2,4}]&=-\frac{1568}{225}+\frac{128}{15}\log 2+\frac{128}{9}C_{\mathrm{IR}},\\
\mathrm{Re}[c_{2,6}]&=\frac{14656}{1125}-\frac{102176}{7875}\log 2-\frac{11168}{525}C_{\mathrm{IR}}.
\end{aligned} \tag{22}$$

$$\begin{aligned}
\mathrm{Re}[d_0]&=-8\pi,\\
\mathrm{Re}[d_2]&=8\pi,\\
\mathrm{Re}[d_4]&=-\frac{74}{9}\pi.
\end{aligned} \tag{23}$$

with

$$C_{\mathrm{IR}}=-\frac{1}{2}\left(\frac{1}{\epsilon}-\log\frac{m^2}{4\pi\mu_{\mathrm{IR}}^2}-\gamma_E\right). \tag{24}$$

In the following discussion, we define $\mathrm{Re}[c_{J,i}^{fin}]$ as the finite part of $\mathrm{Re}[c_{J,i}]$ with the C_{IR} related parts being subtracted.

IV. COMPARE THE RESULTS WITH NRQCD

The asymptotic behavior of the amplitude in pQCD can be used to get the coefficients of the NRQCD by matching. Comparing our Eq. (19) with the expressions in Appendix B of Ref. [10], one can find the following relations:

$$\begin{aligned}
\mathrm{Im}[f_1(^3P_J)]&=\frac{1}{8}c_f^{(2g)}\alpha_s^2\mathrm{Im}[c_{J,2}],\\
\mathrm{Im}[g_1(^3P_J)]&=\frac{1}{8}c_f^{(2g)}\alpha_s^2\mathrm{Im}[c_{J,4}]
\end{aligned} \tag{25}$$

where $f_1(^3P_J)$ and $g_1(^3P_J)$ are the coefficients of the four-fermion couplings $\mathcal{O}_1(^3P_J)$ and $\mathcal{P}_1(^3P_J)$ in the NRQCD effective Lagrangian [10], respectively. This relation is natural since the two calculations should give the same expressions for the decay width except for the global factor difference between the wave functions and the matrix elements.

It is sure that the real parts of the coefficients obey the same relations as follows:

$$\begin{aligned}
\mathrm{Re}[f_1(^3P_J)]&=\frac{1}{8}c_f^{(2g)}\alpha_s^2\mathrm{Re}[c_{J,2}^{fin}],\\
\mathrm{Re}[g_1(^3P_J)]&=\frac{1}{8}c_f^{(2g)}\alpha_s^2\mathrm{Re}[c_{J,4}^{fin}].
\end{aligned} \tag{26}$$

Furthermore, the coefficients $c_{J,6}$ are corresponding to the sum of the coefficients of next order in NRQCD which means

$$\begin{aligned}
\mathrm{Im}[h_1(^3P_J)+h_1'(^3P_J)]&=\frac{1}{8}c_f^{(2g)}\alpha_s^2\mathrm{Im}[c_{J,6}],\\
\mathrm{Re}[h_1(^3P_J)+h_1'(^3P_J)]&=\frac{1}{8}c_f^{(2g)}\alpha_s^2\mathrm{Re}[c_{J,6}^{fin}],
\end{aligned} \tag{27}$$

where the coefficients $h_1(^3P_J)$ and $h_1'(^3P_J)$ are defined as

$$\mathcal{L}_{4\mathrm{f}}^{\mathrm{NRQCD}}\supset\frac{h_1(^3P_J)}{m^8}\mathcal{Q}_1(^3P_J)+\frac{h_1'(^3P_J)}{m^8}\mathcal{Q}_1'(^3P_J), \tag{28}$$

with

$$\mathcal{Q}_1(^3P_0) = \frac{1}{12}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\psi + \mathrm{H.c.},$$
$$\mathcal{Q}_1(^3P_1) = \frac{1}{8}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\psi + \mathrm{H.c.},$$
$$\mathcal{Q}_1(^3P_2) = \frac{1}{4}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\chi\chi^\dagger\left(-\frac{i}{4}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\psi + \mathrm{H.c.},$$
$$\mathcal{Q}'_1(^3P_0) = \frac{1}{12}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\psi + \mathrm{H.c.},$$
$$\mathcal{Q}'_1(^3P_1) = \frac{1}{8}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\psi + \mathrm{H.c.},$$
$$\mathcal{Q}'_1(^3P_2) = \frac{1}{4}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\chi^\dagger\left(-\frac{i}{4}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\psi + \mathrm{H.c.}, \quad (29)$$

similarly with the operators $\mathcal{O}_1(^3P_J)$ and $\mathcal{P}_1(^3P_J)$ [10].

V. THE MASS SHIFTS OF $H(^3P_J)$ STATES

To discuss the mass shifts of the $H(^3P_J)$ states, in principle, one can take the Green's function as a part of the interaction kernel of the Bethe-Salpeter equation, then solve the Bethe-Salpeter equation to get the corrected energy spectrum and the mass shifts. Since such an approach is unconquerable at present, usually the perturbative form is used. One can do this order by order by matching NRQCD into pNRQCD [11] under the frame of effective theory. In this work for simplicity we directly match the amplitude in pQCD with that in the quantum mechanism order by order to extract the effective potential in the quantum mechanism. In the LO-α_s one has the following simple relation for the corresponding effective potential:

$$\langle V_{\mathrm{eff},J}\rangle \triangleq \langle H(^3P_J)|V_{\mathrm{eff}}|H(^3P_J)\rangle = -(\mathcal{M}(^3P_J)+\mathcal{M}(^3S_1)). \quad (30)$$

In the LO-v, the corresponding decay widths of $H(^3P_J)$ to the light hadrons ($l.h$) from the above diagrams, which are labeled as $\Gamma(^3P_J \to l.h)$, are expressed as

$$\Gamma(^3P_J\to l.h) = -2\mathrm{Im}[\langle V_{\mathrm{eff},J}\rangle] = \frac{3}{4\pi}\alpha_s^2\mathrm{Im}[c_{J,2}]\frac{|R_1^{(1)}(0)|^2}{m^4}, \quad (31)$$

and the corresponding mass shifts labeled as $\Delta M(^3P_J)$ are expressed as

$$\Delta M(^3P_J) = \mathrm{Re}[\langle V_{\mathrm{eff},J}\rangle]$$
$$= -\frac{3}{8\pi}\alpha_s^2\mathrm{Re}[c_{J,2}^{fin}]\frac{|R_1^{(1)}(0)|^2}{m^4} + \pi\alpha_s\left[\frac{8}{3}\frac{\alpha_s}{\pi^2}C_{\mathrm{IR}}\frac{|R_1^{(1)}(0)|^2}{m^4} + \frac{1}{4\pi}\frac{|R_0(0)|^2}{m^2}\right], \quad (32)$$

where we have used the relation

$$\int \phi_1(p)p^{2n+3}dp = (-1)^n\frac{2n+3}{4\pi}R_1^{(2n+1)}(|\mathbf{r}|)\Big|_{|\mathbf{r}|=0},$$
$$\int \phi_0(p)p^{2n+2}dp = (-1)^n\frac{1}{4\pi}R_0^{(2n)}(|\mathbf{r}|)\Big|_{|\mathbf{r}|=0}. \quad (33)$$

Equation (32) shows when one goes to discuss the mass shifts of the $H(^3P_J)$ states due to the two-gluon annihilation effects in the LO-α_s, the color octet contribution should also be considered. The IR divergence in $\mathcal{M}(^3P_J)$ can be absorbed in a unitary way by $\mathcal{M}(^3S_1)$ in the LO-v. The absorbed form is unique and is same as the case of the one-loop radiative corrections to the decay width $\Gamma(^3P_J \to l.h)$. This is natural if one goes to match the above results with the corresponding NRQCD coefficients.

In the literature, the decay widths $\Gamma(^3P_J \to l.h)$ in the LO-v and the NLO-α_s can be expressed as follows [12]:

$$\Gamma(\chi_0\to LH) = \frac{4}{3}\pi\alpha_s^2H_1\left[1+\frac{\alpha_sC_0}{\pi}\right] + n_f\frac{\pi}{3}\alpha_s^2\left[\frac{16}{27}\frac{\alpha_s}{\pi}H_1\log\frac{m}{\mathcal{E}}+H_8\right]$$
$$\Gamma(\chi_2\to LH) = \frac{16}{45}\pi\alpha_s^2H_1\left[1+\frac{\alpha_s}{\pi}C_2\right] + n_f\frac{\pi}{3}\alpha_s^2\left[\frac{16}{27}\frac{\alpha_s}{\pi}H_1\log\frac{m}{\mathcal{E}}+H_8\right], \quad (34)$$

where n_f is the number of light quarks, $n_f = 3$ for charmonium and $n_f = 4$ for bottomonium states, C_0 and C_2 are expressed as

$$C_0 = \frac{4}{3}\left(\frac{\pi^2}{4} - \frac{7}{3}\right) + 3\left(\frac{454}{81} - \frac{\pi^2}{144} - \frac{11\log 2}{3}\right) + 3\left(\frac{2\log 2}{3} - \frac{16}{27}\right),$$
$$C_2 = -\frac{16}{3} + 3\left(\frac{2239}{216} - \frac{337\pi^2}{384} - 2\log 2\right) + 3\left(\frac{2\log 2}{3} - \frac{11}{8}\right), \tag{35}$$

and H_1 is related to the derivative of the wave function through the relation:

$$H_1 = \frac{9}{2\pi}\frac{|R_1^{(1)}|^2}{m^4}[1 + O(v^2)]. \tag{36}$$

Using the following replacement, which is similar with that in [13],

$$\log\frac{m}{\mathcal{E}} \sim \frac{1}{-2\epsilon} \sim C_{\mathrm{IR}}, \tag{37}$$

we can see that the IR divergences in Eq. (32) and Eq. (34) are absorbed in the same form.

The property of the color octet matrix element H_8 has been discussed in the literature such as in Ref. [11] where how H_8 reabsorbs the IR divergence coming from the singlet sector are shown. In our case, we can get the following relation from the comparison between Eq. (32) and Eq. (34):

$$H_8 = \frac{1}{4\pi}\frac{|R_0(0)|^2}{m^2}. \tag{38}$$

Combing Eq. (32) with Eq. (34), finally one can get

$$\Delta M(^3P_J) = -\frac{1}{12}\mathrm{Re}[c_{J,2}]\alpha_s^2 H_1 + \pi\alpha_s\bar{H}_8, \tag{39}$$

with

$$\bar{H}_8 \triangleq \frac{8}{3}\frac{\alpha_s}{\pi^2}C_{\mathrm{IR}}\frac{|R_1^{(1)}(0)|^2}{m^4} + \frac{1}{4\pi}\frac{|R_0(0)|^2}{m^2} \sim \frac{16}{27}\frac{\alpha_s}{\pi}H_1\log\frac{m}{\mathcal{E}} + H_8. \tag{40}$$

Finally, Eq. (39) can be used to estimate the mass shifts of $H(^3P_J)$ in the LO-α_s and the LO-v due to the two-gluon annihilation effect.

VI. NUMERICAL RESULT AND CONCLUSION

The main results of our calculation are the expressions of the coefficients $c_{J,i}$ and the mass shifts $\Delta M(^3P_J)$. In the LO-v and the LO-α_s, if one assumes that the contribution from the $\bar{H}_8$ related term is small, then the ratios between the mass shifts and the decay widths can be expressed as

$$\frac{\Delta M(^3P_0)}{\Gamma(^3P_0)} = \frac{1 + 6\log 2}{6\pi} \approx 0.27,$$
$$\frac{\Delta M(^3P_2)}{\Gamma(^3P_2)} = \frac{2\log 2 - 1}{2\pi} \approx 0.06,$$
$$\frac{\Delta M_{(^1S_0)}}{\Gamma(^1S_0)} = \frac{\log 2 - 1}{\pi} \approx -0.098, \tag{41}$$

where the similar result for the $^1S_0$ state is also presented. We can see that the ratio for the $^3P_0$ state is much larger than the ratios for the $^3P_2$ and $^1S_0$ states and the ratios are positive for the ^3P_J states and negative for the $^1S_0$ state.

Furthermore one can extract the parameters H_1 and $\bar{H}_8$ from the experimental data by Eq. (34) in the NLO-α_s and the LO-v, and then one can use the extracted parameters to estimate the mass shifts $\Delta M(^3P_J)$ using Eq. (39) in the LO-α_s and the LO-v. The corresponding numerical results of $\Delta M(^3P_J)$ for χ_{cJ} are listed in Table I where the experimental data are taken from Ref. [14]. The similar estimation can be applied to the bottomonium. Comparing these numerical results with the corresponding results of η_c [5], we can find that the mass shifts of χ_{cJ} are very different. These properties mean the corrections to different states cannot be subtracted or hidden in a unified way. Combing the numerical results, one can get $\Delta M(^3P_0) - \Delta M(^1S_0) \approx 9.0 \sim 8.6$ MeV with $\alpha_s \approx 0.25 \sim 0.35$, correspondingly. This numerical result suggests that the annihilation effects should be considered seriously when we try to understand the spectrum of heavy quarkonia precisely, especially when some decay channels with large decay widths are opened.

Another interesting property is that although the decay width of the $^3P_1$ states to the two-gluon intermediated state is zero, the corresponding mass shift is nonzero.

TABLE I. The numerical results for $\Delta M(^3P_J)$ which refer to the mass shifts of χ_{cJ} in the leading order. The experimental decay widths are taken from Ref. [14]; the values of H_1 and $\bar{H}_8$ are extracted by using Eq. (34) with α_s taking as $0.25 \sim 0.35$, correspondingly.

	$\Gamma_{l.h}^{Ex}$ (MeV)	H_1 (MeV)	$\bar{H}_8$ (MeV)	$\Delta M(^3P_J)$ (MeV)
$\chi_{c0}(1P)$	10.8			$5.92 \sim 5.45$
$\chi_{c1}(1P)$	⋯	$69.8 \sim 29.3$	$1.18 \sim 1.21$	$1.57 \sim 1.86$
$\chi_{c1}(2P)$	1.6			$1.23 \sim 1.58$

In summary, the real part of the nonrelativistic asymptotic behavior of the amplitudes of $q\bar{q} \to 2g \to q\bar{q}$ in the ^3P_J channels is discussed in the LO-α_s. By expanding the expressions on the three-momentum of quarks and antiquarks, the expressions are calculated to order 6. The imaginary part of the first two terms of our results are the same as those given in the references. The real part of our results can be used to estimate the mass shifts of the ^3P_J heavy quarkonia due to the two-gluon annihilation effect. In the LO-α_s and the LO-v, we get the following properties: (1) the mass shifts of the ^3P_J states are positive which are different from the $^1S_0$ case where the mass shifts are negative; (2) the mass shifts of the $^3P_1$ states are nonzero although their decay widths are zero; (3) the numerical estimation shows the contributions to the mass shifts of $\chi_{c0,c1,c2}$ are about $1.23 \sim 1.58$ MeV, $1.57 \sim 1.86$ MeV, and $5.92 \sim 5.45$ MeV when taking $\alpha_s \approx 0.25 \sim 0.35$.

ACKNOWLEDGMENTS

The author Hai-Qing Zhou would like to thank Wen-Long Sang, Zhi-Yong Zhou, and Dian-Yong Chen for their kind and helpful discussions. This work is supported by the National Natural Science Foundations of China under Grant No. 11975075.

APPENDIX

In this appendix, the manifest expressions for $P(J,X,n)$ and $Q(X,n)$ are listed. From the definition of $P(J,X,n)$ and $Q(X,n)$ which are expressed as

$$P(J,X,n) \triangleq \sum_{s_i,s_f} \langle 1s_i;1m_i|JJ_z\rangle\langle 1s_f;1m_f|JJ_z\rangle \int d\Omega_{\mathbf{p}_i} d\Omega_{\mathbf{p}_f} Y_{1m_i}(\Omega_i) Y^*_{1m_f}(\Omega_f)(p_i\cdot p_f)^n X,$$
$$Q(X,n) \triangleq \langle 1J_z|1s_f\rangle\langle 1s_i|1J_z\rangle \int d\Omega_{\mathbf{p}_i} d\Omega_{\mathbf{p}_f} Y_{00}(\Omega_i) Y^*_{00}(\Omega_f)(\hat{p}_i\cdot\hat{p}_f)^n X, \tag{A1}$$

with $0 \le n \le 3$, we have

$$P(J,\epsilon(s_i)\cdot\epsilon^*(s_f),1) = \frac{4\pi}{3}, \qquad P(J,\epsilon(s_i)\cdot\epsilon^*(s_f),3) = \frac{4\pi}{5}, \tag{A2}$$

$$P(0,\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_f,0) = 4\pi, \qquad P(0,\epsilon(s_i)\cdot p_i\epsilon^*(s_f)\cdot p_f,2) = \frac{4\pi}{3},$$
$$P(2,\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_f,2) = \frac{16\pi}{75}, \tag{A3}$$

$$P(0,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,0) = \frac{4\pi}{3}, \qquad P(0,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,2) = \frac{4\pi}{5},$$
$$P(1,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,0) = -\frac{4\pi}{3}, \qquad P(1,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,2) = -\frac{4\pi}{15},$$
$$P(2,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,0) = \frac{4\pi}{3}, \qquad P(0,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,2) = \frac{12\pi}{25}, \tag{A4}$$

$$P(0,\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_i,1) = -\frac{4\pi}{3}, \qquad P(0,\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_i,3) = -\frac{4\pi}{5},$$
$$P(2,\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_i,1) = -\frac{8\pi}{15}, \qquad P(2,\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_i,3) = -\frac{8\pi}{25}, \tag{A5}$$

$$P(0,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_f,1) = -\frac{4\pi}{3}, \qquad P(0,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_f,3) = -\frac{4\pi}{5},$$
$$P(2,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_f,1) = -\frac{8\pi}{15}, \qquad P(2,\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_f,3) = -\frac{8\pi}{25}, \tag{A6}$$

and the results for the other (J,n) are zero,

$$Q(\epsilon(s_i)\cdot\epsilon^*(s_f),0) = -4\pi, \qquad Q(\epsilon(s_i)\cdot\epsilon^*(s_f),2) = -\frac{4\pi}{3}$$
$$Q(\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_f,1) = -\frac{4\pi}{9} \qquad Q(\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_f,3) = -\frac{4\pi}{15},$$
$$Q(\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,1) = -\frac{4\pi}{9}, \qquad Q(\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_i,3) = -\frac{4\pi}{15}$$
$$Q(\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_i,1) = -\frac{4\pi}{9}, \qquad Q(\epsilon(s_i)\cdot\hat{p}_i\epsilon^*(s_f)\cdot\hat{p}_i,3) = -\frac{4\pi}{15}$$
$$Q(\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_f,1) = -\frac{4\pi}{9}, \qquad Q(\epsilon(s_i)\cdot\hat{p}_f\epsilon^*(s_f)\cdot\hat{p}_f,3) = -\frac{4\pi}{15} \quad \text{(A7)}$$

and the results for other n are zero.

[1] D. P. Stanley and D. Robson, Phys. Rev. D **21**, 3180 (1980); S. Ono and F. Schoberl, Phys. Lett. **118B**, 419 (1982); S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985).

[2] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979); E. V. Shuryak, Phys. Rep. **115**, 151 (1984); L. J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rep. **127**, 1 (1985); M. Nielsen, F. S. Navarra, and S. H. Lee, Phys. Rep. **497**, 41 (2010).

[3] P. Jain and H. J. Munczek, Phys. Rev. D **44**, 1873 (1991); H. J. Munczek and P. Jain, Phys. Rev. D **46**, 438 (1992); P. Jain and H. J. Munczek, Phys. Rev. D **48**, 5403 (1993); Y.-B. Dai, C.-S. Huang, and H.-Y. Jin, Z. Phys. C **60**, 527 (1993); K. Kusaka and A. G. Williams, Phys. Rev. D **51**, 7026 (1995); K. I. Aoki, T. Kugo, and M. G. Mitchard, Phys. Lett. B **266**, 467 (1991); C. R. Munz, J. Resag, B. C. Metsch, and H. R. Petry, Nucl. Phys. **A578**, 418 (1994); P. Maris and C. D. Roberts, Phys. Rev. C **56**, 3369 (1997); P. Maris and P. C. Tandy, Phys. Rev. C **60**, 055214 (1999); R. Alkofer, P. Watson, and H. Weigel, Phys. Rev. D **65**, 094026 (2002); A. Krassnigg, Phys. Rev. D **80**, 114010 (2009); T. Hilger, M. Gmez-Rocha, and A. Krassnigg, Phys. Rev. D **91**, 114004 (2015); C. S. Fischer, S. Stanislav, and R. Williams, Eur. Phys. J. A **51**, 10 (2015); C. Popovici, T. Hilger, M. Gomez-Rocha, and A. Krassnigg, Few-Body Syst. **56**, 481 (2015).

[4] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **46**, R1914 (1992); **51**, 1125 (1995); **55**, 5853(E) (1997).

[5] H.-Y. Cao and H.-Q. Zhou, Phys. Rev. D **99**, 074007 (2019).

[6] J. H. Kuhn, J. Kaplan, and E. G. O. Safiani, Nucl. Phys. **B157**, 125 (1979).

[7] G. T. Bodwin and A. Petrelli, Phys. Rev. D **66**, 094011 (2002); **87**, 039902(E) (2013).

[8] V. Shtabovenko, R. Mertig, and F. Orellana, Comput. Phys. Commun. **207**, 432 (2016); R. Mertig, M. Bohm, and A. Denner, Comput. Phys. Commun. **64**, 345 (1991).

[9] A. V. Smirnov, Comput. Phys. Commun. **204**, 189 (2016); **185**, 2090 (2014).

[10] N. Brambilla, E. Mereghetti, and A. Vairo, Phys. Rev. D **79**, 074002 (2009); **83**, 079904(E) (2011).

[11] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. **B566**, 275 (2000); N. Brambilla, D. Eiras, A. Pineda, J. Soto, and A. Vairo, Phys. Rev. Lett. **88**, 012003 (2001); B. A. Kniehl, A. A. Penin, and V. A. Smirnov, Nucl. Phys. **B635**, 357 (2002); N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Rev. Mod. Phys. **77**, 1423 (2005).

[12] A. Petrelli, Phys. Lett. B **380**, 159 (1996).

[13] H.-W. Huang and K.-T. Chao, Phys. Rev. D **54**, 6850 (1996).

[14] M. Tanabashi *et al.* (Particle Data Group), Phys. Rev. D **98**, 030001 (2018).