New relation between entanglement and geometry from M(atrix) theory

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In the context of Matrix/light-cone gauge M theory, we develop a new approach for computing quantum entanglement between a probe gravitating in the vicinity of a source mass and the source mass. We demonstrate that this entanglement is related to the gravitational potential energy between the two objects. We then show that the von Neumann entropy is a function of two derivatives of the gravitational potential. We conjecture a relation between the entropy and the local Riemann tensor sampled by the probe, establishing a general scheme to relate entropy to local geometric data. This relation connects the rate of change, rotation, and twist of a small volume element at the probe's location to the quantum entanglement of the probe with the source.

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I. INTRODUCTION AND HIGHLIGHTS

Various relations between quantum information and spacetime geometry seem to hint at the need for a fundamental rethinking of gravity. In this program, the general theme appears to be that gravity is an emergent phenomenon; and that underlying microscopic quantum degrees of freedom (d.o.f.) weave—through quantum entanglement—a fabric that we effectively perceive as space. In this paper, we want to analyze these ideas in the context of Matrix theory, a nonperturbative formulation of string theory and quantum gravity [1]. We will consider a simple setup where a massive source pulls gravitationally on a probe and where it is well known that the effective quantum potential that arises from Matrix theory matches exactly with the expected gravitational potential that the probe experiences in light-cone gauge M theory [2–11]. This effective potential arises from integrating out fast off-diagonal matrix modes that correspond to strings stretched between the two objects. In this work, we add the slower diagonal excitations and derive their quantum effective potential. We then demonstrate that the quantum vacuum of these modes is an entangled state in such a way that the entanglement entropy between source and probe is generally a function of derivatives of their gravitational potential. We compute the von Neumann entropy and, based on the result we obtain, we conjecture a relation between the entropy and the local Riemann tensor sampled by the probe. Essentially, this entanglement entropy is shown to be directly related to local tidal forces. This connects the entropy to the rate of change, rotation, and twist of a small volume element at the location of the probe. The setup is reminiscent of entropy-area relations, except the statement we obtain is local.

In the first section, we describe the setup and outline the computation of the entanglement entropy. In the second section, we present a conjecture relating this entropy to local geometry. The Conclusion discusses the more general implications of these results and future directions.

II. QUANTUM ENTANGLEMENT AND GRAVITY

Matrix theory is a $0+1$-dimensional $U(N)$ super Yang-Mills (SYM) theory that is purported to be dual to light-cone gauge M theory. The rank of the gauge group N maps onto light-cone momentum in M theory. Our starting point is the Matrix theory action in the background field gauge[1]:

$$S = \frac{1}{g_{\rm YM}^2}\int dt \mathrm{Tr}\bigg[D_t X^i D_t X^i + \frac{1}{2}[X^i, X^j]^2 - (\partial_t A_0 - i[X^i_{\rm bg}, X^i])^2 + i\Psi_\alpha D_t \Psi_\alpha - \Psi_\alpha \Gamma^i_{\alpha\beta}[X^i, \Psi_\beta] + i\bar{G}\partial_t D_t G + \bar{G}[X^i_{\rm bg}, [X^i, G]]\bigg]. \quad (1)$$

All fields are in the adjoint of $U(N)$, and the spinor fields Ψ_α are 10-dimensional Majorana-Weyl spinors. The last term in the first line is a gauge fixing term for the condition

$$\partial_t A_0 - i[X^i_{\rm bg}, X^i] = 0, \quad (2)$$

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[1]We will try to follow, as much as possible, the notation and conventions used in [2,12].

and G is a matrix of Faddeev-Popov ghosts. The Yang-Mills coupling is given by $g_{\rm YM}^2 = 2R$ where R is the radius of the M-theory light-cone circle. We work in string units, $\ell_s = 1$.

We will be employing the background field method to integrate out fluctuations about a background that represents two widely separated blocks of matrices. This is a technique that is well studied and developed in the literature [2,6–12]. The first part of our computations is based on works such as [2,12]; indeed, we could borrow results from [2] and modify them for the computation of entanglement. However, with the goal of presenting a self-contained exposition, and also pinpointing the nontrivial parallels in computing the effective potential and entanglement entropy in Matrix theory, we will review the process of integrating out perturbations using the background field method—closely following [2]. We take the background as

$$X^i_{\rm bg} = \begin{pmatrix} \bar{X}^i_1(t) & 0 \\ 0 & \bar{X}^i_2(t) \end{pmatrix} \tag{3}$$

with all other fields vanishing. This is a block diagonal configuration with $\bar{X}^i_1$ being an $N_1 \times N_1$ matrix, and $\bar{X}^i_2$ being an $N_2 \times N_2$ matrix; we have $N = N_1 + N_2$. In M-theory language, $\bar{X}^i_1$ is to represent an object that carries N_1 units of light-cone momentum, such as a spherical mass or a graviton, while $\bar{X}^i_2$ represents another object with N_2 units of light-cone momentum. We then want to write down an effective action by perturbing this background by

$$A_0 = \begin{pmatrix} a_1(t) & a(t) \\ \bar{a}(t) & a_2(t) \end{pmatrix} \qquad X^i = X^i_{\rm bg} + \begin{pmatrix} x^i_1(t) & x^i(t) \\ x^{i\dagger}(t) & x^i_2(t) \end{pmatrix}$$
$$\Psi_\alpha = \begin{pmatrix} \psi_{1\alpha}(t) & \psi_\alpha(t) \\ \psi^\dagger{}_\alpha(t) & \psi_{2\alpha}(t) \end{pmatrix}. \tag{4}$$

The centers of mass of the two background objects are given by

$$\bar{x}^i_{1,2} \equiv \frac{{\rm Tr}\bar{X}^i_{1,2}}{N_{1,2}} \tag{5}$$

while the size of each object might naturally be represented by the second moments:

$$R^2_{1,2} \equiv \frac{{\rm Tr}(\bar{X}^i_{1,2})^2}{N_{1,2}} - (\bar{x}^i_{1,2})^2. \tag{6}$$

We assume that the two background objects are widely separated from each other so that their gravitational potential energy is small compared to their kinetic energies. We also assume that their sizes are much smaller than the distance between them. In this regime, the off-diagonal perturbations in (4) are heavy or high frequency modes. One can then integrate them out and discover that, for large $N_{1,2}$ and while setting all diagonal perturbations to zero, the resulting effective potential for the background variables $\bar{X}^i_1$ and $\bar{X}^i_2$ agrees with the Newtonian gravitational potential between the two objects in light-cone gauge M theory [2]. This is a remarkable result in support of the Matrix theory–M theory correspondence.

Our task is to add to this computation the lighter, slower perturbations on the diagonal: the $x_{1,2}$'s, $a_{1,2}$'s, and $\psi_{1,2}$'s. We then want to write the effective potential for the $x_{1,2}$ and $\psi_{1,2}$ after the fast modes are integrated out. We write the effective potential, after integrating out the heavy off-diagonal modes, as

$$S_{\rm eff} = S_0 + S_V \tag{7}$$

where the first term S_0 comes from the part of the action that does *not* involve the off-diagonal perturbations and takes the form

$$S_0 = \int dt {\rm Tr}((\partial_t x^i_1)^2 + (\partial_t x^i_2)^2 + [x^i_1, x^j_1][\bar{X}^i_1, \bar{X}^j_1] + [x^i_1, \bar{X}^j_1][x^i_1, \bar{X}^j_1] - [x^i_1, \bar{X}^j_1][x^j_1, \bar{X}^i_1] + [x^i_2, x^j_2][\bar{X}^i_2, \bar{X}^j_2] + [x^i_2, \bar{X}^j_2][x^i_2, \bar{X}^j_2]$$
$$- [x^i_2, \bar{X}^j_2][x^j_2, \bar{X}^i_2] - (\partial_t a_1)^2 - (\partial_t a_2)^2 + 2i(\partial_t a_1)[\bar{X}^i_1, x^i_1] + 2i(\partial_t a_2)[\bar{X}^i_2, x^i_2] + i\psi_{1\alpha}\partial_t\psi_{1\alpha} + i\psi_{2\alpha}\partial_t\psi_{2\alpha}$$
$$+ \psi_{1\alpha}\Gamma^i_{\alpha\beta}[\psi_{1\beta}, x^i_1] + \psi_{2\alpha}\Gamma^i_{\alpha\beta}[\psi_{2\beta}, x^i_2]). \tag{8}$$

An important observation here is that there are no x_1-x_2 couplings in S_0; hence, the coupling between the two objects, and thus any entanglement between them, can come only from S_V. Furthermore, there are no ψ_1-ψ_2 coupling terms in S_0; nor will there be any in S_V: to leading order in small perturbations, given the action's quadratic form in the fermions, there is no entanglement to be considered between the fermionic diagonal modes.

The second piece of (7), S_V, involves the off-diagonal perturbations that can be integrated out in the regime of interest. The computation of S_V proceeds as in [2] where the diagonal perturbations were set to zero, except now $\bar{X}^i_1$ and $\bar{X}^i_2$ are shifted by x^i_1 and x^i_2; we get from the ground state energy of the oscillators [2]

$$S_V = -\int dt \Bigg({\rm Tr'}\sqrt{M_{0b} + M_{1b}} - \frac{1}{2}{\rm Tr'}\sqrt{M_{0f} + M_{1f}}$$
$$- 2{\rm Tr'}\sqrt{M_g} \Bigg), \tag{9}$$

where we define the "mass matrices" along [2]: from the bosonic sector involving x and a, we have

$$M_{0b}=\sum_i K^{i2}\otimes\mathbf{1}_{10\times10},\qquad M_{1b}=\begin{pmatrix}\mathbf{0} & -2\partial_t K^j\\ 2\partial_t K^i & 2[K^i,K^j]\end{pmatrix};\tag{10}$$

from the fermionic sector involving ψ, we have

$$M_{0f}=\sum_i K^{i2}\otimes\mathbf{1}_{16\times16},$$
$$M_{1f}=i\partial_t K^i\otimes\Gamma^i+\frac{1}{2}[K^i,K^j]\otimes\Gamma^{ij};\tag{11}$$

and from the ghost sector, we get

$$M_g=\sum_i K^{i2}.\tag{12}$$

In these expressions, we have defined the matrix

$$K^i=(\bar{X}_1^i+x_1^i)\otimes\mathbf{1}_{N_2\times N_2}-\mathbf{1}_{N_1\times N_1}\otimes(\bar{X}_2^{iT}+x_2^{iT}).\tag{13}$$

In (9), Tr' corresponds to tracing over both group and Lorentz spaces. Throughout, we are assuming, as in [2], that the background satisfies the equations of motion, and hence all terms linear in the perturbations should be dropped. Hence, it is implicit in (9) that we drop linear terms in x_1^i and x_2^i once the expression is expanded further. Given the similarities between (9) and the result in [2], with the only modification coming from the shifts by x_1^i and x_2^i in (13), the computations proceed along similar steps: we write the square root of the matrices using a Dyson perturbation series in M_{1b} and M_{1f}, where M_{1b} and M_{1f} are smaller than M_{0b} and M_{0f}. The zeroth order corresponds to zero point energy and cancels by supersymmetry once we include the contribution from the ghosts (M_g only contributes to zeroth order); the cancellations carry over to linear, quadratic, and third order in M_1. The first nonzero contribution arises at fourth order and we get

$$\begin{aligned}S_V=\frac{1}{2\sqrt{\pi}}\mathrm{Tr}\int dt\Bigg(&\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{d\tau_1 d\tau_2 d\tau_3 d\tau_4 d\tau_5}{(\tau_1+\tau_2+\tau_3+\tau_4+\tau_5)^{3/2}}\\ &\times e^{-(\tau_1+\tau_2+\tau_3+\tau_4+\tau_5)M_{0b}}\mathrm{Tr}_L[M_{1b}(\tau_2+\tau_3+\tau_4+\tau_5)M_{1b}(\tau_3+\tau_4+\tau_5)M_{1b}(\tau_4+\tau_5)M_{1b}(\tau_5)]\\ &-\frac{1}{2}e^{-(\tau_1+\tau_2+\tau_3+\tau_4+\tau_5)M_{0f}}\mathrm{Tr}_L[M_{1f}(\tau_2+\tau_3+\tau_4+\tau_5)M_{1f}(\tau_3+\tau_4+\tau_5)M_{1f}(\tau_4+\tau_5)M_{1f}(\tau_5)]\Bigg)\end{aligned}\tag{14}$$

where we defined

$$M_1(\tau)\equiv e^{\tau M_0}M_1e^{-\tau M_0}.\tag{15}$$

Tr_L involves tracing over Lorentz space, while Tr refers to tracing over group space as usual. Let us then write

$$K^i=\bar{K}^i+\Delta K^i\tag{16}$$

where we define

$$\bar{K}^i\equiv\bar{X}_1^i\otimes\mathbf{1}_{N_2\times N_2}-\mathbf{1}_{N_1\times N_1}\otimes\bar{X}^{iT}\tag{17}$$

so that all diagonal perturbations are in the ΔK^i matrix. To proceed further, we will focus on a subsector of diagonal perturbations that perturb the location of the centers of masses of the two objects. We write

$$x_1^i=\epsilon_1^i\mathbf{1}_{N_1\times N_1},\qquad x_2^i=\epsilon_2^i\mathbf{1}_{N_2\times N_2},\tag{18}$$

where ϵ_1^i and ϵ_2^i are now the small perturbations associated with blocks 1 and 2 respectively. Beyond being a physically natural choice, these perturbations also decouple from other perturbations as they drop out of the commutators appearing in (1). This means that truncating to this sector of perturbations is mathematically consistent. The first part of the action given by S_0 in (8) then becomes

$$\begin{aligned}S_0=\int dt(&N_1(\partial_t\epsilon_1^i)^2+N_2(\partial_t\epsilon_2^i)^2+\mathrm{Tr}[-(\partial_t a_1)^2\\ &-(\partial_t a_2)^2+i\psi_{1\alpha}\partial_t\psi_{1\alpha}+i\psi_{2\alpha}\partial_t\psi_{2\alpha}]).\end{aligned}\tag{19}$$

We also have

$$\Delta K^i=(\epsilon_1^i-\epsilon_2^i)\mathbf{1}_{N_1\times N_1}\otimes\mathbf{1}_{N_2\times N_2}.\tag{20}$$

We then get

$$\begin{aligned}K^{i2}&=(\bar{K}^i+\Delta K^i)^2\\ &=\bar{K}^{i2}+2(\epsilon_1^i-\epsilon_2^i)\bar{K}^i+(\epsilon_1^i-\epsilon_2^i)^2\mathbf{1}_{N_1\times N_1}\otimes\mathbf{1}_{N_2\times N_2}.\end{aligned}\tag{21}$$

Assuming that the size of each object R_1 and R_2 is much smaller than the separation distance between them, the eigenvalues of $\bar{K}^{i2}$ scale as r^2 where we define the relative position vector between the centers of mass of the two objects as

$$r^i = \bar{x}_1^i - \bar{x}_2^i. \tag{22}$$

More generally, we expect that

$$\bar{K}^i = r^i \mathbf{1}_{N_1\times N_1} \otimes \mathbf{1}_{N_2\times N_2} + \kappa^i \tag{23}$$

where κ^i is a matrix whose entries scale at most as R_1^2 and R_2^2, the characteristic sizes of the two objects—independent of the distance r separating them. As long as $R_{1,2} \ll r$, we can then approximately write

$$K^{i2} \simeq r^2 + 2(\epsilon_1^i - \epsilon_2^i) r^i + r^2 = (r^i + \epsilon_1^i - \epsilon_2^i)^2 \tag{24}$$

which is large, scaling as r^2 with large r. Looking back at (14), we focus first on the exponential factor in the integrand. Whether for bosons or fermions, we have a structure of the form

$$e^{-(\tau_1+\tau_2+\tau_3+\tau_4+\tau_5)K^{i2}}. \tag{25}$$

For large r, this implies that the predominant contribution to the integral in (14) comes from the region where the τ's are zero. As a result, we can approximately write, as in [2],

$$M_1(\tau) = e^{\tau M_0} M_1 e^{-\tau M_0} \simeq M_1. \tag{26}$$

This leads to a very similar expression to the effective Newtonian potential computed in [2], now given by

$$S_V = \int dt \frac{5}{128((\bar{x}_1^i - \bar{x}_2^i + \epsilon_1^i - \epsilon_2^i)^2)^{7/2}} \times \mathrm{Tr}[8F^\mu{}_\nu F^\nu{}_\lambda F^\lambda{}_\sigma F^\sigma{}_\mu + 16F_{\mu\nu}F^{\mu\lambda}F^{\nu\sigma}F_{\lambda\sigma} - 4F_{\mu\nu}F^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} - 2F_{\mu\nu}F_{\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}] \tag{27}$$

where we define

$$F_{0i} = \partial_t K^i, \qquad F_{ij} = i[K^i, K^j]. \tag{28}$$

Notice that, given that the center of mass perturbations commute with all matrices, we have

$$F_{ij} = i[\bar{K}^i, \bar{K}^j]. \tag{29}$$

And we also have

$$F_{0i} = \partial_t \bar{K}^i + \partial_t(\epsilon_1^i - \epsilon_2^i)\mathbf{1}_{N_1\times N_1} \otimes \mathbf{1}_{N_2\times N_2}. \tag{30}$$

These time derivatives of $\epsilon_{1,2}$ are subleading to the kinetic terms of the perturbations arising in (19) as they will be multiplied by $\sim r^{-7}$. The terms involving $\partial_t \epsilon_{1,2}$ can then be dropped as long as the distance between the two objects is large. We then get

$$F_{ij} = i[\bar{K}^i, \bar{K}^j] = \bar{F}_{ij}, \qquad F_{0i} \simeq \partial_t \bar{K}^i = \bar{F}_{0i}. \tag{31}$$

Note next that the $\bar{F}_{ij}$ and $\bar{F}_{0i}$ are independent of r^i, the separation vector between the two objects. To see this, we have from (23)

$$\bar{F}_{ij} = i[\kappa^i, \kappa^j] \tag{32}$$

where the matrix entries of κ^i scale as the size of each object, independent of r^i. As for $\bar{F}_{0i}$, we have from (23)

$$\bar{F}_{0i} = \partial_t r^i \mathbf{1}_{N_1\times N_1} \otimes \mathbf{1}_{N_2\times N_2} + \partial_t \kappa^i \tag{33}$$

demonstrating that $\bar{F}_{0i}$ is also r^i independent—but of course it depends on $\partial_t r^i$. Putting things together, we can then write

$$S_V \simeq -\int dt \frac{1}{2} \epsilon_a^i \epsilon_b^j \frac{\partial^2 V}{\partial \bar{x}_a^i \partial \bar{x}_b^j} \tag{34}$$

where a and b sum over 1 and 2, and where V is the potential from [2]

$$V = -\frac{5}{128 r^7} W \tag{35}$$

with

$$W = \mathrm{Tr}[8\bar{F}^\mu{}_\nu \bar{F}^\nu{}_\lambda \bar{F}^\lambda{}_\sigma \bar{F}^\sigma{}_\mu + 16\bar{F}_{\mu\nu}\bar{F}^{\mu\lambda}\bar{F}^{\nu\sigma}\bar{F}_{\lambda\sigma} - 4\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}\bar{F}_{\lambda\sigma}\bar{F}^{\lambda\sigma} - 2\bar{F}_{\mu\nu}\bar{F}_{\lambda\sigma}\bar{F}^{\mu\nu}\bar{F}^{\lambda\sigma}]. \tag{36}$$

Note that, as promised, we dropped terms linear in ϵ. In [2], it was shown that V matches precisely (including numerical coefficient) with the expected Newtonian gravitational potential averaged over the light-cone direction between the two objects as long as $N_{1,2}$ are large:

$$V = -\frac{15}{4} \frac{R^4}{N_1 N_2 r^7} \left((p_1 \cdot p_2)^2 - \frac{1}{9} p_1^2 p_2^2 \right) \tag{37}$$

where p_1 and p_2 are the 11-dimensional momenta of the two objects.

Combining this result with the rest of the action from (19), we then have the effective action for ϵ_1 and ϵ_2 that represent diagonal perturbations of the two objects:

$$\int dt \left(N_1 (\partial_t \epsilon_1^i)^2 + N_2 (\partial_t \epsilon_2^i)^2 - \left(\frac{1}{2}\epsilon_1^i \epsilon_1^j + \frac{1}{2}\epsilon_2^i \epsilon_2^j - \epsilon_1^i \epsilon_2^j \right) \frac{\partial^2 V}{\partial r^i \partial r^j} \right) \tag{38}$$

where we used the fact that, in the regime of large distance r between the objects, the potential V depends only on $r^i = x_1^i - x_2^i$.

To proceed further, we need to set up a particular scenario where one of the two objects is treated as the heavy source and the other is a light probe. This sets the stage to interpreting the soon-to-be-computed quantum entanglement as a measure of the local curved geometry experienced by the probe due to the source. Let us take object 1 to be the massive "star" whose geometry object 2 is probing; for example, we might write

$$\bar{X}_1^i = \bar{x}_1^i \mathbf{1}_{N_1\times N_1} + \frac{2r_1}{N_1} J_i \tag{39}$$

where the J_i are the angular momentum matrices, satisfying the $SU(2)$ algebra and the Casimir relation

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \qquad \mathrm{Tr} J_i^2 \simeq \frac{N_1^3}{4}, \tag{40}$$

where we assumed that $N_1 \gg 1$. Similarly, we can take object 2 to be a spherical "planet" with N_2 units of light-cone momentum that is much lighter and smaller. Each object has a nonzero size $R_{1,2}$ which is, at the least, the radius of the corresponding black hole. However, spatially localized configurations like the one given by (39) do not solve the equations of motion without an additional infrared cutoff—i.e., we may not assume that the background is on shell as we have done so. If object 1 were to be a black hole, we expect that the chaotic nature of Matrix theory would admit a metastable spherical configuration that is long lived as it evaporates away slowly via Hawking radiation [13]. It has been shown that this stochastic short timescale dynamics can be effectively modeled by adding by hand a quadratic mass term to the action. Alternatively, one can imagine a background flux that stabilizes the configuration like in the case of the giant gravitons of the Berenstein-Maldacena-Nastase (BMN) Matrix model [14]. In either scenario, object 1 maintains a finite size due to some additional terms in the action, either due to effective stochastic physics or due to a nonflat background that essentially puts the system in a box. Here, we account for this by adding by hand a generic stabilizing term, the simplest of which would be

$$S \to S - \int dt \alpha_1 \left(\mathrm{Tr} X_1^i X_1^i - \frac{\mathrm{Tr} X_1^i \mathrm{Tr} X_1^i}{N_1} \right) - \int dt \alpha_2 \left(\mathrm{Tr} X_2^i X_2^i - \frac{\mathrm{Tr} X_2^i \mathrm{Tr} X_2^i}{N_2} \right) \tag{41}$$

where $\alpha_{1,2}$ are positive constants that are tuned to assure given stable sizes $R_{1,2}$ for objects 1 and 2.[2] The important general observation is that α_1 and α_2 must be positive to assure stability, and they are larger for larger objects. To see this, for the configuration given by (39), we can check that the size of object 1 is $R_1 = r_1$, and its mass scales as $M \sim r_1^2 \sim \alpha_1 N_1^2$ (the area of the spherical membrane). For fixed light-cone momentum N_1, large α_1 corresponds to larger energy. Treating object 2 as the light probe, we henceforth assume that $\alpha_2 \ll \alpha_1$. In fact, as we shall see, it does not matter which one of the two objects is the lighter probe–the entanglement entropy of either one is the same as the other's, as expected from the fact that the combined system of diagonal perturbations is in a pure state.

The result of this is that one ends up adding additional terms to the effective action (38) of the form $-N_1\alpha_1(\varepsilon_1^i)^2$ and $-N_2\alpha_2(\varepsilon_2^i)^2$ which dominate the corresponding $(\varepsilon_1^i)^2$ and $(\varepsilon_2^i)^2$ terms in (38). We then have the modified effective action

$$\int dt \left(N_1(\partial_t \varepsilon_1^i)^2 + N_2(\partial_t \varepsilon_2^i)^2 - N_1\alpha_1(\varepsilon_1^i)^2 - N_2\alpha_2(\varepsilon_2^i)^2 + \varepsilon_1^i \varepsilon_2^j \frac{\partial^2 V}{\partial r^i \partial r^j} \right). \tag{42}$$

We rescale the perturbations so as to canonically normalize the kinetic terms

$$z_{1,2} \equiv \sqrt{N_{1,2}}\, r \varepsilon_{1,2}. \tag{43}$$

We end up with the final effective action for the perturbations[3]:

$$S_{\mathrm{eff}} = \int dt \left((\partial_t z_1)^2 + (\partial_t z_2)^2 - \alpha_1 z_1^2 - \alpha_2 z_2^2 + \frac{z_1^i z_2^j}{\sqrt{N_1 N_2}} \partial_i \partial_j V \right). \tag{44}$$

We write $\partial_i = \partial/\partial z_2^i$, derivatives with respect to the probe's location. This is the effective action that describes the diagonal perturbations, to leading order R_1/r and R_2/r, between blocks 1 and 2 of the matrices—in a regime where object 2 is a light probe under the influence of a massive object 1 that curves the spacetime around it. We next compute the quantum entanglement in the vacuum of the z_1-z_2 system arising from the $z_1 z_2$ coupling term in this effective action.

We have a system with two d.o.f. with a Hamiltonian

$$H = (\partial_t z_1)^2 + (\partial_t z_2)^2 + \alpha_1 z_1^2 + \alpha_2 z_2^2 - \frac{1}{\sqrt{N_1 N_2}} z_1^i z_2^j \partial_i \partial_j V \equiv (\partial_t z_a)^2 + z_a^i W_{(ai)(bj)} z_b^j, \tag{45}$$

where a, b sum over 1,2. Following [15], we define the matrix ω as

[2] For example, it is easy to check that, for a spherical configuration of radius R_1 given by (39), one needs $\alpha_1 = 8R_1^2/N_1^2$.

[3] One can also consider the probe to be a graviton. As a result, $\alpha_2 \to 0$ and we must keep the ε_2^2 term from (38). The subsequent computation is then slightly modified and the general pattern persists as long as the probe is much lighter than the source.

$$\omega = W^{1/2} \quad \text{where}$$
$$W = \begin{pmatrix} \alpha_1 \delta_{kl} & -\frac{1}{2\sqrt{N_1 N_2}} \partial_i \partial_k V \\ -\frac{1}{2\sqrt{N_1 N_2}} \partial_l \partial_j V & \alpha_2 \delta_{ij} \end{pmatrix}, \quad (46)$$

in 2×2 block diagonal form. The density matrix for the vacuum state takes the form

$$\rho(z, z') = \sqrt{\frac{\det \omega}{\pi}} e^{-\frac{1}{2} z^T \omega z} e^{-\frac{1}{2} z'^T \omega z'}. \quad (47)$$

In our case, we get

$$\omega = \begin{pmatrix} \sqrt{\alpha_1} & -\frac{1}{2\sqrt{\alpha_1}\sqrt{N_1 N_2}} \partial_i \partial_k V \\ -\frac{1}{2\sqrt{\alpha_1}\sqrt{N_1 N_2}} \partial_l \partial_j V & \sqrt{\alpha_2} \end{pmatrix}, \quad (48)$$

where we have evaluated the square root of the matrix in the regime where (a) the off-diagonal entries of W are much smaller than the diagonal ones and where (b) we have $\alpha_2 \ll \alpha_1$ since object 2 is the probe.

We are interested in computing the entanglement entropy of object 2 with object 1 by tracing over the Hilbert space of object 1 and computing the von Neumann entropy of the resulting reduced density matrix. Following [15], we then define

$$\Lambda \equiv \omega_{22}^{-1/2} \cdot \omega_{21} \cdot \omega_{11}^{-1} \cdot \omega_{12} \cdot \omega_{22}^{-1/2} \quad (49)$$

where ω_{11} is the sub-block of the matrix on the diagonal referring to object 1, ω_{12} is the sub-block between objects 1 and 2, etc… Λ is then a 9×9 matrix in the tangent space of the probe's location—parametrized by z_2^i. For our case, we have

$$\Lambda_{ij} = \frac{1}{4\alpha_1 N_1 N_2} \frac{(\partial_i \partial_k V)(\partial_k \partial_j V)}{\sqrt{\alpha_1 \alpha_2}} \equiv \gamma^2 (\partial_i \partial_k V)(\partial_k \partial_j V). \quad (50)$$

We have defined γ to absorb all constants that refer to information about the individual objects, such as their sizes, masses, and equations of state. Note also that the eigenvalues of Λ are much smaller than one in the regime we have been working in.

The von Neumann entropy of interest is then given by [15]

$$S_{ent}(\Lambda) = \mathrm{Tr}\left(\ln \frac{1 - \Lambda/2 + \sqrt{1-\Lambda}}{1 - \Lambda + \sqrt{1-\Lambda}} - \frac{\Lambda}{2} \frac{\ln \frac{\Lambda}{2 - \Lambda + 2\sqrt{1-\Lambda}}}{1 - \Lambda + \sqrt{1-\Lambda}} \right)$$
$$\simeq -\mathrm{Tr}\left(\frac{\Lambda}{4} \ln \frac{\Lambda}{4} \right) + \mathrm{Tr} \frac{\Lambda}{4}, \quad (51)$$

where the simpler form on the second line is valid when the eigenvalues of Λ are much smaller than 1, as is the case for us. Note than all $U(N)$ matrix structure has disappeared and the relevant object lives in the tangent space of the probe's position—the vector space over which the expression traces. We have hence computed the entanglement entropy of the two objects in the quantum vacuum of perturbations of their centers of mass, and we have shown how this entropy is a function of the gravitational potential that the probe experiences due to the presence of the source.

III. A NEW ENTROPY-GEOMETRY RELATION

The gravitational potential V encodes information about the curvature of the spacetime at the probe's location. This means that we must be able to relate the entanglement entropy of the probe to local spacetime geometry. We start on the general relativity side with the light-cone gauge M theory probe evolving along a timelike geodesic with tangent denoted by u^μ, where $\mu = 0, 1, \ldots, 10$. Indices $1, \ldots, 9$ are the transverse directions to the light cone, mapping onto the Matrix theory target space indices, while the theory is boosted in light-cone direction x^{10}. Let z_i^μ be nine spacelike vectors tangent to u^μ, so we have $i, j = 1, \ldots, 9$. One can project onto this subspace using

$$h^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu. \quad (52)$$

We can then relate the Newtonian gravitational potential V of the probe to the local Riemann tensor that it samples by [16]

$$z_i^\mu z_j^\nu V_{;\mu\nu} \simeq z_i^\mu z_j^\nu R_{\mu\rho\nu\sigma} u^\rho u^\sigma. \quad (53)$$

Looking back at (50), we see that the entropy is expressed as a function of the double derivatives of the potential, instead of covariant derivatives. This is natural in the context of Matrix theory as the Matrix theory formulation is background dependent, built up on top of a flat Minkowski background. This suggests that the probe coordinates on the Matrix theory side of the correspondence cannot map onto general coordinates that the dual M-theory geometry might be written in. We then conjecture that one is required to interpret the Matrix theory coordinates as *locally flat* coordinates at the location of the probe on the M-theory side.[4] Matrix theory would then build up geometry locally through probe tidal acceleration that the Matrix effective potential can naturally determine. In locally flat coordinates at the location of the probe, the Christoffel symbols vanish and we have

[4]Note that (53) does *not* map onto the desired form involving simple derivatives at asymptotic infinity where curvatures are weak and where our computation is designed to hold. Hence, there is no alternative to locally flat coordinates, where the Christoffel symbols vanish. Note also that this is a more general coordinate system than Riemann normal or Fermi normal coordinates, and there is still infinite freedom *globally* in fixing locally flat coordinates. At the location of the probe, the freedom consists of local rotations $SO(9)$, a subgroup of the gauge group of 11-dimensional gravity given the restriction to light-cone gauge. As required, this is also the symmetry group on the Matrix theory side.

$$z_i^\mu z_j^\nu V_{,\mu\nu} = z_i^\mu z_j^\nu \partial_\mu \partial_\nu V = z_i^\mu z_j^\nu R_{\mu\rho\nu\sigma} u^\rho u^\sigma. \tag{54}$$

We also have at the location of the probe $\eta_{\mu\nu} z_i^\mu u^\nu = 0$ for $i = 1, \ldots, 9$. It is easy to check that we can write

$$z_i^\mu = \delta_i^\mu - \frac{u_i}{u_-} \delta_-^\mu, \tag{55}$$

where we use the light-cone metric such that $-2u^+ u^- + (u^i)^2 = -1$. Note that the light-cone momentum $p^+ = N_2/R$, and the light-cone energy is $p^- = (m^2 + (p^i)^2)/(2p^+)$. We also have $\partial_- V = 0$ from the fact that V is averaged over x^- since no longitudinal momentum is exchanged between source and probe. We then can write

$$\partial_i \partial_j V = \left(\delta_i^\mu - \frac{u_i}{u_-}\delta_-^\mu\right)\left(\delta_j^\nu - \frac{u_j}{u_-}\delta_-^\nu\right) R_{\mu\rho\nu\sigma} u^\rho u^\sigma \equiv \mathcal{R}_{ij}, \tag{56}$$

defining the new quantity $\mathcal{R}_{ij}$ built out of the local Riemann tensor, or equivalently tidal forces.

Putting things together, we write

$$S_{\rm ent} \simeq -\gamma^2 \mathrm{Tr}\left(\frac{\mathcal{R}^2}{4} \ln \frac{\mathcal{R}^2}{4}\right). \tag{57}$$

This is a local relation between the curvature sampled by the probe and the quantum entanglement between the center of mass d.o.f. of the source and probe. Note that this entropy is finite, not surprisingly given that we are working in a UV complete theory of quantum gravity. In the limit where the curvature vanishes, so does this expression for entropy. Next, we consider the expression

$$\theta_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta u_{(\alpha;\beta)} \tag{58}$$

which is a measure of deformations of the shape and orientation of a small sphere at the probe along its trajectory. We then have a version of Raychaudhuri's equation

$$\frac{d}{d\tau} \theta_{\mu\nu} \simeq -R_{\mu\rho\nu\sigma} u^\rho u^\sigma \tag{59}$$

where we have dropped higher order terms that are smaller than the leading contribution at weak curvatures. Using locally flat coordinates, and projecting onto the nine-dimension subspace using the z_i^μ's, we have

$$\frac{d}{d\tau} \theta_{ij} \simeq -z_i^\mu z_j^\nu R_{\mu\rho\nu\sigma} u^\rho u^\sigma = -\mathcal{R}_{ij}. \tag{60}$$

In particular $d\theta/d\tau$, where θ is the trace of θ_{ij} using the metric h_{ij}, is the rate of change of a volume element along the probe's geodesic. Hence, Eq. (57) establishes a relation between the source-probe entanglement entropy and the rate at which a small volume of space shrinks, rotates, and twists along the geodesic of the probe. If we were to choose the probe to be massless, one can easily show that one obtains a similar relation but now involving an *area* transverse to a congruence of *null* geodesics associated with the probe. All this is somewhat reminiscent of the entropy-area relations we encounter in other settings [17–19] with one significant difference being that our relation is a *local* statement.

It is useful to reflect on the implications of all this. We have demonstrated that the computation of a certain von Neumann entanglement entropy closely parallels the computation of the effective potential in Matrix theory. At leading order, the information in both quantities is the same. This is sensible as entanglement generally arises because of interactions in any quantum system. Demonstrating the similarities in computing a certain entanglement entropy and effective potential in Matrix theory, and the details of the relation between the effective potential and entropy—in particular the explicit form (57), are nontrivial and new. However, we emphasize that this is a first step involving one example, and to be able to make a general conjecture about entropy and geometry in Matrix theory, more case studies are needed to confirm and/or extend relation (57). Despite this, the following is significant: we essentially were able to feed the computation of an effective potential into the computation of a quantum entanglement in Matrix theory in a rather general scheme. The resulting entropy-geometry relation arising from Matrix theory then has quite general overtones.

IV. CONCLUSION

We have demonstrated that von Neumann entanglement entropy between two blocks of matrices in Matrix theory—that represent a probe gravitating near a source—can quite generically be written as a function of derivatives of their mutual gravitational potential. We also presented arguments and a conjecture for expressing this relation as a map between entanglement entropy and local spacetime geometry as sampled by the probe in the background of the source.

It is useful to summarize the key ingredients that allow the mapping of the computation of von Neumann entropy onto that of the effective gravitational potential or geometry. The essence lies in realizing that the entanglement of perturbations on the diagonal of the matrices arises from the mass matrix of these perturbations, and the latter in turn comes from the dependence of the effective potential for off-diagonal modes on the distance between the diagonal matrix blocks. Roughly speaking, the effective Matrix potential generically looks like

$$V \sim \frac{r\text{-independent terms}}{r^7} \tag{61}$$

where r is the distance between the two lumps of energy (dependence of the numerator on dr/dt is inconsequential to the analysis). And we have shown that the mass matrix of diagonal perturbations in the $U(1)$ would generally receive contributions only from r-dependent terms of this effective action. These two conclusions then necessarily result in a relation between the second derivative of the potential and entanglement entropy.

We considered a particular scenario and worked consistently only to leading order in weak gravitational potential energy. Yet, the analysis introduces a new general way to develop maps between quantum information and spacetime geometry in Matrix theory. This involves looking at diagonal matrix fluctuations and focusing on the ground state density matrix of these d.o.f. As a result, when one focuses on a sub-block of a matrix, the resulting reduced density matrix and quantum entanglement will be related to the effective Matrix potential between the two matrix sub-blocks arising from integrating out fast off-diagonal modes. This mechanism appears general and might hint as to why, at least to leading order in weak gravity, one expects a relation between entanglement entropy and spacetime geometry.

Entanglement entropy by nature is multifaceted. It depends on how one slices parts of a larger system and on what quantum state the entire system lives in. These freedoms are very much reflected in the analysis, where we made a series of choices to set a computationally accessible setup. There are many more settings to explore, and a catalog of case studies can help develop intuition on the general pattern of expected relations between entropy and local geometry. We end by pointing out a couple of particularly interesting cases: the case involving massless probes, where one has the promise to connect with ideas from holography and the entropy of light-sheets developed from different perspectives [20–22], and the case where the approach is used in BMN theory that admits stable giant gravitons and hence the need to add stabilizing terms to the action is avoided [23].

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