Holographic dual of conformal field theories with very special $T\bar{J}$ deformations

Yu Nakayama

Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan

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Very special $T\bar{J}$ deformations of a conformal field theory are irrelevant deformations that break the Lorentz symmetry but preserve the twisted Lorentz symmetry. We construct a holographic description of very special $T\bar{J}$ deformations. We give a holographic recipe to study the double trace as well as single trace deformations. The former is obtained from the change of the boundary condition, while the latter is obtained from the change of the supergravity background.

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I. INTRODUCTION

Inspired by exactly marginal and integrable $J\bar{J}$ deformations (e.g., see [1–5]) and irrelevant but integrable $T\bar{T}$ deformations of two-dimensional conformal field theories [6–9], the Lorentz breaking, irrelevant but integrable $T\bar{J}$ deformations of conformal field theories, which break Lorentz symmetry, have been vigorously studied in recent works [10–22]. A better understanding of ultraviolet completions of power-counting nonrenormalizable theories is theoretically important—in particular, with Lorentz violations because they might give a hint toward (Lorentz violating) ultraviolet completions of quantum gravity in higher dimensions (such as Horava-Lifshitz gravity [23])—and we expect that $T\bar{J}$ deformations play a role in such toy models.

The generic $T\bar{J}$ deformations break the global conformal symmetries of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ down to $SL(2,\mathbb{R})\times\mathbb{R}$ (i.e., left-moving translation, left-moving dilatation, left-moving special conformal transformation, and right-moving translation). In the subsequent work [15], we have proposed the "very special" types of $T\bar{J}$ deformations which preserve the additional (right-moving) twisted Lorentz transformation while still breaking the right-moving special conformal transformation. The four-dimensional analogue of the symmetry structure here is the so-called very special conformal symmetry (based on the earlier idea of very special relativity proposed by Cohen and Glashow [24–28]), and it is considered very special (in addition to the original meaning of very special relativity) and peculiar because unitarity or locality is typically sacrificed for its realization.

The success of field theoretic ultraviolet completions of $T\bar{J}$ deformations has naturally led to holographic studies of the $T\bar{J}$ deformations.[1] By construction, the asymptotic behavior of gravity should be different from that of the AdS space-time that is dual to the undeformed theory, and this may lead to novel types of holographic setups. Since we do not know the very definition of quantum gravity beyond the asymptotically AdS space-time, the holographic construction of the $T\bar{J}$ deformations may give a new direction to understand the nature of quantum gravity in more general settings.

There are two different types of holographic $T\bar{J}$ deformations. The first is given by the double trace $T\bar{J}$ deformations [11]. This is a natural realization of the original field theoretic idea of $T\bar{J}$ deformations in the sense that we study the composite operators of T and $\bar{J}$ as a double trace operator in the large N setup. The construction is ubiquitous in every (holographic) conformal field theory with a conserved (chiral) current and holographically realized by changing the boundary conditions of asymptotic infinity. The other type is the so-called single trace $T\bar{J}$ deformation [12,13]. This is given by the operator which has the same quantum number as the double trace $T\bar{J}$ but is a single trace operator. The existence of such operators is not necessarily guaranteed in generic holographic conformal field theories with a conserved current, but in many concrete examples of the AdS/CFT setups, the Kaluza-Klein towers of metric play the role of such operators. The crucial difference of the single trace $T\bar{J}$ deformations is that the actual asymptotic behaviors of the metric (rather than the boundary conditions) are modified.

[1]The analogous (holographic) studies of $T\bar{T}$ deformations can be found in [29–47].

In this paper, we propose holographic descriptions of double trace as well as single trace very special $T\bar{J}$ deformations from effective field theory approaches in bulk gravity. In dual conformal field theories, the very special $T\bar{J}$ deformations are accompanied by the twisted Lorentz symmetry [15], and we implement a similar mechanism in holography. The holographic realizations are, to some extent, phenomenological and based on the effective bulk theories (rather than the full string embedding) because the realizations of noncompact symmetries in the bulk that we need are not straightforward in the full string setup. It will be an interesting future direction to see if we can embed our constructions in a full string theory background.

The organization of the paper is as follows. In Sec. II, we briefly review field theoretic aspects of very special $T\bar{J}$ deformations of two-dimensional conformal field theories. In Sec. III, we first construct holographic dual descriptions of double trace $T\bar{J}$ deformations, and then we construct holographic dual descriptions of single trace $T\bar{J}$ deformations. In Sec. IV, we conclude the paper with discussions.

II. VERY SPECIAL $T\bar{J}$ DEFORMATION OF CONFORMAL FIELD THEORIES

Let us briefly review the very special $T\bar{J}$ deformations of the two-dimensional conformal field theories proposed in [15]. We use the light-cone notations, and our convention is

$$x^{\pm} = \frac{1}{\sqrt{2}}(t \pm x). \tag{1}$$

We call the fields that are only dependent on x^- left moving and the fields only dependent on x^+ as right moving. For example, a component of the (traceless) energy-momentum tensor $T^+_- = -T_{--}$ satisfying $\partial_+ T^+_- = 0$ is left moving. In the Euclidean setup $T_{--} = -T^+_-$ is identified with the holomorphic (left moving in our convention) energy-momentum tensor $T(z)$ with $\bar{\partial}T(z) = 0$. Similarly, a component of the chiral conserved current J^- satisfying $\partial_- J^- = 0$ is identified with the right-moving or antiholomorphic current $\bar{J}(\bar{z})$ with $\partial\bar{J}(\bar{z}) = 0$.

Suppose there is a conformal field theory with a (chiral) conserved current $\bar{J}(\bar{z})$. The $T\bar{J}$ deformation is formally defined as perturbing the conformal field theory by adding

$$\delta S = \int d^2 z \mu T\bar{J} \tag{2}$$

to the action, where $T(z)$ is the holomorphic energy-momentum tensor of the undeformed theory. This added term has the conformal dimensions of $(h, \bar{h}) = (2, 1)$, and it is irrelevant with respect to the conventional dilatation $D = h + \bar{h}$; however, we may study the ultraviolet completion. Note that the deformation breaks the Lorentz symmetry, but it preserves the right-moving conformal (or Virasoro) symmetry together with the left-moving translation.

The idea of very special $T\bar{J}$ deformation is to further assume that the current $\bar{J}(z)$ is charged under a (noncompact) symmetry generator K:

$$i[K, \bar{J}(z)] = \bar{J}(z). \tag{3}$$

Note that the charge is “pure imaginary” [similar to the ghost charge or dilatation charge because we assume $\bar{J}(z)$ is Hermitian] in contrast to compact $U(1)$ symmetry like the electric charge. With a nonzero weight for $\bar{J}(\bar{z})$ under K, we may define the twisted Lorentz transformation $J_{\text{twisted}} = L_0 - \bar{L}_0 + K$ such that the deformation is invariant under the twisted Lorentz symmetry. This is the idea of the very special $T\bar{J}$ deformation of conformal field theories.

Under the typical very special $T\bar{J}$ deformations, the left-moving special conformal symmetry is broken while the right-moving conformal symmetry is intact as can be seen from the study of the energy-momentum tensor (see [15] for the detailed study). This is a generic feature of the twisted conformal field theories with deformations. The current $\bar{K}(\bar{z})$ associated with the symmetry generator K plays the role of the virial current.

One may study correlation functions of very special $T\bar{J}$ deformed conformal field theories as studied in the $T\bar{J}$ deformed conformal field theories in [18]. The additional feature is that we have twisted Lorentz symmetry. We just emphasize here that the existence of twisted Lorentz symmetry drastically simplifies the computation when we study the perturbation theory with respect to μ. Since the K charge is conserved and the perturbed operator is only positively charged under K, a single term out of infinite perturbative series gives nonzero results in the perturbative computation of the correlation functions.

III. HOLOGRAPHIC VERY SPECIAL $T\bar{J}$ DEFORMATION

A. Double trace deformation

In holographic descriptions of large N conformal field theories, a bulk supergravity field directly corresponds to a single trace operator of the dual conformal field theories. The single trace deformation of the conformal field theory therefore corresponds to a change of the bulk background. In contrast, in the double trace deformation, we do not change the bulk background, but we change the boundary conditions of the supergravity fields [11,48]. In the holographic description of the double trace deformations corresponding to the very special $T\bar{J}$ deformations, we therefore change the boundary conditions for the metric and a bulk vector field so that they break the full isometry of the AdS space-time but preserve the twisted Lorentz symmetry.

We focus on the effective bulk theory of Einstein gravity coupled with $SL(2,\mathbb{R})$ Chern-Simons theory.

The ingredients are more or less minimal in the sense that the resulting theory contains the energy-momentum tensor and the $SL(2,\mathbb{R})$ chiral current algebra, so the very special $T\bar{J}$ deformations based on twisted Lorentz symmetry are possible. The bulk action is given by

$$S=\frac{k}{4\pi}\int d^3x\sqrt{|g|}(R+2)+\frac{\tilde{k}}{4\pi}\int \mathrm{Tr}\left(AdA+\frac{2}{3}A^3\right). \quad (4)$$

Here, the Chern-Simons gauge field A takes the value in generators of $SL(2,\mathbb{R})$ (i.e., $A=A_M^{(i)}T^{(i)}dx^M$ with $i=+,0,-$), and the trace is over the $SL(2,\mathbb{R})$ algebra. In the asymptotic AdS space-time with the standard Dirichlet boundary condition, the bulk action (4) describes the holographic dual of a conformal field theory with $SL(2,\mathbb{R})$ current algebra. Although the classical equations of motion do not depend on k and $\tilde{k}$, we implicitly assume large k and large $\tilde{k}$ limits to suppress the quantum corrections.

In order to make a holographic interpretation of the bulk theory, we have to specify the boundary condition. Let us first consider the dual description of the undeformed conformal field theory. For the metric we choose the Fefferman-Graham gauge

$$ds^2=\frac{dz^2}{z^2}+\left(\frac{g_{\mu\nu}^{(0)}}{z^2}+g_{\mu\nu}^{(2)}+\cdots\right)dx^\mu dx^\nu \quad (5)$$

with the fixed boundary metric $g_{\mu\nu}^{(0)}$ which is identified with the source for the energy-momentum tensor in the dual conformal field theory.

For the bulk gauge field, we would like to have a holographic realization of the right-moving $SL(2,\mathbb{R})$ current algebra (in the undeformed theory). For this purpose, we first put the boundary term

$$S_{\text{boundary}}=-\frac{\tilde{k}}{16\pi}\int d^2x\sqrt{|\gamma|}\gamma^{\mu\nu}\mathrm{Tr}(A_\mu A_\nu) \quad (6)$$

to make the variation principle well defined. Then, we choose the Dirichlet boundary condition $\lim_{z\to 0}A_-=a_-$. Here, a_- will be identified with the source for J^- (or $\bar{J}$ in the holomorphic coordinate) of the dual conformal field theory. We use the radial gauge $A_z=0$, and the remaining component A_+, which will be identified with J^- of the dual conformal field theory, is determined from the bulk equations of motion.

Reference [11] proposed the boundary condition that corresponds to the $T\bar{J}$ deformations. In our case, we take the $U(1)$ current $\bar{J}$ as the null current $J^{-(+)}$ out of the three generators of $SL(2,\mathbb{R})$. Suppose, before the deformation, we have a recipe to compute the partition function $Z[e,a]$ with respect to the undeformed source, e.g., two-dimensional vielbein e_a^α and the two-dimensional background $SL(2,\mathbb{R})$ gauge field $a_\alpha^{(i)}$.[2] Then, in the very special $T\bar{J}$ deformed theory, the new boundary condition becomes

$$\begin{aligned} e_a^\alpha &= \tilde{e}_a^\alpha+\mu_a J^{\alpha(+)} \\ a_\alpha^{(+)} &= \tilde{a}_\alpha^{(+)}+\mu_a T_\alpha^a \\ a_\alpha^{(0)} &= \tilde{a}_\alpha^{(0)} \\ a_\alpha^{(-)} &= \tilde{a}_\alpha^{(-)}, \end{aligned} \quad (7)$$

with $\mu_-=\mu$ and $\mu_+=0$ corresponding to the $T\bar{J}$ deformation.

Here, tilted quantities are sources in the very special $T\bar{J}$ deformed theory. On the other hand, $J^{\alpha(i)}$ and T_α^a are expectation values of the current and the energy-momentum tensor (in the original theory), so once we somehow know the original partition function $Z[e,a]$, this gives a (possibly nonlinear) relation between the expectation values of $J^{\alpha(i)}$ and T_α^a and the source $\tilde{e}_a^\alpha$ and $\tilde{a}_\alpha^{(i)}$ (or e_a^α and $a_\alpha^{(i)}$), which can be, at least in principle, solved.

In the holographic background in the large N limit (or large k limit), we compute $Z[e,a]$ by using the standard GKP-Witten prescription with the Dirichlet boundary condition for the metric and the gauge field. Since T_α^a and $J^{\alpha(i)}$ are computed from the holography by deriving the on-shell action with respect to e_a^α and $a_\alpha^{(i)}$, the boundary condition employed here is essentially the Robin-type boundary condition in the asymptotically AdS space-time. In the large μ limit, it effectively becomes the Neumann boundary condition.

Eventually, we set $\tilde{e}_a^\alpha=\delta_a^\alpha$ and $\tilde{a}_\alpha^{(i)}=0$ to describe the flat background in the very special $T\bar{J}$ deformed theory. Let us see how the boundary condition breaks the Lorentz symmetry while it preserves the twisted Lorentz symmetry in this particular background. Under the Lorentz transformation, $J^{-(+)}$ is charged, so the first two conditions in (7) break the Lorentz symmetry. However, if we define the twisted Lorentz symmetry as the sum of the original Lorentz transformation and the Cartan of $SL(2,\mathbb{R})$, i.e., $J^{(0)}$ as $J_{\text{twisted}}=L_0-\bar{L}_0+J^{(0)}$, then $J^{-(+)}$ becomes neutral under the twisted deformation. In other words, the new boundary condition (7) with $\tilde{e}_\alpha=\delta_a^\alpha$ and $\tilde{a}_\alpha^{(i)}=0$ is invariant under the twisted Lorentz transformation, so the entire formalism preserves the twisted Lorentz symmetry.

One may also study the conservation of the holographic energy-momentum tensor as in [11]. With $\tilde{e}_\alpha=\delta_a^\alpha$ and $\tilde{a}_\alpha=0$, the two most important equations are

[2] We use the convention that $\alpha,\beta\cdots$ refer to the two-dimensional space-time indices, while $a,b\cdots$ are local Lorentz indices. In the bulk, we use $M,N,\cdots$ and $A,B,\cdots$.

$$\partial_+ T_-^{+} + \partial_-(\mu J^{-(+)} T_-^{+}) = 0 \tag{8}$$

and

$$\partial_- J^{-(0)} = \mu J^{-(+)} T_-^{+}, \tag{9}$$

which enable us to verify the conservation of the twisted Lorentz current [15]

$$\partial_+(x^- T_-^{+}) + \partial_-(x^- T_-^{-} - J^{-(0)}) = 0. \tag{10}$$

The latter equation (9) is a consequence of the Chern-Simons equations of motion

$$\partial_+ A_-^{(0)} - \partial_- A_+^{(0)} = A_+^{(+)} A_-^{(-)} - A_-^{(+)} A_+^{(-)}. \tag{11}$$

Without the source in the very special $T\bar{J}$ deformed conformal field theory, $\lim_{z\to 0} A_-^{(0)} = 0$ and $\lim_{z\to 0} A_-^{(-)} = 0$, but $\lim_{z\to 0} A_-^{(+)} = \mu T_-^{-}$; thus, we obtain (9) by noting $\lim_{z\to 0} A_+ = J_+ = -J^-$.

At the end of the previous section, we argued that the extra twisted Lorentz symmetry drastically simplifies the perturbative computation of the very special $T\bar{J}$ deformed correlation functions. To see this in our holographic setup with double trace deformations, we note that the undeformed partition function $Z[e, a]$ is invariant under the spurious symmetry of $a^{(\pm)} \to e^{\pm i\theta} a^{(\pm)}$. In other words, $a^{(+)}$ and $a^{(-)}$ are always paired up in the partition function. Particularly, this means that when $a^{(-)} = 0$, the insertion of $J^{(+)}$ is trivial because it is computed by the derivative $\frac{\delta}{\delta a^{(+)}}$ which always accompanies $a^{(-)}$. According to (7), it then follows that the correlation functions of $T_{\mu\nu}$ (without any charged operators) is unaffected by the very special $T\bar{J}$ deformations. A similar selection rule applies to all the correlation functions with nonzero charges.

There is an alternative formulation of the gravity part by using $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory instead of Einstein gravity [49]. While the formulation is completely equivalent to Einstein gravity with the cosmological constant at the classical level, the reformulation may be of theoretical beauty.[3] We replace the Einstein-Hilbert action with the action constructed out of two Chern-Simons terms:

$$S_{\text{gravity}} = \frac{k}{4\pi}\int \mathrm{Tr}\Big(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A}\Big) - \frac{k}{4\pi}\int \mathrm{Tr}\Big(\bar{\mathcal{A}}d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}\bar{\mathcal{A}}\bar{\mathcal{A}}\Big) - \frac{k}{4\pi}\int d\mathrm{Tr}(\mathcal{A}\bar{\mathcal{A}}) \tag{12}$$

where the Chern-Simons gauge fields are related to the three-dimensional vielbein and spin connection as

$$\mathcal{A}_M = (w_M^{(i)} + e_M^{(i)}) T^{(i)}$$
$$\bar{\mathcal{A}}_M = (w_M^{(i)} - e_M^{(i)}) T^{(i)}, \tag{13}$$

where we have identified the $SL(2,\mathbb{R})$ indices with the local Lorentz indices. The bulk metric is therefore given by

$$g_{MN} = \frac{1}{2}\mathrm{Tr}((\mathcal{A}-\bar{\mathcal{A}})_M(\mathcal{A}-\bar{\mathcal{A}})_N). \tag{14}$$

In order to construct the holographic dictionary, we need to specify the boundary conditions. The Chern-Simons gauge field in the Graham-Fefferman gauge [50,51] corresponds to

$$\mathcal{A}_z = -z^{-2} b^{-1}\partial_z b$$
$$\mathcal{A}_\mu = b^{-1}\alpha_\mu(x^\pm) b$$
$$\bar{\mathcal{A}}_z = -z^{-2}\bar{b}^{-1}\partial_z \bar{b}$$
$$\bar{\mathcal{A}}_\mu = \bar{b}^{-1}\bar{\alpha}_\mu(x^\pm)\bar{b} \tag{15}$$

with $b = e^{-(\log z)T^{(0)}}$, $\bar{b} = b^{-1}$, where $T^{(0)}$ is the Cartan of the $SL(2,\mathbb{R})$ generator. We also supplement the condition [52,53]

$$\mathrm{Tr}[(A-\bar{\mathcal{A}})_\mu T^{(0)}] = 0. \tag{16}$$

One may directly check that this ansatz leads to the Graham-Fefferman form of the three-dimensional metric ansatz. In particular, we may identify the boundary vielbein as

$$e_\mu^- = \alpha_\mu^-$$
$$e_\mu^+ = -\bar{\alpha}_\mu^+. \tag{17}$$

The computed Chern-Simons functional $Z[\alpha^-, \bar{\alpha}^+]$ will be identified with the gravitational partition function $Z[e]$. With given boundary values of α_μ^- and $\bar{\alpha}_\mu^+$, we may solve the sub-leading normalizable terms α_μ^+ and $\bar{\alpha}_\mu^-$ by using the Chern-Simons equations of motion [in the gauge (15) and (16)]. Then, they will determine the expectation values of the boundary energy-momentum tensor T_α^a. This procedure corresponds to solving the Einstein equation in the Graham-Fefferman gauge.

Now, in order to discuss the holographic description of the very special $T\bar{J}$ deformations in the Chern-Simons gravity, we first introduce another $SL(2,\mathbb{R})$ Chern-Simons theory in the bulk whose action and the (undeformed) boundary conditions are specified as before. Then, we introduce the deformed boundary conditions

[3] There has been an ongoing discussion about whether or not the Chern-Simons formulation may make sense at the quantum level with the holographic interpretation, which we will not address in this paper.

$$
\begin{aligned}
e_a^\alpha &= \tilde{e}_a^\alpha + \mu_a J^{\alpha(+)} \\
a_\alpha^{(+)} &= \tilde{a}_\alpha^{(+)} + \mu_a T_\alpha^a \\
a_\alpha^{(0)} &= \tilde{a}_\alpha^{(0)} \\
a_\alpha^{(-)} &= \tilde{a}_\alpha^{(-)},
\end{aligned}
\tag{18}
$$

with $\mu_- = \mu$ and $\mu_+ = 0$ corresponding to the very special $T\bar{J}$ deformation. These can be realized in our Chern-Simons formulation with our Graham-Fefferman gauge choice (15) and (16) and the identification (17) at the boundary. In particular, the first two equations in (18) give the mixed boundary conditions for the three Chern-Simons gauge fields.

We do not analyze how these boundary conditions lead to the symmetry that is compatible with the very special $T\bar{J}$ deformations because it is classically identical to the Einstein formulation. Note that although the three $SL(2,\mathbb{R})$ Chern-Simons theories are more or less identical in the bulk (except for the choice of the level), the boundary conditions make them behave differently.

B. Single trace deformation

Let us now consider a holographic description of the single trace very special $T\bar{J}$ deformations. As emphasized in the Introduction, it is not always guaranteed that a given (holographic) conformal field theory with a $SL(2,\mathbb{R})$ current algebra possesses the single trace operator with the same quantum number as the double trace $T\bar{J}$ operator.[4] For this purpose, we need a further assumption that the bulk theory has a vector field whose scaling dimension is $(h,\bar{h}) = (2,1)$. It is typical that such a vector field is given by a Kaluza-Klein tower of the graviton and Kalb-Ramond field, as we will see.

Let us take the background AdS_3 and an "internal space" M with $SL(2,\mathbb{R})$ isometry as a starting point before the $T\bar{J}$ deformation. For instance, we discuss the case with $M = \mathrm{AdS}_3$ or H_3^+. The bulk AdS_3 may admit a world-sheet string realization with the NS-NS flux. Then, the AdS space-time is a classical solution of the Einstein equation and may be embedded in the full string theory (once the central charge is properly chosen). More precisely, the world-sheet sigma model is given by

$$
S = \int d^2 w 2k \left(\frac{\partial z \bar{\partial} z - 2\partial x^+ \bar{\partial} x^-}{z^2} \right), \tag{19}
$$

[4] Similar to the case with the single trace $T\bar{T}$ deformations, the single trace $T\bar{J}$ deformations are not the deformations defined by the leading operator product expansions of the energy-momentum tensor and the (antiholomorphic) current. This may cause the loss of integrability or universal features of such deformations from the field theory perspective. However, our focus is on the twisted Lorentz symmetry preserved by the deformations, which will not be affected by this nonuniversality.

where the target space metric and the Kalb-Ramond field are

$$
\begin{aligned}
ds^2 &= 2k \frac{dz^2 - 2dx^+ dx^-}{z^2} \\
B &= -4k \frac{dx^+ dx^-}{z^2}.
\end{aligned}
\tag{20}
$$

The world-sheet theory is conformal invariant and can be part of the full string theory background. Here, the level k determines the size of the AdS space-time.

This bulk geometry has the isometry of $SO(2,3)$ generated by the translation in x^+ and x^-, the Lorentz transformation rotating $x^\pm$, dilatation $(z, x^\pm) \to \lambda(z, x^\pm)$, as well as the special conformal transformation

$$
\begin{aligned}
\delta x^+ &= -z^2 \\
\delta x^- &= -2(x^-)^2 \\
\delta z &= -2(x^-) z
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
\delta x^+ &= -2(x^+)^2 \\
\delta x^- &= -z^2 \\
\delta z &= -2(x^+) z.
\end{aligned}
\tag{22}
$$

Correspondingly, the world-sheet theory has the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ current algebra [12,13] whose left-moving part is

$$
\begin{aligned}
j^+ &= -k \frac{\sqrt{2}}{z^2} \partial x^- \\
j^0 &= -k \left(-\frac{2}{z^2} x^+ \partial x^- + \frac{1}{z} \partial z \right) \\
j^- &= -k \left(\frac{2\sqrt{2}}{z^2} (x^+)^2 \partial x^- - \frac{2\sqrt{2}}{z} x^+ \partial z + \sqrt{2} \partial x^+ \right)
\end{aligned}
\tag{23}
$$

and similarly for the right-moving part. The operator product expansion is given by

$$
\begin{aligned}
j^+(w) j^-(0) &= \frac{k}{w^2} + \frac{2j^3(0)}{w} \\
j^3(w) j^3(0) &= -\frac{k/2}{w^2} \\
j^3(w) j^\pm(0) &= \pm \frac{j^\pm(0)}{w}.
\end{aligned}
\tag{24}
$$

We use lower script to refer to the world-sheet current algebra (rather than the current algebra of the dual conformal field theory), and we use w (rather than z) for the world-sheet coordinate to avoid possible confusion.

In order to realize the very special single trace $T\bar{J}$ deformations, we need an additional assumption for the world-sheet theory describing the internal space M. Let us suppose that the internal space world-sheet conformal field theory has another right-moving $SL(2,\mathbb{R})$ current algebra $\bar{k}^i$ with the OPE

$$\begin{aligned}\bar{k}^+(\bar{w})\bar{k}^-(0) &= \frac{k}{\bar{w}^2} + \frac{2\bar{k}^3(0)}{\bar{w}}\\ \bar{k}^3(\bar{w})\bar{k}^3(0) &= -\frac{k/2}{\bar{w}^2}\\ \bar{k}^3(\bar{w})\bar{k}^\pm(0) &= \pm\frac{\bar{k}^\pm(0)}{\bar{w}}.\end{aligned} \tag{25}$$

Following the idea of the single trace $T\bar{J}$ deformations realized in string theory [12,13], we propose that the world-sheet description of the single trace very special $T\bar{J}$ deformation is obtained by adding the world-sheet marginal current-current deformations of

$$\delta S = \mu \int d^2 w j^+ \bar{k}. \tag{26}$$

The deformations break the world-sheet symmetry of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ down to $SL(2,\mathbb{R})\times \mathrm{diag}(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))$ (in addition to the Virasoro symmetries that are always preserved). In particular, although the original target space Lorentz symmetry (rotations of $x^\pm$) is broken, the twisted target space Lorentz symmetry is preserved.

Let us study explicit realizations of M. Let us first take the case with $M = \mathrm{AdS}_3$. Then, the wordsheet sigma model is

$$S = \int d^2 w 2\tilde{k}\left(\frac{\partial\tilde{z}\,\bar{\partial}\,\tilde{z} - 2\partial\tilde{x}^+\bar{\partial}\tilde{x}^-}{\tilde{z}^2}\right), \tag{27}$$

with

$$\begin{aligned}\bar{k}^+ &= -\tilde{k}\frac{1}{\tilde{z}^2}\bar{\partial}\tilde{x}^+\\ \bar{k}^0 &= -\tilde{k}\left(\frac{1}{\tilde{z}^2}\tilde{x}^-\bar{\partial}\tilde{x}^+ + \frac{1}{\tilde{z}}\bar{\partial}\,\tilde{z}\right)\\ \bar{k}^- &= -\tilde{k}\left(\frac{1}{\tilde{z}^2}(\tilde{x}^-)^2\bar{\partial}\tilde{x}^+ + \frac{2}{\tilde{z}}\tilde{x}^-\bar{\partial}\,\tilde{z} - \bar{\partial}\tilde{x}^-\right),\end{aligned} \tag{28}$$

which generates the world-sheet $SL(2,\mathbb{R})$ current algebra. Now, the world-sheet current-current deformation is given by

$$\delta S = \mu\int d^2 x\left(\frac{\partial x^+\bar{\partial}\tilde{x}^+}{z^2\tilde{z}^2}\right). \tag{29}$$

The space-time interpretation is that the metric is deformed,

$$ds^2 = 2k\frac{dz^2 + dx^+dx^-}{z^2} + 2\tilde{k}\frac{d\tilde{z}^2 + d\tilde{x}^+d\tilde{x}^-}{\tilde{z}^2} + \mu\frac{dx^+d\tilde{x}^+}{z^2\tilde{z}^2}, \tag{30}$$

and the Kalb-Ramond field is deformed,

$$B = 2k\frac{dx^+dx^-}{z^2} + 2\tilde{k}\frac{d\tilde{x}^+d\tilde{x}^-}{\tilde{z}^2} + \mu\frac{dx^+d\tilde{x}^+}{z^2\tilde{z}^2}. \tag{31}$$

It is immediate to see that the background breaks the original Lorentz symmetry but preserves the twisted one given by $(x^+,x^-,\tilde{x}^+,\tilde{x}^-)\to(\lambda x^+,\lambda^{-1}x^-,\lambda^{-1}\tilde{x}^+,\lambda\tilde{x}^-)$.

In the double trace very special $T\bar{J}$ deformations, we discuss the simple structure of the deformed correlation functions due to the selection rule. We see a similar selection rule in the single trace deformation from the world-sheet conservation of the twisted Lorentz symmetry. For example, the boundary correlation functions with no charged operators under the internal boost of $\tilde{x}^\pm$ are unaffected by the very special $T\bar{J}$ deformations. More generally, the selection rule of the world-sheet $J\bar{J}$ deformations directly give the selection rule of the single trace very special $T\bar{J}$ deformations.

The case with the H_3^+ is more intricate. Let us begin with the H_3^+ sigma model

$$S = \int d^2 x 2\tilde{k}\left(\frac{\partial\tilde{z}\,\bar{\partial}\,\tilde{z} + \partial\xi\bar{\partial}\bar{\xi}^-}{\tilde{z}^2}\right), \tag{32}$$

where $\xi = \tilde{x}_1 + i\tilde{x}_2$ and $\bar{\xi} = \tilde{x}_1 - i\tilde{x}_2$ are complex conjugate. Therefore, the model has the pure imaginary NS-NS flux

$$B = i\frac{d\tilde{x}_1 d\tilde{x}_2}{z^2}. \tag{33}$$

One may perform the same current-current deformations as in the AdS_3 case with the complex metric (or more precisely, nonreal metric), but the more intricate choice of the deformation

$$\delta S = \mu\int d^2 x\left(\frac{\partial x^+\bar{\partial}w + \partial\bar{w}\,\bar{\partial}\,x^+}{z^2\tilde{z}^2}\right) \tag{34}$$

might look better because the metric and the Kalb-Ramond field satisfy the same "reality condition" as the undeformed theory. This is the combination of two different world-sheet current-current deformations of $j\bar{k}$ and $k\bar{j}$, and it is not immediately obvious if this gives the exact marginal deformations. Since the AdS current part is taken to be the same and null, the first order perturbation is marginal, so it at least passes the first order check.

IV. DISCUSSIONS

In this paper, we have proposed holographic descriptions of very special $T\bar{J}$ deformations of conformal field theories. The very special $T\bar{J}$ deformation is peculiar: It preserves "twisted" Poincaré symmetry as well as dilatation, but it only has chiral conformal transformation. Typically, unitary scale invariant relativistic field theories have the full conformal symmetry [54,55], but this is avoided because the existence of a noncompact symmetry and use of the topological twist resulted in nonunitary quantum field theories.

Our holographic construction is classical, and it is an interesting question if the gravity side can be quantized, e.g., in the full string setup. One of our formulations for the double trace $T\bar{J}$ deformations is based on the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory, and without our complicated boundary conditions, the quantization may be straightforward. However, whether quantum gravity makes sense, in particular, with our mixed boundary conditions is a different story, and it deserves a further study. With this regard, in our discussions, we have restricted ourselves to the parity preserving case with the same Chern-Simons levels for the gravity part, but we may choose three levels of the Chern-Simons theory completely differently. Such choices may be fine for a trivial background source, but they may cause anomalies in the general background.

As for the single trace deformation, the resultant deformed background is similar to the one studied in the $T\bar{J}$ deformations; however, in our case, the internal space is also noncompact, so the further Kaluza-Klein reduction to obtain the warped AdS space-time is not possible. Also, one of our explicit examples has two "timelike" directions in the target space, and the physical significance should be carefully studied beyond the formal analysis done in this paper.

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