Lagrangian formulation for electric charge in a magnetic monopole distribution

G. Marmo,[1,*] Emanuela Scardapane,[1,†] A. Stern,[2,‡] Franco Ventriglia,[1,3,§] and Patrizia Vitale[1,3,∥]

[1]*Dipartimento di Fisica E. Pancini Università di Napoli Federico II, Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Naples, Italy*

[2]*Department of Physics, University of Alabama, Tuscaloosa, Alabama 35487, USA*

[3]*INFN Sez. di Napoli Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Naples, Italy*

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We give a Lagrangian description of an electric charge in a field sourced by a continuous magnetic monopole distribution. The description is made possible thanks to a doubling of the configuration space. The Legendre transform of the nonrelativistic Lagrangian agrees with the Hamiltonian description given recently by Kupriyanov and Szabo [Phys. Rev. D **98**, 045005 (2018)]. The covariant relativistic version of the Lagrangian is shown to introduce a new gauge symmetry, in addition to standard reparametrizations. The generalization of the system to open strings coupled to a magnetic monopole distribution is also given, as is the generalization to particles in a non-Abelian gauge field which does not satisfy Bianchi identities in some region of the space-time.

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I. INTRODUCTION

It is well known that a local Lagrangian description for an electric charge in the presence of fields sourced by an electric charge distribution requires the introduction of potentials on the configuration space, introducing unphysical, or gauge, degrees of freedom (d.o.f.) in the field theory. If the field is sourced by a magnetic monopole, the description can be modified by changing the topology of the underlying configuration space; see, e.g., Refs. [1,2]. On the other hand, this procedure has no obvious extension when the fields are sourced by a continuous distribution of magnetic charge. In that case, auxiliary d.o.f. can be added, possibly introducing additional local symmetries. One possibility is to introduce another set of potentials following work of Zwanziger [3]. Another approach is to enlarge the phase space for the electric charge, and this was done recently by Kupriyanov and Szabo [4]. The result has implications for certain nongeometric string theories and their quantization, which leads to nonassociative algebras; see, e.g., Refs. [5–13].

The analysis of Ref. [4] for the electric charge in a field sourced by magnetic monopole distribution is performed in the Hamiltonian setting. The formulation is made possible thanks to the doubling of the number of phase space variables.[1] In this paper, we give the corresponding Lagrangian description. It naturally requires doubling the number of configuration space variables. So, here, if Q denotes the original configuration space, one introduces another copy, $\tilde{Q}$, and writes down dynamics on $Q \times \tilde{Q}$. While the motion on the two spaces, in general, cannot be separated, the Lorentz force equations are recovered when projecting down to Q. The procedure of doubling the configuration space has a wide range of applications and actually was used long ago in the description of quantum dissipative systems [16–20]. The description in Ref. [4] is nonrelativistic. Here, in addition to giving the associated nonrelativistic Lagrangian, we extend the procedure to the case of a covariant relativistic particle as well as to particles coupled to non-Abelian gauge fields that do not necessarily satisfy the Bianchi identity in a region of space-time. For a further generalization, we consider the case of an open string coupled to a smooth distribution of magnetic monopoles.

The outline of this article is as follows. In Sec. II, we write down the Lagrangian for a nonrelativistic charged particle in the presence of a magnetic field of which the

*marmo@na.infn.it
†emanuela.scardapane@gmail.com
‡astern@ua.edu
§ventri@na.infn.it
∥patrizia.vitale@na.infn.it

[1]The introduction of auxiliary d.o.f. to the phase space has been used in the past for a number of different purposes. For example, in Refs. [14,15], it was found helpful for handling systems of second-class constraints.

divergence field is continuous and nonvanishing in a finite volume of space and show that the corresponding Hamiltonian description is that of Ref. [4]. The relativistic generalization is given in Sec. III. Starting with a fully covariant treatment, we obtain a new time-dependent symmetry, in addition to standard reparametrization invariance. The new gauge symmetry mixes $\tilde{Q}$ with Q. Gauge-fixing constraints can be imposed on the phase space in order to recover the Poisson structure of the nonrelativistic treatment on the resulting constrained submanifold. Further extensions of the system are considered in Sec. IV. In Sec. IV A, we write down the action for a particle coupled to a non-Abelian gauge field which does not satisfy Bianchi identity in some region of space-time, whereas in Sec. IV B, we generalize to field theory, by considering an open string coupled to a magnetic monopole distribution, again violating Bianchi identity. In both cases, we get a doubling of the configuration space variables (which in the case of the particle in a non-Abelian gauge field includes variables living in an internal space) as well as a doubling of the number of gauge symmetries. We note that the doubling of the number of world sheet d.o.f. of the string is also the starting point of Double Field Theory, introduced by Hull and Zwiebach [21] and further investigated by many authors [22–27], in order to deal with the T-duality invariance of the strings dynamics. This has its geometric counterpart in generalized and double geometry (see, e.g., Refs. [28,29] and [30–34], respectively). Moreover, the doubling of configuration space has also been related to Drinfel'd doubles in the context of Lie groups dynamics [35–39] with interesting implications for the mathematical and physical interpretation of the auxiliary variables.

II. NONRELATIVISTIC TREATMENT

We begin with a nonrelativistic charged particle on $\mathbb{R}^3$ in the presence of a continuous magnetic monopole distribution. Say that the particle has mass m and charge e with coordinates and velocities $(x_i, \dot{x}_i)$ spanning $T\mathbb{R}^3$. It interacts with a magnetic field $\vec{B}(x)$ of nonvanishing divergence $\vec{\nabla}\cdot\vec{B}(x) = \rho_M(x)$. In such a case, it is possible to show that the dynamics of the particle, described by the equations of motion

$$m\ddot{x}_i = e\epsilon_{ijk}\dot{x}_j B_k(x), \tag{2.1}$$

cannot be given by a Lagrangian formulation on the tangent space $T\mathbb{R}^3$ because a vector potential for the magnetic field generated by the smooth monopoles distribution cannot be defined, even locally. (A detailed discussion of this issue will appear in Ref. [40].) On the other hand, a Lagrangian description is possible if one enlarges the configuration space to $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$, and this description leads to Kupriyanov and Szabo's Hamiltonian formulation [4]. For this, one extends the tangent space to $T(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) \simeq T\mathbb{R}^3 \times T\tilde{\mathbb{R}}^3$. We parametrize $T\tilde{\mathbb{R}}^3$ by $(\tilde{x}_i, \dot{\tilde{x}}_i)$, $i = 1, 2, 3$. A straightforward calculation shows that the Lagrangian function

$$L = m\dot{x}_i\dot{\tilde{x}}_i + e\epsilon_{ijk}B_k(x)\tilde{x}_i\dot{x}_j \tag{2.2}$$

correctly reproduces Eq. (2.1), together with an equation of motion for the auxiliary d.o.f. $\tilde{x}_i$,

$$m\ddot{\tilde{x}}_i = e\epsilon_{ijk}\dot{\tilde{x}}_j B_k(x) + e\left(\epsilon_{jk\ell}\frac{\partial}{\partial x_i}B_k - \epsilon_{ik\ell}\frac{\partial}{\partial x_j}B_k\right)\dot{x}_j\tilde{x}_\ell, \tag{2.3}$$

which are not decoupled from the motion of the physical d.o.f. Here, we do not ascribe any physical significance to the auxiliary dynamics. There are analogous d.o.f. for dissipative systems, and they are associated with the environment. Since our system does not dissipate energy, the same interpretation does not obviously follow. The Lagrangian (2.2) can easily be extended to include electric fields. This, along with the relativistic generalization, is done in the following section.

In passing to the Hamiltonian formalism, we denote the momenta conjugate to x_i and $\tilde{x}_i$ by

$$p_i = m\dot{\tilde{x}}_i - e\epsilon_{ijk}\tilde{x}_j B_k(x)$$
$$\tilde{p}_i = m\dot{x}_i, \tag{2.4}$$

respectively. Along with x_i and $\tilde{x}_i$, they span the 12-dimensional phase space $T^*(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. The nonvanishing Poisson brackets are

$$\{x_i, p_j\} = \{\tilde{x}_i, \tilde{p}_j\} = \delta_{ij}. \tag{2.5}$$

Instead of the canonical momenta (2.4), one can define

$$\pi_i = p_i + e\epsilon_{ijk}\tilde{x}_j B_k(x) \qquad \tilde{\pi}_i = \tilde{p}_i, \tag{2.6}$$

which have the nonvanishing Poisson brackets:

$$\{x_i, \pi_j\} = \{\tilde{x}_i, \tilde{\pi}_j\} = \delta_{ij}$$
$$\{\pi_i, \tilde{\pi}_j\} = e\epsilon_{ijk}B_k$$
$$\{\pi_i, \pi_j\} = e\left(\epsilon_{jk\ell}\frac{\partial}{\partial x_i}B_k - \epsilon_{ik\ell}\frac{\partial}{\partial x_j}B_k\right)\tilde{x}_\ell. \tag{2.7}$$

The Hamiltonian when expressed in these variables is

$$H = \frac{1}{m}\tilde{\pi}_i\pi_i. \tag{2.8}$$

Equations (2.7) and (2.8) are in agreement with the Hamiltonian formulation in Ref. [4].

Concerning the issue of the lack of a lower bound for H, one can follow the perspective in Ref. [41], in which a very similar Hamiltonian dynamics is derived. Namely, while it is true that H generates temporal evolution, it cannot be regarded as a classical observable of the particle. Rather, such observables should be functions of only the particle's coordinates x_i and its velocities $\tilde{\pi}_i/m$, the dynamics of which is obtained from their Poisson brackets with H,

$$\dot{x}_i = \{x_i, H\} = \frac{1}{m}\tilde{\pi}_i$$
$$\dot{\tilde{\pi}}_i = \{\tilde{\pi}_i, H\} = \frac{e}{m}\epsilon_{ijk}\tilde{\pi}_j B_k. \qquad (2.9)$$

The usual expression for the energy, $\frac{1}{2m}\tilde{\pi}_i\tilde{\pi}_i$, is, of course, an observable, which is positive definite and a constant of motion.

III. RELATIVISTIC COVARIANT TREATMENT

The extension of the Lagrangian dynamics of the previous section can straightforwardly be made to a covariant relativistic system. In the usual treatment of a covariant relativistic particle, written on $T\mathbb{R}^4$, one obtains a first-class constraint in the Hamiltonian formulation, which generates reparametrizations. Here, we find that the relativistic action for a charged particle in a continuous magnetic monopole distribution, which is now written on $T\mathbb{R}^4 \times \widetilde{T\mathbb{R}^4}$, yields an additional first-class constraint, generating a new gauge symmetry. When projecting the Hamiltonian dynamics onto the constrained submanifold of the phase space, and taking the nonrelativistic limit, we recover the Hamiltonian description of Ref. [4].

As stated above, our action for the charged particle in a continuous magnetic monopole distribution is written on $T\mathbb{R}^4 \times \widetilde{T\mathbb{R}^4}$. Let us parametrize $T\mathbb{R}^4$ by space-time coordinates and velocity 4-vectors $(x^\mu, \dot{x}^\mu)$ and $\widetilde{T\mathbb{R}^4}$ by $(\tilde{x}^\mu, \dot{\tilde{x}}^\mu)$, $\mu = 0, 1, 2, 3$. So, here, we have included two "time" coordinates, x^0 and $\tilde{x}^0$. Now, the dot denotes the derivative with respect to some variable τ which parametrizes the particle world line in $\mathbb{R}^4 \times \widetilde{\mathbb{R}^4}$. The action for a charged particle in an electromagnetic field $F_{\mu\nu}(x)$, which does *not* in general satisfy the Bianchi identity $\frac{\partial}{\partial x^\mu}F_{\nu\rho} + \frac{\partial}{\partial x^\nu}F_{\rho\mu} + \frac{\partial}{\partial x^\rho}F_{\mu\nu} = 0$ is[2]

$$S = \int d\tau \left\{ m\frac{\dot{x}_\mu \dot{\tilde{x}}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} + eF_{\mu\nu}(x)\tilde{x}^\mu \dot{x}^\nu + a\sqrt{-\dot{x}^\nu \dot{x}_\nu} \right\}, \qquad (3.1)$$

where a is an arbitrary constant. Note that dynamics on the physical subspace $\mathbb{R}^4$ results from variations of S with respect to $\tilde{x}^\mu$. Therefore, the equations of motion on the physical subspace $\mathbb{R}^4$ are unaffected by the presence of the third term in the Lagrangian (although the dynamics on $\mathbb{R}^4 \times \widetilde{\mathbb{R}^4}$ gets modified by this term). Indices $\mu, \nu, \ldots$ are raised and lowered with the Lorentz metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

The action is invariant under Lorentz transformations and arbitrary reparametrizations of τ, $\tau \to \tau' = f(\tau)$. The action is also invariant under a local transformation that mixes $\tilde{\mathbb{R}}^4$ with $\mathbb{R}^4$,

$$x^\mu \to x^\mu \qquad \tilde{x}^\mu \to \tilde{x}^\mu + \frac{\epsilon(\tau)\dot{x}^\mu}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}}, \qquad (3.2)$$

for an arbitrary real function $\epsilon(\tau)$. The first term in the integrand of (3.1) changes by a τ derivative under (3.2), while the remaining terms in the integrand are invariant. The existence of such an additional gauge symmetry was to be expected, since it, along with the reparametrization symmetry, reduces the total number of gauge-invariant d.o.f. to 6, in agreement with the nonrelativistic system discussed in the previous section. The gauge symmetry (3.2) means that there will be a corresponding first-class constraint in the Hamiltonian description of the theory, as we shall discuss. For the case of rigid transformations (3.2), i.e., where ϵ is independent of τ, the associated Nöther charge is the mass m.

Upon extremizing the action with respect to arbitrary variations $\delta\tilde{x}^\mu$ of $\tilde{x}^\mu$, we recover the standard Lorentz force equation on $T\mathbb{R}^4$,

$$\dot{\tilde{p}}_\mu = eF_{\mu\nu}(x)\dot{x}^\nu, \qquad (3.3)$$

while arbitrary variations δx^μ of x^μ lead to

$$\dot{p}_\mu = e\frac{\partial F_{\rho\sigma}}{\partial x^\mu}\tilde{x}^\rho \dot{x}^\sigma. \qquad (3.4)$$

p_μ and $\tilde{p}_\mu$ are the momenta canonically conjugate to x^μ and $\tilde{x}^\mu$, respectively,

$$p_\mu = \frac{m}{(-\dot{x}^\rho \dot{x}_\rho)^{3/2}}(\dot{x}_\mu \dot{\tilde{x}}_\nu - \dot{x}_\nu \dot{\tilde{x}}_\mu)\dot{x}^\nu - eF_{\mu\nu}\tilde{x}^\nu - \frac{a\dot{x}_\mu}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}}$$
$$\tilde{p}_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}}. \qquad (3.5)$$

The momenta p_μ and $\tilde{p}_\mu$, along with coordinates x^μ and $\tilde{x}^\mu$, parametrize a 16-dimensional phase space, which we denote simply by T^*Q. x^μ, $\tilde{x}^\mu$, p_μ, and $\tilde{p}_\mu$ satisfy canonical Poisson brackets relations, the nonvanishing ones being

$$\{x^\mu, p_\nu\} = \{\tilde{x}^\mu, \tilde{p}_\nu\} = \delta^\mu_\nu. \qquad (3.6)$$

[2] If one prefers not to deal with square roots in the action, one can introduce yet another auxiliary d.o.f., or einbein, $E(\tau)$, and replace the first term in the integrand of (3.1) by $\frac{1}{2}m\dot{x}_\mu\dot{\tilde{x}}^\mu(\frac{1}{E} - \frac{E}{(\dot{x}^\nu \dot{x}_\nu)})$, The action (3.1) can then be recovered upon eliminating E via its equation of motion.

$\tilde{p}_\mu$ satisfies the usual mass shell constraint

$$\Phi_1 = \tilde{p}_\mu \tilde{p}^\mu + m^2 \approx 0, \quad (3.7)$$

where $\approx$ means "weakly" zero in the sense of Dirac. Another constraint is

$$\Phi_2 = p_\mu \tilde{p}^\mu + eF_{\mu\nu}(x)\tilde{p}^\mu \tilde{x}^\nu \approx 0, \quad (3.8)$$

where from now on we set the arbitrary coefficient a equal to zero.

The 3-momenta π_i and $\tilde{\pi}_i$ of the previous section can easily be generalized to 4-vectors according to

$$\pi_\mu = p_\mu + eF_{\mu\nu}(x)\tilde{x}^\nu \qquad \tilde{\pi}_\mu = \tilde{p}_\mu. \quad (3.9)$$

Their nonvanishing Poisson brackets are

$$\begin{aligned} &\{x^\mu, \pi_\nu\} = \{\tilde{x}^\mu, \tilde{\pi}_\nu\} = \delta^\mu_\nu \\ &\{\pi_\mu, \tilde{\pi}_\nu\} = eF_{\mu\nu} \\ &\{\pi_\mu, \pi_\nu\} = -e\left(\frac{\partial}{\partial x^\mu}F_{\nu\rho} + \frac{\partial}{\partial x^\nu}F_{\rho\mu}\right)\tilde{x}^\rho. \end{aligned} \quad (3.10)$$

Then, the constraints (3.7) and (3.8) take the simple form

$$\Phi_1 = \tilde{\pi}_\mu \tilde{\pi}^\mu + m^2 \approx 0 \qquad \Phi_2 = \pi_\mu \tilde{\pi}^\mu \approx 0. \quad (3.11)$$

From (3.10), one has $\{\Phi_1, \Phi_2\} = 0$, and therefore Φ_1 and Φ_2 form a first-class set of constraints. They generate the two gauge (i.e., τ-dependent) transformations on T^*Q. Unlike in the standard covariant treatment of a relativistic particle, the mass shell constraint Φ_1 does not generate reparametrizations. Φ_1 instead generates the transformations (3.2), while a linear combination of Φ_1 and Φ_2 generates reparametrizations. After imposing (3.7) and (3.8) on T^*Q, one ends up with a gauge-invariant subspace that is 12 dimensional, which is in agreement with the dimensionality of the nonrelativistic phase space.

Alternatively, one can introduce two additional constraints on T^*Q which fix the two time coordinates x^0 and $\tilde{x}^0$ and thus break the gauge symmetries. The set of all four constraints would then form a second-class set, again yielding a 12-dimensional reduced phase space, which we denote by $\overline{T^*Q}$. The dynamics on the reduced phase space is then determined from Dirac brackets and some Hamiltonian H. We choose H to be

$$H = p_0 = \pi_0 - eF_{0i}(x)\tilde{x}^i. \quad (3.12)$$

p_0 differs from π_0 in the presence of an electric field. The latter can be expressed as a function of the spatial momenta π_i and $\tilde{\pi}_i$, $i = 1, 2, 3$, after solving the constraints (3.11). The result is

$$\pi_0 = \frac{\pi_i \tilde{\pi}_i}{\sqrt{\tilde{\pi}_j^2 + m^2}}. \quad (3.13)$$

π_0 correctly reduces to the nonrelativistic Hamiltonian (2.8) in the limit $\tilde{\pi}_j^2 \ll m^2$.

In addition to recovering the nonrelativistic Hamiltonian of the previous section, the gauge-fixing constraints, which we denote by $\Phi_3 \approx 0$ and $\Phi_4 \approx 0$, can be chosen such that the Dirac brackets on $\overline{T^*Q}$ agree with the Poisson brackets (2.7) of the nonrelativistic treatment. For this, take

$$\Phi_3 = x^0 - g(\tau) \qquad \Phi_4 = \tilde{x}^0 - h(\tau), \quad (3.14)$$

where g and h are unspecified functions of the proper time. By definition, the Dirac brackets between two functions A and B of the phase space coordinates are given by

$$\{A, B\}_{\mathrm{DB}} = \{A, B\} - \sum_{a,b=1}^{4} \{A, \Phi_a\} M^{-1}_{ab} \{\Phi_b, B\}, \quad (3.15)$$

where M^{-1} is the inverse of the matrix M with elements $M_{ab} = \{\Phi_a, \Phi_b\}, a, b = 1, \ldots, 4.$ From the constraints (3.11) and (3.14), we get

$$M^{-1} = \frac{1}{2(\tilde{\pi}^0)^2}\begin{pmatrix} 0 & 0 & -\pi^0 & \tilde{\pi}^0 \\ 0 & 0 & 2\tilde{\pi}^0 & 0 \\ \pi^0 & -2\tilde{\pi}^0 & 0 & 0 \\ -\tilde{\pi}^0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.16)$$

Substituting into (3.15) gives

$$\begin{aligned} \{A, B\}_{\mathrm{DB}} = \{A, B\} - \frac{1}{2(\tilde{\pi}^0)^2} &(\pi^0(\{A, x^0\}\{\tilde{\pi}_\mu \tilde{\pi}^\mu, B\} \\ &- \{B, x^0\}\{\tilde{\pi}_\mu \tilde{\pi}^\mu, A\}) \\ &- \tilde{\pi}^0(\{A, \tilde{x}^0\}\{\tilde{\pi}_\mu \tilde{\pi}^\mu, B\} - \{B, \tilde{x}^0\}\{\tilde{\pi}_\mu \tilde{\pi}^\mu, A\}) \\ &- 2\tilde{\pi}^0(\{A, x^0\}\{\tilde{\pi}_\mu \pi^\mu, B\} - \{B, x^0\}\{\tilde{\pi}_\mu \pi^\mu, A\})). \end{aligned} \quad (3.17)$$

It shows that the Dirac brackets $\{A, B\}_{\mathrm{DB}}$ and their corresponding Poisson brackets $\{A, B\}$ are equal if both functions A and B are independent of π^0 and $\tilde{\pi}^0$. We need to evaluate the Dirac brackets on the constrained subsurface, which we take to be $T\mathbb{R}^3 \times \widetilde{T\mathbb{R}^3}$, parametrized by x_i, $\tilde{x}_i$, π_i, and $\tilde{\pi}_i$, $i = 1, 2, 3$. It is then sufficient to compute their Poisson brackets. The nonvanishing Poisson brackets of the coordinates of $T\mathbb{R}^3 \times \widetilde{T\mathbb{R}^3}$ are

$$\{x_i, \pi_j\} = \{\tilde{x}_i, \tilde{\pi}_j\} = \delta_{ij}$$
$$\{\pi_i, \tilde{\pi}_j\} = e\epsilon_{ijk} B_k$$
$$\{\pi_i, \pi_j\} = e\left(\epsilon_{jk\ell}\frac{\partial}{\partial x_i} B_k - \epsilon_{ik\ell}\frac{\partial}{\partial x_j} B_k\right)\tilde{x}_\ell + e\left(\frac{\partial}{\partial x_i}E_j - \frac{\partial}{\partial x_j}E_i\right)h(\tau), \tag{3.18}$$

where $F_{ij} = \epsilon_{ijk}B_k$, $F_{0i} = E_i$, and we have imposed the constraint $\Phi_4 = 0$. These Poisson brackets agree with those of the nonrelativistic treatment (2.7) in the absence of the electric field.

IV. FURTHER EXTENSIONS

Here, we extend the dynamics of the previous sections to 1) the case of a particle coupled to a non-Abelian gauge field violating Bianchi identities and 2) the case of an open string coupled to a smooth distribution of magnetic monopoles. Of course, another extension would be the combination of both of these two cases, i.e., where an open string interacts with a non-Abelian gauge field that does not satisfy the Bianchi identities in some region of the space-time. We shall not consider that here.

A. Particle in a non-Abelian magnetic monopole distribution

Here, we replace the underlying Abelian gauge group of the previous sections, with an N-dimensional non-Abelian Lie group G. We take it to be compact and connected with a simple Lie algebra. Given a unitary representation Γ of G, let t_A, $A = 1, 2, \ldots N$ span the corresponding representation $\bar{\Gamma}$ of the Lie algebra, satisfying $t_A^\dagger = t_A$, $\mathrm{Tr} t_A t_B = \delta_{AB}$, and $[t_A, t_B] = ic_{ABC}t_C$, c_{ABC} being totally antisymmetric structure constants. In Yang-Mills field theory, the field strengths now take values in $\bar{\Gamma}$, $F_{\mu\nu}(x) = f^A_{\mu\nu}(x)t_A$. A particle interacting with a Yang-Mills field carries d.o.f. $I(\tau)$ associated with the non-Abelian charge, in addition to space-time coordinates $x^\mu(\tau)$. These new d.o.f. live in the internal space $\bar{\Gamma}$, $I(\tau) = I^A(\tau)t_A$. Under gauge transformations, $I(\tau)$ transforms as a vector in the adjoint representation of G, just as the field strengths $F_{\mu\nu}(x)$ do, i.e., $I(\tau) \to h(\tau)I(\tau)h(\tau)^\dagger$, $h(\tau) \in \Gamma$.

The standard equations of motion for a particle in a non-Abelian gauge field were given long ago by Wong [42]. They consist of two sets of coupled equations. One set is a straightforward generalization of the Lorentz force law

$$\dot{\tilde{p}}_\mu = \mathrm{Tr}(F_{\mu\nu}(x)I(\tau))\dot{x}^\nu, \tag{4.1}$$

where $\tilde{p}_\mu$ is again given in (3.5). The other set consists of first-order equations describing the precession of $I(\tau)$ in the internal space $\bar{\Gamma}$. Yang-Mills potentials are required in order to write these equations in a gauge-covariant way.

The Wong equations were derived from action principles using a number of different approaches. The Yang-Mills potentials again play a vital role in all of the Lagrangian descriptions. In the approach of coadjoint orbits, one takes the configuration space to be $Q = \mathbb{R}^4 \times \Gamma$ and writes [2,43]

$$I(\tau) = g(\tau)Kg(\tau)^\dagger, \tag{4.2}$$

where $g(\tau)$ takes values in Γ and K is a fixed direction in $\bar{\Gamma}$. Under gauge transformations, $g(\tau)$ transforms with the left action of the group, $g(\tau) \to h(\tau)g(\tau)$, $h(\tau) \in \Gamma$. The two sets of Wong equations result from variations of the action with respect to $g(\tau)$ and $x^\mu(\tau)$.

Now, in the spirit of Ref. [4], we imagine that there is a region of space-time where the Bianchi identity does not hold, and so the usual expression for the field strengths in terms of the Yang-Mills potentials is not valid. So, we cannot utilize the known actions which yield Wong's equations, as they require the existence of the potentials. We can instead try a generalization of (3.1), which doubles the number of space-time coordinates. This appears, however, to be insufficient. To have a gauge-invariant description for the particle, we claim that it is necessary to double the number of internal variables as well. Thus, we double the entire configuration space, $Q \to Q \times \tilde{Q}$. Proceeding along the lines of the coadjoint orbits approach, we take $\tilde{Q}$ to be another copy of $\mathbb{R}^4 \times \Gamma$. Let us denote all the dynamical variables in this case to be $x^\mu(\tau)$, $\tilde{x}^\mu(\tau)$, $g(\tau)$, and $\tilde{g}(\tau)$, where both $g(\tau)$ and $\tilde{g}(\tau)$ take values in Γ and gauge transformation with the left action of the group, $g(\tau) \to h(\tau)g(\tau)$, $\tilde{g}(\tau) \to h(\tau)\tilde{g}(\tau)$, $h(\tau) \in \Gamma$.

We now propose the gauge-invariant action for the particle

$$S = \int d\tau\Big\{\mathrm{Tr}Kg(\tau)^\dagger\dot{g}(\tau) - \mathrm{Tr}I(\tau)\dot{\tilde{g}}(\tau)\tilde{g}(\tau)^\dagger + m\frac{\dot{x}_\mu\dot{\tilde{x}}^\mu}{\sqrt{-\dot{x}^\nu\dot{x}_\nu}} + \mathrm{Tr}(F_{\mu\nu}(x)I(\tau))\tilde{x}^\mu\dot{x}^\nu\Big\}, \tag{4.3}$$

where $I(\tau)$ is defined in (4.2). To see that the action is gauge invariant, we note that the first two terms in the integrand can be combined as $\mathrm{Tr}Kg(\tau)^\dagger\tilde{g}(\tau)\frac{d}{d\tau}(\tilde{g}(\tau)^\dagger g(\tau))$, $\tilde{g}(\tau)^\dagger g(\tau)$ being gauge invariant. Variations of $\tilde{x}^\mu$ in the action yield the Wong equation (4.1). Variations of x^μ in the action give a new set of equations defining motion on the enlarged configuration space

$$\dot{p}_\mu = \mathrm{Tr}\left(\frac{\partial F_{\rho\sigma}}{\partial x^\mu}I(\tau)\right)\tilde{x}^\rho\dot{x}^\sigma,$$
where $$p_\mu = \frac{m}{(-\dot{x}^\rho\dot{x}_\rho)^{3/2}}(\dot{x}_\mu\dot{\tilde{x}}_\nu - \dot{x}_\nu\dot{\tilde{x}}_\mu)\dot{x}^\nu - \mathrm{Tr}(F_{\mu\nu}I(\tau))\tilde{x}^\nu. \tag{4.4}$$

These equations are the non-Abelian analogs of (3.4). The remaining equations of motion result from variations of the

$g(\tau)$ and $\tilde{g}(\tau)$ and describe motion in $\Gamma \times \Gamma$. Infinitesimal variations of $g(\tau)$ and $\tilde{g}(\tau)$ may be performed as follows: for $\tilde{g}(\tau)$, it is simpler to consider variations resulting from the right action on the group, $\delta\tilde{g}(\tau) = i\tilde{g}(\tau)\tilde{\epsilon}(\tau)$, $\tilde{\epsilon}(\tau) \in \bar{\Gamma}$. The action (4.3) is stationary with respect to these variations when

$$\frac{d}{d\tau}(\tilde{g}(\tau)I(\tau)\tilde{g}(\tau)^\dagger) = 0, \tag{4.5}$$

thus stating that $\tilde{g}(\tau)I(\tau)\tilde{g}(\tau)^\dagger$ is a constant of the motion. For $g(\tau)$, consider variations resulting from the left action on the group, $\delta g(\tau) = i\epsilon(\tau)g(\tau)$, $\epsilon(\tau) \in \bar{\Gamma}$. These variations lead to the equations of motion

$$\dot{I}(\tau) = [I(\tau), \dot{\tilde{g}}(\tau)\tilde{g}(\tau)^\dagger - F_{\mu\nu}(x)\tilde{x}^\mu \dot{x}^\nu]. \tag{4.6}$$

The consistency of both (4.5) and (4.6) leads to the following constraint on the motion:

$$[I(\tau), F_{\mu\nu}(x)]\tilde{x}^\mu \dot{x}^\nu = 0. \tag{4.7}$$

This condition on $TQ \times T\tilde{Q}$ is a feature of the non-Abelian gauge theory and is absent from the Abelian gauge theory.

B. Open string coupled to a magnetic monopole distribution

Finally, we generalize the case of a particle interacting with a smooth magnetic monopole distribution to that of a string interacting with the same monopole distribution. Just as we doubled the number of particle coordinates in the previous sections, we now double the number of string coordinates. We note that a doubling of the world sheet coordinates of the string, originally limited to the compactified coordinates, also occurs in the context of Double Field Theory [22], with the original purpose of making the invariance of the dynamics under T duality a manifest symmetry of the action. The approach has been further extended to strings propagating in so called nongeometric backgrounds [11,12,44,45], which leads to quasi-Posson brackets, violating the Jacobi identity. The resolution involves a doubling of the world sheet coordinates, similar to what happens in the case under study.

Whereas the configuration space for a Nambu-Goto string moving in d dimensions is $\mathbb{R}^d$, which can have an indefinite signature, here we take it to be $\mathbb{R}^d \times \widetilde{\mathbb{R}^d}$. Denote the string coordinates for $\mathbb{R}^d$ and $\widetilde{\mathbb{R}^d}$ by $x^\mu(\sigma)$ and $\tilde{x}^\mu(\sigma)$, $\mu = 0, 1, \ldots, d-1$, respectively, where $\sigma = (\sigma^0, \sigma^1)$ parametrizes the string world sheet, $\mathcal{M}$. σ^0 is assumed to be a timelike parameter, and σ^1 is assumed to be a spatial parameter. In addition to writing down the induced metric g on $T\mathbb{R}^d$,

$$g_{\mathsf{ab}} = \partial_{\mathsf{a}} x^\mu \partial_{\mathsf{b}} x_\mu, \tag{4.8}$$

where $\partial_{\mathsf{a}} = \frac{\partial}{\partial\sigma^{\mathsf{a}}}$, $\mathsf{a}, \mathsf{b}, \ldots = 0, 1$, we define a nonsymmetric matrix $\tilde{g}$ on $T\mathbb{R}^d \times \widetilde{T\mathbb{R}^d}$,

$$\tilde{g}_{\mathsf{ab}} = \partial_{\mathsf{a}} x^\mu \partial_{\mathsf{b}} \tilde{x}_\mu. \tag{4.9}$$

For the free string action, we propose replacing the usual Nambu-Goto action by

$$S_0 = \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \sqrt{-\det g}\, g^{\mathsf{ab}}\tilde{g}_{\mathsf{ab}}, \tag{4.10}$$

where g^{ab} denote matrix elements of g^{-1} and α' is the string constant.

The action (4.10), together with the interaction term given below, is a natural generalization of the point-particle action Eq. (3.1) because:

(i) Just as with the case of the relativistic point-particle action in Sec. III, it is relativistically covariant.

(ii) Just as with the case of the relativistic point-particle action in Sec. III, there is a new gauge symmetry, in addition to reparametrizations, $\sigma^{\mathsf{a}} \to \sigma'^{\mathsf{a}} = f^{\mathsf{a}}(\sigma)$, leading to new first-class constraints in the Hamiltonian formalism. This new gauge symmetry mixes $\tilde{\mathbb{R}}^d$ with $\mathbb{R}^d$. Infinitesimal variations are given by

$$\delta x^\mu = 0 \qquad \delta\tilde{x}^\mu = \frac{\epsilon^{\mathsf{a}}(\sigma)\partial_{\mathsf{a}} x^\mu}{\sqrt{-\det g}}, \tag{4.11}$$

where $\epsilon^{\mathsf{a}}(\sigma)$ are arbitrary functions of σ, which we assume vanish at the string boundaries. This is the natural generalization of the τ-dependent symmetry transformation (3.2) for the relativistic point particle. Invariance of S_0 under variations (4.11) follows from

$$\begin{aligned}
\delta S_0 &= \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \sqrt{-\det g}\, g^{\mathsf{ab}}\partial_{\mathsf{a}} x_\mu \partial_{\mathsf{b}}\left(\frac{\epsilon^{\mathsf{c}}\partial_{\mathsf{c}} x^\mu}{\sqrt{-\det g}}\right) \\
&= \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\, g^{\mathsf{ab}}\left(g_{\mathsf{ac}}\partial_{\mathsf{b}}\epsilon^{\mathsf{c}} + \partial_{\mathsf{a}} x_\mu \partial_{\mathsf{b}}\partial_{\mathsf{c}} x^\mu \epsilon^{\mathsf{c}} - \frac{\partial_{\mathsf{b}}\det g}{2\det g} g_{\mathsf{ac}}\epsilon^{\mathsf{c}}\right) \\
&= \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\left(\partial_{\mathsf{c}}\epsilon^{\mathsf{c}} + g^{\mathsf{ab}}\left(\partial_{\mathsf{a}} x_\mu \partial_{\mathsf{b}}\partial_{\mathsf{c}} x^\mu - \frac{1}{2}\partial_{\mathsf{c}} g_{\mathsf{ab}}\right)\epsilon^{\mathsf{c}}\right) \\
&= \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\sigma^{\mathsf{a}}\epsilon_{\mathsf{a}},
\end{aligned} \tag{4.12}$$

which vanishes upon requiring $\epsilon_{\mathsf{a}}|_{\partial\mathcal{M}} = 0$.

(iii) The action (4.10) leads to the standard string dynamics when projecting the equations of motion to $\mathbb{R}^d$. Excluding for the moment interactions, variations of the action S_0 with respect to $\tilde{x}^\mu(\sigma)$ away from the boundary $\partial\mathcal{M}$ give the equations of motion

$$\partial_{\mathsf{a}}\tilde{p}^{\mathsf{a}}_\mu = 0, \quad \tilde{p}^{\mathsf{a}}_\mu = \frac{1}{2\pi\alpha'}\sqrt{-\det g}\, g^{\mathsf{ab}}\partial_{\mathsf{b}}x_\mu. \tag{4.13}$$

These are the equations of motion for a Nambu string. In addition to recovering the usual string equations on $\mathbb{R}^d$, variations of S_0 with respect to $x^\mu(\sigma)$ lead to another set of the equation of motion on $\mathbb{R}^d \times \tilde{\mathbb{R}}^d$:

$$\begin{aligned}&\partial_{\mathsf{a}}p^{\mathsf{a}}_\mu = 0,\\ &\quad p^{\mathsf{a}}_\mu = \frac{1}{2\pi\alpha'}\sqrt{-\det g}\{(g^{\mathsf{ab}}g^{\mathsf{cd}} - g^{\mathsf{ad}}g^{\mathsf{bc}} - g^{\mathsf{ac}}g^{\mathsf{bd}})\\ &\qquad \times \tilde{g}_{\mathsf{cd}}\partial_{\mathsf{b}}x_\mu + g^{\mathsf{ab}}\partial_{\mathsf{b}}\tilde{x}_\mu\}.\end{aligned} \tag{4.14}$$

Of course, Eq. (4.10) can be used for both a closed string and an open string. We now include interactions with the electromagnetic field. They occur at the boundaries of an open string and are standardly expressed in terms of the electromagnetic potential, which again is not possible in the presence of a continuous magnetic monopole charge distribution. So, here, we take instead

$$S_I = e\int_{\partial\mathcal{M}} d\sigma^{\mathsf{a}} F_{\mu\nu}(x)\tilde{x}^\mu\partial_{\mathsf{a}}x^\nu, \tag{4.15}$$

where $F_{\mu\nu}(x)$ is not required to satisfy the Bianchi identity in a finite volume of $\mathbb{R}^d$. We take $-\infty < \sigma^0 < \infty$, $0 < \sigma^1 < \pi$, with $\sigma^1 = 0, \pi$ denoting the spatial boundaries of the string. Then, the boundary equations of motion resulting from variations of $\tilde{x}^\mu(\sigma)$ in the total action $S = S_0 + S_I$ are

$$(\tilde{p}^1_\mu + eF_{\mu\nu}(x)\partial_0 x^\nu)|_{\sigma^1=0,\pi} = 0, \tag{4.16}$$

which are the usual conditions in $\mathbb{R}^d$. The boundary equations of motion resulting from variations of $x^\mu(\sigma)$ in the total action $S = S_0 + S_I$ give some new conditions in $\mathbb{R}^d \times \tilde{\mathbb{R}}^d$,

$$\left(p^1_\mu + e\left(\frac{\partial}{\partial x^\mu}F_{\rho\sigma} + \frac{\partial}{\partial x^\sigma}F_{\mu\rho}\right)\tilde{x}^\rho\partial_0 x^\sigma + eF_{\mu\nu}\partial_0\tilde{x}^\nu\right)\Bigg|_{\sigma^1=0,\pi} = 0. \tag{4.17}$$

In the Hamiltonian formulation of the system, $\pi_\mu = p^0_\mu$ and $\tilde{\pi}_\mu = \tilde{p}^0_\mu$ are canonically conjugate to x^μ and $\tilde{x}^\mu$, respectively, having equal-time Poisson brackets

$$\begin{aligned}\{x^\mu(\sigma^0,\sigma^1), \pi_\nu(\sigma^0,\sigma'^1)\} &= \{\tilde{x}^\mu(\sigma^0,\sigma^1), \tilde{\pi}_\nu(\sigma^0,\sigma'^1)\}\\ &= \delta^\mu_\nu\delta(\sigma^1 - \sigma'^1),\end{aligned} \tag{4.18}$$

for $0 < \sigma^1,\ \sigma'^1 < \pi$, with all other equal-time Poisson brackets equal to zero. The canonical momenta are subject to the four constraints:

$$\begin{aligned}\Phi_1 &= \tilde{\pi}_\mu\tilde{\pi}^\mu + \frac{1}{(2\pi\alpha')^2}\partial_1 x^\mu\partial_1 x_\mu \approx 0\\ \Phi_2 &= \tilde{\pi}_\mu\partial_1 x^\mu \approx 0\\ \Phi_3 &= \pi_\mu\tilde{\pi}^\mu + \frac{1}{(2\pi\alpha')^2}\partial_1 x^\mu\partial_1\tilde{x}_\mu \approx 0\\ \Phi_4 &= \pi_\mu\partial_1 x^\mu + \tilde{\pi}_\mu\partial_1\tilde{x}^\mu \approx 0.\end{aligned} \tag{4.19}$$

It can be verified that they form a first-class set. Φ_1 and Φ_2 generate the local symmetry transformations (4.11), while linear combinations of the four constraints generate reparametrizations.

V. CONCLUSIONS

We have considered the problem of the existence of a Lagrangian description for the motion of a charged particle in the presence of a smooth distribution of magnetic monopoles. The magnetic field does not admit a potential on the physical configuration space. Auxiliary variables are employed in order to solve the problem, following a procedure commonly used to deal with dissipative dynamics. This is the Lagrangian counterpart of the Hamiltonian problem, addressed in Ref. [4], in which the Bianchi identity–violating magnetic field entails a quasi-Poisson algebra on the physical phase space which does not satisfy Jacobi identity unless one doubles the number of d.o.f. The problem was further extended to the relativistic case as well as the non-Abelian case. In the last section, we performed the generalization of the relativistic point-particle action (3.1) to that of an open-string interacting, once again, with a Bianchi identity–violating magnetic field. To circumvent the problem of the lack of a potential vector, the world sheet d.o.f. were doubled analogously to the case in double field theory. Many interesting issues can be addressed, such as a possible relationship with double field theory or the quantization problem, which relates Jacobi violation to nonassociativity of the quantum algebra. We plan to investigate these aspects in a forthcoming publication.

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