# Dihedral symmetry in $SU(N)$ Yang-Mills theory

Kyle Aitken,[1,*] Aleksey Cherman,[2,†] and Mithat Ünsal[3,‡]
[1]*Department of Physics, University of Washington, Seattle, Washington 98105, USA*
[2]*Institute for Nuclear Theory, University of Washington, Seattle, Washington 98105, USA*
[3]*Department of Physics, North Carolina State University, Raleigh, North Carolina 27607, USA*

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We point out that charge conjugation and coordinate reflection symmetries do not commute with the center symmetry of $SU(N)$ Yang-Mills (YM) theory when $N > 2$. As a result, for generic values of the $\theta$ angle, the group of discrete 0-form symmetries of YM theory on, e.g., the spacetime manifold $\mathbb{R}^3 \times S^1$ includes the dihedral group $D_{2N}$, which is non-Abelian for $N > 2$. At $\theta = \pi$, the non-Abelian factor in the symmetry group is enhanced to $D_{4N}$ due to discrete 't Hooft anomaly considerations. We illustrate these results in YM theory as well as in a simple quantum mechanical model, where we study representation theory as a function of the $\theta$ angle.

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## I. INTRODUCTION

Internal symmetries are a familiar feature in quantum field theory with many established properties. For example, places where symmetry realizations change can be associated with the emergence of gapless excitations. Often, the realizations of internal symmetries are constrained by 't Hooft anomaly matching. Additionally, in relativistic quantum field theories (QFTs), the Coleman-Mandula theorem [1] implies that continuous internal symmetries commute with the Poincaré group so that the full symmetry group of the theory, $G$, is a direct product: $G = G_{\mathrm{Poincare}} \times G_{\mathrm{internal}}$.

All of these features are illustrated in QCD. QCD with $N \geq 3$ colors has a $G_{\mathrm{internal}} = [SU(N_F)_V \times SU(N_F)_A \times U(1)_Q]/(\mathbb{Z}_{N_F} \times \mathbb{Z}_N)$ flavor symmetry in the chiral limit where $m_q = 0$, and $G_{\mathrm{QCD}} = G_{\mathrm{Poincare}} \times G_{\mathrm{internal}}$. The $SU(N_F)_A$ part of the internal symmetry group has a 't Hooft anomaly. This can be used to argue that when $m_q = 0$ the low-energy effective theory describing fluctuations about the thermodynamic ground state must include some gapless degrees of freedom. For some values of $N_F$ and $N$, these gapless degrees of freedom are associated with spontaneous breaking of the $SU(N_F)_A$ symmetry.

Here, our focus will be on pure $SU(N)$ Yang-Mills (YM) theory

$$S = \frac{1}{4g^2}\int d^4x F^a_{\mu\nu} F^{a\mu\nu} + i\frac{\theta}{16\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}, \qquad (1.1)$$

with $\mu, \nu = 1, \ldots, 4$ and $a = 1, \ldots, N$. Pure YM theory has no conventional internal symmetries which would act on local operators. It has long been known, however, that it does have a subtler type of internal symmetry, $G_{\mathrm{internal}} = \mathbb{Z}_N$ center symmetry [2–5]. Center symmetry acts nontrivially on certain line operators, but it does not act on local operators. In the language of Ref. [6], center symmetry is a "1-form symmetry," which can be contrasted with, e.g., the chiral symmetry of QCD, which is a "0-form symmetry" of which the natural charged objects are local operators. It turns out that, just as with more familiar 0-form symmetries, center symmetry can participate in 't Hooft anomalies [7]. In particular, there is a mixed 't Hooft anomaly between center symmetry and $CP$ symmetry at $\theta = \pi$ for even $N$ and a closely related notion of "global inconsistency" for odd N [7,8].

If the conclusions of the Coleman-Mandula theorem were to apply to center symmetry, then center symmetry would commute with $G_{\mathrm{Poincare}}$. However, one cannot appeal to this theorem for two reasons. First, the Coleman-Mandula theorem is derived for continuous internal symmetries, while the center symmetry of $SU(N)$ YM theory is discrete. Second, the Coleman-Mandula theorem follows from working out the constraints of symmetries on the S-matrix for relativistic particle scattering, while the charged objects for center symmetry are associated to stringlike extended operators. Indeed, we find that for pure $SU(N)$ YM theory on $\mathbb{R}^{3,1}$ the full symmetry group $G_{\mathrm{YM}}$ is generally *not* a direct product:

$$G_{\mathrm{YM}} \neq G_{\mathrm{Poincare}} \times G_{\mathrm{internal}}. \qquad (1.2)$$

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[*] kaitken17@gmail.com
[†] aleksey.cherman.physics@gmail.com
[‡] unsal.mithat@gmail.com

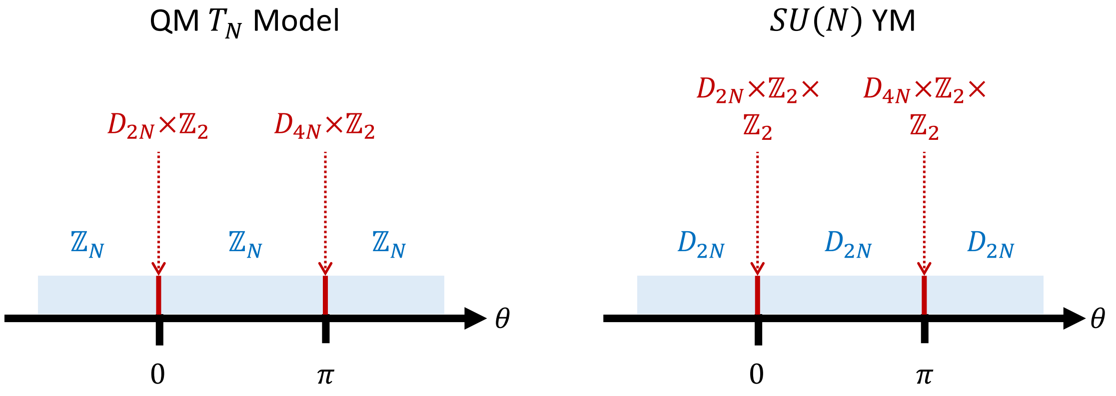

FIG. 1. A summary of the symmetries of $SU(N)$ YM theory (right) and of a related $T_N$ toy model from quantum mechanics (left), as a function of $\theta$.

In particular, when $N \geq 3$ center symmetry transformations do not commute with a simultaneous transformation of parity and time reversal, $PT$, or with charge conjugation $C$.[1] $PT$, $C$, and center transformations are symmetries of YM theory for all values of $g$ and $\theta$, so these two symmetries generate a discrete non-Abelian subgroup of $G_{\mathrm{YM}}^{\mathrm{disc}} \subset G_{\mathrm{YM}}$ for $N \geq 3$. However, we will see that the nature of $G_{\mathrm{YM}}^{\mathrm{disc}}$ depends on $\theta$.

We will show that when $SU(N)$ YM theory is compactified on $\mathbb{R}^3 \times S^1$ the discrete 0-form symmetries fit into the group

$$G_{\mathrm{YM}}^{\mathrm{discrete}} = \begin{cases} D_{2N} \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \theta = 0 \mod 2\pi \\ D_{4N} \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \theta = \pi \mod 2\pi \\ D_{2N} & \text{otherwise.} \end{cases} \tag{1.3}$$

Here, $D_{2N}$ is the dihedral group of symmetries of a regular planar $N$-gon. The dihedral group involves the 0-form part of center symmetry, which acts on Wilson loops which wind around $S^1$, as well as charge conjugation. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ factors are related to parity and time-reversal symmetries. Compactification on a circle simplifies the discussion but is not essential; see Sec. II for a discussion concerning the symmetries on $\mathbb{R}^4$.

The rest of the paper is concerned with illustrating how these symmetries behave in two different calculable settings. First, we discuss a simple quantum-mechanical toy model in Sec. III, where many of the ideas can be appreciated in the simplest possible context. In Sec. IV, we then explore the symmetries of a calculable deformation of YM theory obtained by a compactification on a small circle with stabilized center symmetry. This semiclassically calculable regime was uncovered in Ref. [10] and intensively explored in related works; see, e.g., Refs. [9,11–42]. A comparison of the symmetries between our quantum mechanical (QM) toy model and $SU(N)$ YM theory is given in Fig. 1. Our results are summarized in Sec. V and end with some Appendixes with details on some of our calculations.

In a companion paper [43], we further explore the vacuum properties of the deformed YM theory as a function of $\theta$.

## II. NON-ABELIAN GLOBAL SYMMETRY OF YM THEORY

In this section, we argue that the discrete part of the symmetry group of YM theory $G_{\mathrm{YM}}^{\mathrm{discrete}}$ includes the dihedral group $D_{2N}$. This involves showing that center symmetry does not commute with charge conjugation $C$. Equivalently, center symmetry does not commute with $PT$ symmetry; our discussion below will only explicitly refer to $C$ for simplicity.

Since center symmetry does not act on any local operators, a nontrivial check of the symmetry group generated by center symmetry and charge conjugation will involve consideration of line operators. For simplicity of exposition, we work in Euclidean space. We will first discuss the symmetries on $\mathbb{R}^3 \times S^1$ and then comment on the generalization to $\mathbb{R}^4$.

First, take spacetime to be $\mathbb{R}^3 \times S^1$, with $S^1$ the $x_4$ direction. The 0-form part of center symmetry acts nontrivially on "Polyakov loops"—Wilson loops wrapping the circle $\mathrm{tr}\Omega = \mathrm{tr}P\exp(i\oint dx_4 A_4)$, which are local with respect to $\mathbb{R}^3$. The action of center symmetry is[2]

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[1] This was noted but not explored in Ref. [9].

[2] To see where this transformation rule comes from, let us define gauge transformations to be periodic on $S^1$, as is appropriate for an $SU(N)$ gauge theory. Then, consider a "gauge transformation" $g(x_4, x)$ that is aperiodic by an element of the center of $SU(N)$, $g(x_4 + L, x) = e^{2\pi i/N} g(x_4, x)$. Under this not-quite-gauge transformation, $A_\mu \to g A_\mu g^\dagger - i g^\dagger \partial_\mu g$, which implies the transformation rule (2.1). If instead one chooses to view center-aperiodic transformations $g$ as genuine gauge transformations, one would be working with $SU(N)/\mathbb{Z}_N$ gauge theory. In $SU(N)/\mathbb{Z}_N$ gauge theory, Eq. (2.1) would be a gauge transformation, and $\mathrm{tr}\Omega$ would not be a gauge-invariant operator.

$$\mathcal{S}\cdot \mathrm{tr}\Omega = \omega \mathrm{tr}\Omega, \tag{2.1}$$

where the exponent of $\omega = e^{2\pi i/N}$ is the charge of $\mathrm{tr}\Omega$ and we have denoted the operator implementing center symmetry transformations by $\mathcal{S}$. Of course, $\mathcal{S}^N = 1$.

The theory is invariant under charge conjugation symmetry $\mathcal{C}$ at an arbitrary $\theta$ angle since the topological term respects $\mathcal{C}$. Charge conjugation maps $\mathcal{C}: \Omega \to \Omega^\dagger = \Omega^{-1}$ so that $\mathcal{C}^2 = 1$. Let us now work out the group obeyed by $\mathcal{S}$ and $\mathcal{C}$. One can then verify that

$$\mathcal{C}\cdot\mathcal{S}\cdot\mathcal{C}\cdot \mathrm{tr}\Omega = \mathcal{S}^{-1}\cdot \mathrm{tr}\Omega. \tag{2.2}$$

Thus, $\mathcal{C}$ and $\mathcal{S}$ do not commute. In fact, they obey the defining relations of the dihedral group of symmetries of a regular planar $N$-gon,

$$D_{2N} = \{\mathcal{S},\mathcal{C} | \mathcal{S}^N = 1, \mathcal{C}^2 = 1, \mathcal{CSC} = \mathcal{S}^{-1}\}. \tag{2.3}$$

At $\theta = 0$, YM theory has parity $\mathcal{P}: x_j \to -x_j, j = 1,2,3$ and $x_4$-reflection $\mathcal{R}: x_4 \to L - x_4$ symmetries. There is also an $SO(3)$ Lorentz symmetry associated with the noncompact directions. It is easy to see that center symmetry also does not commute with $\mathcal{R}$ because its behavior when acting on $\mathrm{tr}\Omega$ is analogous to charge conjugation, $\mathcal{R}\cdot\mathrm{tr}\Omega = \mathrm{tr}\Omega^{-1}$. But $[\mathcal{S},\mathcal{P}] = [\mathcal{C},\mathcal{R}] = [\mathcal{C},\mathcal{P}] = 0$. However, while $\mathcal{R}$ and $\mathcal{P}$ are manifestly symmetries at $\theta = 0$, they are not symmetries for generic $\theta \neq 0, \pi$.

At $\theta = \pi$, there is either a mixed 't Hooft anomaly or a global inconsistency between center and $CP$ symmetries [7], depending on whether $N$ is even or odd. Assuming that center symmetry is not spontaneously broken for all $\theta$, when there is mixed 't Hooft anomaly at $\theta = \pi$, there are two possibilities for the vacuum structure: (1) $CP$ is spontaneously broken, or (2) there is a nontrivial topological field theory which matches the anomaly in the IR limit. A global inconsistency condition at $\theta = \pi$ is slightly weaker and, in addition to the two options above, can also be satisfied if there are phase transitions away from $\theta = 0$, $\pi$ [7,8].

Especially for large $N$, spontaneous breaking of $CP$ at $\theta = \pi$ seems like the most probable way these anomaly/inconsistency conditions would be satisfied, and we assume this is the case in writing expressions in the $\mathbb{R}^4$ limit. On $\mathbb{R}^3\times S^1$, $CP$ breaking can be shown explicitly. We demonstrate that the anomaly/global inconsistency conditions at $\theta = \pi$ imprint themselves on the symmetry group by leading to a central extension. So, at $\theta = \pi$, the discrete global symmetry contains a factor of $D_{4N}$, the double-cover of $D_{2N}$.[3] Taken together, these considerations imply the claim from the Introduction in (1.3).

Note that the dihedral group $D_4$ is isomorphic to the Abelian group $\mathbb{Z}_2\times\mathbb{Z}_2$, but $D_{2N}$ is non-Abelian for all $N > 2$. So, when $N = 2$, the discrete symmetry group of YM theory is Abelian for $\theta \neq \pi$ and becomes non-Abelian only when $\theta = \pi$. However, for all $N > 2$, $G_{\mathrm{YM}}^{\mathrm{discrete}}$ is non-Abelian for all $\theta$, and the group of 0-form symmetries $\mathcal{S}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{C}$ will be shown to take the form (1.3).

We now turn our attention to $\mathbb{R}^4$. Here, it is helpful to adopt the language of Ref. [6], in which center symmetry is viewed as a $p$-form symmetry with $p = 1$. The charges of $p$-form symmetries are measured by integrating conserved $d-p-1$ currents on closed $d-p-1$-dimensional manifolds and are associated to charges of operators supported on manifolds of dimension $p$. The charge of such an operator is nonzero when its world volume manifold has nonvanishing linking number with some $d-p-1$-dimensional manifold where one puts the operator generating the symmetry.

In the case of 1-form center symmetry, the basic charged operators are Wilson lines with appropriate topological properties. In particular, consider an open Wilson line defined on a curve $\gamma$ of which the ends go off to infinity in different directions, for instance, along $x_4 \to \pm\infty$. One can think of such a line operator $\Omega(\gamma)$ as being associated with a probe fundamental quark-antiquark pair with separation taken to infinity, with, e.g., the quark going to $x_4 \to +\infty$ and the antiquark going to $x_4 \to -\infty$. We note that if the $x_4$ direction is compactified to $S^1$ this open Wilson line becomes precisely the Polyakov loop considered earlier. Since the Polyakov loop is a local operator in $\mathbb{R}^3$, the center symmetry acting on the Polyakov loop can be thought of as a 0-form symmetry from the perspective of an effective field theory (EFT)on $\mathbb{R}^3$. But even in an EFT on $\mathbb{R}^3$, there is a 1-form center symmetry acting on Wilson loops in $\mathbb{R}^3$. So, it is useful to keep track of the fact that center symmetry is most generally defined as 1-form symmetry.

For our purposes, it will be useful to associate the operator generating the 1-form center symmetry with the closed two-dimensional surface $\Sigma_2$ which spans the $x_1$-$x_2$ plane. In this case, center symmetry acts on $\mathrm{tr}\Omega(\gamma)$ as [6]

$$\mathcal{S}\cdot\mathrm{tr}\Omega(\gamma) = \omega^{\ell(\gamma,\Sigma_2)}\mathrm{tr}\Omega(\gamma) = \omega^{+1}\mathrm{tr}\Omega(\gamma), \tag{2.4}$$

where $\ell(\gamma,\Sigma_2)$ is the linking number of $\gamma$ with $\Sigma_2$ [6], which is $+1$ in the case above.

Now, consider charge conjugation. This symmetry interchanges quarks and antiquarks, so it acts on $\Omega(\gamma)$ as

$$\mathcal{C}\cdot\mathrm{tr}\Omega(\gamma) = \mathrm{tr}\Omega(\gamma)^\dagger = \mathrm{tr}\Omega(\gamma^{-1}), \tag{2.5}$$

so $\mathcal{C}$ flips the orientation of $\gamma$. Flipping the orientation of $\gamma$ flips the sign of the linking number of $\gamma$ with $\Sigma_2$, $\ell(-\gamma,\Sigma_2) = -\ell(\gamma,\Sigma_2)$. The operator group then follows as before,

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[3] The fact that the symmetry group of $SU(2)$ YM theory involves a $D_8$ factor at $\theta = \pi$ was discussed in Ref. [7].

$$\mathcal{C}\cdot\mathcal{S}\cdot\mathcal{C}\cdot \mathrm{tr}\Omega(\gamma)=\omega^{-1}\mathrm{tr}\Omega(\gamma)=\mathcal{S}^{-1}\cdot \mathrm{tr}\Omega(\gamma). \quad (2.6)$$

Thus, $\mathcal{C}$ and $\mathcal{S}$ do not commute on $\mathbb{R}^4$. It is also easy to see that $\mathcal{S}$ does not commute with $\mathcal{R}$, the $\theta = 0$, $\pi$ symmetry operator which now maps $x_4 \to -x_4$.

Rather trivially, symmetries of quantum systems can be associated with groups because, given some state $|\psi\rangle$ in Hilbert space which transforms nontrivially under a symmetry, one can verify that the symmetry action obeys the group axioms. In our case, choosing $|\psi\rangle = \mathrm{tr}\Omega|0\rangle$, our remarks above imply that the actions of the $\mathcal{C}$ and $\mathcal{S}$ transformations obey the group axioms and combine into the symmetry group $D_{2N}$. Nevertheless, we are dealing with the somewhat unusual situation of considering the combination of a 0-form symmetry and a 1-form symmetry. Recently, Refs. [44,45], in which it is argued that the general algebraic structure appropriate to discuss the mixture of 0-form and 1-form symmetries is a “2-group” [46], appeared (see also, e.g., Ref. [47]).[4]

### A. Physical consequences

We now comment on some physical consequences of the existence of the dihedral non-Abelian symmetry in $SU(N)$ YM theory. The fact that charge conjugation and center symmetry do not commute means that the associated charge operators cannot be simultaneously diagonalized. This means that if one considers a state that transforms nontrivially under both center and charge conjugation symmetry one cannot simultaneously specify its center symmetry and charge conjugation quantum numbers. Of course, this means that the existence of the $D_{2N}$ symmetry does not imprint itself on the correlation functions of local operators. One must consider correlation functions of appropriate line operators to see the symmetry.

For example, consider $SU(N)$ YM theory with $N > 2$ on $\mathbb{R}^3 \times S^1$. Finite-energy states transforming under center symmetry can be built out of Wilson loops wrapping $S^1$. Then, one can consider scattering amplitudes involving such states, for example at $\theta = 0$. Suppose we choose to specify the center labels of the states. Then, the fact that one cannot simultaneously specify the center and charge conjugation quantum numbers—which is due to the existence of the $D_{2N}$ symmetry—means that one has to sum over the $C$ quantum numbers for both incoming and outgoing states in computing the scattering amplitudes.

At high temperature, center symmetry is spontaneously broken in pure YM theory. It would be interesting to understand the physical implications of the noncommutativity of center symmetry and, e.g., $PT$ symmetry in this setting.

## III. DIHEDRAL SYMMETRIES IN A QUANTUM MECHANICAL MODEL

As a warm-up for studying the symmetries and dynamics of $SU(N)$ gauge theory as a function of $\theta$, we will first consider the QM system of a particle on a circle, $q(t) = q(t) + 2\pi$, in the presence of a potential with $N$ degenerate minima. This class of models is referred to as $T_N$ models in Ref. [24], in which their nonperturbative properties were examined semiclassically. The Euclidean action of the model is

$$S_{T_N}(g,\theta)=\frac{1}{g^2}\int dt\left[\frac{1}{2}\dot{q}^2-\cos(Nq)\right]-i\frac{\theta}{2\pi}\int dt\dot{q}. \quad (3.1)$$

The potential has $N$ degenerate minima at $q_n = \frac{2\pi n}{N}$, $n = 0, 1, \ldots N-1$. But the system does not have $N$ degenerate ground states; tunneling/instanton effects typically lift the degeneracies seen in perturbation theory. However, this does not mean that the ground state is always unique. For some values of $\theta$, it turns out to be doubly degenerate. We discuss the ground state structure below from a perspective that we will find useful in YM theory.

Analogies between the one-dimensional $T_N$ model and four-dimensional $SU(N)$ YM theory were previously explored in Ref. [24], and a detailed analysis of the symmetries of a very closely related model appears in Ref. [48]. The discussion in Sec. III A thus has overlap with Ref. [48], but the subsequent representation-theoretic perspective presented in Sec. III B is new. A discussion of the symmetries of the $T_2$ model as a function of $\theta$ appears in an Appendix of Ref. [49], but our focus will be on features that appear once $N > 2$. Also, a discussion of ’t Hooft anomalies from the path integral perspective is given in Appendix A. The material in this Appendix closely follows the presentation of Ref. [48], and we include it here for completeness.

### A. Symmetry group as a function of θ

Consider the symmetry group of the $T_N$ theory. Classically, there is a shift symmetry $\mathcal{S}$ as well as “charge conjugation” $\mathcal{C}$ and “time-reversal” $\mathcal{T}$ symmetries acting as

$$\mathcal{S}\colon\ q \to q - 2\pi/N \quad (3.2)$$

$$\mathcal{C}\colon\ q \to -q \quad (3.3)$$

$$\mathcal{T}\colon\ t \to -t. \quad (3.4)$$

In the quantum theory, the shift symmetry can represented by the operator

$$\mathcal{S} = e^{\frac{2\pi i}{N}\hat{p}}, \quad (3.5)$$

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[4] We are grateful to K. Jensen and S. Gukov for discussions on this point.

where $\hat{p}$ is the momentum operator obeying the canonical commutation relation $[\hat{q},\hat{p}]=i$. As befits a symmetry operator, $\mathcal{S}$ commutes with the Hamiltonian

$$\hat{H}=\frac{1}{2}\left(\hat{p}-\frac{\theta}{2\pi}\right)^2-\cos(N\hat{q}). \tag{3.6}$$

Demanding that $\mathcal{T}$ and $\mathcal{C}$ leave the Hamiltonian invariant in, e.g., the coordinate basis, one sees that for $\theta=0$ the $\mathcal{T}$ and $\mathcal{C}$ symmetries both act by sending $\hat{p}\to-\hat{p}$, while at $\theta=\pi$, the $\mathcal{T}$ and $\mathcal{C}$ symmetries both act by sending $\hat{p}\to-\hat{p}+1$. Thus, e.g., $\mathcal{T}\hat{p}\mathcal{T}^{-1}=-\hat{p}$ at $\theta=0$, but $\mathcal{T}\hat{p}\mathcal{T}^{-1}=-\hat{p}+1$ at $\theta=\pi$. One can check that at both $\theta=0$ and $\theta=\pi$ the $\mathcal{C}$ and $\mathcal{T}$ operators commute,

$$[\mathcal{T},\mathcal{C}]=0. \tag{3.7}$$

In Minkowski space, time reversal is an antiunitary operation, so in addition to sending $t\to-t$, $\mathcal{T}$ acts by complex conjugation $\mathcal{T}i\mathcal{T}^{-1}=-i$, in contrast to $\mathcal{C}$, which is unitary. One can check that this implies that $[\mathcal{T},\mathcal{S}]=0$ in Minkowski space.

$\mathcal{C}$ does not commute with $\mathcal{S}$ at $\theta=0$. To see this, note the lowest-lying states of the system can be thought of as being associated with nodeless wave functions $|q_n\rangle$ localized near the $N$ minima. These states are called Wannier states. From $|q_n\rangle$, one can build states with good quantum numbers $|k\rangle$ under $\mathcal{S}$ by a discrete Fourier transform,

$$|k\rangle=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\omega^{-nk}|q_n\rangle. \tag{3.8}$$

The states $|k\rangle$ are called Bloch states. Then,

$$\mathcal{S}|k\rangle=\omega^k|k\rangle, \tag{3.9}$$

with $|k\rangle=|k \bmod N\rangle$. Then, one can check that

$$\mathcal{C}|k\rangle=\begin{cases}|N-k\rangle & \theta=0\\ |N-k+1\rangle & \theta=\pi\end{cases}. \tag{3.10}$$

As a result, at $\theta=0$, the symmetry operators obey the group

$$\mathcal{C}\mathcal{S}\mathcal{C}^{-1}=\mathcal{S}^{-1}. \tag{3.11}$$

Given that $\mathcal{T}^2=\mathcal{C}^2=1$, the complete symmetry group is isomorphic to

$$G_{T_N}^{\theta=0}=D_{2N}\times\mathbb{Z}_2. \tag{3.12}$$

On the other hand, at $\theta=\pi$, we instead obtain

$$\mathcal{C}\mathcal{S}\mathcal{C}^{-1}=\omega^{-1}\mathcal{S}^{-1}. \tag{3.13}$$

The appearance of the factor $\omega^{-1}$ on the right-hand side means that the group is not closed in terms of $\mathcal{C}$, $\mathcal{T}$, and $\mathcal{S}$. This is a symmetry-group-level indication of a 't Hooft anomaly or global inconsistency between these symmetries. As a result, one of these symmetries must be spontaneously broken at $\theta=\pi$, or there must be a phase transition at some $\theta$ between 0 and $\pi$.[5]

One can try to redefine the operators to get a closed group, for example, by $\tilde{\mathcal{S}}\equiv\omega^p\mathcal{S}$. We will refer to $p$ as a Chern-Simons coefficient, since this is how it appears in a path integral description of this system; see Refs. [7,48] and Appendix A. This will not spoil the relation $\tilde{\mathcal{S}}^N=1$ so long as $p\in\mathbb{Z} \bmod N$. With such a redefinition, Eq. (3.13) becomes

$$\mathcal{C}\tilde{\mathcal{S}}\mathcal{C}^{-1}=\omega^{2p-1}\tilde{\mathcal{S}}^{-1}. \tag{3.14}$$

To keep (3.14) isomorphic to (3.11) requires $2p-1=0 \bmod N$.

Now, consider the case of odd and even $N$ separately. For even $N$, there is no solution to $2p-1=0 \bmod N$ for $p\in\mathbb{Z}$. Nevertheless, we can get a closed group by taking $p=1/2$. In the path integral description, this gives a Chern-Simons term with an improperly quantized coefficient. This is associated with a mixed 't Hooft anomaly. In the operator description, the choice $p=1/2$ gives

$$\mathcal{C}\tilde{\mathcal{S}}\mathcal{C}^{-1}=\tilde{\mathcal{S}}^{-1}. \tag{3.15}$$

But now the operator $\tilde{\mathcal{S}}$ satisfies

$$\tilde{\mathcal{S}}^N=-1,\qquad\tilde{\mathcal{S}}^{2N}=1. \tag{3.16}$$

As a result, the symmetry group is now isomorphic to $D_{4N}\times\mathbb{Z}_2$, the central extension of $D_{2N}\times\mathbb{Z}_2$.[6] The central extension is the operator-group realization of the anomaly.

For odd $N=2m-1$, $2p-1=0 \bmod N$ is satisfied with the choice $p=(N+1)/2$. Hence, if we define $\tilde{\mathcal{S}}\equiv\omega^{(N+1)/2}\mathcal{S}$, this also reduces to (3.15) since

$$\tilde{\mathcal{S}}^N=\omega^{(N+1)N/2}\mathcal{S}^N=\omega^{2mN/2}(\omega^N)^m=1. \tag{3.17}$$

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[5] To decide which of these two symmetries are "actually" broken, it is helpful to note that there is no way to explicitly break $\mathcal{T}$ at $\theta=k=0$ while preserving $\mathcal{S}$. But if we change the potential $V=\cos[Nq]\to\cos[N(q+\alpha)]$, then for any fixed $\alpha\neq0$, the $\mathcal{C}$ symmetry $q\to-q$ is explicitly broken, but $\mathcal{S}$ and $\mathcal{T}$ are preserved. One can then verify that these $\mathcal{T}$ and $\mathcal{S}$ remain globally inconsistent at $\theta=\pi$ so that one of them must be broken, and this turns out to be $\mathcal{T}$ [48]. Then, taking $\alpha$ to 0, we conclude that it is the $\mathcal{T}$ symmetry which is spontaneously broken at $\theta=\pi$ in our variant of the $T_N$ model defined by (3.1).

[6] One can think of $D_{4N}$ as the spin group of $D_{2N}$, in the sense that under a $2\pi$ shift of $q$ (which is a rotation in target space) states go to minus themselves and only go back to themselves under a $4\pi$ shift.

However, if we insist on preserving the $D_{2N} \times \mathbb{Z}_2$ symmetry at $\theta = 0$, then we must choose the value of Chern-Simons coefficient to be $p = 0$ (i.e., the original operator definition). This is the manifestation of the inconsistency condition and results in a centrally extended group, $D_{4N} \times \mathbb{Z}_2$ at $\theta = \pi$.

Collecting our results, the symmetry group of the $T_N$ model as a function of $\theta$ is isomorphic to

$$G_{T_N} = \begin{cases} D_{2N} \times \mathbb{Z}_2 & \theta = 0 \\ D_{4N} \times \mathbb{Z}_2 & \theta = \pi \\ \mathbb{Z}_N & \text{otherwise.} \end{cases} \tag{3.18}$$

## B. Representations of the dihedral group for $\theta = 0$ and $\theta = \pi$

We now explain how the states of the $T_N$ model fit into the representations of the dihedral group. The value of this discussion is that it relies on the symmetry-group structure, rather than the underlying physics, and thus can later be applied almost verbatim to the YM case.

One can construct the $N$-dimensional representation of a dihedral group based on the behavior of the $N$ vacua of the $T_N$ model under the action of charge conjugation (equivalently, time-reversal) and $\mathbb{Z}_N$ shift symmetry. The decomposition of this representation into irreducible representations (irreps) will show us the form of the energy spectrum and provide us another means to see how the degeneracy of the ground state changes between $\theta = 0$ and $\pi$. For both $D_{2N}$ and $D_{4N}$, we find results consistent with the operator analysis above.

Let us start by briefly reviewing a few properties of dihedral groups. A more detailed review and discussion of our results are given in Appendix B. We will work with a standard presentation of the dihedral group $D_{2M}$, which is given by

$$D_{2M} = \{r, s | r^M = s^2 = 1, srs^{-1} = r^{-1}\}. \tag{3.19}$$

The representations of this group differ for even and odd $M$, so we will consider the two cases separately in what follows.

Below, we will consider the representations which correspond to the low-lying states of the $T_N$ model, i.e., the $N$ low-lying Bloch states $|k\rangle$, for the cases of even and odd $N$. Our goal is to understand the representation of the $N$ low-lying states. The results are visually summarized in Figs. 2 and 3, which plot the energies of these states as a function of the $\theta$ angle, and are compatible with mixed anomalies/global inconsistencies as well as semiclassics.

### 1. Even N

For $M = N = 2k$, the $k + 3$ conjugacy classes are

$$\begin{aligned} &\{1\}, \{r^{\pm 1}\}, \{r^{\pm 2}\}, \ldots, \{r^{\pm (k-1)}\}, \{r^k\}, \\ &\{sr^{2b} | b = 1, \ldots, k\}, \{sr^{2b-1} | b = 1, \ldots, k\}, \end{aligned} \tag{3.20}$$

where the number of elements in the conjugacy classes is given by

$$\{1, \underbrace{2, 2, \ldots, 2}_{k-1}, 1, k, k\}. \tag{3.21}$$

A character table for the representations of $D_{2M}$ is given in Table I.

At $\theta = 0$, the $N$ low-lying states labeled by $|k\rangle$ transform under the action of $D_{2N}$ group elements. The conjugacy classes and number of elements in each class are given by (3.20) and (3.21) with $M = N$. It is straightforward to construct the $N$-dimensional representation associated with $N$ low-lying states under the actions of $\mathcal{S}$ and $\mathcal{C}$. $\mathcal{S}$ simply introduces a vacuum-dependent phase to each of the states, while $\mathcal{C}$ permutes them. The characters corresponding to the conjugacy classes listed in (3.21) are

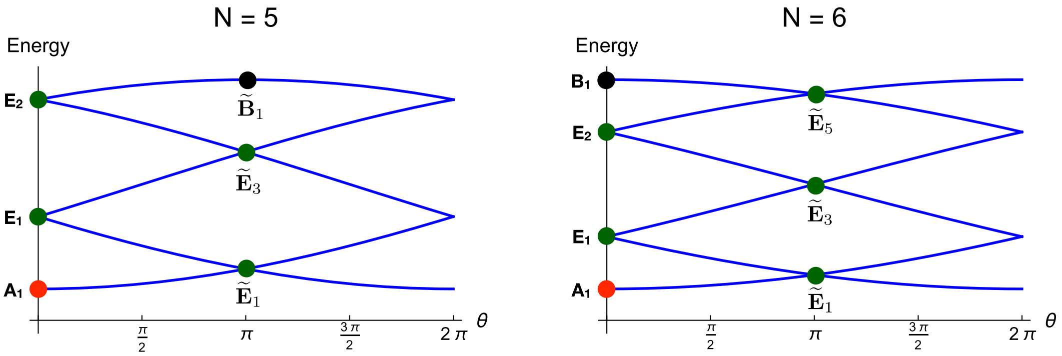

FIG. 2. An illustration of the energy levels of the $T_N$ model for $N = 5$ and $N = 6$. At $\theta = 0$, the ground state is unique and fits into the one-dimensional $\mathbf{A_1}$ representation of $D_{2N}$, while the excited states fit into either the $\mathbf{E_k}$ representations (which are all two dimensional) or the $\mathbf{B_1}$ representation, which is one dimensional. At $\theta = \pi$, on the other hand, the ground state is always in the two-dimensional $\tilde{\mathbf{E}}_1$ representation of $D_{4N}$.

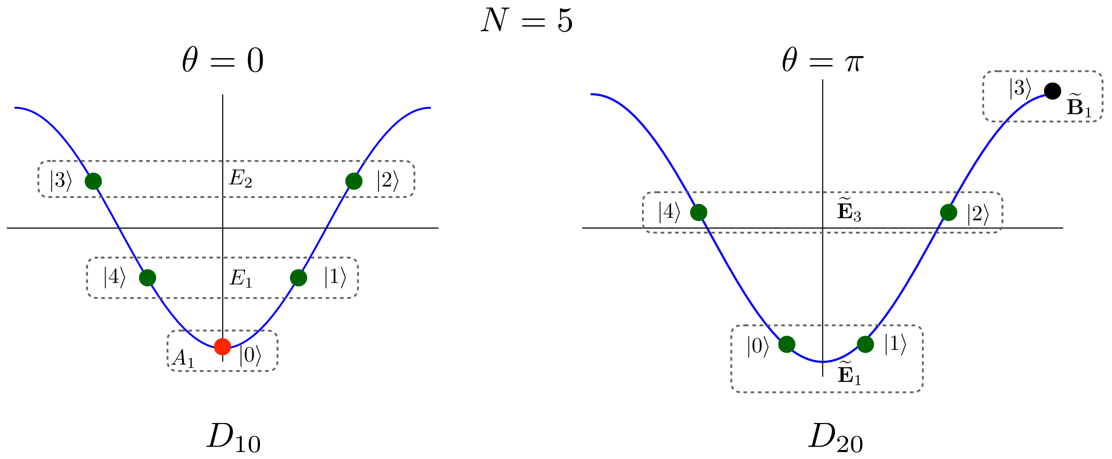

FIG. 3. A sketch of how the states of the $T_N$ model with $N = 5$ and $\theta = 0$ and $\theta = \pi$ fit into the dihedral group $D_{10}$ and $D_{20}$ representations. The Bloch states $|k\rangle$ are defined in (3.8).

$$\chi_{\text{even}}^{\theta=0} = \{N, \underbrace{0, 0, \ldots, 0}_{k-1}, 0, 2, 0\}. \tag{3.22}$$

Character orthogonality then gives the decomposition in terms of irreps,

$$R_{\text{even}}^{\theta=0} = \mathbf{A}_1 \oplus \mathbf{E_1} \oplus \mathbf{E_2} \oplus \ldots \oplus \mathbf{E_{k-1}} \oplus \mathbf{B_1}, \tag{3.23}$$

where $\mathbf{A_1}$ and $\mathbf{B_1}$ are one-dimensional irreps and $\mathbf{E_i}$ is a two-dimensional irrep (a doublet). $\mathbf{A_1}$ represents the unique ground state of this system which transforms trivially under all group operations.

At $\theta = \pi$, per our results of the previous subsection, the symmetry group is now $D_{4N}$. However, we should still construct an $N$-dimensional representations which tells us how the $N$ vacua now transform under this centrally extended group. The characters of this representation are

$$\chi_{\text{even}}^{\theta=\pi} = \{N, \underbrace{0, 0, \ldots, 0}_{N-1}, -N, 0, 0\}. \tag{3.24}$$

The decomposition in terms of irreps is now given by

$$R_{\text{even}}^{\theta=\pi} = \mathbf{\tilde{E}_1} \oplus \mathbf{\tilde{E}_3} \oplus \ldots \oplus \mathbf{\tilde{E}_{2k-1}}, \tag{3.25}$$

(with $\mathbf{\tilde{E}_i}$ now irreps of $D_{4N}$). The fact that the ground state exhibits twofold degeneracy in this simple quantum mechanics example is a manifestation of the 't Hooft anomaly between $\mathbb{Z}_N$ and $\mathbb{Z}_2$ and is tied with the spontaneous breaking of the $\mathbb{Z}_2$ symmetry.

### *2. Odd N*

For odd $M = 2k + 1$, the $k + 2$ conjugacy classes are

$$\{1\}, \{r^{\pm 1}\}, \{r^{\pm 2}\}, \ldots, \{r^{\pm k}\}, \{sr^b | b = 1, \ldots, M\}, \tag{3.26}$$

where the number of elements in each conjugacy class is now

$$\{1, \underbrace{2, 2, \ldots, 2}_{k}, N\}. \tag{3.27}$$

The corresponding character table is given in Table II.

At $\theta = 0$, the $N$ low-lying states transform under the action of $D_{2N} = D_{2(2k+1)}$ group elements. The characters of the $N$-dimensional representation are given by

TABLE I. Character table for $D_{2M} = D_{2(2k)}$. Here, $c_n = \cos(\frac{2\pi n}{M})$. The first row shows the number of elements in the respective conjugacy classes.

| | $1\{1\}$ | $2\{r^{\pm 1}\}$ | $2\{r^{\pm 2}\}$ | $\cdots$ | $2\{r^{\pm(k-1)}\}$ | $1\{r^k\}$ | $k\{sr^{2b}\}$ | $k\{sr^{2b-1}\}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{A_1}$ | 1 | 1 | 1 | $\cdots$ | 1 | 1 | 1 | 1 |
| $\mathbf{A_2}$ | 1 | 1 | 1 | $\cdots$ | 1 | 1 | $-1$ | $-1$ |
| $\mathbf{B_1}$ | 1 | $-1$ | 1 | $\cdots$ | $(-1)^{k-1}$ | $(-1)^k$ | 1 | $-1$ |
| $\mathbf{B_2}$ | 1 | $-1$ | 1 | $\cdots$ | $(-1)^{k-1}$ | $(-1)^k$ | $-1$ | 1 |
| $\mathbf{E_1}$ | 2 | $2c_1$ | $2c_2$ | $\cdots$ | $2c_{k-1}$ | $2c_k$ | 0 | 0 |
| $\mathbf{E_2}$ | 2 | $2c_2$ | $2c_4$ | $\cdots$ | $2c_{2k-2}$ | $2c_{2k}$ | 0 | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\mathbf{E_{k-1}}$ | 2 | $2c_{k-1}$ | $2c_{2k-2}$ | $\cdots$ | $2c_{(k-1)^2}$ | $2c_{(k-1)k}$ | 0 | 0 |

TABLE II. Character table for $D_{2M} = D_{2(2k+1)}$. Here, $c_m = \cos(\frac{2\pi m}{M})$.

| | $1\{1\}$ | $2\{r^{\pm 1}\}$ | $2\{r^{\pm 2}\}$ | $\cdots$ | $2\{r^{\pm k}\}$ | $N\{sr^{2b}\}$ |
|---|---|---|---|---|---|---|
| $\mathbf{A_1}$ | 1 | 1 | 1 | $\cdots$ | 1 | 1 |
| $\mathbf{A_2}$ | 1 | 1 | 1 | $\cdots$ | 1 | −1 |
| $\mathbf{E_1}$ | 2 | $2c_1$ | $2c_2$ | $\cdots$ | $2c_k$ | 0 |
| $\mathbf{E_2}$ | 2 | $2c_2$ | $2c_4$ | $\cdots$ | $2c_{2k}$ | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\mathbf{E_k}$ | 2 | $2c_k$ | $2c_{2k}$ | $\cdots$ | $2c_{k^2}$ | 0 |

$$\chi^{\theta=0}_{\text{odd}} = \{N, \underbrace{0, 0, \ldots, 0}_{k}, 1\}. \tag{3.28}$$

In this case, the decomposition is given by

$$R^{\theta=0}_{\text{odd}} = \mathbf{A_1} \oplus \mathbf{E_1} \oplus \mathbf{E_2} \oplus \ldots \oplus \mathbf{E_k}, \tag{3.29}$$

where $\mathbf{A_1}$ is again a one-dimensional irrep and $\mathbf{E_i}$ are 2 two-dimensional irreps of $D_{2N}$ with odd $N$. $\mathbf{A_1}$ again represents the unique ground state of this system at $\theta = 0$.

For the $T_N$ model with odd $N$, there is a global inconsistency condition at $\theta = \pi$ between $\mathcal{S}$ and $\mathcal{T}$ (or $\mathcal{C}$). As a result, the vacuum cannot remain trivial, which indicates either a nontrivial vacuum or a phase transition between $\theta = 0$ and $\pi$. We will assume that the inconsistency implies the former such that the $N$ low-lying states transform under the action of $D_{4N} = D_{2(4k+2)}$. This is the group that will give rise to the twofold degenerate ground state we find as a result of the global inconsistency condition. We also assume the central extension comes about in the same manner as the even $N$ case, where $\tilde{\mathcal{S}} \equiv \omega^{1/2}\mathcal{S}$. Semiclassical instanton analysis [24] and numerical diagonalization of (3.1) found in, e.g., Ref. [48] support the resulting degeneracies and $\theta$ dependence from this assumption.

At $\theta = \pi$, the $N$ low-lying states transform under the action of $D_{4N} = D_{2(4k+2)}$ group elements. The characters of the conjugacy classes in this case are

$$\chi^{\theta=\pi}_{\text{odd}} = \{N, \underbrace{0, 0, \ldots, 0}_{N-1}, -N, 1, -1\}, \tag{3.30}$$

and the corresponding decomposition is

$$R^{\theta=\pi}_{\text{odd}} = \tilde{\mathbf{E}}_\mathbf{1} \oplus \tilde{\mathbf{E}}_\mathbf{3} \oplus \ldots \oplus \tilde{\mathbf{E}}_\mathbf{2k-1} \oplus \tilde{\mathbf{B}}_\mathbf{1}. \tag{3.31}$$

$\tilde{\mathbf{E}}_\mathbf{1}$ denotes the ground state and exhibits twofold degeneracy. Other $\tilde{\mathbf{E}}$ states are excited states, and $\tilde{\mathbf{B}}_\mathbf{1}$ is the highest-energy state (of the low-lying states), which is a singlet.

## IV. DIHEDRAL SYMMETRIES IN YANG-MILLS THEORY ON $\mathbb{R}^3 \times S^1$

We would now like to illustrate Eq. (1.3) by explicitly looking at symmetry properties of the vacua and excitations of Yang-Mills theory. As is well known, $SU(N)$ YM theory on $\mathbb{R}^4$ is asymptotically free and as such becomes strongly coupled at energy scales small compared to the inverse strong scale, $\Lambda^{-1}$. Hence, methods of studying the explicit vacuum structure of the theory are limited. Instead, we choose to study YM theory on $\mathbb{R}^3 \times S^1$, with a circle size of $L$. In this system, the vacuum dynamics are calculable via weak-coupling methods, specifically in the limit where $NL\Lambda \ll 1$ and center symmetry is preserved. There has been significant evidence [9,11–42] that YM depends smoothly on the parameter $NL\Lambda$, and hence it is conjectured that one can recover results for the theory on $\mathbb{R}^4$ in the large $L$ limit. We will begin by briefly reviewing such a system. Those concerned only with our analysis of the vacuum can skip to Sec. IV B.

### A. Weak-coupling setup

Consider pure $SU(N)$ Yang-Mills theory on $\mathbb{R}^3 \times S^1$. For small $S^1$, it is known that the $\mathbb{Z}_N$ center symmetry is spontaneously broken [4,5], while for large $S^1$, the symmetry is expected to be restored. The order parameter for the associated phase transition is the expectation value of the trace of powers of

$$\Omega(x^\mu) = P\exp\left[i\int_0^L dx_4 A_4(x^\mu, x_4)\right], \tag{4.1}$$

where we have changed conventions slightly and will henceforth use $\mu$, $\nu = 1, 2, 3$. At large $L$, $\langle \mathrm{tr}\Omega^k \rangle = 0$ for $k \neq 0 \bmod N$, while at small $L$, $\langle \mathrm{tr}\Omega \rangle \neq 0$. However, if one is not interested in interpreting $S^1$ as a Euclidean thermal circle, this phase transition can be avoided by "center-stabilizing" deformations. One example of such a deformation is the addition of $N_F > 1$ massive Majorana adjoint fermions with mass $m_a \lesssim 1/(NL)$ [19]. Another example is the addition of a double-trace deformation [10]. With either deformation, it is believed that center symmetry is then preserved for all $L$, with the benefit that at small $L$ the physics becomes analytically calculable.

We choose to explore the behavior of the symmetries in the center-symmetric phase of the theory that follows from either of deformations referenced above. At small $L$, where quantum fluctuations become small, the holonomy takes the form

$$\langle \Omega \rangle = \omega^{-(N-1)/2}\mathrm{diag}(1, \omega, \ldots, \omega^{N-1}), \quad \omega = e^{2\pi i/N}, \tag{4.2}$$

up to gauge transformations.

We will analyze the theory at distances large compared to $L$, where the system can be described by a three-dimensional

(3D) EFT. From (4.1), the holonomy eigenvalues above imply that (in a standard gauge-fixed sense) $\langle A_4 \rangle \neq 0$, which acts as an adjoint Higgs field in the 3D EFT and breaks the gauge group down to $U(1)^{N-1}$. The lightest W-bosons have the tree-level mass

$$m_W \equiv \frac{2\pi}{NL}. \tag{4.3}$$

So, when $m_W \gg \Lambda$—equivalently, when $NL\Lambda \ll 1$—the gauge coupling stops running at the scale $m_W$, and the long-distance 3D EFT becomes weakly coupled. We focus on this tractable limit for the remainder of this paper.

The lightest fields in the 3D EFT are the $U(1)^{N-1}$ gauge bosons, the "photons." It is useful to note that the associated field strength operators $F^a_{\mu\nu}$, $a = 1, \ldots, N-1$, have a gauge-invariant four-dimensional interpolating operator representation given by

$$F^a_{\mu\nu}(x_\mu) \sim \frac{1}{N}\frac{1}{L}\int dx_4 \sum_{q=1}^{N-1} \omega^{-qa} \mathrm{tr}\Omega^q(x^\mu) F_{\mu\nu}(x_4, x_\mu), \tag{4.4}$$

with $F_{\mu\nu}$ the 3D part of the $SU(N)$ non-Abelian field strength. This representation makes it clear that the "color" index can actually be thought of as the discrete Fourier transform of the winding number of a topologically nontrivial state.

In terms of these fields, the tree-level action of the 3D EFT can be written as

$$S_{\mathrm{tree}} = \frac{L}{4g^2}\int d^3x \sum_{a=1}^{N} F^a_{\mu\nu} F^{a\mu\nu}. \tag{4.5}$$

For later notational convenience, we have introduced a fictitious $N$th photon in writing this expression. This extra field can be thought of as the diagonal component of a $U(N)$ field strength and exactly decouples from the physical adjoint fields in our system. Using Eq. (4.4), one can show that center symmetry acts as

$$\mathcal{S}\colon\ F^a_{\mu\nu} \to F^{a+1}_{\mu\nu}. \tag{4.6}$$

In order to analyze the nonperturabtive dynamics of our system, we follow Refs. [10,50] and rewrite (4.5) by dualizing the photon, trading $F^a_{\mu\nu}$ for a pseudoscalar field $\sigma^a$ via the relation

$$F^a_{\mu\nu} \equiv \frac{\lambda}{4\pi^2}\epsilon_{\mu\nu\rho}\partial^\rho \sigma^a, \qquad \lambda = g^2 N. \tag{4.7}$$

This allows us to rewrite (4.5) as

$$\begin{aligned} S_{\mathrm{tree,dual}} &= \lambda m_W \int d^3x \sum_{a=1}^{N} (\partial_\mu \sigma^a)(\partial^\mu \sigma^a) \\ &\equiv \lambda m_W \int d^3x (\partial_\mu \vec{\sigma})^2, \end{aligned} \tag{4.8}$$

where we have defined the $N$-component vector of dual photon fields $\vec{\sigma} = (\sigma^1, \ldots, \sigma^N)$.

The dual photons in (4.8) have no potential to all orders in perturbation theory. So, there is no mass gap in perturbation theory. However, the theory has finite-action field configurations that generate a nonperturbative potential for $\vec{\sigma}$. In Appendix C, we review the finite-action solutions of this theory with the smallest action. They come in $N$ distinct types and are usually called monopole instantons. They carry topological charge $Q_T = 1/N$, action $S_0 = 8\pi^2/\lambda$, and magnetic charges associated to the simple (co)roots $\vec{\alpha}_a$ of the affine extension of the $\mathfrak{su}(N)$ Lie algebra. For more details on the nonperturbation solutions and their transformations under the symmetries of the theory, see Appendix C.

As explained in Ref. [10], summing over the contributions of the monopole-instanton solutions to the path integral using a dilute-gas approximation (which is well justified when $NL\Lambda \ll 1$) produces a potential for the dual photons, so that

$$S_{\vec{\sigma}} = \int d^3x [\lambda m_W (\partial_\mu \vec{\sigma})^2 + V(\vec{\sigma})], \tag{4.9}$$

where the nonperturbative potential is given by

$$V(\vec{\sigma}) = -\frac{A}{\lambda^2} m_W^3 e^{-S_0} \sum_{a=1}^{N} \cos\left[\vec{\alpha}_a \cdot \vec{\sigma} + \frac{\theta}{N}\right] + \ldots. \tag{4.10}$$

The "…" represent higher-order contributions which we will neglect here. Here, $A > 0$ is an $\mathcal{O}(1)$ scheme-dependent dimensionless constant which will not be important in what follows. The monopole-generated potential depends on the $\theta$ angle because the monopole instantons have nonvanishing topological charge.

We now show how the YM symmetry group in (1.3) acts in the EFT associated to (4.9).

### B. Extrema and symmetries as a function of $\theta$

We now begin our analysis of the vacuum structure of (4.9), with the leading-order potential explicitly shown in (4.10). The dual photon fields live in the weight lattice of $\mathfrak{su}(N)$. The potential has $N$ extrema in the unit cell of the weight lattice at

$$\vec{\sigma}_k = \frac{2\pi k}{N}\vec{\rho}, \quad \text{with} \quad \vec{\rho} \equiv \sum_{i=1}^{N-1} \vec{\mu}_i. \tag{4.11}$$

where $k = 0, \ldots, N-1$. Here, $\vec{\mu}_i$ are the $SU(N)$ fundamental root vectors and satisfy $\vec{\alpha}_i \cdot \vec{\mu}_j = \delta_{ij}$, and $\vec{\rho}$ is the Weyl vector satisfying $\vec{\alpha}_i \cdot \vec{\rho} = 1$ for $i = 1, \ldots, N-1$ and $\vec{\alpha}_N \cdot \vec{\rho} = 1 - N$. For example, in a basis where $(\alpha_a)_b = \delta_{a,b} - \delta_{a+1,b}$, $1 \leq a < N$, $\vec{\sigma}_k$ takes the form

$$\vec{\sigma}_k = \frac{2\pi k}{N}(N, N-1, \ldots, 2, 1). \tag{4.12}$$

The nonperturbative 3D energy density evaluated at each of these extrema is

$$\begin{aligned} V_k &\equiv V(\vec{\sigma} = \vec{\sigma}_k) \\ &= -N \frac{A}{\lambda^2} m_W^3 e^{-S_0} \cos\left(\frac{2\pi k + \theta}{N}\right) + \mathcal{O}(e^{-2S_0}). \end{aligned} \tag{4.13}$$

For any given $\theta$, the integer $k$ labeling the globally stable ground state is determined by minimizing (4.13). The metastable states of the system will correspond to the subset of extrema with positive curvature in all directions in $\vec{\sigma}$ space. On any fixed branch, the physics is periodic in $2\pi N$. However, the $k$ that minimizes $V_k$ depends on $\theta$. Thus, just from the form of (4.13), one can see that as $\theta$ varies in $[0, 2\pi)$ the value of $k$ associated with the minimal energy extremum will change in such a way that the physics of the complete system in its ground state has a $\theta$ periodicity of $2\pi$. However, the observables are nonanalytic functions of $\theta$, which is associated with jumps in the value of $k$ which minimize the ground state energy density. This is consistent with Witten's conjectured picture [51,52] for the $\theta$ dependence of YM theory. Earlier discussions of how $2\pi$ periodicity emerges in the present context were presented in, e.g., Refs. [10,22,24–26,33,39].

Let us now understand how center and coordinate reflection symmetries act on the extrema of (4.11). To do this, it is useful to work out how these transformations act in compactified YM theory more generally; see Appendix C and also Ref. [9]. Here, we will focus on reflections of the compactified coordinate $\mathcal{R}$, charge conjugation $\mathcal{C}$, and (0-form) center transformations $\mathcal{S}$. The EFT on $\mathbb{R}^3 \times S^1$ is built from the dual photon fields $\sigma_a$, and the action of these transformations which follows from (4.4) and (4.6) is

$$\mathcal{S}\colon\ \sigma_a \to \sigma_{a+1} \tag{4.14}$$

$$\mathcal{C}\colon\ \sigma_a \to -\sigma_{N-a+1} \tag{4.15}$$

$$\mathcal{R}\colon\ \sigma_a \to \begin{cases} \sigma_{N-a+1}, & \theta = 0 \\ \sigma_{N-a+1} - \frac{2\pi(N-a+1)}{N}, & \theta = \pi. \end{cases} \tag{4.16}$$

Looking at the form of the effective action (4.9), it is clear that $\mathcal{S}$ and $\mathcal{C}$ are symmetries for any $\theta$, as one would expect. The $\mathcal{R}$ coordinate-reflection transformation is a symmetry only if $\theta = 0$ or $\theta = \pi$. Note that when acting on $\vec{\alpha}_a \cdot \vec{\sigma}$ at $\theta = \pi$ the reflection symmetry transformation gives

$$\mathcal{R}\colon\ (\vec{\alpha}_a \cdot \vec{\sigma}) \to -\vec{\alpha}_{N-a} \cdot \vec{\sigma} - \frac{2\pi}{N}. \tag{4.17}$$

The resulting shift in the phase of monopole operators is necessary because a coordinate reflection must be accompanied by a $2\pi$ shift in the $\theta$ angle to be a symmetry of the theory.

One can now easily work out the symmetry group. To do so, consider the action of the symmetry transformations on an operator of the form $e^{i\sigma_a}$. It can be checked that $\mathcal{C}^{-1}\mathcal{S}\mathcal{C} = \mathcal{S}^{-1}$, corresponding to a $D_{2N}$ symmetry group, just as one would expect from the general arguments in Sec. II. For the $\mathcal{R}$ and $\mathcal{S}$ symmetries, we obtain

$$\mathcal{R}^{-1}\mathcal{S}\mathcal{R} = \begin{cases} \mathcal{S}^{-1}, & \theta = 0 \\ \omega\mathcal{S}^{-1}, & \theta = \pi. \end{cases} \tag{4.18}$$

This corresponds to a $D_{2N}$ group for $\theta = 0$ and a $D_{4N}$ group for $\theta = \pi$. As in our discussion of the $T_N$ model, for even $N$, we interpret the $\theta = \pi$ commutation relations in (4.18) to imply the existence of a mixed 't Hooft anomaly between center and time-reversal symmetries, while for odd $N$, we interpret them to imply a global inconsistency between these symmetries. In total, we find precisely the expected 0-form symmetries of (1.3), reproduced here for convenience,

$$G_{\text{YM}}^{\text{discrete}} = \begin{cases} D_{2N} \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \theta = 0 \mod 2\pi \\ D_{4N} \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \theta = \pi \mod 2\pi \\ D_{2N} & \text{otherwise.} \end{cases} \tag{4.19}$$

Note that a benefit of our approach is that we get a simple picture for how the mixed center-$CP$ 't Hooft anomaly of Ref. [7] arises (as a central extension of the symmetry group, just like in toy QM examples). Moreover, given that we work in a regime where the dynamics is calculable, we can fully determine the vacuum structure. On the other hand, the general nature of the considerations of Ref. [7] have their own benefits. In particular, they are valid regardless of the strength of the coupling in the system. We explore further features of the vacuum structure of (4.10) and higher-order corrections in a companion paper [43].

Turning back to the symmetry transformations of the extrema of the potential, we find that $\mathcal{R}$ acts as

$$\mathcal{R}\colon\ \vec{\sigma}_k \to \begin{cases} \vec{\sigma}_{-k} & \theta = 0 \\ \vec{\sigma}_{-k+1} & \theta = \pi, \end{cases} \tag{4.20}$$

while the center transformation rule is

$$\mathcal{S}\colon\ \vec{\sigma}_k \to \vec{\sigma}_k + \frac{2\pi k}{N}\vec{c}, \tag{4.21}$$

where the $N$ vector $\vec{c}$ obeys the relations

$$\vec{\alpha}_a \cdot \vec{c} = \begin{cases} -N & a = 1 \\ 0 & 1 < a < N \\ N & a = N. \end{cases} \tag{4.22}$$

For example, in the basis of Eq. (4.12), $\vec{c} = (1, 1, \ldots, 1, 1 - N)$. The condition that $\mathcal{S}^N \cdot \vec{\sigma}_k = \vec{\sigma}_k$ is related to the periodicity of the $\sigma_a$ fields and the quantization of the coefficient of $\vec{c}$ in (4.21).

## V. CONCLUSIONS

We have examined the global symmetries and ground state properties of $SU(N)$ YM theory as a function of the topological $\theta$ angle. The global symmetries were argued to include non-Abelian discrete groups—specifically, dihedral groups—for all $\theta$ when $N \geq 3$ due to a noncommutativity between center symmetry and charge conjugation. We then examined the vacuum structure of YM theory as a function of $\theta$. First, we warmed up by considering a simple quantum mechanics example of which the symmetries also include dihedral groups. We then used the technique of adiabatic circle compactification of YM theory on $\mathbb{R}^3 \times S^1$ to illustrate the symmetry structure and some ground state properties in a systematically calculable setting.

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## APPENDIX A: PATH INTEGRAL FORMULATION OF THE $T_N$ MODEL

In this Appendix, we consider the path integral description of anomalies and global inconsistency conditions in the quantum-mechanical $T_N$ model. Our exposition is based on Ref. [48]; see also Ref. [7].

A mixed 't Hooft anomaly is when $G = G_1 \times G_2$ and gauging of one of the symmetries results in the loss of the other. This motivates turning on a background gauge field associated to $\mathcal{S}$. $\mathcal{S}$ is a discrete 0-form $\mathbb{Z}_N$ symmetry, and gauging it involves coupling the $T_N$ model to a topological field theory [53,54]. For this, it turns out that it is most efficient to work with two background gauge fields $A$ and $B$, where $A$ is 1-form and $B$ is 0-form, related by a constraint induced by some Lagrange multiplier field $F$. The action of the $T_N$ model with background fields associated to $\mathcal{S}$ is

$$S_{T_N}(A, B; g, \theta, p) = \frac{1}{g^2} \int \left[\frac{1}{2}(dq + A)^2 - \cos(Nq + B)\right] - \frac{i\theta}{2\pi} \int (dq + A) \tag{A1}$$

$$+ i \int F \wedge (dB - NA) + ip \int A, \tag{A2}$$

and the partition function is

$$Z_{T_N}(A, B; g, \theta, p) = \int d[q] d[F] e^{-S_{T_N}(A,B;g,\theta,p)}. \tag{A3}$$

Note that integrating out $F$ enforces the on-shell identity $dB = NA$. The (background) 0-form gauge transformation properties are

$$q \to q - \lambda \tag{A4}$$

$$B \to B + N\lambda \tag{A5}$$

$$A \to A + d\lambda \tag{A6}$$

$$F \to F. \tag{A7}$$

One can check that the action is invariant under these gauge transformations as long as the coefficient of the one-dimensional Chern-Simons term $p$ is an integer. The integer $p$ can be interpreted as a hidden parameter in the theory in addition to the obvious parameters $g$, $\theta$, and to define the theory for any value of the background fields, we must specify all *three* parameters $g$, $\theta$, $p$.

The fact that the action (A2) is gauge invariant means that there is no direct 't Hooft anomaly for $\mathcal{S}$. However, since the system has additional discrete symmetries at $\theta = 0$ and $\pi$, these points of parameter space are potentially problematic and should be checked for mixed 't Hooft anomalies.

At $\theta = 0$, it is easy to check that, as long as $2p = 0 \bmod N$, $\mathcal{C}$ and $\mathcal{T}$ are symmetries with the transformation rules

$$\mathcal{C}\colon \{t \to +t, q \to -q, A \to -A, B \to -B, F \to -F\} \tag{A8}$$

$$\mathcal{T}\colon \{t \to -t, q \to +q, A \to -A, B \to +B, F \to -F\}. \tag{A9}$$

At $\theta = \pi$, on the other hand, the $\mathcal{C}$ and $\mathcal{T}$ transformations are symmetries as long as $2p - 1 = 0 \bmod N$.

The $\theta = 0$ symmetry condition $2p = 0 \bmod N$ can always be satisfied without violating the integrality of $p$ by setting $p = 0$ (or $N/2$ for even $N$). But the $\theta = \pi$ symmetry condition $2p - 1 = 0 \bmod N$ has much stronger consequences. For even $N$, it cannot be satisfied at all with integer $p$. This can be interpreted as a mixed 't Hooft anomaly between the discrete shift symmetry and both $\mathcal{C}$ and $\mathcal{T}$. Consequently, either one of the $\mathcal{C}$ and $\mathcal{T}$ symmetries must be spontaneously broken, or the shift symmetry $\mathcal{S}$ must be broken.

For odd $N$, the $\theta = \pi$ symmetry condition can be satisfied by. e.g., $p = (N-1)/2$, so one can preserve $\mathcal{C}$ and $\mathcal{T}$. This means that with appropriate choices of $p$ one can preserve $\mathcal{C}$ and $\mathcal{T}$ at either $\theta = 0$ or $\theta = \pi$. But the values of the discrete parameter $p$ necessary to keep $\mathcal{C}$ and $\mathcal{T}$ symmetries at $\theta = 0$ and $\theta = \pi$ are not the same. So, if one *defines* the theory with a fixed choice of $p$ which preserves $\mathcal{C}$ and $\mathcal{T}$ symmetries at $\theta = 0$, then one cannot trivially maintain all three discrete symmetries $\mathcal{C}$, $\mathcal{T}$, and $\mathcal{S}$ at $\theta = \pi$. The simplest possibility is that one of these symmetries should be spontaneously broken at $\theta = \pi$. In this sense, there is always a global inconsistency between the $\mathcal{C}$ and $\mathcal{T}$ symmetries and the $\mathcal{S}$ symmetry for any $N > 1$, but there is the slightly stronger condition of a mixed 't Hooft anomaly for even $N$.

Of course, this is a simple QM system, so one can back up the claims of the preceding paragraphs and verify the degeneracy of the ground states by either diagonalizing the Hamiltonian numerically or solving it semiclassically. Indeed, at $\theta = \pi$, time-reversal/charge conjugation breaks spontaneously for all $N > 1$.

## APPENDIX B: REPRESENTATIONS OF THE DIHEDRAL GROUP

In order to find the decomposition of states in terms of irreducible representation, we calculate the character associated with the conjugacy classes of $D_{2N}$. Recall that the character of a group element $g$ in a representation $R$ is given by $\chi_R(g) = \mathrm{tr} D_R(g)$, with $D_R(g)$ the group element $g$ in representation $R$. Expressing this character in terms of characters of the irreducible representations via orthogonality relations then allows us to find the decomposition of $R$.

### 1. Even $N$: $T_{N=2k}$ model

To find the characters, we want to find the general form of the $N$-dimensional representation, $R$, corresponding to how the $N$ translation eigenstates (Bloch states) $|k\rangle$ transform under $\mathcal{S} = s$ and $\mathcal{C} = r$. For example, in the $N = 4$ case, which corresponds to $D_8$ and $k = 2$, using (3.9) and (3.10) gives

$$r = \begin{pmatrix} \omega^{-1} & & & \\ & \omega^{-2} & & \\ & & \omega^{-3} & \\ & & & \omega^{-4} \end{pmatrix}, \quad s = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}. \tag{B1}$$

Generalizing the form of $r$ and $s$ above, it is straightforward to find the characters for arbitrary $N$. Note that nonzero contributions to a transformation's character correspond to states which are mapped back to themselves under such a transformation. Identifying such states will often be a useful tool in finding characters for arbitrary $N$. Obviously the identity element has character $N$. The generalization of $r$ to arbitrary $N$ is a diagonal matrix with all $N$th roots of unity, and as such the $\mathrm{tr} r = 0$. This also holds for $r^m$ for any $m = 1, \ldots, N-1$, since $r^m$ correspond to the $N/\gcd(N, m)$th roots of unity.

We see that $s$ maps precisely two minima back to themselves, and so it will have character 2. This holds for arbitrary $N = 2k$ since there will always be two elements where $N - p \bmod N = p$, namely $\frac{N}{2} = k$ and $N$. For the $N = 4$ case, $sr^2$ also maps two minima back to themselves and hence also has character 2. This follows more easily from the fact all members of a conjugacy class have the same character, and hence if the character of $s$ is 2, so, too, must $\{sr^{2b}\}$ be. However, $sr$ and $sr^3$ have character 0, since the two nonzero diagonal elements of $s$ will always pick out elements of $r^{2m+1}$ which are $\pi$ out of phase on the unit circle (i.e., $\omega^{-2}$ and $\omega^{-4}$ for $N = 4$ and $m = 0$ case). Explicitly, the nonzero elements correspond to the $N$th and $\frac{N}{2}$th positions, and the $N$th position is always $1^{2m+1} = 1$, and the $\frac{N}{2}$th position is $\omega^{-(N/2)(2m+1)} = \omega^{-Nm-N/2} = \omega^{-N/2} = -1$. Hence, the characters of the conjugacy classes (3.21) for arbitrary $N = 2k$ are

$$\chi_{\text{even}}^{\theta=0} = \{N, \underbrace{0, 0, \ldots, 0}_{k-1}, 0, 2, 0\}. \tag{B2}$$

The general character table for $D_{2N}$ is given in, e.g., Ref. [55]. We can use character orthogonality to find the decomposition of a representation. Namely, for a given representation $R$ with characters $\chi_R$, the number of a given irrep $R_1$ with characters $\chi_{R_1}$ is given by

$$\frac{1}{n}\sum_{i=1}^{K} s_i \chi_R \chi_{R_1} = \text{ of a } R_1 \text{ irrep in arbitrary representation,} \tag{B3}$$

where $n$ is the number of elements in the group (i.e., $2N$ for $D_{2N}$), $K$ is the number of conjugacy classes, and $s_i$ is the size of the $i$th conjugacy class. Since the only nonzero terms in the character table are those corresponding to classes $\{1\}$ and $\{sr^{2b}\}$, it is straightforward to perform the projections and find the decomposition of (3.23).

At $\theta = \pi$, the $N$ low-lying states transform under the action of $D_{4N}$ group elements (so, now $k = N$). The conjugacy classes and number of elements in each class are again given by (3.20) and (3.21), but now, $2N = M = k$. The representation here is slightly more complicated because we need to find the $N$-dimensional representation of $D_{4N}$. However, a natural definition is motivated by our definition $\tilde{\mathcal{S}} = \omega^{1/2}\mathcal{S}$, so we can take $\tilde{r} = \omega^{1/2} r$ where $r$ is the $N$-dimensional representation of $D_{2N}$. The form of $s$ follows from (3.10). For example, for $N = 4$, the $N$-dimensional representation given by

$$\tilde{r} = \begin{pmatrix} \omega^{-1/2} & & & \\ & \omega^{-3/2} & & \\ & & \omega^{-5/2} & \\ & & & \omega^{-7/2} \end{pmatrix}, \quad \tilde{s} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \tag{B4}$$

does the trick.

Once more, generalization to arbitrary $N$ is not difficult. The identity again has $\chi_I = N$; meanwhile, all $\tilde{r}$ still have $\chi = 0$ (since shifting the roots of unity uniformly by $\omega^{1/2}$ does not change their cancellation) with the exception of $\tilde{r}^{2N} = -1$, which has character $-N$. Now, $s$ maps $|p\rangle \to |N - p + 1\rangle$, and hence no elements are mapped back to themselves corresponding to zero character. Multiplication of $\tilde{s}$ by any combination of $\tilde{r}$ does not change the location of nonzero elements, so any combination $\tilde{s}\tilde{r}^i$ for $i = 1, \ldots, 2N$ also has zero trace. Thus, the characters are

$$\chi_{\text{even}}^{\theta=\pi} = \{N, \underbrace{0, 0, \ldots, 0}_{N-1}, -N, 0, 0\}. \tag{B5}$$

Once more, for the decomposition, it is only the nonzero components we should worry about, this time corresponding to the 1-element conjugacy classes $\{1\}$ and $\{r^{2N}\}$. Using Table I, $c_{2N(2m-1)} = -1$, and $c_{2N(2m)} = 1$, the decomposition in terms of irreducible characters yields (3.25).

### 2. Odd $N$: $T_{N=2k+1}$ model

The $N$-dimensional representation follows in a very similar manner as before. For example, $N = 5$ yields

$$r = \begin{pmatrix} \omega^{-1} & & & & \\ & \omega^{-2} & & & \\ & & \omega^{-3} & & \\ & & & \omega^{-4} & \\ & & & & \omega^{-5} \end{pmatrix}, \quad s = \begin{pmatrix} & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ & & & & 1 \end{pmatrix}. \tag{B6}$$

The characters for $r^m$ with $m = 1, \ldots, N - 1$ follow similarly. The primary difference here is the fact that $s$ will only bring a single element back to itself, and this is unchanged when multiplying by any power of $r$ since the $N$th diagonal position will always be $\omega^{-Nm} = 1$. Hence, the characters are given by

$$\chi_{\text{odd}}^{\theta=0} = \{N, \underbrace{0, 0, \ldots, 0}_{k}, 1\}. \tag{B7}$$

The characters for arbitrary odd $N$ are given in Table II. Using the orthogonality of characters (B3) for the nonzero elements, we find Eq. (3.29).

For $\theta = \pi$, we found a global inconsistency condition, which implied the group was centrally extended to $D_{4N=2(4k+2)}$. Note this has switched us from conjugacy classes of the form (3.20) with $M = 2N$ instead of those of (3.26), so we should use the character table of Table I. Building an $N$-dimensional representation for $D_{4N}$ from the $N$-dimensional representation of $D_{2N}$ follows in an analogous manner as before. For $N = 5$,

$$\tilde{r} = \omega^{1/2} \begin{pmatrix} \omega^{-1} & & & & \\ & \omega^{-2} & & & \\ & & \omega^{-3} & & \\ & & & \omega^{-4} & \\ & & & & \omega^{-5} \end{pmatrix}, \quad s = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}. \tag{B8}$$

Again, the identity and $\tilde{r}^{2N}$ yield $N$ and $-N$, respectively. We see from the above representation that $s$ will map one element back to itself. This generalizes for $sr^{2b}$ with $b = 1, \ldots, 2k + 1$ since this element will always be that which corresponds to $\omega^{1/2-(N+1)/2}$ and

$$\omega^{[1/2-(N+1)/2]m} = \omega^{-Nm/2} = \begin{cases} 1 & m \text{ even} \\ -1 & m \text{ odd} \end{cases}.$$

Hence, the characters of the conjugacy classes are

$$\chi_{\text{odd}}^{\theta=\pi} = \{N, \underbrace{0, 0, \ldots, 0}_{N-1}, -N, 1, -1\}. \tag{B9}$$

The character orthogonality takes slightly more work but follows in a similar manner. Using Table I, $(-1)^N = (-1)^{2k+1} = -1$, $c_{2N(2m-1)} = -1$, and $c_{2N(2m)} = 1$, the decomposition of (3.31) is found.

## APPENDIX C: DISCRETE SYMMETRIES OF YM ON $\mathbb{R}^3 \times S^1$

In this Appendix, we investigate the discrete symmetries of deformed YM in greater detail and justify why $CP$ is indeed the symmetry which interchanges extrema with the same $V_k$. Since the potential from which we derive these symmetries is a result of a nonperturbative dilute gas summation of monopole-instanton solutions, this necessarily requires a closer investigation of how such solutions transform under discrete symmetries. For completeness, we first review the monopole-instanton solutions of deformed YM. We then investigate how the degenerate extrema are related to one another and see how discrete symmetries act on these solutions.

### 1. Monopole-instanton solutions

Monopole-instanton solutions are found using the usual Bogomol'nyi-Prasad-Sommerfield trick on the Euclidean action (4.5) [23]. For simplicity, suppose $N = 2$. Then, we can express the action in terms of the chromoelectric and chromomagnetic fields from the non-Abelian field strengths (recall that $x^4$ is the compact direction and $\mu, \nu = 1, 2, 3$)

$$E^a_\mu = F^a_{\mu 4} = D^{ba}_\mu A^b_4 \qquad B^a_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}F^a_{\nu\rho} \tag{C1}$$

with $D^{ba}_\mu = \partial^\mu \delta^{ab} + \epsilon^{abc} A^c_\mu$ and $F^a_{\mu\nu} = \partial_{i\mu} A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu$, and so (4.5) with a nonzero $\theta$-term becomes

$$S_{\text{tree}} = \frac{L}{2g^2}\int d^3x (E^a_\mu \mp B^a_\mu)^2 + \left(\frac{i\theta L}{16\pi^2} \pm \frac{L}{g^2}\right)\int d^3x E^a_\mu B^{a\mu}, \tag{C2}$$

where the top (bottom) corresponds to the monopole (antimonopole) solution. We see the monopole and antimonopole then satisfy

$$E^a_\mu = \pm B^a_\mu \quad \Leftrightarrow \quad F^a_{MN} = \pm \tilde{F}^a_{MN} \tag{C3}$$

with $M, N = 1, 2, 3, 4$. The monopole solutions carry magnetic and topological charge, defined by

$$Q_T \sim \int d^3x E^a_\mu B^{a\mu}, \qquad Q^a_M \sim \int d^2x \hat{n}^\mu B^a_\mu. \tag{C4}$$

The standard $\mathbb{R}^4$ monopole solutions which arise from (C3) can be constructed such that they are independent of one spacetime coordinate and thus have the properties of pseudoparticles (codimension 1). When we dimensionally reduce from $\mathbb{R}^3 \times S^1$ to $\mathbb{R}^3$, as long as we choose the compactified direction to correspond to the direction in which our monopole solutions are independent, we will end up with a "monopole-instanton solution" (codimension 0). Monopole/antimonopole (instanton) solutions derived in this way are $x_4$ independent. It is also possible to find $x_4$-dependent solutions with the same action by allowing $\pm 1$ units of Kaluza-Klein momentum [56,57]. This results in a total of $N$ monopole solutions with action $S_0 \equiv 8\pi^2/g^2 N$, magnetic charge $\vec{\alpha}_a$, topological charge $1/N$, and $N$ antimonopoles with opposite magnetic and topological charges.

### 2. Monopole transformation properties

We now consider how the monopole and antimonopole solutions are changed under discrete transformations. This will allow us to understand how the Abelian $\sigma_a$ fields transform and ultimately the behavior of $\vec{\sigma}_k$ under these symmetries. Our results are summarized in Table III.

The monopole and antimonopole solutions are flipped under a parity transformation in $\mathbb{R}^3$, which we will denote $\mathcal{P}_{x_\mu}$. This takes $x_\mu \to -x_\mu$ and $A_\mu \to -A_\mu$, which flips the $E$-field but not the $B$-field. However, because of the $\hat{n}^\mu$ in the definition of $Q^a_M$ (C4), which must also flip under $\mathcal{P}_{x_\mu}$, this transformation *does* flip the magnetic charge. Note that a flip of the magnetic charge of the monopoles is equivalent to a transformation of $\vec{\sigma} \to -\vec{\sigma}$. Hence, since both magnetic and topological charge are flipped, this amounts to an interchange of monopoles and antimonopoles. Since the $\theta$-term is proportional to $Q_T$, this is a symmetry at only $\theta = 0$ and $\theta = \pi$. At $\theta = 0$, the invariance is trivial since the topological charge has no effect on the path integral. However, at $\theta = \pi$, the symmetry must be accompanied by a $2\pi$ shift of $\theta$. We can implement such a shift via our $\sigma_a$ variables by defining the action of $\mathcal{P}_{x_\mu}$ to be $\theta$ dependent,

$$\mathcal{P}_{x_\mu}\colon \sigma_a \to \begin{cases} -\sigma_a & \theta = 0 \\ -\sigma_a + \frac{2\pi(N-a+1)}{N} & \theta = \pi. \end{cases} \tag{C5}$$

TABLE III. Various discrete symmetries and how they transform $Q^a_M \sim \int d^2x \hat{n}^\mu B^a_\mu$ and $Q_T \sim \int d^3x E^a_\mu B^a_\mu$. A "+" sign denotes the charge is unchanged under the corresponding transformation, while a "−" sign indicates a flip in sign.

| Transformation | Definition | $Q_M$ | $Q_T$ | Holonomy eigenvalues |
|---|---|---|---|---|
| $\mathcal{P}_{x_\mu}$ | $x_\mu \to -x_\mu$, $A_\mu \to -A_\mu$ | − | − | Unchanged |
| $\mathcal{P}_t$ | $x_3 \to -x_3$, $A_3 \to -A_3$ | − | − | Unchanged |
| $\mathcal{R}$ | $x_4 \to -x_4$, $A_4 \to -A_4$ | + | − | $a \to N - a + 1$ |
| $\mathcal{C}$ | $A_M \to -A_M$ | − | + | $a \to N - a + 1$ |
| $\mathcal{CPR}$ | $x^M \to -x^M$ | + | + | Unchanged |

Similarly, consider the parity transformation in a single direction[7] of $\mathbb{R}^3$, which we will take to be $x_3$ and denote $\mathcal{P}_t$. This takes $x_3 \to -x_3$ and $A_3 \to -A_3$ and hence flips $B_1$, $B_2$, and $E_3$ but leaves the other components of the electric and magnetic field untouched. Since this flips each of the three terms showing up in the topological charge, the net effect is to flip the total topological charge. Additionally, since this flips $\hat{n}_3 \to -\hat{n}_3$, this also flips all three terms showing up in the magnetic charge and hence takes $Q_M^a \to -Q_M^a$. Thus, the net effect of $\mathcal{P}_t$ is identical to that of $\mathcal{P}_{x_\mu}$, just as one would expect. We will collectively refer to the two noncompact parity transformations as $\mathcal{P}$.

Charge conjugation takes $A_M \to -A_M$, which from (C2) flips both the electric and magnetic fields. As such, the magnetic charges of the monopoles are flipped, but the topological charges are unchanged. The symmetry thus leaves the $\theta$-term untouched, and hence this symmetry persists for all $\theta$. However, from our definition of the holonomy in (4.2), $\Omega$ is also affected charge conjugation. More specifically, charge conjugation has the net effect of rearranging the holonomy eigenvalues. In order to leave the theory unchanged, we define charge conjugation to be accompanied by rearrangement of the holonomy eigenvalues so that the net effect of the transformation is to leave the holonomy unchanged (see Ref. [9] for more details). At the level of the monopoles, the rearranging of said eigenvalues interchanges monopole labels as $a \to N - a + 1$. The combined effect of rearranging labels and flipping the charge means charge conjugation acts on $\sigma_a$ as $\sigma_a \to -\sigma_{N-a+1}$.

Finally, consider the transformation which takes $x_4 \to L - x_4$ and $A_4 \to -A_4$, which we call $\mathcal{R}$. From (C1), this flips the $E$-field but not the $B$-field and hence takes $Q_M^a \to Q_M^a$ and $Q_T \to -Q_T$. However, since a flip in the compact direction transforms $A_4$, it will also affect the holonomy in the same way that charge conjugation acted. Hence, we will also define the $\mathcal{R}$ transformation to come with $a \to N - a + 1$ relabeling [9]. As with the $\mathcal{P}$ transformations, since $\mathcal{R}$ flips the topological charge, we must accompany the transformation at $\theta = \pi$ with an appropriate shift,

$$\mathcal{R}\colon \sigma_a \to \begin{cases} \sigma_{N-a+1} & \theta = 0 \\ \sigma_{N-a+1} - \frac{2\pi(N-a+1)}{N} & \theta = \pi. \end{cases} \tag{C6}$$

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[7] It is tempting to identify this direction as "time" to match the existing literature. But the considerations here can be phrased in Euclidean space, and all one needs to derive, e.g., anomalies is to consider reflections which involve an odd number of directions. So, an identification of $x_4$ with time is possible but not necessary. In particular, we find it helpful to think of the $x_4$ direction as a spatial one.

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