Petrov type *D* equation on horizons of nontrivial bundle topology

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We consider three-dimensional isolated horizons (IHs) generated by null curves that form nontrivial $U(1)$ bundles. We find a natural interplay between the IH geometry and the $U(1)$-bundle geometry. In this context, we consider the Petrov type D equation introduced and studied in previous works [D. Dobkowski-Ryłko, J. Lewandowski, and T. Pawłowski, The Petrov type D isolated null surfaces, Classical Quantum Gravity **35**, 175016 (2018); D. Dobkowski-Ryłko, J. Lewandowski, and T. Pawłowski, Local version of the no-hair theorem, Phys. Rev. D **98**, 024008 (2018); J. Lewandowski and A. Szereszewski, The axial symmetry of Kerr without the rigidity theorem, Phys. Rev. D **97**, 124067 (2018); D. Dobkowski-Ryłko, W. Kamiński, J. Lewandowski, and A. Szereszewski, The Petrov type D equation on genus >0 sections of isolated horizons, Phys. Lett. B **783**, 415 (2018)]. From the four-dimensional spacetime point of view, solutions to that equation define isolated horizons embeddable in vacuum spacetimes (with cosmological constant) as Killing horizons to the second order such that the spacetime Weyl tensor at the horizon is of the Petrov type D. From the point of view of the $U(1)$-bundle structure, the equation couples a $U(1)$ connection, a metric tensor defined on the base manifold and the surface gravity in a very nontrivial way. We focus on the $U(1)$ bundles over two-dimensional manifolds diffeomorphic to 2-sphere. We have derived all the axisymmetric solutions to the Petrov type D equation. They set a four-dimensional family of horizons and there is a four-dimensional family of the Kerr-NUT-dS (AdS) spacetimes in the literature. A surprising result is, that generically, our horizons do not correspond to those spacetimes. It means that among the exact type D spacetimes there exists a new four-dimensional family of spacetimes that generalize the properties of the Kerr-(anti-)de Sitter black holes on one hand and the Taub-NUT spacetimes on the other hand.

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I. INTRODUCTION

The theory of nonexpanding horizons (NEHs) is often used to describe black holes [1]. It is, however, far more general and may also be applied to spacetimes containing cosmological horizons, null boundaries of the conformally completed asymptotically flat spacetimes [2], or black hole holograph construction of spacetimes about isolated horizons [3–5]. Properties of NEHs find their analogs in the black hole spacetimes, such as the black hole "thermodynamics" [6], uniqueness theorems [7], and the rigidity theorem [8]. The long-term program is to understand conditions satisfied by the geometry of NEHs that distinguish the horizons of physical black holes. In a case of NEHs embeddable in spacetime as a Killing horizon to the second order, the vacuum Einstein equations (possibly with a cosmological constant) and the Petrov type D of the spacetime Weyl tensor at the horizon amount to an equation on the Riemann geometry induced on the two-dimensional space of null generators and the 2-form representing the rotation [9,10]. The equation and solutions were investigated in the case of horizons that have the structure of a trivial principal fiber bundle. For rotating solutions, the only allowed topology of a cross section is that of a 2-sphere [11]. For bifurcated horizons, the type D equation implies the axial symmetry [12,13]. All the axially symmetric solutions were derived [10,14]. For every value of the cosmological constant, they form a two-dimensional family that can be parametrized by the area and angular momentum. In that sense, the equation has the properties of rigidity and no hair so well known in the global black hole theory [15]. Even though the Petrov type D equation was derived for nonextremal horizons, it is also an integrability condition for the condition satisfied by the geometry and rotation 1-form potential induced on two-dimensional spaces of null generators of extremal Killing horizons to

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the first order [1,7,9]. This condition is also known as the near horizon geometry (NHG) equation [16,17]. That relation between the type D and the NHG equation was used to show that in the case of nonzero genus the only solutions to the NHG equation are geometries of constant Gauss curvature and zero rotation 1-form potential [18].

In this paper, we consider the Petrov type D equation and the vacuum Einstein equations with cosmological constant for isolated horizons (IHs) of the structure of a nontrivial bundle; the Hopf bundle; or, more generally, the Dirac monopol bundle. Hence, the space of the null generators is topologically a 2-sphere; however, there is no global spacelike cross section. An example of spacetime containing such a horizon is the Taub-NUT spacetime [19]. We derive all the axisymmetric solutions to the Petrov type D equation. They set a three-dimensional family for every value of the cosmological constant. As could be expected, there emerges a new parameter: the topological charge times the surface gravity. The final result, however, is surprising. In the previous, trivial bundle case, the axisymmetric Petrov type D horizons are embeddable in the Kerr–(anti-)de Sitter spacetimes. The generic horizons we find in the current case turn out not to be embeddable in the known in the literature Kerr-NUT-dS (AdS) generalizations of the Kerr spacetimes. That means that our horizons define a new family of suitably generalized black hole spacetimes.

II. ISOLATED HORIZONS OF NONTRIVIAL $U(1)$-BUNDLE TOPOLOGY

In this section, we introduce a general definition of three-dimensional isolated horizons of which the null generators have the structure of nontrivial fibration. While eventually the horizons are surfaces in four-dimensional spacetimes, their intrinsic structure can be considered on its own, independently of an embedding. That is what we do in the first subsection below. In the second subsection we discuss the embedded IHs in the context of four-dimensional spacetime, the assumed symmetries, the Einstein constrains and also recall the Petrov type D equation. For a detailed derivation of the Petrov type D equation for IHs, see Ref. [9]. The derivation is local and applies also to the current case.

In this paper, we use the same abstract index notation [20] as in Ref. [9]:

(i) Indices of four-dimensional spacetime M tensors are denoted by lower greek letters: $\alpha, \beta, \gamma, \ldots = 1, 2, 3, 4$.

(ii) Tensors defined in three-dimensional space H carry indices denoted by lower latin letters: $a, b, c, \ldots = 1, 2, 3$.

(iii) Capital latin letters $A, B, C, \ldots = 1, 2$ are for the tensors defined on the two-dimensional space S of the null generators of H.

A. IH structure on a $U(1)$ bundle

A nontrivial bundle structure is a new element introduced in the IH theory in the current paper. Let

$$\Pi : H \to S \tag{1}$$

be a principal fiber bundle with the structure group $U(1)$. Denote by ℓ the fundamental vector field on H, that is, such that its flow coincides with the action of $U(1)$ on H. We normalize ℓ such that the parameter of the flow ranges the interval $[0, 2\pi]$.

Throughout this paper,

$$\dim H = 3. \tag{2}$$

On H, we introduce an IH geometry compatible with the bundle structure. It consists of the following:

(i) a degenerate metric tensor g_{ab} of the signature $0{+}{+}$, such that

$$\ell^a g_{ab} = 0 = \mathcal{L}_\ell g_{ab}; \tag{3}$$

(ii) a covariant derivative ∇_a on $T(H)$, torsion free and such that

$$\nabla_a g_{bc} = 0, \qquad [\mathcal{L}_\ell, \nabla_a] = 0. \tag{4}$$

The second condition means that ∇_a is invariant with respect to theaction of the $U(1)$ group on H. The same is true about g_{ab} due to the second equality in (3).

It follows that

$$\ell^a \nabla_a \ell^b = \kappa \ell^b, \tag{5}$$

and through out this paper, we are assuming that

$$\kappa = \text{const} \neq 0. \tag{6}$$

After assuming the Einstein constraints, the constancy of κ will be a necessary property, and the nonvanishing means that H is a nonextremal (nondegenerate) IH.

The key ingredient of the covariant derivative (for our paper) is the rotation 1-form potential ω_a defined as

$$\nabla_a \ell^b = \omega_a \ell^b, \tag{7}$$

and by (4), it satisfies

$$\mathcal{L}_\ell \omega_a = 0. \tag{8}$$

It follows from (6) that the 1-form

$$\tilde{\omega} := \frac{1}{\kappa} \omega \tag{9}$$

is a connection 1-form on the $U(1)$ bundle (1). Indeed, due to (7) and the second equation in (4), ω_a satisfies

$$\ell^a \tilde{\omega}_a = 1, \qquad \mathcal{L}_\ell \tilde{\omega}_a = 0. \tag{10}$$

The degenerate metric tensor g_{ab} induces on S a (genuine) metric tensor g_{AB} such that g_{ab} is its pullback,

$$g_{ab} = \Pi^*{}_{ab}{}^{AB} g_{AB}. \tag{11}$$

The area 2-form η_{AB} defined on S and corresponding to g_{AB} (and some orientation of S) may also be pulled back to H,

$$\eta_{ab} \coloneqq \Pi^*{}_{ab}{}^{AB} \eta_{AB}. \tag{12}$$

We use it to define a rotation pseudoscalar Ω,

$$\Omega \eta_{ab} \coloneqq d\omega_{ab} = \kappa d\tilde{\omega}_{ab}. \tag{13}$$

It satisfies

$$\ell^a \Omega_{,a} = 0; \tag{14}$$

hence, we consider Ω as a function on S. The rotation 1-form potential ω_a can be represented by locally defined in a neighborhood of each point $x \in S$, 1-forms ω_A such that

$$d\omega_{AB} = \Omega \eta_{AB}, \tag{15}$$

where Ω is a scalar function globally defined and regular on the entire manifold S.

B. Embedded IHs and the Petrov type D equation

In four-dimensional spacetime $(M, g_{\mu\nu})$ of the signature $-+++$, every IH $(H, \ell^a, g_{ab}, \nabla_a)$ introduced in the previous subsection is a null surface such that the intrinsic geometry (g_{ab}, ∇_a) coincides with the spacetime metric tensor $g_{\mu\nu}$ and the covariant derivative ∇_μ restricted to $T(H)$ (preserved by ∇_μ). Because of the intrinsic symmetries [the second equation in (3) and the second equation in (4)], IH H can be called a Killing horizon to the first order. Indeed, there exists an extension t of the vector field ℓ to a neighborhood of H in M, such that

$$\mathcal{L}_t g_{\mu\nu}{}_{|_H} = 0 = [\mathcal{L}_t, \nabla_\mu]_{|_H}. \tag{16}$$

Throughout this paper, we assume that the spacetime metric tensor $g_{\mu\nu}$ satisfies the vacuum Einstein equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{17}$$

where Λ is a cosmological constant and $G_{\mu\nu}$ is the Einstein tensor.

The constraints induced on $(H, \ell^a, g_{ab}, \nabla_a)$ by Einstein's equations are soluble explicitly in the nonextremal case (6) considered in this paper. The degenerate metric tensor g_{ab} and the rotation 1-form ω_a can be set freely on H [modulo (3), (6), and (8)], and they determine the remaining ingredients of ∇_a.

Henceforth, about the vector field t (16) and the spacetime Weyl tensor $C_{\mu\nu\alpha\beta}$, we assume a stronger condition, namely,

$$\mathcal{L}_t C_{\mu\nu\alpha\beta}{}_{|_H} = 0. \tag{18}$$

That property of H may be called a Killing (or stationary isolated) horizon to the second order. That assumption does not constraint the intrinsic nonextremal IH H geometry (g_{ab}, ∇_a). Instead, via the Einstein equations, it determines all the components of the spacetime Weyl tensor at H by g_{ab} and ω_a. Therefore, the assumption that the Petrov type of the Weyl tensor at H is D turns into an equation on (g_{AB}, Ω). We consider that equation in the next section. However, before doing that, we would like to briefly give the idea on the spacetime elements of the problem and sketch the derivation presented in Ref. [9].

In the IH framework, we use adapted null frames (see Ref. [9], Sec. III B). Each of them consists of a real null vector field ℓ^μ tangent to H and coinciding thereon with the vector field t^μ (16), another real null vector field n^μ orthogonal to a foliation of H by space like 3-sections, and two complex valued vector fields m^μ tangent to the foliation. The foliation and the frame are preserved by the flow generated by the vector field ℓ^μ. The Weyl tensor complex components (see Ref. [9], Sec. III C) Ψ_0, Ψ_1, Ψ_2, and Ψ_3 are automatically guaranteed to be constant along the null generators of H (with the first two vanishing). So, the condition (18) is on Ψ_4 only. Moreover, in the consequence of the Bianchi identities, the components Ψ_3 and now also Ψ_4 can be expressed by Ψ_2 and its first and second derivatives. Next, the Weyl tensor is of the Petrov type D iff it admits two double principal null directions (PNDs). The vector field ℓ^μ is already tangent to a double PND (the vanishing of Ψ_0 and Ψ_1). Hence, the type D assumption concerns the remaining transversal PNDs. For a generic type D horizon, the transversal null vector n^μ is not a PND for any adapted null frame. However, the existence of a second double PND turns into an algebraic condition, namely,

$$3\Psi_2\Psi_4 - 2{\Psi_3}^2 = 0. \tag{19}$$

After expressing Ψ_3 and Ψ_4 by Ψ_2 and its derivatives, the equation somewhat magically takes a compact form. We remind the reader of it in the next section.

III. PETROV TYPE D EQUATION

Given an IH structure $(H, \ell^a, g_{ab}, \nabla_a, S)$, introduced above, the Petrov type D equation is imposed on the Riemaniann metric g_{AB} and the rotation pseudoscalar Ω defined on S. We will also use the Gauss curvature K of g_{AB},

$$^{(2)}R_{AB} =: K g_{AB}, \tag{20}$$

where $^{(2)}R_{AB}$ is the Ricci tensor of the metric tensor g_{AB}. To write the equation, we introduce a complex null coframe m_A such that the metric g_{AB} and area 2-form η_{AB} take the following form:

$$g_{AB} = m_A \bar{m}_B + m_B \bar{m}_A, \quad \eta_{AB} = i(\bar{m}_A m_B - \bar{m}_B m_A). \tag{21}$$

The Weyl tensor is of the type D along the generator $\Pi^{-1}(x)$ of the horizon H, if and only if Eq. (22), which we refer to as the type D equation, holds true at the point $x \in S$,

$$\bar{m}^A \bar{m}^{B\,(2)}\nabla_A {}^{(2)}\nabla_B \left(K - \frac{\Lambda}{3} + i\Omega\right)^{-\frac{1}{3}} = 0, \tag{22}$$

where $^{(2)}\nabla_A$ is the torsion-free, metric covariant derivative defined by g_{AB}, and the term in the brackets does not vanish at the point x, namely,

$$K - \frac{\Lambda}{3} + i\Omega \neq 0. \tag{23}$$

This function is related to the only nonzero invariant (given ℓ) component

$$\Psi_2 = -\frac{1}{2}(K + i\Omega) + \frac{\Lambda}{6}. \tag{24}$$

of the type D Weyl tensor, and if that component vanishes, then all the Weyl tensor vanishes at that point.

We will solve Eq. (22) by assuming that the base manifold S (1) is diffeomorphic to the 2-sphere,

$$S = S_2. \tag{25}$$

In that case, all the $U(1)$ bundles are numbered by integers. An integer m corresponding to H can be calculated from the curvature of the $U(1)$-connection 1-form $\tilde{\omega}$, which passes to a condition on the rotation pseudoinvariant Ω,

$$\int_{S_2} \Omega \eta_{AB} = 2\pi m \kappa =: 2\pi n. \tag{26}$$

For each Ω, there exist 1-forms ω^+ and ω^- defined on S_2 apart from the southern and northern poles, respectively, such that

$$d\omega^{\pm}_{AB} = \Omega \eta_{AB}. \tag{27}$$

Incidentally, from the mathematical point of view, the case $\kappa = 1$ seems to be the most interesting. However, we do not see any reason implied by GR that would restrict κ to that value.

We also assume that the metric tensor g_{AB} and the rotation pseudoscalar Ω invariantly defined on S admit an axial symmetry. Consequently, we choose the coordinates adapted to the symmetry (see the Appendix) in which the 2-metric tensor g_{AB} reads

$$g_{AB} dx^A dx^B = R^2 \left(\frac{1}{P(x)^2} dx^2 + P(x)^2 d\varphi^2\right), \tag{28}$$

where $x \in [-1; 1]$, $\varphi \in [0; 2\pi]$ and R is the area parameter [7,21]. The frame vector and its dual take the form

$$m^A \partial_A = \frac{1}{R\sqrt{2}} \left(P(x)\partial_x + i\frac{1}{P(x)}\partial_\varphi\right),$$
$$\bar{m}_A dx^A = \frac{R}{\sqrt{2}} \left(\frac{1}{P(x)} dx - iP(x) d\varphi\right). \tag{29}$$

The above coordinate system is not well defined at $x = \pm 1$; therefore, to derive the regularity conditions,[1] we use the relation between the metric (28) and the standard 2-sphere coordinates, namely,

$$R^2 \left(\frac{1}{P(x)^2} dx^2 + P(x)^2 d\varphi^2\right) = \Sigma^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{30}$$

For an axisymmetric scalar function f defined globally on S_2 (as one of the functions K, Ω, and Ψ_2), the differentiability condition at the poles $x = \pm 1$ reads

$$\partial_\theta f = 0. \tag{31}$$

This condition is equivalent to

$$P \partial_x f = 0, \tag{32}$$

and it will be assumed for functions K and Ω (or in other words for Ψ_2). For the metric (28) to be twice differentiable at the poles, the following boundary conditions must be satisfied:

$$P^2|_{x=\pm 1} = 0, \tag{33}$$

$$\partial_x P^2|_{x=\pm 1} = \mp 2. \tag{34}$$

[1] For more details, see the Appendix.

The condition (26) boils down to

$$\int_{-1}^{1} \Omega R^2 dx = m\kappa = n. \tag{35}$$

Equation (22) in the coordinates adapted to the axial symmetry reads

$$\partial_x^2 \Psi_2 = 0, \tag{36}$$

and its general solution is of the form

$$\Psi_2 = (c_1 x + c_2)^{-\frac{1}{3}}, \tag{37}$$

where c_1 and c_2 are complex constants. Now, comparing it to Eq. (24) and expressing the Gaussian curvature in the introduced coordinates yields

$$\frac{1}{(c_1 x + c_2)^3} = \frac{1}{4R^2}\partial_x^2 P^2 - \frac{1}{2} i\Omega + \Lambda'. \tag{38}$$

This equation can be solved for the values of the complex parameters c_1 and c_2 that satisfy solubility conditions.

IV. SOLUTION TO THE PETROV TYPE D EQUATION ON THE NONTRIVIAL BUNDLE TOPOLOGY

In the following section, we will solve Eq. (38), first for the case in which the constant c_1 vanishes and later for c_1 taking arbitrary (nonzero) complex values. In the case in which c_2 vanishes, the geometry is not well defined (see below); therefore, we will exclude it from our considerations. We have used a similar approach in Ref. [14], in which we solved the type D equation for the trivial bundle, that is, $n = 0$.

A. Solution for vanishing c_1

The type D equation (38) with the vanishing complex constant c_1 reads

$$\frac{4R^2}{c_2^3} = \partial_x^2 P^2 - 2iR^2\Omega + 4R^2\Lambda', \tag{39}$$

where Λ' (as in Ref. [14]) denotes a rescaled cosmological constant:

$$\Lambda' \coloneqq \Lambda/6. \tag{40}$$

Integrating both sides of Eq. (39) and using boundary conditions (34) and (35) yields

$$c_2^3 = \frac{4R^2}{-2 - in + 4\Lambda' R^2}. \tag{41}$$

We then find the solution to Eq. (39),

$$P^2 = 1 - x^2, \tag{42}$$

and

$$\Omega = \frac{n}{2R^2}. \tag{43}$$

Now, we find the rotation 1-form potential $\omega^{\pm}$. Since $\omega^{\pm}$ has to satisfy Eq. (27) and the regularity conditions at $x = \pm 1$, namely,

$$\omega^{+}|_{x=1} = 0 = \omega^{-}|_{x=-1}, \tag{44}$$

it follows that

$$\omega^{\pm}_A dx^A = \frac{n}{2}(x \mp 1)d\varphi. \tag{45}$$

Consequently, in case of $c_1 = 0$, the solution to the type D equation can be parametrized by two parameters: R^2 and n. Moreover, if $n = 0$, then $\omega^{\pm}$ vanishes. The found solution is embeddable in the Taub–NUT–(anti-)de Sitter spacetime, which is of the type D and is defined by the static spacetime metric tensors satisfying the vacuum Einstein equations with the cosmological constant Λ [19], namely,

$$ds^2 = -\frac{Q}{\rho^2}\left[dt - 4l\sin^2\left(\frac{1}{2}\theta\right)d\phi\right]^2 + \frac{\rho^2}{Q}dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{46}$$

where

$$\rho^2 = r^2 + l^2; \tag{47}$$

$$Q = r^2 - 2Mr - l^2 - \Lambda\left(-l^4 + 2l^2r^2 + \frac{1}{3}r^4\right). \tag{48}$$

Its extension contains Killing horizons, which are parametrized by the roots of the equation:

$$r_H^2 - 2Mr_H - l^2 - \Lambda\left(-l^4 + 2l^2r_H^2 + \frac{1}{3}r_H^4\right) = 0. \tag{49}$$

Each of such horizons, that is, not nonextremal, is one of the type D horizons that we consider. The 2-metric on the (space of the null generators of) Killing horizon admits spherical symmetry,

$$ds_2^2 = \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{50}$$

and the relation between coordinates x, φ, and θ, ϕ is the following:

$$x(\theta) = -\cos\theta, \qquad \varphi(\phi) = \phi. \tag{51}$$

Furthermore, we express the parameters R^2 and n in terms of the parameters of the Taub-NUT horizon, namely, r and l. From the comparison of the area of the S_2 metrics on the horizon, we find that

$$R^2 = r_H^2 + l^2. \tag{52}$$

The Killing vector field

$$\xi = M \frac{\partial}{\partial t} \tag{53}$$

on the horizon defines our generator of the null symmetry (we have introduced the factor M to make the vector field dimensionless as above). It is such that on the horizon

$$\ell = \xi_{|H}. \tag{54}$$

Next, we use the formula

$$(\xi^\mu \xi_\mu)_{;\nu|H} = -2\kappa \xi_\nu \tag{55}$$

to calculate the surface gravity κ on the horizon:

$$\kappa = \frac{-M}{2r_H}\left(\frac{-\Lambda r_H^4 + (1-2\Lambda l^2) r_H^2 + (1-\Lambda l^2) l^2}{r_H^2 + l^2}\right). \tag{56}$$

In case of the Taub–NUT–(anti-)de Sitter metric (46), the 1-form $\tilde{\omega}^+$ reads

$$\tilde{\omega}^-_A dx^A = -\frac{4l}{M}\sin^2\left(\frac{1}{2}\theta\right) d\phi. \tag{57}$$

Furthermore, we use the obtained expressions for κ (56) and $\tilde{\omega}$ (57) and plug them into (9), (26), and (27) to find the relation between n and the parameters r and l, namely,

$$\begin{aligned} n &= \frac{-4l\kappa}{M} \\ &= \frac{2l}{r_H}\left(\frac{-\Lambda r_H^4 + (1-2\Lambda l^2) r_H^2 + (1-\Lambda l^2) l^2}{r_H^2 + l^2}\right). \end{aligned} \tag{58}$$

To conclude, the found horizon for the vanishing c_1 is embeddable in the quotient of Taub–NUT–(anti-)de Sitter spacetime by the symmetry $t \mapsto t + 2\pi M$, and the correspondence between our parameters and those of the Taub–NUT–(anti-)de Sitter horizon is listed in (52) and (58). The embedding, obviously, is not unique and depends on the chosen symmetry.

B. Solution for arbitrary nonzero c_1

Now, assuming that neither complex constant vanishes, we integrate Eq. (38) twice to obtain

$$P^2 = 2R^2 \mathrm{Re}\left[\frac{1}{c_1^2 (c_1 x + c_2)}\right] - 2R^2 \Lambda' x^2 + Cx + D. \tag{59}$$

Using the boundary conditions (33) and (34), we find integration constants C and D:

$$\begin{aligned} C &= -2 + 4R^2\Lambda' + 2R^2 \mathrm{Re}\left[\frac{1}{c_1(c_1+c_2)^2}\right] \\ &= -2R^2 \mathrm{Re}\left[\frac{1}{c_1(c_1^2 - c_2^2)}\right], \end{aligned} \tag{60}$$

$$D = 2R^2 \mathrm{Re}\left[\frac{2c_1 + c_2}{c_1^2 (c_1+c_2)^2}\right] + 2R^2\Lambda' - 2. \tag{61}$$

Moreover, integrating Eq. (38) once and using (34) and (35) yields the relation between R^2, Λ', and n and parameters c_1 and c_2:

$$R^2 = \frac{-2 - in}{4\left(\frac{c_2}{(c_1^2 - c_2^2)^2} - \Lambda'\right)}. \tag{62}$$

The area radius R^2 has to be real (and positive); therefore, the following equation must be satisfied:

$$Im\left[\frac{4\left(\frac{c_2}{(c_1^2 - c_2^2)^2} - \Lambda'\right)}{2 + in}\right] = 0. \tag{63}$$

Consequently, we can choose the parametrization

$$\frac{c_2}{(c_1^2 - c_2^2)^2} = \frac{1}{\gamma} - i\frac{1}{2} n\left(\Lambda' - \frac{1}{\gamma}\right), \tag{64}$$

where γ is a real parameter. The last equality in (60) yields

$$\frac{1}{2R^2} - \Lambda' = \mathrm{Re}\left[\left(\frac{c_1}{c_2} - 1\right)\frac{c_2}{(c_1^2 - c_2^2)^2}\right], \tag{65}$$

which we use to introduce yet another real parameter η,

$$\frac{c_2}{c_1} = \frac{\eta n}{4\Lambda' R^2 - 2} + i\eta = \frac{1}{2}\eta n(\Lambda'\gamma - 1) + i\eta, \tag{66}$$

where we have assumed that

$$1\text{–}2\Lambda' R^2 \neq 0. \tag{67}$$

Using such parametrization, the expression for P^2 reads

$$P^2 = \frac{(1-x^2)\left((x - \frac{1}{2}\eta n(1-\Lambda'\gamma))^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma}\right)}{(x - \frac{1}{2}\eta n(1-\Lambda'\gamma))^2 + \eta^2}. \tag{68}$$

For $n = 0$, Eq. (68) reduces to the case known from Refs. [9,14], namely,

$$P^2 = \frac{(1-x^2)(\eta^2(1-\Lambda'\gamma)-\Lambda'\gamma x^2+1)}{(1-\Lambda'\gamma)(x^2+\eta^2)}. \tag{69}$$

We can now calculate the rotation 1-form potential $\omega^{\pm}$, just as we previously did for $c_1 = 0$. Taking the imaginary part of both sides of the type D equation (38) yields

$$\Omega = \mathrm{Im}\left[\frac{-2}{c_1^3(x+\frac{c_2}{c_1})^3}\right] = \mathrm{Im}\left[\frac{2i(1-\eta^2(\frac{1}{2}n(\Lambda'\gamma-1)+i)^2)}{\eta\gamma(x+\frac{1}{2}\eta n(\Lambda'\gamma-1)+i\eta)^3}\right], \tag{70}$$

and therefore

$$\omega^{\pm} = \mathrm{Im}\left[\frac{i(1-\eta^2(\frac{n}{2}(\Lambda'\gamma-1)+i)^2)^2}{2\eta(1-\Lambda'\gamma)(x+\eta(\frac{n}{2}(\Lambda'\gamma-1)+i))^2}+iC^{\pm}\right]d\phi. \tag{71}$$

Since $\omega^{\pm}$ satisfies the boundary conditions (44), it follows that

$$C^{\pm} = \frac{1}{2\eta(1-\gamma\Lambda')}\left[1-\eta^2+\frac{n^2\eta^2}{4}(1-\gamma\Lambda')^2 \mp n\eta(1-\gamma\Lambda')\right]. \tag{72}$$

Taking all into consideration, the family of solutions (for $c_1 \neq 0$) to the type D equation (38) can be expressed in terms of three real parameters: η, γ, and n.

In case of

$$1-2\Lambda' R^2 = 0, \tag{73}$$

we introduce the following parametrization:

$$\frac{c_2}{(c_1^2-c_2^2)^2} = -\frac{1}{2}in\Lambda' \qquad \frac{c_2}{c_1} = -\frac{n\Lambda'}{2\alpha}. \tag{74}$$

It is easy to see that both expressions vanish for $n = 0$, which is consistent with the result obtained in Ref. [14], in which the $R^2 = \frac{1}{2\Lambda'}$ case has been excluded for the geometry to be well defined. The frame coefficient P^2 takes the following form:

$$P^2 = 1-x^2, \tag{75}$$

and

$$\Omega = -\frac{2\alpha(1-(\frac{n\Lambda'}{2\alpha})^2)^2}{(x-\frac{n\Lambda'}{2\alpha})^3}. \tag{76}$$

The 1-form $\omega^{\pm}$ reads

$$\omega^{\pm} = \left[\frac{\alpha(1-(\frac{n\Lambda'}{2\alpha})^2)^2}{2\Lambda'(x-\frac{n\Lambda'}{2\alpha})^2}+C^{\pm}\right]d\varphi, \tag{77}$$

where

$$C^{\pm} = -\frac{\alpha}{2\Lambda'}\left(1\pm\frac{n\Lambda'}{2\alpha}\right)^2. \tag{78}$$

V. SUMMARY

We have considered three-dimensional IHs generated by null curves that form nontrivial $U(1)$ bundles. In the nonextremal IH case, the rotation 1-form potential corresponds to a connection on the bundle times the surface gravity. Hence, there is a natural interplay between the IH geometry and the $U(1)$-bundle geometry. In this context, we have considered the Petrov type D equation (22) introduced and studied in previous works (Refs. [9,11,12,14]). From the four-dimensional spacetime point of view, solutions of that equation define isolated horizons embeddable in vacuum spacetimes (with cosmological constant) as Killing horizons to the second order such that the spacetime Weyl tensor at the horizon is of the Petrov type D. From the point of view of the $U(1)$-bundle structure, the equation couples a $U(1)$ connection, a metric tensor defined on the base manifold, and the surface gravity in a very nontrivial way. An example of known spacetime containing an IH of the nontrivial bundle structure is the Taub-NUT solution. The Killing horizon in that spacetime has the structure of the Hopf fibration of S_3 over S_2, and it is of the Petrov type D (along with all the spacetime). In the current paper, we have focused on the $U(1)$ bundles over two-dimensional manifolds diffeomorphic to the 2-sphere. A general bundle of that type is characterized by an integer topological charge and is mathematically equivalent to the Dirac monopole; however, the role of the electromagnetic vector potential of the original Dirac monopole in our case is played by the rotation 1-form potential divided by the surface gravity. We have derived all the axisymmetric solutions to the Petrov type D equation. Below, we summarize our results. The analysis is followed by our final comments.

The solutions we have derived are determined by the cosmological constant Λ, the area radius R^2, a function P [R and P give rise to the metric (28)], and the rotation pseudoscalar Ω (13). From g_{AB} and Ω, one can reconstruct the remaining element of the IH, namely, derivative ∇_a (see Ref. [9]). The topological charge m (integer) of the $U(1)$-bundle structure of the horizon and the surface gravity κ set the parameter n that features in Table I. The list of $(\Lambda, R^2, P, \Omega)$ we have found is divided into three classes. We discuss them now.

The first class consists of the metric tensors g_{AB} of constant Gaussian curvature

TABLE I. Solutions to the type D equation on horizons of nontrivial bundle topology divided into three classes.

Possible solutions to type D equation		
Class I	Class II	Class III
$R^2 > 0$	$R^2 = \frac{1}{2\Lambda'}$ and $\Lambda' > 0$	$R^2 \neq \frac{1}{2\Lambda'}$
$P^2 = 1 - x^2$	$P^2 = 1 - x^2$	$P^2 = \frac{(1-x^2)((x-\frac{1}{2}\eta n(1-\Lambda'\gamma))^2+\eta^2+\frac{1-x^2}{1-\Lambda'\gamma})}{(x-\frac{1}{2}\eta n(1-\Lambda'\gamma))^2+\eta^2}$
$\Omega = \frac{n}{2R^2}$	$\Omega = -\frac{2\alpha(1-(\frac{n\Lambda'}{2\alpha})^2)^2}{(x-\frac{n\Lambda'}{2\alpha})^3}$	$\Omega = \mathrm{Im}[\frac{2i(1-\eta^2(\frac{1}{2}n(\Lambda'\gamma-1)+i)^2)}{\eta\gamma(x+\frac{1}{2}\eta n(\Lambda'\gamma-1)+i\eta)^3}]$

$$K = \frac{1}{R^2} \tag{79}$$

and constant rotation scalar Ω related in the table with n and R^2 and is embeddable in the Taub–NUT–(anti-)de Sitter spacetime. The cosmological constant is arbitrary in this class and unrelated to K and Ω. Hence, that class is parametrized freely by three real parameters $\Lambda', R^2 > 0$, and n.

Class 2 in Table I is characterized by the special relation between R^2 and $\Lambda =: 6\Lambda'$,

$$R^2 = \frac{1}{2\Lambda'}, \tag{80}$$

and by the condition

$$\partial_A \Omega \neq 0. \tag{81}$$

The class is parametrized by real parameters Λ', n, and α constrained by certain conditions discussed now. It follows that here we can only consider positive Λ' for the area radius R^2 to be positive:

$$\Lambda' > 0. \tag{82}$$

Furthermore, the frame coefficient takes the form (75), and it is clear that it is non-negative for all x in the domain. However, one has to pay attention to the behavior of Ψ_2, Eq. (37), on the domain $x \in [-1, 1]$ and require it to be well defined, which means

$$\left|\frac{n\Lambda'}{2\alpha}\right| > 1. \tag{83}$$

The third class is parametrized by real parameters Λ', η, γ, and n. First, we specify their domains, in which the metric g_{AB} is well defined and at least four times differentiable, including the poles of the sphere. We want the area radius R^2 to be positive, and therefore

$$R^2 = \frac{1}{2}\frac{\gamma}{\Lambda'\gamma - 1} > 0 \Leftrightarrow \Lambda' > \frac{1}{\gamma}. \tag{84}$$

Also, the frame coefficient P^2 has to be positive for $x \in (-1, 1)$,

$$P^2 > 0 \Leftrightarrow \left(x - \frac{1}{2}\eta n(1 - \Lambda'\gamma)\right)^2 + \eta^2 + \frac{1 - x^2}{1 - \Lambda'\gamma} > 0, \tag{85}$$

and that occurs whenever one of the following is satisfied:

(i) $\gamma < 0$;

(ii) $(\gamma > 0) \wedge \left(\eta^2 > \frac{-\Lambda'\gamma}{(1-\Lambda'\gamma)((1-\Lambda'\gamma)^2\frac{n^2}{4}+\Lambda'\gamma)}\right)$;

(iii) $(\gamma > 0) \wedge \left(\eta^2 < \frac{-\Lambda'\gamma}{(1-\Lambda'\gamma)((1-\Lambda'\gamma)^2\frac{n^2}{4}+\Lambda'\gamma)}\right)$

$$\wedge \left(|\eta n| < \frac{\Lambda'\gamma + \sqrt{((1-\Lambda'\gamma)\eta^2+1)\Lambda'\gamma+\frac{1}{4}\eta^2 n^2(1-\Lambda'\gamma)^3}}{\frac{1}{2}(1-\Lambda'\gamma)}\right).$$

Moreover, c_2 in (37) has to be nonzero, because otherwise the Ψ_2 component of the Weyl tensor would be ill defined at $x = 0$; it follows that η cannot vanish at any case:

$$\eta \neq 0. \tag{86}$$

Several remarks are in order.

The first remark concerns reconstruction of a $U(1)$ bundle and the IH structure from the data provided above. Let us fix arbitrarily a topological charge,

$$m \neq 0, \tag{87}$$

and a corresponding $U(1)$ bundle $\Pi : H \to S_2$. Then, for all data from the table such that

$$n \neq 0, \tag{88}$$

we set the surface gravity κ to be

$$\kappa = \frac{n}{m}, \tag{89}$$

and we can reconstruct a unique (modulo automorphisms of H) IH structure g_{ab}, ∇_a.

For

$$n = 0, \tag{90}$$

on the other hand, Table I reduces to the earlier-derived [14] axisymmetric solutions to the Petrov type D equation the horizon of the $\mathbb{R} \times S^2$ topology. Those horizons can be defined by a subgroup of $\mathbb{R}$ and become the trivial $U(1)$ bundle $U(1) \times S_2$.

The last remark concerns the issue of embedding the generic IHs of the Petrov type D found in the current paper in the known exact solutions to Einstein's equations. We derived a four-dimensional family of vacuum Petrov type D horizons, and there is known four-dimensional family of the Kerr-NUT-dS (AdS) spacetimes in the literature. A surprising observation is, that generically, our horizons do not correspond to those spacetimes. It means that among the exact type D spacetimes there exists a new four dimensional family of spacetimes that generalise the properties of the Kerr-(anti) de Sitter black holes on one hand and the Taub-NUT spacetimes on the other hand. In the trivial bundle case of $H = \mathbb{R} \times S_2$ considered in previous works [14], a generic axisymmetric Petrov type D IH (H, g_{ab}, ∇_a) can be embedded in one of the non-extremal Schwarzschild–(anti-)de Sitter/Kerr–(anti-)de Sitter spacetimes. In the current case, however, for the nonzero values of n and a generic solution in Table I, we were not able to identify any generalized black hole solution that can accommodate it. That result requires a better understanding. It may be an indication of an existence of a new family of spacetimes like Kerr–(anti-) de Sitter–Taub–NUT ones.

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APPENDIX: COORDINATES ADAPTED TO THE AXIAL SYMMETRY

The conditions (33) and (34) are necessary for the metric tensor (28) to be continuous and differentiable at the poles [14]. Now, we will show (33) and (34) are also sufficient. Consider a 2-sphere metric with a conformal factor Σ (independent of φ because of the symmetry),

$$g_{AB}dx^A dx^B = \Sigma^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{A1}$$

and the transformation

$$dx = \frac{\Sigma^2 \sin\theta}{R^2} d\theta, \tag{A2}$$

where R^2 is defined to be the area radius satisfying

$$A = 4\pi R^2.$$

We now introduce the frame coefficient:

$$P^2 = \frac{\Sigma^2 \sin^2\theta}{R^2}.$$

Calculating the area of the transformed metric g_{AB} yields

$$A = R^2(x_1 - x_0)2\pi. \tag{A3}$$

Since x has been defined up to an additive constant, by setting $x_1 = 1$ from the above equation, we obtain that $x_0 = -1$. The coordinate φ is such that the (normalized) infinitesimal axial symmetry equals ∂_φ and the curves $\varphi =$ const are orthogonal to the infinitesimal symmetry. The metric tensor g_{AB} reads

$$g_{AB}dx^A dx^B = R^2\left(\frac{1}{P(x)^2}dx^2 + P(x)^2 d\varphi^2\right). \tag{A4}$$

Next, we check whether condition for the lack of conical singularity, namely,

$$\lim_{x\to\pm 1} \partial_x P^2 = \mp 2, \tag{A5}$$

implies that the metric (A1) is differentiable, that is, if $\Sigma_{,\theta} = 0$ is satisfied on the poles. Using the relation between P and Σ, we obtain

$$\Sigma_{,\theta} = \partial_\theta\left(\frac{PR}{\sin\theta}\right) = \frac{RP_{,\theta} - \Sigma\cos\theta}{\sin\theta} = \frac{\frac{\Sigma^2\sin\theta}{R}P_{,x} - \Sigma\cos\theta}{\sin\theta} = \Sigma\frac{PP_{,x} - \cos\theta}{\sin\theta}. \tag{A6}$$

Now, taking a limit as θ approaches 0 (or π) and using l'Hôpital's rule, we find

$$\Sigma_{,\theta}|_{\theta=0,\pi} = \lim_{\theta\to 0,\pi} \frac{RP(1 + \frac{1}{2}\frac{P^2}{\sin^2\theta}\partial_x^2 P^2)}{\cos\theta}. \tag{A7}$$

As long as the limit of $\frac{P}{\sin\theta}$ as θ approaches 0 (or π) is finite, the expression on the right-hand side will vanish. To calculate this limit, we will use the obtained expression for P^2,

$$P^2 = \frac{(1-x^2)((x - \frac{1}{2}\eta n(1-\Lambda'\gamma))^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma})}{(x - \frac{1}{2}\eta n(1-\Lambda'\gamma))^2 + \eta^2}, \tag{A8}$$

and plug it in the following:

$$\frac{1}{P^2}dx = \frac{1}{\sin\theta}d\theta. \tag{A9}$$

We can use new parameters, $b \coloneqq -\frac{1}{2}\eta n(1-\Lambda'\gamma)$ and $g = \frac{1}{1-\Lambda'\gamma}$, and simplify $\frac{1}{P^2}$ as follows:

$$\frac{1}{P^2} = \frac{1}{1-x^2} - \frac{g}{x^2(1-g) + 2bx + \eta^2 + b^2 + g}. \tag{A10}$$

That way, integrating the left-hand side of Eq. (A9) yields

$$\begin{aligned} L &= \int \left(\frac{1}{1-x^2} - \frac{g}{x^2(1-g) + 2bx + \eta^2 + b^2 + g} \right) dx \\ &= \int \left(\frac{1}{1-x^2} - \frac{g}{(1-g)} \frac{1}{x^2 + 2x\frac{b}{1-g} + \frac{\eta^2+b^2+g}{1-g}} \right) dx \\ &= \log\left(\sqrt{\frac{x+1}{1-x}} \right) - \frac{2G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) + C \\ &= \log\left(C' \sqrt{\frac{x+1}{1-x}} \right) - \frac{2G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right), \end{aligned} \tag{A11}$$

where

$$G = \frac{g}{1-g}; \qquad A = \frac{\eta^2 + b^2 + g}{1-g}; \qquad B = \frac{2b}{1-g},$$

and we assumed that $4A - B^2 > 0$; otherwise, the term under the square root would take the form $-4A + B^2$ and the sign in front of the arctan function would change. Next, we integrate the right-hand side of Eq. (A9) to obtain

$$\begin{aligned} R &= \int \frac{1}{\sin\theta} d\theta = -\log\left(\cot\theta + \frac{1}{\sin\theta} \right) + D \\ &= \log\left(\frac{\sin\theta}{\cos\theta + 1} \right) + D. \end{aligned} \tag{A12}$$

Using expressions (A11) and (A12), we find θ as a function of x:

$$\theta(x) = 2\arctan\left(C'' \sqrt{\frac{x+1}{1-x}} \exp\left(\frac{-2G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) \right). \tag{A13}$$

Next, we write $\sin^2\theta$ in terms of x:

$$\begin{aligned} \sin^2\theta = 4\Bigg(& C''^2 \frac{x+1}{1-x} \exp\left(\frac{-4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) \\ & + \frac{1}{C''^2} \frac{1-x}{x+1} \exp\left(\frac{4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) + 2 \Bigg)^{-1}. \end{aligned} \tag{A14}$$

Finally, we use (68) and (A14) to find

$$\begin{aligned} \frac{P^2}{\sin^2\theta} &= \frac{(x+a)^2 + \eta^2 + g(1-x^2)}{4((x+a)^2 + \eta^2)} \left(C''^2 (x+1)^2 \exp\left(\frac{-4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) \right. \\ &\quad \left. + \frac{1}{C''^2} (1-x)^2 \exp\left(\frac{4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) \right)^{-1} + 2(1-x^2) \Bigg) \\ &= \frac{(x + \frac{1}{2}B(1+G)^{-1})^2 + (A(1+G) - G(1+G) - \frac{1}{4}B^2)(1+G)^{-2} + G(1+G)^{-1}(1-x^2)}{4((x + \frac{1}{2}B(1+G)^{-1})^2 + \eta^2)} \\ &\quad \times \left(C''^2 (x+1)^2 \exp\left(\frac{-4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) \right. \\ &\quad \left. + \frac{1}{C''^2} \left(1-x \right)^2 \exp\left(\frac{4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}} \right) \right) \right)^{-1} + 2(1-x^2) \Bigg); \end{aligned}$$

therefore, the term $\frac{P^2}{\sin\theta}$ is finite at the poles, and in consequence, the right-hand side of Eq. (A7) vanishes.[2]

[2]It is easy to see that for P^2 of the form (42) we get the same conclusion.

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