Semiclassical behavior of spinfoam amplitude with small spins and entanglement entropy

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Spinfoam amplitudes with *small* spins can have interesting semiclassical behavior and relate to semiclassical gravity and geometry in four dimensions. We study the generalized spinfoam model [spinfoams for all loop quantum gravity (LQG) [Kaminski *et al.*, Spin-foams for all loop quantum gravity, Classical Quantum Gravity **27**, 095006 (2010); Erratum, Classical Quantum Gravity **29**, 049502(E) (2012), Ding *et al.*, Generalized spinfoams, Phys. Rev. D **83**, 124020 (2011)] with small spins j but a large number of spin degrees of freedom (d.o.f.), and find that it relates to the simplicial Engle-Pereira-Rovelli-Livine-Freidel-Krasnov model with large spins and Regge calculus by coarse-graining spin d.o.f. Small-j generalized spinfoam amplitudes can be employed to define semiclassical states in the LQG kinematical Hilbert space. Each of these semiclassical states is determined by a four-dimensional Regge geometry. We compute the entanglement Rényi entropies of these semiclassical states. The entanglement entropy interestingly coarse grains spin d.o.f. in the generalized spinfoam model, and satisfies an analog of the thermodynamical first law. This result possibly relates to the quantum black hole thermodynamics in [Ghosh and Perez, Black Hole Entropy and Isolated Horizons Thermodynamics, Phys. Rev. Lett. **107**, 241301 (2011); **108**, 169901(E) (2012)].

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I. INTRODUCTION

Loop quantum gravity (LQG) is a candidate of nonperturbative and background-independent theory of quantum gravity. A covariant approach of LQG is developed by the spinfoam formulation, in which the quantity playing the central role is the *spinfoam amplitude* [1,2]. Four-dimensional spinfoam amplitudes give transition amplitudes of boundary three-dimensional (3D) quantum geometry states in LQG and formulate the LQG version of the quantum gravity path integral. The spinfoam formulation is a successful program for demonstrating the semiclassical consistency of LQG. The recent progress on the semiclassical analysis reveals that spinfoam amplitudes relate to the semiclassical Einstein gravity in the large spin regime, e.g., [3–10].

Although the analysis of the large spin spinfoam amplitude has been fruitful for demonstrating the semiclassical behavior, there are good reasons to expect that some even more interesting semiclassical behavior of spinfoams, or in general LQG, should appear in the regime where spins are all small. There are two motivations for the semiclassical analysis in the small spin regime:

First, recall that the large spin semiclassicality is motivated by requiring the geometrical surface area $\mathbf{a}_S$ to be semiclassical, i.e., $\mathbf{a}_S \gg \ell_P^2$ where ℓ_P is the Planck length. The requirement leads to the spin $j \gg 1$ provided the area spectrum $\mathbf{a}_S = 8\pi\gamma\ell_P^2\sqrt{j(j+1)}$, if we assume that there is only a single spin-network link colored by j intersecting the surface S. Large j is a sufficient condition for the semiclassical area but clearly not necessary. Indeed if we relax this assumption and allow more than one intersecting link l, the area spectrum may become $\mathbf{a}_S = 8\pi\gamma\ell_P^2\sum_{l=1}^N\sqrt{j_l(j_l+1)}$ which sums "area elements" $8\pi\gamma\ell_P^2\sqrt{j_l(j_l+1)}$ at l. N is the total number of intersecting links. Then the semiclassical surface area $\mathbf{a}_S \gg \ell_P^2$ can be achieved not only by large j and small N but also by small j and large N. For instance, all $j = 1/2$ and $N \gg 1$ lead to $\mathbf{a}_S \gg \ell_P^2$. Therefore we anticipate that small spins (with a large number of intersecting links) should also lead to semiclassical behaviors of LQG.

The second motivation comes from the statistical interpretation of black hole entropy in LQG: The black hole horizon with a fixed total area punctured by a large number of spin-network links l. The punctures are colored by spins j_l, each of which contributes area element $8\pi\gamma\ell_P^2\sqrt{j_l(j_l+1)}$ to the horizon. The black hole entropy

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counts the total number of microstates which give the same total horizon area [11–13]. It turns out that the total number of states is dominant by states at punctures with small j_l, while the number of states decays exponentially as j becomes large. The fact that small j's dominate the semiclassical horizon area and entropy suggests that small spins should play an important role in the semiclassical analysis of LQG.

This work takes the first step to study systematically the semiclassical behavior of LQG in the small spin regime, in particular in the spinfoam formulation. From the above motivation, given a surface S punctured by N spin-network links, the semiclassical area of S can be given not only by small N and large j but also by large N and small j. Section II generalizes this observation to quantum polyhedra represented by intertwiners [SU(2) invariant tensors] at spin-network nodes. We find among intertwiners with a fixed large rank $N \gg 1$ (quantum polyhedra with N facets f), there are a subclass of small-j and large-N coherent intertwiners $\|\{j_f\},\{\xi_f\}\rangle_N$ $(f = 1,\ldots,N)$ relating to the large-J and rank-4 coherent intertwiner $\|\{J_\Delta\},\{\xi_\Delta\}\rangle_4$ $(\Delta = 1,\ldots,4)$ and having the semiclassical behavior as geometrical flat tetrahedra. Δ are four groups of intertwiner legs f, and every Δ contains a large number $N_\Delta \gg 1$ of f's. The subclass of coherent intertwiners exhibiting semiclassical behaviors are defined by the *parallel restriction* on ξ_f's

$$\xi_f = \xi_{f'} \equiv \xi_\Delta \text{ up to a phase} \quad \forall f, \quad f' \in \Delta, \qquad (1.1)$$

i.e., ξ_f, $\xi_{f'}$ give the same unit 3-vector $\vec{n}_\Delta = \langle\xi_\Delta|\vec{\sigma}|\xi_\Delta\rangle$ where $\vec{\sigma}$ are Pauli matrices. Geometrical tetrahedra resulting from these intertwiners have face areas proportional to $J_\Delta = \sum_{f\in\Delta} j_f$ and face normals $\vec{n}_\Delta$. J_Δ is large since $N_\Delta \gg 1$ and $j_f \neq 0$. This result has a simple geometrical picture: Given a classical flat tetrahedron, we may partition every face Δ into N_Δ facets f, while the face area sums the facet areas and the facet normals are parallel among facets in a Δ. By partitioning tetrahedron faces into facets, the tetrahedron becomes a polyhedron with a total number of $N = \sum_{\Delta=1}^4 N_\Delta$ facets, each of which has a small area [see Fig. 1(a)]. The correspondence between polyhedra and intertwiners in LQG [14] relates f to intertwiner legs (and tetrahedron faces Δ to four groups of intertwiner legs) and facet areas and normals to coherent intertwiner labels [see Fig. 1(b)]. These parallel normals motivates the above parallel restriction. Beyond the semiclassical behaviors of these intertwiners, quantum corrections to semiclassical tetrahedron geometries are of $O(1/J_\Delta) = O(1/N_\Delta)$, which thus is suppressed by large rank N (or N_Δ). The above result demonstrates that at the level of quantum polyhedra, we can trade small j_f and large rank $N \gg 1$ for large J_Δ and small rank $N = 4$ to obtain the semiclassicality.

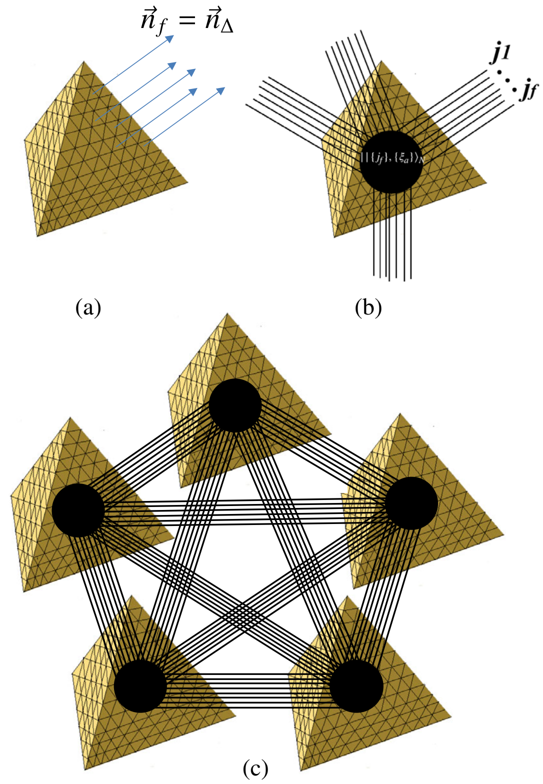

FIG. 1. (a) The classical tetrahedron geometry emergent from a rank-N coherent intertwiner $\|\{j_f\},\{\xi_\Delta e^{i\varphi_f}\}\rangle_N$ with small spins but large rank. The tetrahedron with four large faces is also a polyhedron with N small facets, while normals $\vec{n}_f$ of small polyhedron facets f's are parallel if f's are in the same large tetrahedron face. The flat large tetrahedron faces are composed by many small facets. Each tetrahedron face area $J_{a=1,\ldots,4}$ is a sum of small areas j_f. (b) The rank-N coherent intertwiner $\|\{j_f\},\{\xi_\Delta e^{i\varphi_f}\}\rangle_N$ with small spins j_f can be illustrated as a spin-network node connecting to N links, where each link is dual to a polyhedron facet f and colored by j_f. (c) A spinfoam vertex amplitude defined by a spin-network with five nodes $(\alpha = 1, \ldots, 5)$, connected as shown in the figure. Nodes are colored by intertwiners $\|\{j_f,\{\xi_{\alpha f}\}\rangle_{N_\alpha}$ of large rank but small spins. Geometrically, each node corresponds to a polyhedron of many facets as in (a), and the vertex amplitude glues five polyhedra to form a close boundary of a 4D region. $\{j_f\}$, $\{\xi_{\alpha f}\}$ are boundary data of the vertex amplitude.

Note that the above semiclassical result still holds if we replace the tetrahedron by polyhedra in case their numbers of faces Δ are still small. A similar idea as the above is applied in [15] to relate LQG states to holographic tensor networks, and relates to [16].

Section III generalizes the small-j semiclassical analysis to the spinfoam vertex amplitude in four dimensions. The vertex amplitude A_v is associated with a four-dimensional cell B_4 whose boundaries are closed and made by gluing five polyhedra $\alpha = 1,\ldots,5$, each of which has a large number N_α of facets [see Fig. 1(c)]. Every pair of polyhedra share a large number N_Δ of facets, where $\Delta = \alpha \cap \beta$ is the face made by facets shared by two polyhedra α, β. Ignoring the fine partition of Δ, B_4 relates to a 4-simplex where Δ relates to triangles of the 4-simplex. A_v depends on the boundary data which contain small spins j_f and five intertwiners $\|\{j_f\},\{\xi_{\alpha f}\}\rangle_{N_\alpha}$ of quantum polyhedra. To be concrete, we consider A_v to be the generalized spinfoam vertex [17,18] (in the Euclidean signature with $0<\gamma<1$) which admits nonsimplicial cells. We write A_v in terms of coherent intertwiners and impose the parallel restriction Eq. (1.1) to boundary data $\xi_{\alpha f}$ with $f\in\Delta$. We find that up to an overall phase, A_v with small j_f and large N_Δ is identical to the Engle-Pereira-Rovelli-Livine-Freidel-Krasnov (EPRL-FK) vertex amplitude of 4-simplex with large spins $J_\Delta \gg 1$, where ten Δ become triangles of the 4-simplex and $J_\Delta=\sum_{f\in\Delta} j_f$ similar to the case of polyhedra. Because of large J_Δ, the same asymptotic analysis as in [19] can be applied to A_v and gives the following asymptotic formula relating to the 4-simplex Regge action $\sum_\Delta \gamma J_\Delta \Theta_\Delta$ (the triangle area $\mathbf{a}_\Delta = 8\pi\gamma\ell_P^2 J_\Delta$):

$$A_v = (\text{overall phase})\left(\frac{2\pi}{N}\right)^{12}\left[2\mathcal{N}^\gamma_{+-}\cos\left(\sum_\Delta \gamma J_\Delta\Theta_\Delta\right)+\mathcal{N}^\gamma_{++}e^{\sum_\Delta J_\Delta\Theta_\Delta}+\mathcal{N}^\gamma_{--}e^{-\sum_\Delta J_\Delta\Theta_\Delta}\right]\left(1+O\left(\frac{1}{N}\right)\right). \tag{1.2}$$

We refer the reader to [19] for expressions of $\mathcal{N}^\gamma_{+-},\mathcal{N}^\gamma_{++},\mathcal{N}^\gamma_{--}$. The expansion parameter N is the order of magnitude of $N_\Delta \sim J_\Delta$.

Section IV generalizes the discussion to spinfoam amplitude $A(\mathcal{K})$ on cellular complexes $\mathcal{K}$ in four dimensions. The 4D cell of $\mathcal{K}$ is B_4 to define vertex amplitudes A_v as above. We again apply the generalized spinfoam formulation to define the amplitude on $\mathcal{K}$. By the above relation between B_4 and 4-simplex, $\mathcal{K}$ relates to a unique simplicial complex $\mathcal{K}_s$, where decomposing triangles $\Delta\in\mathcal{K}_s$ into facets f gives $\mathcal{K}$. In the above analysis of a single A_v, the parallel restriction can be applied since $\xi_{\alpha f}$ are boundary data. However, for the spinfoam amplitude $A(\mathcal{K})$ we do need to consider internal $\xi_{\alpha f}$ beyond the parallel restriction since individual $\xi_{\alpha f}$'s are integrated independently in $A(\mathcal{K})$. We write the spinfoam amplitude as a sum over spins $A(\mathcal{K})=\sum_{\{j_f\}}A_{\{j_f\}}(\mathcal{K})$ and focus on $A_{\{j_f\}}(\mathcal{K})$ in Sec. IV. $A_{\{j_f\}}(\mathcal{K})$ has the standard integral expression:

$$A_{\{j_f\}}(\mathcal{K})=\prod_f A_\Delta(j_f)\int[\mathrm{d}\xi_{\alpha f}\mathrm{d}g^\pm_{v\alpha}]e^S,$$
$$S=\sum_\pm\sum_{v,f}2j_f^\pm \ln\langle\xi_{\alpha f}|g^{\pm -1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta f}\rangle, \tag{1.3}$$

where the face amplitude $A_\Delta(j_f)$ is $2j_f+1$ to a certain power depending only on Δ. It turns out that the stationary phase analysis can still be applied to $A_{\{j_f\}}(\mathcal{K})$ with small nonzero j_f but large N_Δ. It is clear from the discussion in the last paragraph that $A_{\{j_f\}}(\mathcal{K})$ reduces to the simplicial EPRL-FK spinfoam amplitude with large spins $J_\Delta=\sum_{f\in\Delta} j_f$ if we impose by hand the parallel restriction to internal $\xi_{\alpha f}$'s. We prove that all critical points of the large J_Δ simplicial EPRL-FK amplitude give critical points of $A_{\{j_f\}}(\mathcal{K})$ if we relate the critical data by $J_\Delta=\sum_{f\in\Delta} j_f$, internal $\xi_{\alpha\Delta}=\xi_{\alpha f}$ (up to a phase), and $g^\pm_{v\alpha}$ is identified between simplicial EPRL-FK and $A_{\{j_f\}}(\mathcal{K})$. We denote these critical points by $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$. Some of these critical points relate to Regge geometries in four dimensions similar to the simplicial EPRL-FK amplitude [9,20]. At these critical points, J_Δ is identified to be the area of the triangle Δ. The application of critical points to the stationary phase analysis is discussed in Sec. IX.

The relation between the simplicial EPRL-FK amplitude and $A(\mathcal{K})$ suggests a new viewpoint that the EPRL-FK model with spins J_Δ can be an effective theory emergent from a more fundamental theory formulated by $A(\mathcal{K})$ with j_f. The EPRL-FK model is obtained from $A(\mathcal{K})$ by coarse graining from j_f to J_Δ and imposing the parallel restriction [more rigorously, the EPRL-FK model appears as a partial amplitude in $A(\mathcal{K})$ after integrating out the nonparallel $\xi_{\alpha f}$ as shown in Sec. IX]. The EPRL-FK amplitude with given J_Δ is a collection of a large number of microdegrees of freedom $\{j_f\}$ satisfying $J_\Delta=\sum_{f\in\Delta} j_f$ at all Δ. Critical points from the EPRL-FK model and Regge geometries are "macrostates" which contain $\{j_f\}$ as "microstates." This picture is interesting and turns out to be important in the computation of entanglement entropy.

Before the analysis of the full amplitude $A(\mathcal{K})$ in Sec. IX, Secs. V–VII make a modification of the amplitude by imposing weakly the parallel restriction to internal $\xi_{\alpha f}$'s, and apply the modified amplitude to the study of entanglement entropy in LQG (see, e.g., [15,21–26] for some existing studies of entanglement entropy in LQG). The modified amplitude is used to define a class of states in the LQG Hilbert space: Given a 4-manifold $\mathcal{M}_4$ with boundary Σ and consider $\mathcal{K}$ (whose 4-cells are B_4) as a cellular decomposition of $\mathcal{M}_4$ (e.g., Fig. 2). The boundary complex $\partial\mathcal{K}\subset\Sigma$ gives the dual graph $\partial\mathcal{K}^*\subset\Sigma$. $\mathcal{H}_\Sigma$ is defined as the LQG kinematical Hilbert space on $\partial\mathcal{K}^*$ and is spanned by the spin-network states $|T_{\vec{j},\vec{i}}\rangle$ with spins $\vec{j}$ and intertwiners $\vec{i}$ on links and nodes of $\partial\mathcal{K}^*$. In Sec. V, we construct a class of states $|\psi\rangle\in\mathcal{H}_\Sigma$ as finite linear combinations of spin networks $|T_{\vec{j},\vec{i}}\rangle$ weighted by spinfoam

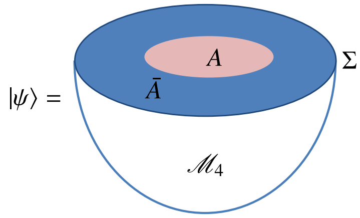

FIG. 2. A 4-manifold (viewed from five dimensions) with a boundary 3-manifold Σ. The state $|\psi\rangle$ given by Eq. (5.7) is constructed by the spinfoam amplitude on a cellular partition $\mathcal{K}$ of $\mathcal{M}_4$. The boundary Σ is subdivided into region A and its complement $\bar{A}$. The subdivision A and $\bar{A}$ is adapted to $\mathcal{K}$, in the sense that the boundary $\mathcal{S}$ between A and $\bar{A}$ is triangulated by Δ's, each of which is made by a large number of facets f in $\mathcal{K}$.

amplitudes whose boundary data are $\vec{j}$, $\vec{i}$. In terms of coherent intertwiners,

$$|\psi\rangle = \sum_{\{j_f\}}' \prod_f A_\Delta(j_f) \int_{\mathfrak{N}_{g,\xi}} [\mathrm{d}g^\pm_{v\alpha} \mathrm{d}\xi_{\alpha f}] e^{S+NV} |T_{\vec{j},\vec{\xi}}\rangle, \tag{1.4}$$

where $|T_{\vec{j},\vec{\xi}}\rangle$ are spin networks with coherent intertwiners. V is a potential which imposes the parallel restriction when $N\to\infty$. $|\psi\rangle$ depends on a choice of the isolated critical point $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ where $\xi_{\alpha f} = \xi_{\alpha\Delta}$ (up to phases) satisfy the parallel restriction. $\sum'_{\{j_f\}}$ is constrained by $\sum_{f\in\Delta} j_f = J_\Delta$ and thus is a finite sum. $\int[\mathrm{d}g^\pm_{v\alpha}\mathrm{d}\xi_{\alpha f}]$ is over a neighborhood $\mathfrak{N}_{g,\xi}$ which contains a unique isolated critical point $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$. $|\psi\rangle$ has a nice semiclassical property: the weight of $|T_{\vec{j},\vec{\xi}}\rangle$ is peaked (in the space of boundary $\vec{\xi}$) at the boundary value $\vec{\xi}$ from the critical data $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$. The implementation of the parallel restriction by V makes the entanglement entropy of $|\psi\rangle$ computable with tools from the stationary phase approximation.

We subdivide Σ into 2 subregions A and $\bar{A}$, such that the boundary $\mathcal{S}$ between A and $\bar{A}$ is triangulated by $\Delta\subset\mathcal{K}_s$. Accordingly the Hilbert space is split by $\mathcal{H}_\Sigma \simeq \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ (here $\mathcal{H}_\Sigma$ has to be suitably enlarged to include some non-gauge-invariant states in order to define the split and entanglement entropy; see Sec. VII for details). The reduced density matrix ρ_A and the nth Rényi entanglement entropy $S_n(A)$ are defined by

$$\rho_A = \mathrm{tr}_{\bar{A}}|\psi\rangle\langle\psi|, \qquad S_n(A) = \frac{1}{1-n}\ln\frac{\mathrm{tr}(\rho_A^n)}{\mathrm{tr}(\rho_A)^n}, \tag{1.5}$$

while the von Neumann entropy is given by $S(A) = \lim_{n\to 1} S_n(A)$. Entanglement entropies characterize the amount of entanglement from $|\psi\rangle$ between d.o.f. in A and $\bar{A}$. Section VII computes the Rényi entropy $S_n(A)$ and shows that $S_n(A)$ is a function of "macrostates" J_Δ, N_Δ:

$$S_n(A) \simeq \sum_{\Delta\subset\mathcal{S}}[\lambda_\Delta(n)J_\Delta + \sigma_\Delta(n)N_\Delta], \tag{1.6}$$

where $\lambda_\Delta(n), \sigma_\Delta(n)$ depend on the ratio J_Δ/N_Δ. When $\mathcal{K}$ and $\mathcal{S}$ are chosen such that all $\Delta\in\mathcal{S}$ are shared by the same number of B_4's, $\lambda_\Delta(n) = \lambda(n), \sigma_\Delta(n) = \sigma(n)$ become independent of Δ. In this case,

$$S_n(A) \simeq \lambda(n)J_\mathcal{S} + \sigma(n)N_\mathcal{S}, \tag{1.7}$$

where $J_\mathcal{S} = \sum_{\Delta\subset\mathcal{S}} J_\Delta$ and $N_\mathcal{S} = \sum_{\Delta\subset\mathcal{S}} N_\Delta$ are total area and total number of facets of $\mathcal{S}$.

Section VI demonstrates an important intermediate step toward $S_n(A)$: Computing $\mathrm{tr}(\rho_A^n)$ reduces to a quantity which can be interpreted as counting microstates $\{j_f\}$ in a statistical ensemble with fixed "macrostate" J_Δ, N_Δ at a given Δ. The computation has an interesting analog to the statistical ensemble of identical systems, in which J_Δ, N_Δ are the total energy and total number of identical systems. This counting of microstates is similar to the black hole entropy counting in LQG [11].

Section VIII points out that the resulting Rényi entanglement entropy $S_n(A)$ and its differential give an analog of the thermodynamical first law:

$$\delta S_n(A) \simeq \sum_{\Delta\subset\mathcal{S}}[\lambda_\Delta(n)\delta J_\Delta + \sigma_\Delta(n)\delta N_\Delta], \tag{1.8}$$

$$\text{or}\quad \delta S_n(A) \simeq \lambda(n)\delta J_\mathcal{S} + \sigma(n)\delta N_\mathcal{S}, \tag{1.9}$$

where in Eq. (1.9), $\mathcal{K}$ and $\mathcal{S}$ are chosen such that all $\Delta\in\mathcal{S}$ are shared by the same number of B_4's. Since $J_\mathcal{S}$ is an analog of the total energy, Eq. (1.9) suggests the analog between $\lambda(n)^{-1}$ and the temperature, as well as between $-\sigma(n)/\lambda(n)$ and the chemical potential. In the most general situation Eq. (1.8), the temperature and chemical potential are not constants over $\mathcal{S}$. $\mathcal{S}$ is in a nonequilibrium state, although every plaquette Δ is in equilibrium. Interestingly, Eq. (1.9) is very similar to the thermodynamical first law derived from the quantum isolated horizon in [11], if we relate $S_n(A)$ to the black hole entropy, $J_\mathcal{S}$ to the horizon area (proportional to the quasilocal energy observed by the near-horizon Unruh observer), and $N_\mathcal{S}$ to the total number of spin-network punctures on the horizon.

The above analogy with thermodynamics is clearly a consequence from coarse graining in the spinfoam model $A(\mathcal{K})$. The entanglement entropy effectively coarse grains the microdegrees of freedom $\{j_f\}$ collected by the macrostate J_Δ, N_Δ.

The above discussion mostly focuses on the spinfoam small-j amplitudes with the implementation of parallel restriction. Section IX studying the full amplitude $A(\mathcal{K})$ in Eq. (1.3) by removing parallel restrictions to all internal $\xi_{\alpha f}$'s, while integrating out explicitly all nonparallel d.o.f. of $\xi_{\alpha f}$ at every Δ. As a result, the amplitude becomes a sum over Ising configurations at all Δ, where at each Δ some $\xi_{\alpha f}$ are parallel $\xi_{\alpha f} = \xi_{\alpha\Delta}$ while others are antiparallel $\xi_{\alpha f} = J\xi_{\alpha\Delta}$ [$J(\xi^1, \xi^2)^T = (-\bar{\xi}^2, \bar{\xi}^1)^T$; that ξ's are antiparallel means that $\vec{n} = \langle\xi|\vec{\sigma}|\xi\rangle$ are antiparallel]. The amplitude constrained by the parallel restriction is identified as a partial amplitude in the sum and relates to the simplicial EPRL-FK amplitude, while all other partial amplitudes are made by flipping a certain number of $\xi_{\alpha f}$ from $\xi_{\alpha\Delta}$ to $J\xi_{\alpha\Delta}$. Importantly, all partial amplitudes in the sum can be studied by stationary phase approximation. All partial amplitudes, whose numbers of antiparallel $\xi_{\alpha f}$ are much less than the numbers of parallel $\xi_{\alpha f}$ at all Δ's, are dominated by contributions from critical points $(g^{\pm}_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ satisfying the parallel restriction. In particular, 4D Regge geometries can still be realized as a subset of critical points in the full amplitude $A(\mathcal{K})$. However, for partial amplitudes whose numbers of antiparallel $\xi_{\alpha f}$ are comparable to the numbers of parallel $\xi_{\alpha f}$ at certain Δ's, they give critical points corresponding to semiclassically degenerate tetrahedron geometries. The 4D geometrical interpretations of these critical points are not clear at the moment.

II. QUANTUM POLYHEDRON AND PARALLEL RESTRICTION

In LQG, polyhedron geometries are quantized by intertwiners $\|i\rangle_N \in \mathrm{Inv}_{\mathrm{SU}(2)}(j_1, \ldots, j_N)$ which are invariant in the tensor product of N SU(2) unitary irreps $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$ (spins $j \geq 1/2$ label the irreps) [27–29]. In this paper we always assume j's to be small but the rank N to be large: $N \gg 1$. Denoting by $\vec{L}_f$ SU(2) generators acting on the fth irrep $\mathcal{H}_{j_f}$ $(f = 1, \ldots, N)$, every invariant tensor $\|i\rangle$ satisfies $\sum_{f=1}^N \vec{L}_f \|i\rangle_N = 0$, which is a quantum analog of the classical closure condition $\sum_{f=1}^N \mathbf{a}_f \vec{n}_f = 0$ ($\mathbf{a}_f \in \mathbb{R}, \vec{n}_f$ unit 3-vectors). $\{\mathbf{a}_f, \vec{n}_f\}_{f=1}^N$ satisfying this condition uniquely determines a geometrical polyhedron with N facets, such that $\mathbf{a}_f$ is the area of the facet f while $\vec{n}_f$ is the unit normal vector of f [30].

An overcomplete basis of $\mathrm{Inv}_{\mathrm{SU}(2)}(j_1, \ldots, j_N)$ can be chosen to be coherent intertwiners [27]

$$\|\{j_f\}, \{\xi_f\}\rangle_N = \int_{\mathrm{SU}(2)} \mathrm{d}h \bigotimes_{f=1}^{N} h|j_f, \xi_f\rangle, \tag{2.1}$$

where $\mathrm{d}h$ is the Haar measure and $|j, \xi\rangle$ is the SU(2) coherent state in spin-j irrep labeled by $\xi = (\xi^1, \xi^2)^T$ normalized by the Hermitian inner product

$$|j, \xi\rangle = g(\xi)|j, j\rangle, \qquad g(\xi) = \begin{pmatrix} \xi^1 & -\bar{\xi}^2 \\ \xi^2 & \bar{\xi}^1 \end{pmatrix}. \tag{2.2}$$

Suppose j are all large, $\|\{j_f\}, \{\xi_f\}\rangle_N$ gives a semiclassical flat polyhedron geometry with N facets, which have areas $\mathbf{a}_f \propto j_f$ and normals $\vec{n}_f = \langle\xi_f|\vec{\sigma}|\xi_f\rangle$ ($\vec{\sigma}$ are Pauli matrices) [14,27]. However, when j are small, this semiclassical geometry is lost, since the quantum fluctuation is of order $1/j$. However, as we see below, some different semiclassical polyhedron geometries can still be found from some $\|\{j_f\}, \{\xi_f\}\rangle_N$ with small j.

An observation is that a subclass of small-spin and large-rank coherent intertwiners relate to large-spin coherent intertwiners with small rank. Let us consider the small rank to be four as an example (generalizations to other small ranks are trivial): we make a partition of $\{1, \ldots, N\}$ into four sets, say $\{1, \ldots, N_1\}, \{N_1 + 1, \ldots, N_1 + N_2\}, \{N_1 + N_2 + 1, \ldots, N_1 + N_2 + N_3\}, \{N_1 + N_2 + N_3 + 1, \ldots, N\}$, where each set has a large number $N_\Delta \gg 1$ elements, and we use $\Delta = 1, \ldots, 4$ to label these four sets. We restrict to a subclass of coherent states denoted by $\|\{j_f\}, \{\xi_\Delta\}\rangle_N$, asking ξ_f's are parallel up to a phase when $f \in \Delta$:

$$\text{Parallel restriction:} \ \xi_f = \xi_\Delta e^{i\varphi_f}, \quad \text{or} \quad \vec{n}_f = \vec{n}_\Delta, \quad \forall f \in \Delta. \tag{2.3}$$

Parallel ξ_f's up to phases make parallel normals $\vec{n}_f$'s. Intuitively, this restriction makes a tetrahedron with four large flat faces from a polyhedron with many small facets [see Fig. 1(a)].

The squared norm of $\|\{j_f\}, \{\xi_\Delta e^{i\varphi_f}\}\rangle_N$ is computed by factorizations of coherent states $|j, \xi\rangle = |\xi\rangle^{\otimes 2j}$ and Eq. (2.3):

$$\|\|\{j_f\}, \{\xi_\Delta e^{i\varphi_f}\}\rangle_N\|^2 = \int \mathrm{d}h \prod_{\Delta=1}^{4} \langle\xi_\Delta|h|\xi_\Delta\rangle^{2J_\Delta}, \qquad J_\Delta \equiv \sum_{f\in\Delta} j_f. \tag{2.4}$$

Although j_f are small, $J_\Delta \gg 1$ because $N_\Delta \gg 1$ and $j_f \geq \frac{1}{2}$. When above J_Δ's satisfy triangle inequalities, Eq. (5.16) is of the same expression as the square norm of the rank-4 coherent intertwiner $\|\{J_\Delta\}, \{\xi_\Delta\}\rangle_4$ if we relate the above J_Δ to the large spins of the rank-4 intertwiner. Thus the same stationary phase analysis in [27] can be applied to Eq. (5.16) and shows that Eq. (5.16) is exponentially suppressed unless the following closure condition holds for the coherent state labels:

$$\sum_{\Delta=1}^{4} J_\Delta \vec{n}_\Delta = \sum_{f=1}^{N} j_f \vec{n}_f = 0, \tag{2.5}$$

where $\vec{n}_\Delta = \langle\xi_\Delta|\vec{\sigma}|\xi_\Delta\rangle$ and thus $\vec{n}_f = \vec{n}_\Delta$ for all $f \in \Delta$. Comparing to the classical closure condition of polyhedron, Eq. (2.5) uniquely determines a classical flat geometrical tetrahedron, whose face areas are proportional to $J_\Delta \gg 1$ and face normals are $\vec{n}_\Delta$. However, here J_Δ emerges from summing many small j_f's. Equation (2.5) may still be interpreted as a classical closure condition of a polyhedron with N facets with small areas j_f's, while facets compose large flat faces of the tetrahedron. The quantum correction of the classical geometry is of $O(1/J_\Delta)$ and thus is suppressed by the large rank.

The above demonstrates that the classical tetrahedron geometry can emerge from intertwiners with small j's but large rank $N_\Delta \to \infty$. The geometrical picture of the tetrahedron/polyhedron is illustrated in Fig. 1(a).

Importantly, rank-N intertwiners have many more d.o.f. than tetrahedra. There are coherent intertwiners with ξ_f's beyond the parallel restriction, while $\|\{j_f\},\{\xi_\Delta\}\rangle_N$ only span a subspace. In addition, the same tetrahedron geometry with areas J_Δ may come from different spin configurations $\{j_f\}$ satisfying $J_\Delta \equiv \sum_{f\in\Delta} j_f$.

Lemma 2.1: Given four J_Δ satisfying the triangle inequality such that $\otimes_{\Delta=1}^4 \mathcal{H}_{J_\Delta}$ has a nontrivial invariant subspace, any spin configuration $\{j_f\}_{f\in\Delta}$ satisfying $J_\Delta \equiv \sum_{f\in\Delta} j_f$ leads to a nontrivial invariant subspace in $\otimes_{\Delta=1}^4 \otimes_{f\in\Delta} \mathcal{H}_{j_f}$.

Proof: It is convenient to consider coherent intertwiners satisfying the parallel restriction Eq. (2.3) and use the factorization property $|j,\xi\rangle = |\xi\rangle^{\otimes 2j}$,

$$\begin{aligned}\|\{j_f\},\{\xi_\Delta e^{i\varphi_f}\}\rangle_N &= e^{i\sum_f 2j_f\varphi_f}\int_{\mathrm{SU}(2)} \mathrm{d}h \bigotimes_{\Delta=1}^4 \bigotimes_{f\in\Delta} h|j_f,\xi_\Delta\rangle = e^{i\sum_f 2j_f\varphi_f}\int_{\mathrm{SU}(2)} \mathrm{d}h \bigotimes_{\Delta=1}^4 \bigotimes_{f\in\Delta} (h|\xi_\Delta\rangle)^{\otimes 2j_f} \\ &= e^{i\sum_f 2j_f\varphi_f}\int_{\mathrm{SU}(2)} \mathrm{d}h \bigotimes_{\Delta=1}^4 (h|\xi_\Delta\rangle)^{\otimes 2J_\Delta} = e^{i\sum_f 2j_f\varphi_f}\int_{\mathrm{SU}(2)} \mathrm{d}h \bigotimes_{\Delta=1}^4 h|J_\Delta,\xi_\Delta\rangle. \end{aligned} \tag{2.6}$$

The right-hand side gives up to a phase the rank-4 coherent intertwiner, which is nonzero by the assumption that J_Δ satisfies the triangle inequality. Therefore $\|\{j_f\}, \{\xi_\Delta e^{i\varphi_f}\}\rangle_N$ is nonzero, and thus the invariant subspace in $\otimes_{\Delta=1}^4 \otimes_{f\in\Delta} \mathcal{H}_{j_f}$ is nontrivial. ∎

III. SPINFOAM VERTEX AMPLITUDE

We extend our discussion of small-j semiclassicality to LQG dynamics in the spinfoam formulation. We first focus on a class of spinfoam vertex amplitudes asssociated with a 4D spacetime region B_4 whose closed boundary is made by gluing five polyhedra (labeled by $\alpha,\beta = 1,\ldots,5$) through facets. Each polyhedron has $N_\alpha \gg 1$ facets, and every pair of polyhedra α, β share a large number $N_\Delta \gg 1$ facets. Δ denotes the interface between α, β made by N_Δ facets f.

We apply the generalized spinfoam formulation to construct amplitude on nonsimplicial B_4 [17,18]. The vertex amplitude of B_4 evaluates a spin network with five nodes (dual to polyhedra), and each pair of nodes α, β is connected by N_Δ links. See Fig. 1(c) for an illustration. Links connecting nodes are dual to f's shared by polyhedra and colored by spins j_f. We color every node α by rank-N_α coherent intertwiners $\|\{j_f\},\{\xi_{\alpha f}\}\rangle_{N_\alpha}$ studied above ($j_f \neq 0$ but small), while making the parallel restriction as in Eq. (2.3):

$$\xi_{\alpha f} = \xi_{\alpha\Delta} e^{i\varphi_{\alpha f}} \quad \forall\, f \in \Delta. \tag{3.1}$$

The vertex amplitude $A_v(j_f,\xi_{\alpha f})$ (in the Euclidean signature) describes a local transition in B_4 of boundary geometrical states $\otimes_{\alpha=1}^5 \|\{j_f\},\{\xi_{\alpha f}\}\rangle_{N_\alpha}$:

$$A_v = \int [\mathrm{d}g_\alpha^\pm] \prod_\pm \prod_{\alpha<\beta} \prod_{f\in(\alpha,\beta)} \langle j_f^\pm, \xi_{\alpha f}| g_\alpha^{\pm -1} g_\beta^\pm | j_f^\pm, \xi_{\beta f}\rangle = \int [\mathrm{d}g_\alpha^\pm] e^{\sum_\pm \sum_f 2j_f^\pm \ln \langle \xi_{\alpha f}| g_\alpha^{\pm -1} g_\beta^\pm |\xi_{\beta f}\rangle}, \tag{3.2}$$

where $(g_\alpha^+, g_\alpha^-) \in \mathrm{Spin}(4)$ associates with each node and $j_f^\pm = (1\pm\gamma)j_f/2$ with $\gamma < 1$. We have applied the factorization property of the coherent state in the above. By the parallel restriction,

$$A_v = \prod_{\Delta,f} e^{2ij_f(\varphi_{\beta f}-\varphi_{\alpha f})} \int [\mathrm{d}g_\alpha^\pm] e^{\sum_\pm \sum_\Delta 2J_\Delta^\pm \ln \langle \xi_{\alpha\Delta}| g_\alpha^{\pm -1} g_\beta^\pm |\xi_{\beta\Delta}\rangle}, \qquad J_\Delta^\pm = \sum_{f\in\Delta} j_f^\pm, \tag{3.3}$$

where ten $J_\Delta = \sum_{f\in(\alpha,\beta)} j_f \gg 1$ emerge as summing j_f over facets $f \in \Delta$. J_Δ are all large since $N_\Delta \gg 1$ and $j_f \geq \frac{1}{2}$. $\prod_{\Delta,f} e^{2ij_f(\varphi_{\beta f}-\varphi_{\alpha f})}$ is an overall phase since Eq. (3.1) restricts $\xi_{\alpha f}$ parallel up to a phase.

Although A_v is a generalized spinfoam vertex with boundary polyhedra and small spins, the integral Eq. (3.3) has the same expression as the EPRL-FK 4-simplex amplitude (boundary states are rank-4 intertwiners)

[19,31,32] if we relate J_Δ to actual spins in the EPRL-FK amplitude.

Definition 3.1: Given an integral $\int_D \mathrm{d}^n x e^{S(x)}$, its stationary points x_0 are solutions of $\vec{\nabla} S(x_0) = 0$, and its critical points are stationary points with $\mathrm{Re}(S(x_0)) = 0$.

Since $\mathrm{Re}\ln\langle\xi'|\xi\rangle = \ln|\langle\xi'|\xi\rangle| \leq \ln(\|\xi'\|\cdot\|\xi\|) = 0$ by the Schwarz inequality, the exponents in Eqs. (3.2) and (3.3) are nonpositive. The critical points of A_v in Eq. (3.2) are solutions of

$$\hat{g}^{\pm}_{v\beta}\vec{n}_{\beta f} = \hat{g}^{\pm}_{v\alpha}\vec{n}_{\alpha f}, \qquad \sum_{f\subset\alpha} j_f \kappa_{\alpha\Delta}\vec{n}_{\alpha f} = 0, \tag{3.4}$$

where the first equation comes from $\mathrm{Re}(S) = 0$. $\kappa_{\alpha\Delta} = \pm 1$ appears when $\partial_{g^{\pm}_{v\alpha}}$ acts on $g^{\pm}_{v\alpha}$ or $g^{\pm -1}_{v\alpha}$. $\hat{g}^{\pm}_{v\alpha} \in \mathrm{SO}(3)$ is the three-dimensional irrep of $g^{\pm}_{v\alpha}$. When the parallel restriction is imposed to boundary data. The critical equations, Eq. (3.4), reduce to

$$\hat{g}^{\pm}_{v\beta}\vec{n}_{\beta\Delta} = \hat{g}^{\pm}_{v\alpha}\vec{n}_{\alpha\Delta}, \qquad \sum_{\Delta\subset\alpha} J_\Delta \kappa_{\alpha\Delta}\vec{n}_{\alpha\Delta} = 0, \tag{3.5}$$

which are also critical equations from Eq. (3.3).

The same asymptotic analysis as in [19] is valid for Eq. (3.3) as $J_\Delta \gg 1$. Here we adapt results in [19] to our A_v: When the boundary data j_f, ξ_Δ satisfy the closure condition as in Eq. (2.5), and give flat geometrical tetrahedra that are glued (with Δ matching in shapes and orientation matching) to form a closed boundary of a flat nondegenerate 4-simplex, the asymptotics of A_v relates to the Regge action of the 4-simplex: If we define N to be the order of magnitude of N_Δ [$N \sim N_\Delta \sim J_\Delta$ since all $j_f \sim O(1)$], then A_v has the following asymptotic formula:

$$A_v = (\text{overall phase})\left(\frac{2\pi}{N}\right)^{12}\left[2\mathcal{N}^{\gamma}_{+-}\cos\left(\sum_\Delta \gamma J_\Delta \Theta_\Delta\right) + \mathcal{N}^{\gamma}_{++} e^{\sum_\Delta J_\Delta\Theta_\Delta} + \mathcal{N}^{\gamma}_{--} e^{-\sum_\Delta J_\Delta \Theta_\Delta}\right]\left(1 + O\left(\frac{1}{N}\right)\right). \tag{3.6}$$

We refer the reader to [19] for expressions of $\mathcal{N}^{\gamma}_{+-}$, $\mathcal{N}^{\gamma}_{++}$, $\mathcal{N}^{\gamma}_{--}$. The asymptotics is dominant by contributions from four critical points $(g^{+}_{v\alpha}, g^{-}_{v\alpha})$, $(g^{-}_{v\alpha}, g^{+}_{v\alpha})$, $(g^{+}_{v\alpha}, g^{+}_{v\alpha})$, $(g^{-}_{v\alpha}, g^{-}_{v\alpha})$ solving Eq. (3.5) with the boundary condition. Θ_Δ is the 4D dihedral angle between a pair of tetrahedra in the geometrical 4-simplex. The quantity inside the cosine is the Regge action of classical gravity when we identify the tetrahedron face area $\mathbf{a}_\Delta$ as

$$\mathbf{a}_\Delta = 8\pi\gamma J_\Delta \ell_P^2 = \sum_{f\in\Delta}\mathbf{a}_f, \qquad \mathbf{a}_f = 8\pi\gamma j_f \ell_P^2. \tag{3.7}$$

The large tetrahedron face area is given by summing small areas of polyhedron facets. ℓ_P is the Planck length.

IV. SPINFOAM AMPLITUDES ON COMPLEXES

Our semiclassical analysis with small spins can be generalized to spinfoam amplitudes on cellular complexes with arbitrarily many cells. We construct a generalized spinfoam amplitude on a complex $\mathcal{K}$ whose cells $\mathcal{C}$ are similar to B_4 (every $\partial\mathcal{C}$ are made by five polyhedra α of large numbers of facets f, though different $\mathcal{C}$ may have different numbers of facets). $N \sim N_\Delta \gg 1$ are assumed. $\mathcal{C}$s are glued in $\mathcal{K}$ by sharing boundary polyhedra. $\mathcal{K}$ determines a simplicial complex $\mathcal{K}_s$ by substituting all polyhedra and $\mathcal{C}$ with tetrahedra and 4-simplices. We associate A_v with every $\mathcal{C}$, and write the spinfoam amplitude on $\mathcal{K}$ by [20,28]

$$A(\mathcal{K}) = \sum_{\{j_f\}}\prod_f A_f(j_f)\int[\mathrm{d}\xi_{\alpha f}\mathrm{d}g^{\pm}_{v\alpha}]e^S, \tag{4.1}$$

$$S = \sum_{\pm}\sum_{v,f} 2j^{\pm}_f \ln\langle\xi_{\alpha f}|g^{\pm -1}_{v\alpha} g^{\pm}_{v\beta}|\xi_{\beta f}\rangle, \tag{4.2}$$

where A_f is the face amplitude given by [33] (see the Appendix for explanations)

$$A_f(j_f) = A_\Delta(j_f) = (2j_f+1)^{n_v(\Delta)+1} \quad \text{for internal } f,$$
$$A_f(j_f) = A_\Delta(j_f) = (2j_f+1)^{n_v(\Delta)+2} \quad \text{for boundary } f, \tag{4.3}$$

$n_v(\Delta)$ is the number of B_4 sharing $f \in \Delta$ in $\mathcal{K}$ and equals the number of 4-simplices sharing Δ in $\mathcal{K}_s$. A_f depends on $n_v(\Delta)$ in the coherent state formulation since $(2j+1)\int \mathrm{d}\xi|j,\xi\rangle\langle j,\xi| = 1$ where $\mathrm{d}\xi$ is the standard normalized measure on the unit sphere. $\sum_{\{j_f\}}$ and $\int[\mathrm{d}\xi_{\alpha f}]$ sum coherent state labels of all internal facets f. Each $\int \mathrm{d}\xi_{\alpha f}$ is over S^2. Different from A_v where we can apply the parallel restriction to boundary data, $A(\mathcal{K})$ sums independently $\xi_{\alpha f}$'s at different internal f's, so we need to take into account fluctuations beyond the parallel restriction. When $\mathcal{K}$ has a boundary, we still make the parallel restriction to boundary $\xi_{\alpha f}$'s.

S has the following gauge symmetry:

(i) *Continuous*: (1) A diagonal Spin(4) action at σ, $g^{\pm}_{v\alpha} \to h^{\pm}_v g^{\pm}_{v\alpha}$ for all α at v by $(h^+_v, h^-_v) \in \mathrm{Spin}(4)$; (2) At any internal α, $|\xi_{\alpha f}\rangle \to h_\alpha|\xi_{\alpha f}\rangle$ and $g^{\pm}_{v\alpha} \to g^{\pm}_{v\alpha}h^{-1}_\alpha$ for all v having α at boundaries; and (3) $|\xi_{\alpha f}\rangle \to e^{i\theta_{\alpha f}}|\xi_{\alpha f}\rangle$ at any internal $|\xi_{\alpha f}\rangle$.

(ii) *Discrete*: $g^{+}_{v\alpha} \to \pm g^{+}_{v\alpha}$ and independently $g^{-}_{v\alpha} \to \pm g^{-}_{v\alpha}$.

If we expand S at $\xi_{\alpha f}$ satisfying the parallel restriction, i.e., $e^{-i\varphi_{\alpha f}}\xi_{\alpha f} = \xi_{\alpha\Delta} + \delta\xi_{\alpha f}$, $\forall f \in \Delta$, then $\delta\xi_{\alpha f}$ are fluctuations of $\xi_{\alpha f}$ away from the parallel restriction. Notice that $\xi_{\alpha f} \to e^{i\varphi_{\alpha f}}\xi_{\alpha f}$ at internal f's are gauge symmetries of S,

$$S = S_0 + \sum_f 2j_f \Phi_\Delta[\xi_{\alpha f}],$$
$$S_0 = \sum_\pm \sum_{v,\Delta} 2J_\Delta^\pm \ln\langle \xi_{\alpha\Delta}|g_{v\alpha}^{\pm-1} g_{v\beta}^\pm|\xi_{\beta\Delta}\rangle,$$
$$\Phi_\Delta = \sum_\pm \frac{1\pm\gamma}{2}\sum_v [\ln\langle\xi_{\alpha f}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|\xi_{\beta f}\rangle - \ln\langle\xi_{\alpha\Delta}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|\xi_{\beta\Delta}\rangle] = O(\delta\xi), \tag{4.4}$$

where $J_\Delta^\pm$ is the same as in Eq. (3.3) and is large by $N_\Delta \gg 1$. J_Δ are assumed to satisfy the triangle inequality. S_0 reduces to Eq. (3.3) at each v and is the same as the EPRL-FK spinfoam action used for large spin asymptotics on the simplicial complex $\mathcal{K}_s$.

A. Critical points satisfying parallel restriction

Critical points of S_0, denoted by $(g_{v\alpha}^\pm, \xi_{\alpha\Delta})_c[J_\Delta]$, are gauge equivalence classes of solutions of critical equations $\mathrm{Re}(S_0) = \partial_{g_{v\alpha}^\pm} S_0 = \partial_{\xi_{\alpha\beta}} S_0 = 0$. These critical equations have been well studied in [19,20,28] and reduce to [it is straightforward to check that $\partial_{\xi_{\alpha\beta}} S_0 = 0$ follows from $\mathrm{Re}(S_0) = 0$]

$$g_{v\beta}^\pm|\xi_{\beta\Delta}\rangle = e^{i\phi_{\alpha v\beta}^\pm} g_{v\alpha}^\pm|\xi_{\alpha\Delta}\rangle, \quad \sum_{\Delta\subset\alpha} J_\Delta \kappa_{\alpha\Delta}(v)\vec{n}_{\alpha\Delta} = 0, \tag{4.5}$$

where $\kappa_{\alpha\Delta}(v) = \pm 1$ when $\partial_{g_{v\alpha}^\pm}$ acts on $g_{v\alpha}^\pm$ or $g_{v\alpha}^{\pm-1}$.

Theorem 4.1: Critical points of S_0 are also critical points of S.

Proof: We check that $\mathrm{Re}(S) = \partial S/\partial g_{v\alpha}^\pm = \partial S/\partial\xi_{\alpha f} = 0$ at all critical points of S_0. First of all, at any critical point of S_0,

$$\mathrm{Re}(S)|_c = \mathrm{Re}(S_0)|_c = 0, \tag{4.6}$$

where $|_c$ means evaluating at any critical point $(g_{v\alpha}^\pm, \xi_{\alpha\Delta})_c[J_\Delta]$ of S_0 where $\xi_{\alpha f} = \xi_{\alpha\Delta}$, $\forall f \in \Delta$.

If we write $\xi = (\xi^1, \xi^2)$ and define $J\xi = (-\bar{\xi}^2, \bar{\xi}^1)$, then ξ and $J\xi$ form an orthonormal basis in $\mathbb{C}^2$ with the Hermitian inner product. When we perturb S, we write $\delta\xi_{\alpha f} = \varepsilon_{\alpha f} J\xi_{\alpha f} + i\eta_{\alpha f}\xi_{\alpha f}$ where $\varepsilon_{\alpha f} \in \mathbb{C}$ and $\eta_{\alpha f} \in \mathbb{R}$. The coefficient in front of $\xi_{\alpha f}$ is purely imaginary because $\xi_{\alpha f}$ is normalized. Since every $\xi_{\alpha f}$ is shared by two terms with neighboring v's

$$\delta_{\xi_{\alpha f}} S|_c = \sum_\pm (1\pm\gamma) j_f \left[\varepsilon_{\alpha f}\frac{\langle\xi_{\beta' f}|(g_{v'\beta'}^\pm)^{-1}g_{v'\alpha}^\pm|J\xi_{\alpha f}\rangle}{\langle\xi_{\beta' f}|(g_{v'\beta'}^\pm)^{-1}g_{v'\alpha}^\pm|\xi_{\alpha f}\rangle} + \varepsilon_{\alpha f}^*\frac{\langle J\xi_{\alpha f}|(g_{v\alpha}^\pm)^{-1}g_{v\beta}^\pm|\xi_{\beta f}\rangle}{\langle\xi_{\alpha f}|(g_{v\alpha}^\pm)^{-1}g_{v\beta}^\pm|\xi_{\beta f}\rangle}\right]_c = 0. \tag{4.7}$$

At the critical point, $\xi_{\alpha f} = \xi_{\alpha\Delta}$, $\xi_{\beta f} = \xi_{\beta\Delta}$ at v and satisfy Eq. (4.5); similarly $\xi_{\beta' f} = \xi_{\beta'\Delta}$ and satisfy Eq. (4.5) at v'. Then $\delta_{\xi_{\alpha f}} S = 0$ by the orthogonality between ξ and $J\xi$.

For derivative in $g_{v\alpha}^\pm$, we use $\delta g_{v\alpha}^\pm = \frac{i}{2}\theta_{v\alpha}^\pm \vec{\sigma} g_{v\alpha}^\pm$ ($\theta_{v\alpha} \in \mathbb{R}$). At the critical point and by Eq. (4.5),

$$\delta_{g_{v\alpha}^\pm} S|_c = \frac{i}{2}\theta_{v\alpha}^\pm \sum_\Delta \kappa_{\alpha\Delta}(1\pm\gamma)\sum_{f\in\Delta} j_f \left.\frac{\langle\xi_{\alpha f}|(g_{v\alpha}^\pm)^{-1}\vec{\sigma}g_{v\beta}^\pm|\xi_{\beta f}\rangle}{\langle\xi_{\alpha f}|(g_{v\alpha}^\pm)^{-1}g_{v\beta}^\pm|\xi_{\beta f}\rangle}\right|_c = \frac{i}{2}\theta_{v\alpha}^\pm(1\pm\gamma)g_{v\alpha}^\pm \cdot \sum_\Delta \kappa_{\alpha\Delta} J_\Delta \hat{n}_{\alpha\Delta}|_c = 0, \tag{4.8}$$

where $\hat{n}_{\alpha\Delta} = \langle\xi_{\alpha\Delta}|\vec{\sigma}|\xi_{\alpha\Delta}\rangle$ is a unit 3-vector. $\kappa_{\alpha\Delta} = \pm 1$ relates to orientations of links in Fig. 1(c). We have chosen orientations such that all links connecting α, β are oriented parallel. ■

Critical points of S_0 have been completely classified in the case that all tetrahedra reconstructed from the closure condition are nondegenerate. We refer the reader to [9,20,28,34] for details of the classification. When J_Δ are areas relating to edge lengths on $\mathcal{K}_s$ by ($\ell_{ij}, \ell_{jk}, \ell_{ik}$ are three edge lengths of a triangle Δ)

$$\gamma J_\Delta(\ell) = \frac{1}{4}\sqrt{2(\ell_{ij}^2\ell_{jk}^2 + \ell_{ik}^2\ell_{jk}^2 + \ell_{ij}^2\ell_{ik}^2) - \ell_{ij}^4 - \ell_{ik}^4 - \ell_{jk}^4}, \tag{4.9}$$

there are a subset $\mathcal{G}$ of critical points $(g_{v\alpha}^\pm, \xi_{\alpha\Delta})_c[J_\Delta]$ of S_0 that can be interpreted as nondegenerate 4D Regge geometries, if the boundary condition of $\xi_{\alpha\Delta}$ gives the boundary 3D Regge geometry. Defining $N_\alpha(v)$ by $N_\alpha^0(v)\mathbf{1} + iN_\alpha^i(v)\sigma_i = g_{v\alpha}^-(g_{v\alpha}^+)^{-1}$ (σ_i are Pauli matrices), $\mathcal{G}$ is defined by critical points $(g_{v\alpha}^\pm, \xi_{\alpha\Delta})_c[J_\Delta]$ with

$$\det(N_{\alpha_1}(v), N_{\alpha_2}(v), N_{\alpha_3}(v), N_{\alpha_4}(v)) \neq 0, \tag{4.10}$$

for all $v \subset \mathcal{K}_s$ and all four $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ out of five α's at v. We have the following one-to-one correspondence [9,34]:

$$\text{Critical points } (g_{v\alpha}^\pm, \xi_{\alpha\Delta})_c[J_\Delta] \in \mathcal{G}$$
$$\updownarrow$$
$$\text{4D nondegenerate Regge geometry on } \mathcal{K}_s \text{ and 4-simplex orientations.} \tag{4.11}$$

Triangles Δ in Regge geometries are made by polyhedron facets as Fig. 1(c), and γJ_Δ is the area of Δ. Different critical points may give the same Regge geometry but different 4D orientations $\mu(v) = \pm 1$ at individual v. We focus on critical points $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta] \in \mathcal{G}$ that are isolated.

Consider infinitesimal deformations $(g^\pm_{v\alpha}, \xi_{\alpha\Delta}) \mapsto (g^\pm_{v\alpha} + \delta g^\pm_{v\alpha}, \xi_{\alpha\Delta} + \delta\xi_{\alpha\Delta})$ (including boundary data $\xi_{\alpha\Delta}$) from $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta] \in \mathcal{G}$ with fixed J_Δ, and ask whether the deformation can reach another critical point [solution of Eq. (4.5)]. Any infinitesimal deformation cannot break the condition Eq. (4.10), so it cannot reach critical points outside $\mathcal{G}$. Moreover the deformation cannot flip the orientation [9]. Therefore if the deformation reaches another critical point $(g'^\pm_{v\alpha}, \xi'_{\alpha\Delta})_c[J_\Delta]$, $(g'^\pm_{v\alpha}, \xi'_{\alpha\Delta})_c[J_\Delta]$ must still belong to $\mathcal{G}$ and correspond to a different nondegenerate Regge geometry with the same set of areas γJ_Δ. In other words, $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ and $(g'^\pm_{v\alpha}, \xi'_{\alpha\Delta})_c[J_\Delta]$ correspond to two different nondegenerate Regge geometries with the same set of areas. At any 4-simplex, Eq. (4.9) with ten fixed areas gives ten quadratic equations for ten squared edge lengths. These two different Regge geometries correspond to two different solutions of these ten quadratic equations with fixed J_Δ at least one 4-simplex. And these two different solutions are infinitesimally close to each other, since one comes from the infinitesimal deformation from the other. Then it implies the 10×10 matrix $\partial J^2_\Delta / \partial \ell^2_{ij}$ is degenerate at $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$. As a result, if $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ gives Regge geometry with nondegenerate $\partial J^2_\Delta / \partial \ell^2_{ij}$ at all 4-simplices, $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ is an isolated critical point. Note that the deformations considered above include deformations of boundary data $\xi_{\alpha\Delta}$, so $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ is isolated in a larger space of $g^\pm_{v\alpha}, \xi_{\alpha\Delta}$ including boundary $\xi_{\alpha\Delta}$. It is easy to find isolated critical points by numerically checking the determinant of $\partial J^2_\Delta / \partial \ell^2_{ij}$. Some experience from numerics suggests that degenerate $\partial J^2_\Delta / \partial \ell^2_{ij}$ might only happen at degenerate 4-simplices.

A critical point $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta] \in \mathcal{G}$ with a uniform orientation $\mu(v) = \mu$ at all v's evaluates

$$S_0|_c = \mu\left(\sum_{\Delta\in\mathcal{K}_s} \gamma J_\Delta \varepsilon_\Delta + \sum_{\Delta\in\partial\mathcal{K}_s} \gamma J_\Delta \Theta_\Delta\right) = \frac{i\mu}{8\pi\ell_P^2}\left(\sum_{\Delta\in\mathcal{K}_s} \mathbf{a}_\Delta \varepsilon_\Delta + \sum_{\Delta\in\partial\mathcal{K}_s} \mathbf{a}_\Delta \Theta_\Delta\right), \qquad \mu = \pm 1, \tag{4.12}$$

and is the Regge action on $\mathcal{K}_s$ [9,20,28,35,36]. $|_c$ means evaluating at any critical point $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ of S_0. ε_Δ and Θ_Δ are the deficit angles and dihedral angles hinged at internal and boundary Δ's. γJ_Δ are interpreted as triangle areas $\mathbf{a}_\Delta = \sum_{f\in\Delta} \mathbf{a}_f$ made by facet areas $\mathbf{a}_f$ as in Eq. (3.7). The validity of Eq. (4.12) has some topological requirements on $\mathcal{K}_s$: (1) all internal Δ are shared by an even number of 4-simplices, and (2) $\mathcal{K}_s$ is a triangulation of manifold $\mathcal{M}$ with trivial second cohomology $H^2(\mathcal{M}, \mathbb{Z}_2) = 0$ [9]. The first requirement is generically satisfied by triangulations used in Regge calculus; see [9,37] for examples. The above result applies to, e.g., $\mathcal{M}$ at S^4, $S^3 \times I$ (where I is an interval in $\mathbb{R}$), or a topologically trivial region in $\mathbb{R}^4$.

Beyond the subset $\mathcal{G}$, there are other critical points with the BF-type and/or vector geometry critical data [9,19,20]. Each of these critical points has critical data of $g^\pm_{v\alpha}$ to satisfy $g^+_{v\alpha} = g^-_{v\alpha}$ or equivalently $\det(N_{\alpha_1}(v), N_{\alpha_2}(v), N_{\alpha_3}(v), N_{\alpha_4}(v)) = 0$ at certain v's. The difference between the BF-type and vector geometry critical data is that the BF-type data still associate with nondegnerate 4-simplices, while vector geometries are degenerate 4-simplices.

B. Critical points violating parallel restriction

The converse of Theorem 4.1 is not true. There exist critical points of S which are not critical points of S_0. Critical points of S satisfy

$$\hat{g}^\pm_{v\beta} \vec{n}_{\beta f} = \hat{g}^\pm_{v\alpha} \vec{n}_{\alpha f}, \qquad \sum_{f\subset\alpha} j_f \kappa_{\alpha\Delta} \vec{n}_{\alpha f} = 0. \tag{4.13}$$

Theorem 4.2: Every critical point of S that is not a critical point of S_0 either (1) relates to a critical point of S_0, $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$, by $g^+_{v\alpha} \neq g^-_{v\alpha}$ and $\xi_{\alpha f} = J\xi_{\alpha\Delta}$ up to a phase at some internal $f \in \Delta$, or (2) satisfies $g^+_{v\alpha} = g^-_{v\alpha}$ for all v, α modulo discrete gauge.

Proof: We write $\hat{g}^\pm_{\alpha\beta} \equiv (\hat{g}^\pm_{v\alpha})^{-1}(\hat{g}^\pm_{v\beta})$, the first equation in (4.13) gives $\hat{g}^+_{\alpha\beta}\vec{n}_{\beta f} = \vec{n}_{\alpha f}$ and $\hat{g}^-_{\alpha\beta}\vec{n}_{\beta f} = \vec{n}_{\alpha f}$ and implies $(\hat{g}^+_{\alpha\beta})^{-1}\hat{g}^-_{\alpha\beta}\vec{n}_{\beta f} = \vec{n}_{\beta f}$ for all $f \in \Delta$; i.e., $\vec{n}_{\beta f}$ at all $f \in \Delta$ are eigenvectors of $(\hat{g}^+_{\alpha\beta})^{-1}\hat{g}^-_{\alpha\beta}$ with a unit eigenvalue. It does not constrain $\vec{n}_{\beta f}$ if $(\hat{g}^+_{\alpha\beta})^{-1}\hat{g}^-_{\alpha\beta} = 1$. But when the SO(3) matrix $(\hat{g}^+_{\alpha\beta})^{-1}\hat{g}^-_{\alpha\beta} \neq 1$, its eigenspace with the unit eigenvalue is at most one dimensional. Therefore in this case, all $\vec{n}_{\beta f}$ are collinear, thus $\vec{n}_{\beta f} = \pm\vec{n}_{\beta f'}$ for any pair of $f, f' \in \Delta$, and Eq. (4.13) reduces to Eq. (4.5) whose solution gives $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$. Hence $\vec{n}_{\beta f} = \pm\vec{n}_{\alpha\Delta}$, i.e., $\xi_{\beta f} = \xi_{\alpha\Delta}$ or $J\xi_{\alpha\Delta}$ up to a phase. At each v, we have to gauge fix $g^\pm_{v\alpha} = 1$ at a certain α, then require that $(\hat{g}^+_{\alpha\beta})^{-1}\hat{g}^-_{\alpha\beta} \neq 1$ is equivalent to $g^+_{v\beta} \neq g^-_{v\beta}$ for all $\beta \neq \alpha$ [$g^+_{v\beta} = -g^-_{v\beta}$ still implies $(\hat{g}^+_{\alpha\beta})^{-1}\hat{g}^-_{\alpha\beta} = 1$, but it is gauge equivalent to $g^+_{v\beta} = g^-_{v\beta}$ by a discrete gauge transformation]. ■

We may generalize the definition Eq. (4.10) of the subclass $\mathcal{G}$ to include all critical points of S. It contains critical points of S_0, $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta] \in \mathcal{G}$, and critical points of S which flip some internal or boundary $\xi_{\alpha f} \to J\xi_{\alpha\Delta}$. Critical points in either class (1) or class (2) in Theorem 4.2 are isolated from $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta] \in \mathcal{G}$ because an infinitesimal deformation from $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ at fixed J_Δ cannot flip $\xi_{\alpha f} \to J\xi_{\alpha\Delta}$ and cannot break the condition Eq. (4.10).

Although we find critical points of S_0 and S, we cannot apply the stationary phase approximation of the integral at the present stage since all j_f's are small. The critical points in Theorem 4.2 seem useless. But we come back to the computation of the integral in Sec. IX and see why these critical points are useful to the stationary phase approximation of integrals.

V. SEMICLASSICAL STATES FROM SPINFOAM AMPLITUDE

Spinfoam amplitudes can be used to construct quantum states in LQG Hilbert space. Given a 4-manifold $\mathcal{M}_4$ with a spatial boundary Σ as in Fig. 2, we make an arbitrary cellular decomposition of $\mathcal{M}_4$. The cellular complex is denoted by $\mathcal{K}$. Spinfoam amplitudes can be defined on $\mathcal{K}$ and denoted by $A(\mathcal{K})_{\vec{j},\vec{i}}$ where $\vec{j},\vec{i}$ are spins and intertwiners coloring the boundary dual complex $\partial\mathcal{K}^*$. On the other hand, Σ associates with a LQG kinematical Hilbert space $\mathcal{H}_\Sigma$ in which spin-network states $T_{\mathcal{G},\vec{j},\vec{i}}(\vec{U})$ for all graphs $\mathcal{G}$ colored by $\vec{j},\vec{i}$. $\vec{U}$ are SU(2) holonomies along links of $\mathcal{G}$. We define a linear combination of $T_{\mathcal{G},\vec{j},\vec{i}}$ by identifying $\mathcal{G}=\partial\mathcal{K}^*$ and letting the coefficients be $A(\mathcal{K})_{\vec{j},\vec{i}}$:

$$\Psi_\mathcal{K}(\vec{U})=\sum_{\vec{j},\vec{i}}A(\mathcal{K})_{\vec{j},\vec{i}}T_{\partial\mathcal{K}^*,\vec{j},\vec{i}}(\vec{U}). \tag{5.1}$$

One may even consider to sum over the cellular decomposition and define $\Psi(\vec{U})=\sum_\mathcal{K}\Psi_\mathcal{K}(\vec{U})$. If we truncate the sum in $\Psi_\mathcal{K}$ (or Ψ) to be finite, $\Psi_\mathcal{K}$ (or Ψ) is a state in the kinematical Hilbert space $\mathcal{H}_\Sigma$. If the sums in $\Psi_\mathcal{K}$ are kept infinite, $\Psi_\mathcal{K}$ may not be normalizable in $\mathcal{H}_\Sigma$, but one may anticipate that $\Psi_\mathcal{K}$ is a physical state living in the dual space of a dense subspace in $\mathcal{H}_\Sigma$. $\Psi_\mathcal{K}$ may be viewed as a spinfoam analog of the Hartle-Hawking wave function.

When $\mathcal{M}_4$ has several disconnected boundaries Σ_1, $\Sigma_2,\ldots,\Sigma_n$ in additional to Σ, a cellular decomposition $\mathcal{K}$ of $\mathcal{M}_4$ induces boundary dual complexes $\partial\mathcal{K}_1^*,\ldots,\partial\mathcal{K}_n^*$. A state $\Psi_\mathcal{K}(\vec{U})$ on Σ can be defined by choices of (initial) states $\phi_a\in\mathcal{H}_{\Sigma_a}$ $(a=1,\ldots,n)$, whose spin-network decompositions are $\phi_a=\sum_{\vec{j}_a,\vec{i}_a}\prod_l(2j_l+1)(\phi_a)_{\vec{j}_a,\vec{i}_a}T_{\partial\mathcal{K}_a^*,\vec{j}_a,\vec{i}_a}$. ϕ_a is based on a single graph $\partial\mathcal{K}_a^*$. $\Psi_\mathcal{K}(\vec{U})$ can be constructed as

$$\Psi_\mathcal{K}(\vec{U})=\sum_{\vec{j},\vec{i}}\sum_{\{\vec{j}_a,\vec{i}_a\}_{a=1}^n}\prod_{a=1}^n(\phi_a)_{\vec{j}_a,\vec{i}_a}A(\mathcal{K})_{\vec{j},\vec{i},\{\vec{j}_a,\vec{i}_a\}_{a=1}^n}\times T_{\partial\mathcal{K}^*,\vec{j},\vec{i}}(\vec{U}). \tag{5.2}$$

It is useful to write Eqs. (5.1) and (5.2) in terms of coherent intertwiners. For instance, if we consider $\mathcal{K}$ whose cells are B_4 as in Sec. IV and apply the spinfoam amplitude $A(\mathcal{K})$ as in Eq. (4.1), $\Psi_\mathcal{K}(\vec{U})$ in Eq. (5.1) can be written as

$$\Psi_\mathcal{K}(\vec{U})=\sum_{\{j_f\}}\prod_f A_\Delta(j_f)\int[\mathrm{d}g^\pm_{v\alpha}\mathrm{d}\xi_{\alpha f}]e^S T_{\partial\mathcal{K}^*,\vec{j},\vec{\xi}}(\vec{U}), \tag{5.3}$$

while Eq. (5.2) can be written analogously. In Eq. (5.3), $\sum_{\{j_f\}}$ and $\int\mathrm{d}\xi_{\alpha f}$ integrate all internal and boundary j_f's and $\xi_{\alpha f}$'s. Gauge symmetries of the integrand

$$g^\pm_{v\alpha}\sim g^\pm_{v\alpha}h_\alpha,\quad \xi_{\alpha f}\sim h_\alpha^{-1}\xi_{\alpha f},\quad \forall\, h_\alpha\in\mathrm{SU}(2),\quad \xi_{\alpha f}\sim e^{i\varphi}\xi_{\alpha f}, \tag{5.4}$$

apply to both internal and boundary α. $T_{\vec{j},\vec{\xi}}(\vec{U})$ are spin-network states with coherent intertwiners (see Appendix for convention):

$$T_{\partial\mathcal{K}^*,\vec{j},\vec{\xi}}(\vec{U})=\mathrm{tr}\Big[\bigotimes_{f\in\partial\mathcal{K}}R^{j_f}(U_f)\bigotimes_{\alpha\in\partial\mathcal{K}}\|\{j_f\},\{\xi_{\alpha f}\}\rangle\bigotimes_{\beta\in\partial\mathcal{K}}\langle\{j_f\},\{\xi_{\beta f}\}\|\Big], \tag{5.5}$$

where $\|\{j_f\},\{\xi_{\alpha f}\}\rangle$ are coherent intertwiners at polyhedra $\alpha\in\partial\mathcal{K}$ and are bras or kets depending on the orientation of the spin-network graph. $R^{j_f}(U_f)$ satisfies the following normalization:

$$\int_{\mathrm{SU}(2)}\mathrm{d}U\overline{R^j_{mn}(U)}R^{j'}_{m'n'}(U)=\frac{1}{\dim(j)}\delta_{j,j'}\delta_{mm'}\delta_{nn'}. \tag{5.6}$$

A. Truncated states $|\psi\rangle$ with parallel restriction

In the following we always consider states constructed by spinfoam amplitudes on a fixed cellular complex $\mathcal{K}$, plus certain truncations. The resulting states are inside $\mathcal{H}_\Sigma$. We again focus on $\mathcal{K}$ whose 4-cells are B_4. The boundary $\partial\mathcal{K}$ is a polyhedral decomposition of Σ.

We apply the following truncations to $\Psi_\mathcal{K}$: (1) The sum $\sum_{\{j_f\}}$ is constrained by $\sum_{f\in\Delta}j_f=J_\Delta$ with fixed J_Δ at every Δ. (2) The integral of $\int[\mathrm{d}g^\pm_{v\alpha}\mathrm{d}\xi_{\alpha f}]$ is over a neighborhood $\mathfrak{N}_{g,\xi}$ (of both internal and boundary variables) at an isolated critical point $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]\in\mathcal{G}$ of S_0 (the critical point is isolated in the space of $g^\pm_{v\alpha},\xi_{\alpha f}$ including boundary $\xi_{\alpha f}$). $\mathfrak{N}_{g,\xi}$ only contains a single critical point.[1] The critical data $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$ are a gauge equivalence class by

[1]ψ contains integral over boundary $\vec{\xi}$; different boundary data $\vec{\xi}$ might lead to different critical points for the integral over $g^\pm_{v\alpha}$ and internal $\xi_{\alpha f}$. Here the assumption that $\mathfrak{N}_{g,\xi}$ only contains a single critical point means that arbitrary changes of boundary data $\vec{\xi}$ within $\mathfrak{N}_{g,\xi}$ do not lead to any other critical point in $\mathfrak{N}_{g,\xi}$ different from $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$. $\mathfrak{N}_{g,\xi}$ satisfying the requirement is nontrivial. Indeed, if an infinitesimal change of boundary data $\vec{\xi}$ leads to another critical point in $\mathfrak{N}_{g,\xi}$ different from $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$, then the new critical point has to be infinitesimally close to $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$. Otherwise this new critical point can be excluded by redefining $\mathfrak{N}_{g,\xi}$. But it violates the assumption that $(g^\pm_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$ is isolated.

Eq. (5.4), and other gauge transformations mentioned in Sec. IV, $(g^{\pm}_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$, include the data of boundary $J_\Delta,\xi_{\alpha\Delta}$. (3) We impose the parallel restriction to $\xi_{\alpha f}$ by a real gauge invariant potential $V_{\alpha,\Delta}(\xi_{\alpha f})$ at every pair of internal and boundary α, Δ, such that the minimum $V_{\alpha,\Delta}(\xi_{\alpha f})=0$ gives the parallel restriction. The truncated state is denoted by ψ:

$$\psi(\vec{U})=\sum_{\{j_f\}}{}'\prod_f A_\Delta(j_f)\int_{\mathfrak{N}_{g,\xi}}[\mathrm{d}g^{\pm}_{v\alpha}\mathrm{d}\xi_{\alpha f}]\times e^S\prod_{\alpha,\Delta}e^{-NV_{\alpha,\Delta}}T_{\partial\mathcal{K}^*,\vec{j},\vec{\xi}}(\vec{U}). \tag{5.7}$$

An example of $V_{\alpha,\Delta}(\xi_{\alpha f})$ may be an analog of the 2D spin-chain Hamiltonian: $V_{\alpha,\Delta}(\xi_{\alpha f})=\sum_{\langle f,f'\rangle}(1-\vec{n}_{\alpha f}\cdot\vec{n}_{\alpha f'})$ where $\langle f,f'\rangle$ are close-neighbor pairs. Our following discussion does not reply on details of $V_{\alpha,\Delta}$. N is of the same order of magnitude as J_Δ. $\sum'_{\{j_f\}}$ only sums nonzero j_f in Eq. (5.7). $\sum'_{\{j_f\}}$ constrained by $\sum_{f\in\Delta}j_f=J_\Delta$ is a finite sum, so $\psi\in\mathcal{H}_\Sigma$.

Sending the coupling constant of $V_{\alpha,\Delta}$ to infinity $N\to\infty$ independent of J_Δ imposes strongly the parallel restriction which reduces the vertex amplitude used in Eq. (5.7) to the EPRL-FK 4-simplex amplitude. Equation (5.7) is a generalization from the following analog using large-J EPRL-FK amplitudes on the simplicial complex $\mathcal{K}_s$:

$$\psi_{\text{EPRL-FK}}(\vec{U})=\prod_\Delta A_\Delta(J_\Delta)\int_{\mathfrak{N}_{g,\xi}}[\mathrm{d}g^{\pm}_{v\alpha}\mathrm{d}\xi_{\alpha\Delta}]e^{S_0}T_{\partial\mathcal{K}_s^*,\vec{J},\vec{\xi}}(\vec{U}). \tag{5.8}$$

The generalization from $\psi_{\text{EPRL-FK}}$ to ψ releases mildly the d.o.f. of nonparallel $\xi_{\alpha f}$'s in Δ, but releases a large number of microdegrees of freedom of small j_f's at every Δ. Spinfoam amplitude with the parallel restriction imposed by $V_{\alpha\Delta}$ is constructed for the purpose of defining ψ which has the semiclassical property discussed below and gives interesting entanglement entropy (see Sec. VII). The computation of the amplitude without the parallel restriction is discussed in Sec. IX.

Given that ψ associates with a unique critical point $(g^{\pm}_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$, when $(g^{\pm}_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]\in\mathcal{G}$ corresponds to a Regge spacetime geometry, ψ may be viewed as a semiclassical state associated with the Regge spacetime geometry. Indeed, if the boundaries $\vec{j}$ and $\vec{\xi}$ in $T_{\partial\mathcal{K}^*,\vec{j},\vec{\xi}}$ are consistent with the boundary data of $(g^{\pm}_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$, its coefficient gives

$$\begin{aligned}\sum_{\{j_f\}}{}'\prod_f A_\Delta(j_f)\int_{\mathfrak{N}_{g,\xi}}[\mathrm{d}g^{\pm}_{v\alpha}\mathrm{d}\xi_{\alpha f}]e^S\prod_{\alpha,\Delta}e^{-NV_{\alpha,\Delta}}&=\int_{\mathfrak{N}_{g,\xi}}[\mathrm{d}g^{\pm}_{v\alpha}\mathrm{d}\xi_{\alpha\Delta}]\left(\frac{2\pi}{N}\right)^{\sum_{\Delta\in i(\mathcal{K})}(N_\Delta-1)}\frac{e^{S_0}}{\sqrt{\det H_V(\xi_{\alpha\Delta})}}\sum_{\{j_f\}}{}'\prod_f A_\Delta(j_f)\left[1+O\left(\frac{1}{N}\right)\right]\\&=\left(\frac{2\pi}{N}\right)^{24N_v+2\sum_{\Delta\in\mathcal{K}}N_\Delta}\frac{e^{S_0|_c}}{\sqrt{\det(H_V|_c)\det(-H_0|_c)}}\sum_{\{j_f\}}{}'\prod_f A_\Delta(j_f)\left[1+O\left(\frac{1}{N}\right)\right].\end{aligned} \tag{5.9}$$

In the first step we choose a f_0 in every Δ and define $\xi_{\alpha f_0}\equiv\xi_{\alpha\Delta}$, then integrate out $\xi_{\alpha f}$'s ($f\neq f_0$) by $N\gg1$, and reduce S to S_0 which depends on j_f only through J_Δ. In the second step we apply the stationary phase approximation of the integral with S_0 in $\mathfrak{N}_{g,\xi}$ which contains a single critical point. H_V and H_0 are Hessian matrices of $\sum_{\alpha,\Delta}V_{\alpha,\Delta}$ and S_0, and are assumed to be nondegenerate. If the boundary $\vec{\xi}$ in $T_{\partial\mathcal{K}^*,\vec{j},\vec{\xi}}$ is away from the boundary data of $(g^{\pm}_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$, critical equations from S_0 have no solution in $\mathfrak{N}_{g,\xi}$, so the integral is suppressed exponentially by large J_Δ. It shows that coefficients in Eq. (5.7) as a function of boundary $\vec{\xi}$ is peaked at the boundary data of $(g^{\pm}_{v\alpha},\xi_{\alpha\Delta})_c[J_\Delta]$. ψ is a spinfoam analog of the Hartle-Hawking state.

In addition, ψ also explicitly depends on the size of the neighborhood $\mathfrak{N}_{g,\xi}$. But as we are going to see in a moment, the squared norm of ψ and entanglement entropy only mildly depend on the size $\mathfrak{N}_{g,\xi}$ through the subleading order.

B. Squared norm of $|\psi\rangle$

The squared norm of $|\psi\rangle$ is computed as follows:

$$\begin{aligned}\langle\psi|\psi\rangle=&\sum_{\{j_f\},\{j'_f\}}{}'\prod_f A_\Delta(j_f)A_\Delta(j'_f)\int_{\mathfrak{N}_{g,\xi}\times\mathfrak{N}_{g,\xi}}[\mathrm{d}g^{\pm}_{v\alpha}\mathrm{d}g'^{\pm}_{v\alpha}\mathrm{d}\xi_{\alpha f}\mathrm{d}\xi'_{\alpha f}]e^{S+\bar{S}}\prod_{\alpha,\Delta}e^{-N[V_{\alpha,\Delta}(\xi_{\alpha f})+V_{\alpha,\Delta}(\xi'_{\alpha f})]}\\&\times\prod_{f\subset\Sigma}\frac{\delta_{j_f,j'_f}}{2j_f+1}\prod_{\alpha\subset\Sigma}\int_{\mathrm{SU}(2)}\mathrm{d}g_\alpha e^{\sum_{f\in\partial\alpha}2j_f\ln\langle\xi'_{\alpha f}|g_\alpha|\xi_{\alpha f}\rangle}\prod_{\beta\subset\Sigma}\int_{\mathrm{SU}(2)}\mathrm{d}g_\alpha e^{\sum_{f\in\partial\alpha}2j_f\ln\langle\xi_{\beta f}|g_\alpha|\xi'_{\beta f}\rangle},\end{aligned} \tag{5.10}$$

where $j'_f, g'^{\pm}_{v\alpha}, \xi'_{\alpha f}$ denote variables from $\langle\psi|$. S and $\bar{S}$ are from $|\psi\rangle$ and $\langle\psi|$, and thus depend on unprimed and primed variables, respectively. $j'_f = j_f$ for $f \subset \Sigma$. $2j_f + 1$ in the denominator comes from the normalization Eq. (5.6). We have applied the integral expressions of inner products between coherent intertwiners:

$$\langle\{j_f\},\{\xi'_{\alpha f}\}||\{j_f\},\{\xi_{\alpha f}\}\rangle = \int_{\mathrm{SU}(2)} \mathrm{d}g_\alpha e^{\sum_{f\in\partial\alpha} 2j_f \ln\langle\xi'_{\alpha f}|g_\alpha|\xi_{\alpha f}\rangle}. \tag{5.11}$$

h_α in the integrand of $\langle\psi|\psi\rangle$ can be removed by a gauge transformation Eq. (5.4).

We may define a total action by collecting all exponents in the integrand:

$$\begin{aligned} S_{\mathrm{tot}} = {} & S[j_f, g^{\pm}_{v\alpha}, \xi_{\alpha f}] + \overline{S[j'_f, g'^{\pm}_{v\alpha}, \xi'_{\alpha f}]} - N\sum_{\alpha,\Delta}[V_{\alpha,\Delta}(\xi'_{\alpha f}) + V_{\alpha,\Delta}(\xi_{\alpha f})] \\ & + \sum_{\alpha\subset\Sigma}\sum_{f\in\partial\alpha} 2j_f \ln\langle\xi'_{\alpha f}|g_\alpha|\xi_{\alpha f}\rangle + \sum_{\beta\subset\Sigma}\sum_{f\in\partial\beta} 2j_f \ln\langle\xi_{\beta f}|g_\beta|\xi'_{\beta f}\rangle. \end{aligned} \tag{5.12}$$

We may choose a f_0 in every Δ and define $\xi_{\alpha f_0} \equiv \xi_{\alpha\Delta}$. The large N implements the parallel restriction and reduces S to S_0, $\sum_{f\in\Delta} 2j_f \ln\langle\xi'_{\alpha f}|\xi_{\alpha f}\rangle$ to $2J_\Delta \ln\langle\xi'_{\alpha\Delta}|\xi_{\alpha\Delta}\rangle$ up to $O(1/N)$ after integrating out nonparallel $\xi_{\alpha f}$'s. The integral in $\langle\psi|\psi\rangle$ reduces to

$$\begin{aligned} & \int_{\mathfrak{N}_{g,\xi}\times\mathfrak{N}_{g,\xi}} [\mathrm{d}g^{\pm}_{v\alpha}\mathrm{d}g'^{\pm}_{v\alpha}\mathrm{d}\xi_{\alpha\Delta}\mathrm{d}\xi'_{\alpha\Delta}\mathrm{d}g_\alpha\mathrm{d}g_\beta] \frac{(2\pi/N)^{2\sum_{\Delta\in\mathcal{K}}(N_\Delta-1)}}{\sqrt{\det H_V(\xi_{\alpha\Delta})\det H_V(\xi'_{\alpha\Delta})}} e^{S'_{\mathrm{tot}}}\left[1 + O\left(\frac{1}{N}\right)\right] \\ & S'_{\mathrm{tot}} = S_0 + \bar{S}_0 + \sum_{\alpha\subset\Sigma}\sum_{\Delta\subset\alpha} 2J_\Delta \ln\langle\xi'_{\alpha\Delta}|g_\alpha|\xi_{\alpha\Delta}\rangle + \sum_{\beta\subset\Sigma}\sum_{\Delta\subset\beta} 2J_\Delta \ln\langle\xi_{\beta\Delta}|g_\beta|\xi'_{\beta\Delta}\rangle, \end{aligned} \tag{5.13}$$

where $H_V(\xi_{\alpha\Delta})$ is the Hessian matrix of $\sum_{\alpha,\Delta} V_{\alpha,\Delta}(\xi_{\alpha f})$ evaluated at the minimum. Equation (5.13) can be computed by stationary phase approximation. The critical equation of this integral is given by Eq. (4.5) from S_0 and in addition

$$g_\beta|\xi'_{\beta\Delta}\rangle = e^{i\varphi_{\beta\Delta}}|\xi_{\beta\Delta}\rangle, \qquad \forall\, \Delta\subset\beta, \qquad \beta\subset\Sigma, \tag{5.14}$$

from $\mathrm{Re}(2J_\Delta \ln\langle\xi_{\beta\Delta}|g_\beta|\xi'_{\beta\Delta}\rangle) = 0$. Equation (5.14) implies that $|\xi'_{\beta\Delta}\rangle$ and $|\xi_{\beta\Delta}\rangle$ are related by a gauge transformation. A critical point $(g^{\pm}_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ of S_0 gives rise to a critical point of S'_{tot} by double copying, i.e., $(g'^{\pm}_{v\alpha}, \xi'_{\alpha\Delta})_c[J_\Delta] = (g^{\pm}_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ modulo gauge equivalence. A gauge transformation $|\xi_{\beta\Delta}\rangle \mapsto e^{-i\varphi_{\beta\Delta}} g_\beta|\xi_{\beta\Delta}\rangle$, $g^{\pm}_{v\beta} \mapsto g^{\pm}_{v\beta}g^{-1}_\beta$ identifies $|\xi_{\beta\Delta}\rangle = |\xi'_{\beta\Delta}\rangle$ by Eq. (5.14). $\mathfrak{N}_{g,\xi}$ contains a single critical point $(g^{\pm}_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$, which implies that $\mathfrak{N}_{g,\xi}\times\mathfrak{N}_{g,\xi}$ contains a single critical point made by double copying. S'_{tot} vanishes at the critical point[2], so Eq. (5.13) is estimated by

$$\left(\frac{2\pi}{N}\right)^{24N_v + 2\sum_{\Delta\in\mathcal{K}} N_\Delta} \frac{1}{\det(H_V|_c)} \frac{1}{\sqrt{\det(-H'_{\mathrm{tot}}|_c)}}\left[1 + O\left(\frac{1}{N}\right)\right], \tag{5.15}$$

where N_v is the total number of B_4 in $\mathcal{K}$ and $24N_v + 2\sum_{\Delta\in\mathcal{K}} N_\Delta$ is the total number of integration variables in $\psi(\vec{U})$. $H'_{\mathrm{tot}}|_c$ is the Hessian matrix of S'_{tot} evaluated at the critical point and is assumed to be nondegenerate.

We observe that the leading order in Eq. (5.15) depends on $\{j_f\}$ only through their sum J_Δ, so it is a constant in the sum over $\{j_f\}$ in $\langle\psi|\psi\rangle$. Therefore inserting the above estimate of the integral,

$$\langle\psi|\psi\rangle = \left(\frac{2\pi}{N}\right)^{24N_v + 2\sum_{\Delta\in\mathcal{K}} N_\Delta} \frac{1}{\det(H_V|_c)} \frac{1}{\sqrt{\det(-H'_{\mathrm{tot}}|_c)}} \prod_{\Delta\in i(\mathcal{K}_s)} \Gamma_\Delta[J_\Delta]^2 \prod_{\Delta\subset\Sigma} \Gamma'_\Delta[J_\Delta]\left[1 + O\left(\frac{1}{N}\right)\right], \tag{5.16}$$

where $i(\mathcal{K}_s)$ is the interior of the simplicial complex $\mathcal{K}_s$ determined by $\mathcal{K}$, and $\Gamma_\Delta, \Gamma'_\Delta$ are given by

[2]At the critical point, we apply the gauge transformation $|\xi_{\beta\Delta}\rangle \mapsto g_\beta|\xi_{\beta\Delta}\rangle$ to boundary $\xi_{\beta\Delta}$'s and set phase conventions such that $|\xi_{\beta\Delta}\rangle = |\xi'_{\beta\Delta}\rangle$ (set $\varphi_{\beta\Delta} = 0$ by gauge transformation). They make $\ln\langle\xi_{\beta\Delta}|g_\beta|\xi'_{\beta\Delta}\rangle$ vanish and identify the complex conjugate of S_0 to be $\bar{S}_0$. $S_0 + \bar{S}_0$ vanishes since S_0 is purely imaginary at the critical point.

$$\Gamma_\Delta[J_\Delta] = \sum_{\{j_{f\in\Delta}\}}{}' \prod_f A_\Delta(j_f) = \sum_{\{j_{f\in\Delta}\}}{}' \prod_f (2j_f+1)^{n_v(\Delta)+1},$$
$$\Gamma'_\Delta[J_\Delta] = \sum_{\{j_{f\in\Delta}\}}{}' \prod_f \frac{A_\Delta(j_f)^2}{2j+1} = \sum_{\{j_{f\in\Delta}\}}{}' \prod_f (2j_f+1)^{2n_v(\Delta)+3}. \tag{5.17}$$

VI. ANALOG WITH MICROSTATE COUNTING

Interestingly, $\Gamma_\Delta[J_\Delta]$ and $\Gamma'_\Delta[J_\Delta]$ are two analogs of counting microstates corresponding to the macrostate (J_Δ, N_Δ), where the microstates are $\{j_f\}$ with degeneracies $(2j_f+1)^{n_v(\Delta)+1}$ and $(2j_f+1)^{2n_v(\Delta)+3}$ at the level j_f. Here we list quantities in $\Gamma_\Delta[J_\Delta]$ or $\Gamma'_\Delta[J_\Delta]$ as analogs with quantities in a statistical ensemble of identical systems:

$$\begin{aligned} N_\Delta &\leftrightarrow \text{total number of identical systems in the ensemble,}\\ J_\Delta &\leftrightarrow \text{total energy of the ensemble,}\\ j &\leftrightarrow \text{energy levels of the system,}\\ (2j+1)^{2n_v(\Delta)+1} \quad \text{or} \quad (2j+1)^{2n_v(\Delta)+3} &\leftrightarrow \text{degeneracy at each energy leve,}\\ \Gamma_\Delta[J_\Delta] \quad \text{or} \quad \Gamma'_\Delta[J_\Delta] &\leftrightarrow \text{total number of microstates in the ensemble.} \end{aligned} \tag{6.1}$$

$\Gamma_\Delta[J_\Delta]$ and $\Gamma'_\Delta[J_\Delta]$ are similar to the black hole entropy counting in LQG [11]

Here we focus on computing the boundary contribution $\Gamma'_\Delta[J_\Delta]$. We define n_j to be the number of facets f carrying the nonzero spin j,

$$\Gamma'_\Delta[J_\Delta] = \sum_{\{j_{f\in\Delta}\}}{}' \prod_{f\in\Delta} g_\Delta(j_f) = \sum_{\{n_j\}}{}' N_\Delta! \prod_{j\neq 0} \frac{g_\Delta(j)^{n_j}}{n_j!}, \qquad g_\Delta(j) = \frac{A_\Delta(j)^2}{2j+1} = (2j+1)^{2n_v(\Delta)+3}, \tag{6.2}$$

where $\sum_{j=1/2}^\infty j n_j = J_\Delta$ and $\sum_{j=1/2}^\infty n_j = N_\Delta$ is imposed to $\sum'_{\{n_j\}}$. $\Gamma_\Delta[J_\Delta]$ is computed by simply replacing $g_\Delta(j)$ by $(2j+1)^{2n_v(\Delta)+1}$. Following the Darwin-Fowler method in statistical mechanics (see, e.g., [38]), we define the generating functional

$$\sum_{J_\Delta=1/2}^\infty \Gamma'_\Delta[J_\Delta] z^{2J_\Delta} = \sum_{\{n_j\}} N_\Delta! \prod_{j=1/2}^\infty \frac{g_\Delta(j)^{n_j} z^{2n_j j}}{n_j!} = \left[\sum_{j=1/2}^\infty z^{2j} g_\Delta(j)\right]^{N_\Delta}, \tag{6.3}$$

where $\sum_{J_\Delta=1/2}^\infty$ relaxes the constraint $\sum_{j=1/2}^\infty j n_j = J_\Delta$ on $\sum_{\{n_j\}}$. $\sum_{\{n_j\}}$ satisfies only one constraint, $\sum_{j=1/2}^\infty n_j = N_\Delta$. $\sum_{j=1/2}^\infty z^{2j} g_\Delta(j)$ has a nonzero radius of convergence, so it is an analytic function of z at a neighborhood at $z=0$. $\Gamma'_\Delta[J_\Delta]$ is given by a contour integral

$$\begin{aligned} \Gamma'_\Delta[J_\Delta] &= \frac{1}{2\pi i} \oint_{z=0} \mathrm{d}z \frac{1}{z^{2J_\Delta+1}} \left[\sum_{j=1/2}^\infty z^{2j} g_\Delta(j)\right]^{N_\Delta} \\ &= \frac{1}{2\pi i} \oint_{z=0} \mathrm{d}z \exp\left(N_\Delta \ln\left[\sum_{j=1/2}^\infty z^{2j} g_\Delta(j)\right] - (2J_\Delta+1)\ln(z)\right). \end{aligned} \tag{6.4}$$

The integration contour is a circle inside the domain where the generating function is analytic. The exponent in the integrand is bounded along the contour. Given that both $N_\Delta, J_\Delta \gg 1$, the above integral can be computed by the method of steepest descent: If we denote the exponent by

$$N_\Delta f(z) \simeq N_\Delta \ln\left[\sum_{j=1/2}^\infty z^{2j} g_\Delta(j)\right] - 2J_\Delta \ln(z), \tag{6.5}$$

then the variational principle $\partial_z f(z_0) = 0$ gives

TABLE I. Solutions z_0 maximizing $f(z_0)$ at different $n_v(\Delta)$ and J_Δ/N_Δ [$f''(z_0)$ are all nonzero].

	$J_\Delta/N_\Delta = 0.6$	$J_\Delta/N_\Delta = 0.7$	$J_\Delta/N_\Delta = 0.8$	$J_\Delta/N_\Delta = 0.9$	$J_\Delta/N_\Delta = 1$
$n_v(\Delta) = 1$	$z_0 = 0.0257781$	$z_0 = 0.0505039$	$z_0 = 0.0742575$	$z_0 = 0.0971007$	$z_0 = 0.119083$
$n_v(\Delta) = 2$	$z_0 = 0.0119832$	$z_0 = 0.0244767$	$z_0 = 0.0374077$	$z_0 = 0.0506988$	$z_0 = 0.0642717$
$n_v(\Delta) = 3$	$z_0 = 0.00552678$	$z_0 = 0.0117148$	$z_0 = 0.0185671$	$z_0 = 0.0260657$	$z_0 = 0.0341736$

$$\frac{\sum_{j=1/2}^{\infty} j z_0^{2j} g_\Delta(j)}{\sum_{j=1/2}^{\infty} z_0^{2j} g_\Delta(j)} = \frac{J_\Delta}{N_\Delta}. \tag{6.6}$$

There is always a solution on the positive real axis, $z_0 > 0$, which maximizes the integrand on the circle [38]. We denote them by

$$z_0 = e^{-\beta_\Delta/2} \quad \text{and} \quad e^{\mu_\Delta} = \sum_{j=1/2}^{\infty} z_0^{2j} g_\Delta(j). \tag{6.7}$$

The integral can be approximated by

$$\Gamma'_\Delta[J_\Delta] = e^{N_\Delta f(z_0)} \left(\frac{1}{2\pi N_\Delta f''(z_0)}\right)^{\frac{1}{2}} \left[1 + O\left(\frac{1}{N_\Delta}\right)\right],$$
$$N_\Delta f(z_0) \equiv \mu_\Delta N_\Delta + \beta_\Delta J_\Delta, \tag{6.8}$$

where

$$f''(z_0) \simeq \frac{\sum_{j=1/2}^{\infty} 2j(2j-1) z_0^{2j-2} g_\Delta(j)}{\sum_{j=1/2}^{\infty} z_0^{2j} g_\Delta(j)} - 4\frac{{J_\Delta}^2/{N_\Delta}^2 - J_\Delta/N_\Delta}{z_0^2}. \tag{6.9}$$

In all following numerical computations of z_0, we always check that $f''(z_0) \neq 0$. Table I gives examples of solutions z_0 at different $n_v(\Delta)$ and J_Δ/N_Δ.

VII. ENTANGLEMENT RÉNYI ENTROPY

A. Second Rényi entropy

We subdivide the boundary slice Σ into two subregions A and $\bar{A}$ [Fig. (2)]. The subdivision is assumed to be compatible to the complexes $\mathcal{K}$ and $\mathcal{K}_s$, in the sense that the boundary $\mathcal{S}$ between A and $\bar{A}$ are triangulated by triangles $\Delta \in \mathcal{K}_s$, each of which is made by a large number of facets $f \in \mathcal{K}$. Thus the spin-network functions $T_{\vec{j},\vec{\xi}}(\vec{U})$ in the definition of $\psi(\vec{U})$ are defined on graph $\mathcal{G}_0 = \partial\mathcal{K}^*$ which have (many) links intersecting $\mathcal{S}$, while $\mathcal{S}$ does not intersect the spin-network nodes.

We improve the spin-network graph $\mathcal{G}_0$ by including all intersecting points $n_\mathcal{S} = l \cap \mathcal{S}$ between $\mathcal{S}$ and links. $n_\mathcal{S}$ breaks l into 2 links l_1, l_2. The improved graph is denoted by $\mathcal{G}$. By the cylindrical consistency, all $T_{\vec{j},\vec{\xi}}(\vec{U})$ are also spin networks on the improved graph $\mathcal{G}$, since all U_l along links intersecting $\mathcal{S}$ can be decomposed into $U_l = U_{l_1} U_{l_2}$.

The boundary Hilbert space $\mathcal{H}_\Sigma$ is defined as follows: We denote by $L(\mathcal{G})$, $L(\mathcal{G}_A)$, and $L(\mathcal{G}_{\bar{A}})$ the set of links in $\mathcal{G}$, $\mathcal{G}_A = \mathcal{G} \cap A$, and $\mathcal{G}_{\bar{A}} = \mathcal{G} \cap \bar{A}$,

$$\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}, \quad \text{where } \mathcal{H}_\Sigma = L^2(\mathrm{SU}(2))^{\otimes|L(\mathcal{G})|}/\mathrm{gauge}(\mathcal{G}_0),$$
$$\mathcal{H}_A = L^2(\mathrm{SU}(2))^{\otimes|L(\mathcal{G}_A)|}/\mathrm{gauge}(\mathcal{G}_A), \qquad \mathcal{H}_{\bar{A}} = L^2(\mathrm{SU}(2))^{\otimes|L(\mathcal{G}_{\bar{A}})|}/\mathrm{gauge}(\mathcal{G}_{\bar{A}}). \tag{7.1}$$

Here gauge($\mathcal{G}_0$) only includes gauge transformations acting on nodes in $\mathcal{G}_0$ (without bivalent nodes $n_\mathcal{S}$'s). The gauge($\mathcal{G}_A$) and gauge($\mathcal{G}_{\bar{A}}$) only include gauge transformations acting on nodes in the interior of A and $\bar{A}$. $T_{\vec{j},\vec{\xi}}(\vec{U})$ and $\psi(\vec{U})$ are also gauge invariant at all $n_\mathcal{S}$'s and thus belong to a proper Hilbert subspace in $\mathcal{H}_\Sigma$. However, this subspace does not admit a factorization into Hilbert spaces associated with A and $\bar{A}$. Therefore in our discussion of quantum entanglement in $|\psi\rangle$, we view $|\psi\rangle$ as a state in the larger Hilbert space $\mathcal{H}_\Sigma$, although some states in $\mathcal{H}_\Sigma$ are not gauge invariant at bivalent nodes $n_\mathcal{S}$'s.

We define a reduced density matrix ρ_A from $|\psi\rangle \in \mathcal{H}_\Sigma$ by tracing out the d.o.f. in $\mathcal{H}_{\bar{A}}$:

$$\rho_A = \mathrm{tr}_{\bar{A}}(\rho), \qquad \rho = |\psi\rangle\langle\psi|. \tag{7.2}$$

The quantum entanglement in $|\psi\rangle$ can be quantified by the nth Rényi entanglement entropy associated with A:

$$S_n(A) = \frac{1}{1-n} \ln \frac{\mathrm{tr}(\rho_A^n)}{\mathrm{tr}(\rho_A)^n}. \tag{7.3}$$

The von Neumann entanglement entropy is given by $S(A) = \lim_{n\to 1} S_n(A)$.

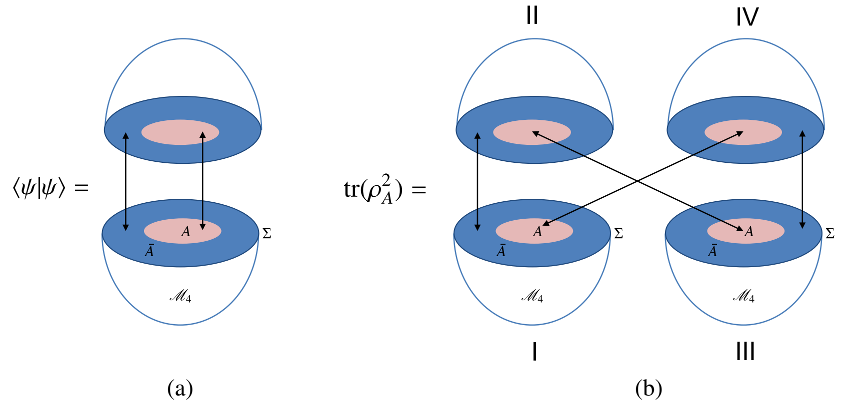

FIG. 3. (a) The inner product $\langle\psi|\psi\rangle$ is taken in both $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ between two copies of $|\psi\rangle$. (b) In $\mathrm{tr}(\rho_A^2)$, the inner products in $\mathcal{H}_{\bar{A}}$ are taken between copies I and II and between III and IV of $|\psi\rangle$, while the inner products in $\mathcal{H}_A$ are taken between copies I and IV and between II and III. If the inner products are understood as gluing manifolds and their path integrals, the manifold for $\mathrm{tr}(\rho_A^2)$ has a branch cut whose branch points make the boundary $\mathcal{S}$ between A and $\bar{A}$.

The $\mathrm{tr}(\rho_A)=\langle\psi|\psi\rangle$ has been computed above. The following task is to compute $\mathrm{tr}(\rho_A^n)$. Let us first focus on the second Rényi entropy at $n=2$. The computation is illustrated graphically in Fig. 3. The $\mathrm{tr}(\rho_A^2)$ is made by inner products among four copies of ψ. The inner products in $\mathcal{H}_{\bar{A}}$ take place between copies I and II and between III and IV, while the inner products in $\mathcal{H}_A$ take place between copies I and IV and between II and III. The inner products of $\mathrm{tr}(\rho_A^2)$ are computed in the same way as the above derivation for $\langle\psi|\psi\rangle$:

$$\begin{aligned}\mathrm{tr}(\rho_A^2) = &\sum_{\{j_f^{(I)}\},\{j_f^{(II)}\},\{j_f^{(III)}\},\{j_f^{(IV)}\}}{}^{\prime} \prod_f \prod_{a=I}^{IV} A_\Delta(j_f^{(a)}) \int_{\mathfrak{N}_{g,\xi}^{\times 4}} \left[\prod_{a=I}^{IV} \mathrm{d}g_{v\alpha}^{(a)\pm}\mathrm{d}\xi_{\alpha f}^{(a)}\right] e^{S^{(I)}+\overline{S^{(II)}}+S^{(III)}+\overline{S^{(IV)}}} \\ &\times \prod_{\alpha,\Delta} e^{-N[V_{\alpha,\Delta}(\xi_{\alpha f}^{(I)})+V_{\alpha,\Delta}(\xi_{\alpha f}^{(II)})+V_{\alpha,\Delta}(\xi_{\alpha f}^{(III)})+V_{\alpha,\Delta}(\xi_{\alpha f}^{(IV)})]} \prod_{f\subset\mathcal{S}} \frac{1}{(2j_f+1)^3} \prod_{f\subset\Sigma\setminus\mathcal{S}} \frac{1}{2j_f+1} \\ &\times \prod_{\alpha\subset A} \langle\{j_f^{(IV)}\},\{\xi_{\alpha f}^{(IV)}\}\Big\|\{j_f^{(I)}\},\{\xi_{\alpha f}^{(I)}\}\rangle\langle\{j_f^{(II)}\},\{\xi_{\alpha f}^{(II)}\}\Big\|\{j_f^{(III)}\},\{\xi_{\alpha f}^{(III)}\}\rangle \\ &\times \prod_{\beta\subset\bar{A}} \langle\{j_f^{(III)}\},\{\xi_{\beta f}^{(III)}\}\Big\|\{j_f^{(IV)}\},\{\xi_{\beta f}^{(IV)}\}\rangle\langle\{j_f^{(I)}\},\{\xi_{\beta f}^{(I)}\}\Big\|\{j_f^{(II)}\},\{\xi_{\beta f}^{(II)}\}\rangle, \end{aligned} \tag{7.4}$$

where $j_f^{(a)}$, $g_{v\alpha}^{(a)\pm}$, and $\xi_{\alpha f}^{(a)}$ are variables in the ath copy of ψ ($a=I,\ldots,IV$), and $S^{(a)}$ depends on the variables labeled by a. We apply the convention in the above formula that $\langle\{j\},\{\xi\}\|\{j'\},\{\xi'\}\rangle=\delta^{jj'}\langle\{j\},\{\xi\}\|\{j\},\{\xi'\}\rangle$. A factor $1/(2j_f+1)^3$ appearing for each $f\subset\mathcal{S}$ comes from the following inner products at f:

$$\begin{aligned}&\int \mathrm{d}U_A\mathrm{d}U_{\bar{A}}\mathrm{d}U_A'\mathrm{d}U_{\bar{A}}' \sum_{k^{(I)},k^{(II)},k^{(III)},k^{(IV)}} R^{j_f^{(I)}}_{m^{(I)}k^{(I)}}(U_A)R^{j_f^{(I)}}_{k^{(I)}n^{(I)}}(U_{\bar{A}})\overline{R^{j_f^{(II)}}_{m^{(II)}k^{(II)}}(U_A')}\,\overline{R^{j_f^{(II)}}_{k^{(II)}n^{(II)}}(U_{\bar{A}})} \\ &\quad\times R^{j_f^{(III)}}_{m^{(III)}k^{(III)}}(U_A')R^{j_f^{(III)}}_{k^{(III)}n^{(III)}}(U_{\bar{A}}')\overline{R^{j_f^{(IV)}}_{m^{(IV)}k^{(IV)}}(U_A)}\,\overline{R^{j_f^{(IV)}}_{k^{(IV)}n^{(IV)}}(U_{\bar{A}}')} \\ &=\left(\frac{1}{2j_f+1}\right)^4 \delta^{j_f^{(I)}j_f^{(II)}}\delta^{j_f^{(II)}j_f^{(III)}}\delta^{j_f^{(III)}j_f^{(IV)}}\delta^{j_f^{(IV)}j_f^{(I)}} \sum_{k^{(I)},k^{(II)},k^{(III)},k^{(IV)}} \delta_{k^{(I)}k^{(II)}}\delta_{k^{(II)}k^{(III)}}\delta_{k^{(III)}k^{(IV)}}\delta_{k^{(IV)}k^{(I)}} \\ &\quad\times \delta_{n^{(I)}n^{(II)}}\delta_{m^{(II)}m^{(III)}}\delta_{n^{(III)}n^{(IV)}}\delta_{m^{(I)}m^{(IV)}} \\ &=\left(\frac{1}{2j_f+1}\right)^3 \delta^{j_f^{(I)}j_f^{(II)}}\delta^{j_f^{(II)}j_f^{(III)}}\delta^{j_f^{(III)}j_f^{(IV)}}\delta^{j_f^{(IV)}j_f^{(I)}}\delta_{n^{(I)}n^{(II)}}\delta_{m^{(II)}m^{(III)}}\delta_{n^{(III)}n^{(IV)}}\delta_{m^{(I)}m^{(IV)}}, \end{aligned} \tag{7.5}$$

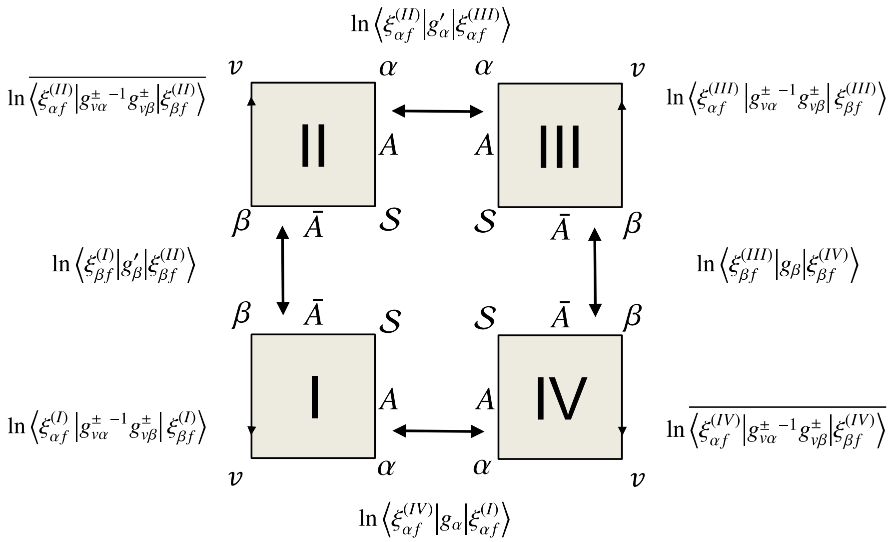

FIG. 4. The situation that $f \subset \mathcal{S}$ is contained in a single B_4 in $\mathcal{K}$, the figure draws four copies of faces in $\mathcal{K}^*$ dual to a $f \subset \mathcal{S}$ from four copies of ψ in computing the second Rényi entropy. $U_A, U_{\bar{A}}, U'_A, U_{\bar{A}}{}'$ in Eq. (7.5) are holonomies along links labeled by $A, \bar{A}$. Integrating these holonomies glues four copies of dual faces.

where $U_A U_{\bar{A}}$ is the holonomy along the link intersecting $\mathcal{S}$ and dual to f in Σ (see Fig. 4). The above inner products identify four spins of f from four different copies of ψ: $j_f^{(I)} = j_f^{(II)} = j_f^{(III)} = j_f^{(IV)} = j_f$. The total action in Eq. (7.4) is given by

$$\begin{aligned} S_{\mathrm{tot}}^{(2)} &= S^{(I)} + \overline{S^{(II)}} + S^{(III)} + \overline{S^{(IV)}} - N\sum_{\alpha,\Delta}[V_{\alpha,\Delta}(\xi_{\alpha f}^{(I)}) + V_{\alpha,\Delta}(\xi_{\alpha f}^{(II)}) + V_{\alpha,\Delta}(\xi_{\alpha f}^{(III)}) + V_{\alpha,\Delta}(\xi_{\alpha f}^{(IV)})] \\ &\quad + \sum_{\alpha\subset A}\sum_{f\subset\alpha} 2j_f^{(IV)} \ln\langle \xi_{\alpha f}^{(IV)}|g_\alpha|\xi_{\alpha f}^{(I)}\rangle + \sum_{\alpha\subset A}\sum_{f\subset\alpha} 2j_f^{(II)} \ln\langle \xi_{\alpha f}^{(II)}|g'_\alpha|\xi_{\alpha f}^{(III)}\rangle \\ &\quad + \sum_{\beta\subset\bar{A}}\sum_{f\subset\alpha} 2j_f^{(III)} \ln\langle \xi_{\beta f}^{(III)}|g_\beta|\xi_{\beta f}^{(IV)}\rangle + +\sum_{\beta\subset\bar{A}}\sum_{f\subset\alpha} 2j_f^{(I)} \ln\langle \xi_{\beta f}^{(I)}|g'_\beta|\xi_{\beta f}^{(II)}\rangle. \end{aligned} \tag{7.6}$$

The situation at $f \subset \mathcal{S}$ is illustrated in Fig. 4. The large N again imposes the parallel restriction to $\xi_{\alpha f}$ and reduces $S_{\mathrm{tot}}^{(2)}$ to

$$\begin{aligned} S'^{(2)}_{\mathrm{tot}} &= S_0^{(I)} + \overline{S_0^{(II)}} + S_0^{(III)} + \overline{S_0^{(IV)}} \\ &\quad + \sum_{\alpha\subset A}\sum_{\Delta\subset\alpha} 2J_\Delta^{(IV)} \ln\langle \xi_{\alpha\Delta}^{(IV)}|g_\alpha|\xi_{\alpha\Delta}^{(I)}\rangle + \sum_{\alpha\subset A}\sum_{\Delta\subset\alpha} 2J_\Delta^{(II)} \ln\langle \xi_{\alpha\Delta}^{(II)}|g'_\alpha|\xi_{\alpha\Delta}^{(III)}\rangle \\ &\quad + \sum_{\beta\subset\bar{A}}\sum_{\Delta\subset\alpha} 2J_\Delta^{(III)} \ln\langle \xi_{\beta\Delta}^{(III)}|g_\beta|\xi_{\beta\Delta}^{(IV)}\rangle + +\sum_{\beta\subset\bar{A}}\sum_{\Delta\subset\alpha} 2J_\Delta^{(I)} \ln\langle \xi_{\beta\Delta}^{(I)}|g'_\beta|\xi_{\beta\Delta}^{(II)}\rangle. \end{aligned} \tag{7.7}$$

A large-J_Δ stationary phase analysis similar to $\langle\psi|\psi\rangle$ shows that the integration domain of Eq. (7.4) again only contains a single critical point, which is four copies of $(g_{v\alpha}^{\pm}, \xi_{\alpha\Delta})_c[J_\Delta]$ with their boundary data identified according to Fig. 3. $S_{\mathrm{tot}}^{(2)}$ vanishes at the critical point.

The asymptotic behavior of the integral depends on j_f only through their sum J_Δ, so similar to the computation of $\langle\psi|\psi\rangle$,

$$\begin{aligned} \mathrm{tr}(\rho_A^2) &\simeq \left(\frac{2\pi}{N}\right)^{48N_v + 4\sum_{\Delta\in\mathcal{K}} N_\Delta} \frac{1}{\det(H_V|_c)^2} \frac{1}{\sqrt{\det(-H'^{(2)}_{\mathrm{tot}}|_c)}} \\ &\quad \times \prod_{\Delta\in i(\mathcal{K}_s)} \Gamma_\Delta[J_\Delta]^4 \prod_{\Delta\subset i(A)} \Gamma'_\Delta[J_\Delta]^2 \prod_{\Delta\subset i(\bar{A})} \Gamma'_\Delta[J_\Delta]^2 \prod_{\Delta\subset\mathcal{S}} \Gamma_\Delta^{(2)}[J_\Delta]\left[1 + O\left(\frac{1}{N}\right)\right], \end{aligned} \tag{7.8}$$

where $H'^{(2)}_{\mathrm{tot}}|_c$ is the Hessian matrix of $S'^{(2)}_{\mathrm{tot}}$ evaluated at the critical point and is assumed to be nondegenerate. $\Delta \subset \mathcal{S}$ are special because they are shared by all four copies of ψ in $\mathrm{tr}(\rho_A^2)$. $\Gamma_\Delta^{(2)}$ for $\Delta \subset \mathcal{S}$ is given by

$$\Gamma_\Delta^{(2)}[J_\Delta] = \sum_{\{j_{f\in\Delta}\}}{}' \prod_{f\in\Delta} g_\Delta^{(2)}(j_f), \qquad g_\Delta^{(2)}(j) = \frac{A_\Delta(j)^4}{(2j+1)^3} = (2j+1)^{4n_v(\Delta)+5}. \tag{7.9}$$

Similar to Γ'_Δ, $\Gamma_\Delta^{(2)}$ can also be viewed as an analog of microstate counting, where $g_\Delta^{(2)}(j)$ corresponds to the degeneracy of microstates at the level j. The label (2) indicates that it is for computing the second Rényi entropy,

$$\Gamma_\Delta^{(2)}[J_\Delta] \simeq e^{N_\Delta f^{(2)}(z_0^{(2)})}\left(\frac{1}{2\pi N_\Delta f^{(2)\prime\prime}(z_0^{(2)})}\right)^{\frac{1}{2}}\left[1+O\left(\frac{1}{N_\Delta}\right)\right], \qquad N_\Delta f^{(2)}(z_0^{(2)}) \equiv \mu_\Delta^{(2)} N_\Delta + \beta_\Delta^{(2)} J_\Delta, \tag{7.10}$$

where $f^{(2)}(z)$ and $z_0^{(2)}$ are given by

$$N_\Delta f^{(2)}(z) \simeq N_\Delta \ln\left[\sum_{j=1/2}^{\infty} z^{2j} g_\Delta^{(2)}(j)\right] - 2J_\Delta \ln(z), \frac{\sum_{j=1/2}^{\infty} j[z_0^{(2)}]^{2j} g_\Delta^{(2)}(j)}{\sum_{j=1/2}^{\infty}[z_0^{(2)}]^{2j} g_\Delta^{(2)}(j)} = \frac{J_\Delta}{N_\Delta}. \tag{7.11}$$

The second equation in Eq. (7.11) comes from the variation principle of $f^{(2)}(z)$. We denote

$$z_0^{(2)} = e^{-\beta_\Delta^{(2)}/2}, \qquad e^{\mu_\Delta^{(2)}} = \sum_{j=1/2}^{\infty}[z_0^{(2)}]^{2j} g_\Delta^{(2)}(j). \tag{7.12}$$

Table II gives examples of solutions z_0 at different $n_v(\Delta)$ and J_Δ/N_Δ.

Combining Eq. (7.8) with Eq. (7.10) for $\mathrm{tr}(\rho_A^2)$ and Eq. (5.16) for $\langle\psi|\psi\rangle = \mathrm{tr}(\rho_A)$ gives the following second Rényi entropy:

$$\begin{aligned} S_2(A) &= -\ln\frac{\mathrm{tr}(\rho_A^2)}{\mathrm{tr}(\rho_A)^2} = -\ln\frac{\prod_{\Delta\subset\mathcal{S}}\Gamma_\Delta^{(2)}}{\prod_{\Delta\subset\mathcal{S}}\Gamma_\Delta'^2}\frac{\det(-H'_{\mathrm{tot}}|_c)}{\sqrt{\det(-H'^{(2)}_{\mathrm{tot}}|_c)}}\left[1+O\left(\frac{1}{N}\right)\right] \\ &\simeq \sum_{\Delta\subset\mathcal{S}} N_\Delta[2f(z_0) - f^{(2)}(z_0^{(2)})] = \sum_{\Delta\subset\mathcal{S}}[(2\beta_\Delta - \beta_\Delta^{(2)})J_\Delta + (2\mu_\Delta - \mu_\Delta^{(2)})N_\Delta], \end{aligned} \tag{7.13}$$

where $\ln\frac{\det(-H'_{\mathrm{tot}}|_c)}{\sqrt{\det(-H'^{(2)}_{\mathrm{tot}}|_c)}}$ is subleading and negligible as $J_\Delta \sim N_\Delta \gg 1$.

The z_0, $z_0^{(2)}$ or $\beta_\Delta, \mu_\Delta, \beta_\Delta^{(2)}, \mu_\Delta^{(2)}$ clearly depend on J_Δ, N_Δ. If we fix J_Δ and let N_Δ vary,

$$\begin{aligned} &\frac{\partial[N_\Delta f(z_0)]}{\partial N_\Delta} = \mu_\Delta + N_\Delta\left(\frac{J_\Delta}{N_\Delta} + \frac{\partial\mu_\Delta}{\partial\beta_\Delta}\right)\frac{\partial\beta_\Delta}{\partial N_\Delta} = \mu_\Delta, \qquad \frac{\partial\mu_\Delta}{\partial\beta_\Delta} = \frac{\sum_j e^{-\beta_\Delta j}(-j) g_\Delta(j)}{\sum_j e^{-\beta_\Delta j} g_\Delta(j)} = -\frac{J_\Delta}{N_\Delta}, \\ &\frac{\partial[N_\Delta f^{(2)}(z_0)]}{\partial N_\Delta} = \mu_\Delta^{(2)} + N_\Delta\left(\frac{J_\Delta}{N_\Delta} + \frac{\partial\mu_\Delta^{(2)}}{\partial\beta_\Delta^{(2)}}\right)\frac{\partial\beta_\Delta^{(2)}}{\partial N_\Delta} = \mu_\Delta^{(2)}, \qquad \frac{\partial\mu_\Delta^{(2)}}{\partial\beta_\Delta^{(2)}} = \frac{\sum_j e^{-\beta_\Delta^{(2)} j}(-j) g_\Delta^{(2)}(j)}{\sum_j e^{-\beta_\Delta^{(2)} j} g_\Delta^{(2)}(j)} = -\frac{J_\Delta}{N_\Delta}. \end{aligned} \tag{7.14}$$

Therefore,

TABLE II. Solutions z_0 maximizing $f^{(2)}(z_0)$ at different $n_v(\Delta)$ and J_Δ/N_Δ [$f^{(2)\prime\prime}(z_0)$ are all nonzero].

	$J_\Delta/N_\Delta = 0.6$	$J_\Delta/N_\Delta = 0.7$	$J_\Delta/N_\Delta = 0.8$	$J_\Delta/N_\Delta = 0.9$
$n_v(\Delta) = 1$	$z_0 = 0.00552678$	$z_0 = 0.0117148$	$z_0 = 0.0185671$	$z_0 = 0.0260657$
$n_v(\Delta) = 2$	$z_0 = 0.0011542$	$z_0 = 0.00260368$	$z_0 = 0.00441412$	$z_0 = 0.00664713$
$n_v(\Delta) = 3$	$z_0 = 0.000236694$	$z_0 = 0.000560573$	$z_0 = 0.00100989$	$z_0 = 0.00163299$

$$\frac{\partial[2N_\Delta f(z_0) - N_\Delta f^{(2)}(z_0^{(2)})]}{\partial N_\Delta} = 2\mu_\Delta - \mu_\Delta^{(2)}. \tag{7.15}$$

$S_2(A)$ is extremized at the value of the ratio J_Δ/N_Δ which gives $2\mu_\Delta = \mu_\Delta^{(2)}$ at every Δ. The extremal value of $S_2(A)$ gives

$$S_2(A) \simeq \sum_{\Delta\subset\mathcal{S}}(2\beta_\Delta - \beta_\Delta^{(2)})J_\Delta. \tag{7.16}$$

If the complex $\mathcal{K}$ and the entangling surface $\mathcal{S}$ are chosen such that $n_v(\Delta)$ is a constant for all $\Delta\subset\mathcal{S}$ (every $\Delta\subset\mathcal{S}$ is shared by the same number of B_4's), β_Δ and $\beta_\Delta^{(2)}$ are constants independent of Δ; in this case, $S_2(A)$ satisfies the area law

$$S_2(A) \simeq c\sum_{\Delta\subset\mathcal{S}} J_\Delta = \frac{c}{8\pi\gamma\ell_P^2}\mathbf{a}_\mathcal{S}, \quad c = 2\beta_\Delta - \beta_\Delta^{(2)}, \tag{7.17}$$

where $\mathbf{a}_\mathcal{S} = 8\pi\gamma\ell_P^2\sum_{\Delta\subset\mathcal{S}}J_\Delta$ is the total area of $\mathcal{S}$. The relation between $\mathbf{a}_\mathcal{S}$ and J_Δ is given by the geometrical interpretation of the critical point $(g_{v\alpha}^\pm, \xi_{\alpha\Delta})_c[J_\Delta]\in\mathcal{G}$. But in general the extremal $S_2(A)$ may satisfy a weighted area law Eq. (7.16) with different weights $2\beta_\Delta - \beta_\Delta^{(2)}$ at different Δ.

To see if $2\mu_\Delta = \mu_\Delta^{(2)}$ maximizes $S_2(A)$, we compute the second derivative:

$$\frac{\partial^2[N_\Delta f(z_0)]}{\partial N_\Delta^2} = \frac{\partial\mu_\Delta}{\partial\beta_\Delta}\frac{\partial\beta_\Delta}{\partial N_\Delta} = -\frac{1}{N_\Delta}\frac{1}{\frac{N_\Delta^2}{J_\Delta^2}\langle j^2\rangle - 1}, \qquad \langle j^2\rangle \equiv \frac{\sum_{j=1/2}^\infty j^2 e^{-\beta j}g_\Delta(j)}{\sum_{j=1/2}^\infty e^{-\beta j}g_\Delta(j)},$$

$$\frac{\partial^2[N_\Delta f^{(2)}(z_0^{(2)})]}{\partial N_\Delta^2} = \frac{\partial\mu_\Delta^{(2)}}{\partial\beta_\Delta^{(2)}}\frac{\partial\beta_\Delta^{(2)}}{\partial N_\Delta} = -\frac{1}{N_\Delta}\frac{1}{\frac{N_\Delta^2}{J_\Delta^2}\langle j^2\rangle^{(2)} - 1}, \qquad \langle j^2\rangle^{(2)} \equiv \frac{\sum_{j=1/2}^\infty j^2 e^{-\beta_\Delta^{(2)} j}g_\Delta^{(2)}(j)}{\sum_{j=1/2}^\infty e^{-\beta_\Delta^{(2)} j}g_\Delta^{(2)}(j)},$$

$$\frac{\partial^2[2N_\Delta f(z_0) - N_\Delta f^{(2)}(z_0^{(2)})]}{\partial N_\Delta^2} = \frac{1}{N_\Delta}\left(\frac{1}{\frac{N_\Delta^2}{J_\Delta^2}\langle j^2\rangle^{(2)} - 1} - \frac{2}{\frac{N_\Delta^2}{J_\Delta^2}\langle j^2\rangle - 1}\right). \tag{7.18}$$

The following list provides some values of J_Δ/N_Δ which give $2\mu_\Delta = \mu_\Delta^{(2)}$ at different $n_v(\Delta)$:

$$n_v(\Delta) = 1:\ J_\Delta/N_\Delta = 0.802182, \qquad 2\beta_\Delta - \beta_\Delta^{(2)} = 2.41769, \qquad N_\Delta\frac{\partial^2[2N_\Delta f(z_0) - N_\Delta f^{(2)}(z_0^{(2)})]}{\partial N_\Delta^2} = -10.3142,$$

$$n_v(\Delta) = 2:\ J_\Delta/N_\Delta = 0.782484, \qquad 2\beta_\Delta - \beta_\Delta^{(2)} = 2.38741, \qquad N_\Delta\frac{\partial^2[2N_\Delta f(z_0) - N_\Delta f^{(2)}(z_0^{(2)})]}{\partial N_\Delta^2} = -11.0869,$$

$$n_v(\Delta) = 3:\ J_\Delta/N_\Delta = 0.762613, \qquad 2\beta_\Delta - \beta_\Delta^{(2)} = 2.35677, \qquad N_\Delta\frac{\partial^2[2N_\Delta f(z_0) - N_\Delta f^{(2)}(z_0^{(2)})]}{\partial N_\Delta^2} = -12.0193.$$

The negative second derivative implies that $2\mu_\Delta = \mu_\Delta^{(2)}$ gives the maximum of $S_2(A)$. Figure 5 plots

$$\mathcal{F}_2\left[n_v(\Delta), \frac{J_\Delta}{N_\Delta}\right] := \frac{N_\Delta}{J_\Delta}[2f(z_0) - f^{(2)}(z_0^{(2)})], \qquad S_2(A) = \sum_{\Delta\subset\mathcal{S}}J_\Delta\mathcal{F}_2\left[n_v(\Delta), \frac{J_\Delta}{N_\Delta}\right] \tag{7.19}$$

at different $n_v(\Delta)$, and suggests that when J_Δ is fixed, $2\mu_\Delta = \mu_\Delta^{(2)}$ indeed gives the global maximum of $S_2(A)$.

The above result shows that fixing J_Δ, the second Rényi entropy $S_2(A)$, as a function of N_Δ is in general bounded by an (weighted) area law,

$$S_2(A) \leq \sum_{\Delta\subset\mathcal{S}}(2\beta_\Delta - \beta_\Delta^{(2)})J_\Delta, \tag{7.20}$$

where the bound is saturated at J_Δ/N_Δ which gives $2\mu_\Delta = \mu_\Delta^{(2)}$. The bound becomes an area law if $n_v(\Delta)$ is a constant for all $\Delta\subset\mathcal{S}$.

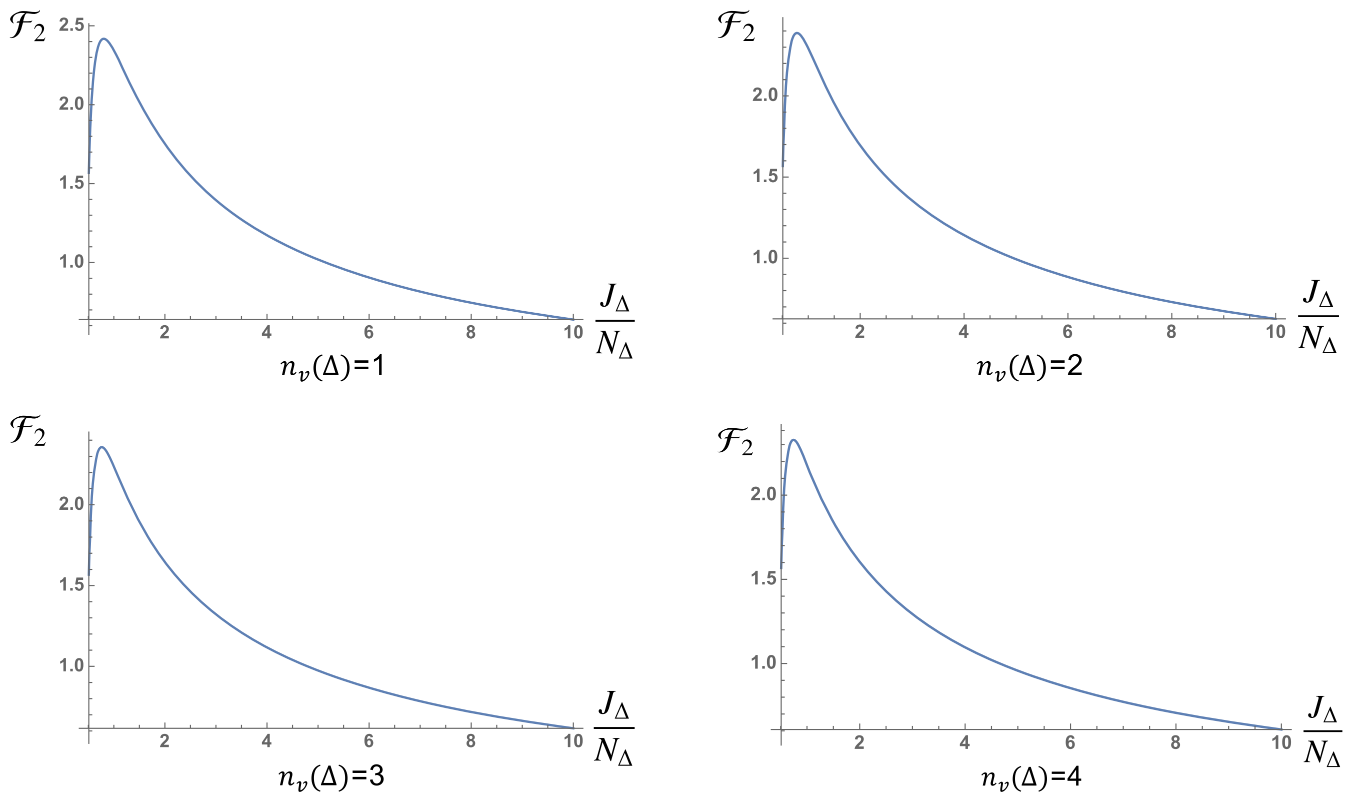

FIG. 5. Plots of $\mathcal{F}_2[n_v(\Delta), J_\Delta/N_\Delta]$ in Eq. (7.27) at $n_v(\Delta) = 1, \ldots, 4$ and $J_\Delta/N_\Delta \in [0.51, 10]$.

B. Higher Rényi entropy

The computation of higher Rényi entropy $S_n(A)$ with $n > 2$ is a simple generalization of the second Rényi entropy computation. The $\mathrm{tr}(\rho_A^n)$ includes $2n$ copies of $|\psi\rangle$ or $\langle\psi|$ in the computation illustrated by Figs. 3 and 4. Equation (7.8) is modified to

$$\mathrm{tr}(\rho_A^n) \simeq \left(\frac{2\pi}{N}\right)^{24nN_v + 2n\sum_{\Delta\in\mathcal{K}} N_\Delta} \frac{1}{\det\left(H_V|_c\right)^n} \frac{1}{\sqrt{\det\left(-H'^{(n)}_{\mathrm{tot}}|_c\right)}} \times \prod_{\Delta\in i(\mathcal{K}_s)} \Gamma_\Delta[J_\Delta]^{2n} \prod_{\Delta\subset i(A)} \Gamma'_\Delta[J_\Delta]^n \prod_{\Delta\subset i(\bar{A})} \Gamma'_\Delta[J_\Delta]^n \prod_{\Delta\subset\mathcal{S}} \Gamma^{(n)}_\Delta[J_\Delta]\left[1 + O\left(\frac{1}{N}\right)\right]. \tag{7.21}$$

Here $\Gamma^{(n)}_\Delta$ for $\Delta\subset\mathcal{S}$ is computed similar to $\Gamma^{(2)}_\Delta$,

$$\Gamma^{(n)}_\Delta[J_\Delta] = \sum_{\{j_{f\in\Delta}\}}{}' \prod_{f\in\Delta} g^{(n)}_\Delta(j_f), \qquad g^{(n)}_\Delta(j) = \frac{A_\Delta(j)^{2n}}{(2j+1)^{2n-1}} = (2j+1)^{2n(n_v(\Delta)+1)+1}. \tag{7.22}$$

As a result,

$$S_n(A) = \frac{1}{1-n}\ln\frac{\mathrm{tr}(\rho_A^n)}{\mathrm{tr}(\rho_A)^n} = \frac{1}{1-n}\ln\frac{\prod_{\Delta\subset\mathcal{S}}\Gamma^{(n)}_\Delta}{\prod_{\Delta\subset\mathcal{S}}\Gamma'^{\,n}_\Delta}\frac{\det(-H'_{\mathrm{tot}}|_c)^n}{\sqrt{\det(-H'^{(n)}_{\mathrm{tot}}|_c)}}\left[1 + O\left(\frac{1}{N}\right)\right]$$
$$\simeq \sum_{\Delta\subset\mathcal{S}}\left[\frac{\beta^{(n)}_\Delta - \beta_\Delta n}{1-n}J_\Delta + \frac{\mu^{(n)}_\Delta - \mu_\Delta n}{1-n}N_\Delta\right], \tag{7.23}$$

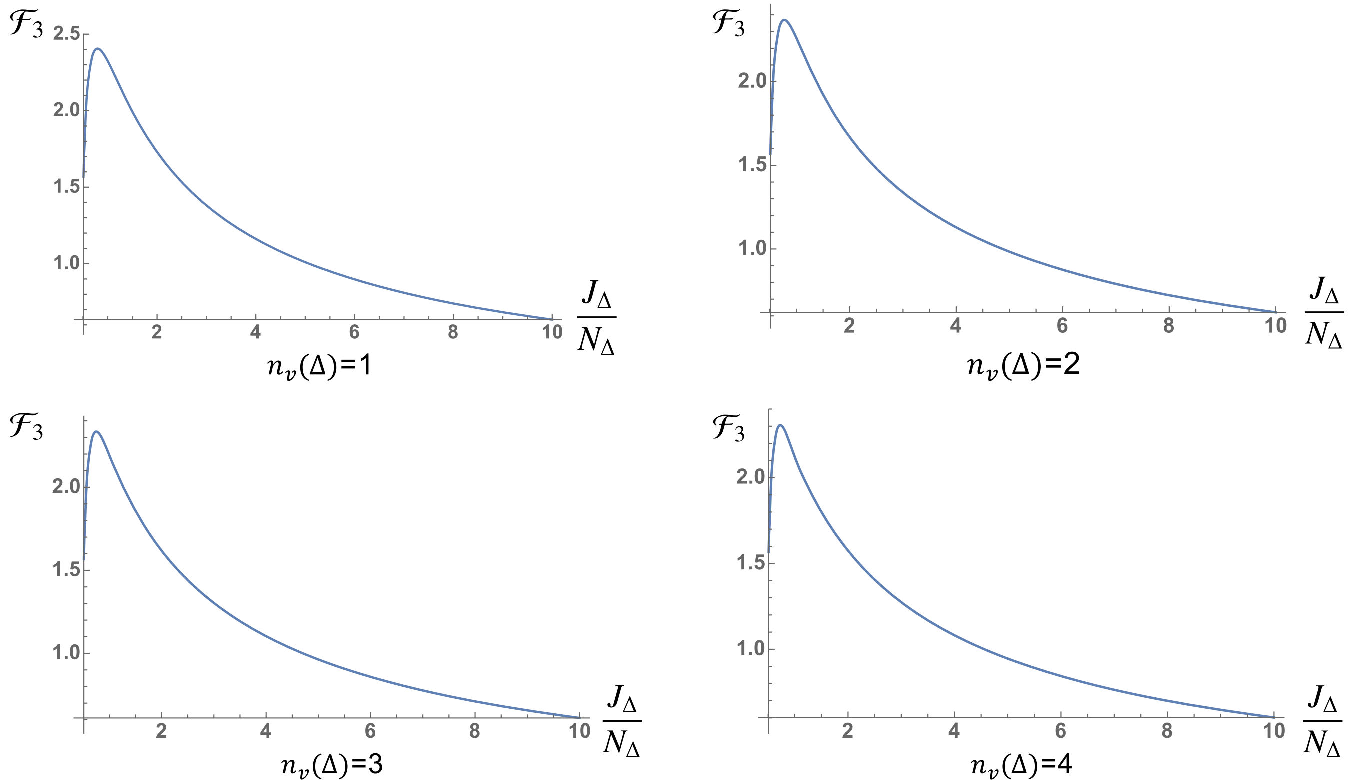

FIG. 6. Plots of $\mathcal{F}_3[n_v(\Delta), J_\Delta/N_\Delta]$ at $n_v(\Delta) = 1, \ldots, 4$ and $J_\Delta/N_\Delta \in [0.51, 10]$.

where $\beta_\Delta^{(n)}, \mu_\Delta^{(n)}$ satisfies

$$e^{-\beta_\Delta^{(n)}/2} = z_0^{(n)}, \qquad e^{\mu_\Delta^{(n)}} = \sum_{j=1/2}^{\infty} [z_0^{(n)}]^{2j} g_\Delta^{(n)}(j), \tag{7.24}$$

and $z_0^{(n)} \in (0,1)$ solves

$$\frac{\sum_{j=1/2}^{\infty} j [z_0^{(n)}]^{2j} g_\Delta^{(n)}(j)}{\sum_{j=1/2}^{\infty} [z_0^{(n)}]^{2j} g_\Delta^{(n)}(j)} = \frac{J_\Delta}{N_\Delta}. \tag{7.25}$$

Similar to $S_2(A)$, if we fix J_Δ and let N_Δ vary, $S_n(A)$ maximizes at $\mu_\Delta^{(n)} = \mu_\Delta n$ and thus is bounded by a weighted area law,

$$S_n(A) \leq \sum_{\Delta \subset \mathcal{S}} \frac{\beta_\Delta^{(n)} - \beta_\Delta n}{1-n} J_\Delta, \tag{7.26}$$

where J_Δ relates to the area of Δ by the geometrical interpretation of the critical point $(g_{v\alpha}^{\pm}, \xi_{\alpha\Delta})_c[J_\Delta]$ in defining $|\psi\rangle$. Figure 7 plots

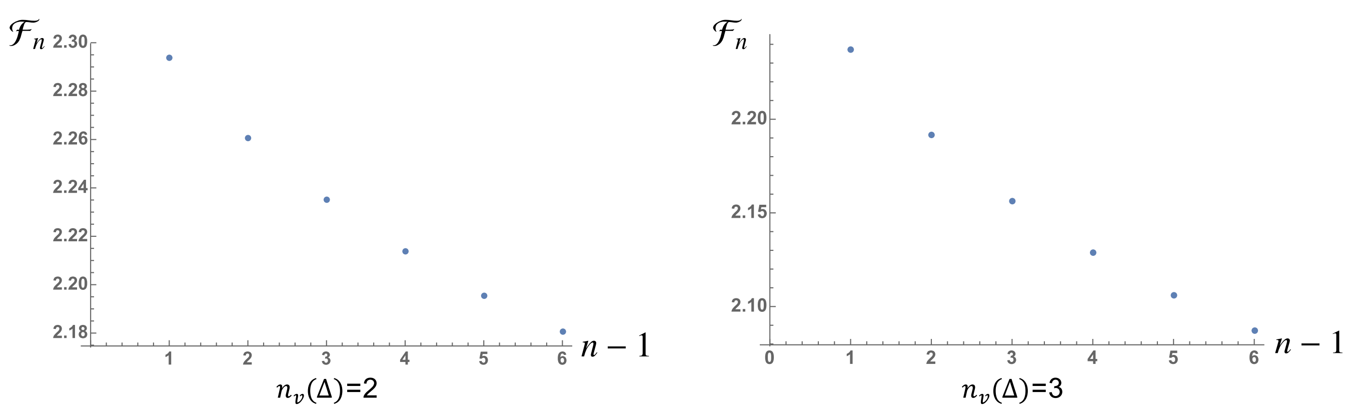

FIG. 7. Plots of $\mathcal{F}_n[n_v(\Delta), J_\Delta/N_\Delta = 1]$ at $n_v(\Delta) = 1$, 2 and $n = 2, \ldots, 7$.

$$\mathcal{F}_n\left[n_v(\Delta),\frac{J_\Delta}{N_\Delta}\right] \coloneqq \frac{N_\Delta}{J_\Delta}\left[\frac{\beta_\Delta^{(n)}-\beta_\Delta n}{1-n}\frac{J_\Delta}{N_\Delta}+\frac{\mu_\Delta^{(n)}-\mu_\Delta n}{1-n}\right],$$
$$S_n(A)=\sum_{\Delta\subset\mathcal{S}}J_\Delta\mathcal{F}_n\left[n_v(\Delta),\frac{J_\Delta}{N_\Delta}\right] \tag{7.27}$$

at $n=3$ and $n_v(\Delta)=1,\ldots,4$. Figures 6 and 7 plot $\mathcal{F}_n$ at $J_\Delta/N_\Delta=1$, $n_v(\Delta)=1$, 2, and $n=2,\ldots,7$.

VIII. ANALOGOUS THERMODYNAMICAL FIRST LAW

The Rényi entanglement entropy $S_n(A)$ derived in the last section is a function of the "macrostate" J_Δ, N_Δ and has an interesting analog with entropy in thermodynamics. In Sec. VI, we give an analog between J_Δ, N_Δ and the total energy and total number of identical systems of a statistical ensemble.

Theorem 8.1: The differential of $S_n(A)$ with respect to J_Δ, N_Δ gives the following analog of the thermodynamical first law:

$$\delta S_n(A)=\sum_{\Delta\subset\mathcal{S}}[\lambda_\Delta(n)\delta J_\Delta+\sigma_\Delta(n)\delta N_\Delta], \tag{8.1}$$

where $\lambda_\Delta(n)=\frac{\beta_\Delta^{(n)}-\beta_\Delta n}{1-n}$ and $\sigma_\Delta(n)=\frac{\mu_\Delta^{(n)}-\mu_\Delta n}{1-n}$. When all $\Delta\in\mathcal{S}$ have the same $n_v(\Delta)$, $\beta_\Delta^{(n)},\beta_\Delta,\mu_\Delta^{(n)},\mu_\Delta$ become independent of Δ. In this case $\lambda_\Delta(n)\equiv\lambda(n)$ and $\sigma_\Delta(n)\equiv\sigma(n)$ becomes independent of Δ, $\delta S_n(A)$ reduces to

$$\delta S_n(A)=\lambda(n)\delta J_{\mathcal{S}}+\sigma(n)\delta N_{\mathcal{S}}, \tag{8.2}$$

where $J_{\mathcal{S}}=\sum_{\Delta\subset\mathcal{S}}J_\Delta$ and $N_{\mathcal{S}}=\sum_{\Delta\subset\mathcal{S}}N_\Delta$ are total area and total number of facets in $\mathcal{S}$.

Proof: Equation (8.1) can be checked by computing $\partial S_n(A)/\partial J_\Delta$ and $\partial S_n(A)/\partial N_\Delta$:

$$\frac{\partial S_n(A)}{\partial J_\Delta}=\frac{1}{1-n}\left(\frac{\partial\beta_\Delta^{(n)}}{\partial J_\Delta}J_\Delta+\frac{\partial\mu_\Delta^{(n)}}{\partial\beta_\Delta^{(n)}}\frac{\partial\beta_\Delta^{(n)}}{\partial J_\Delta}N_\Delta+\beta_\Delta^{(n)}\right)-\frac{n}{1-n}\left(\frac{\partial\beta_\Delta}{\partial J_\Delta}J_\Delta+\frac{\partial\mu_\Delta}{\partial\beta_\Delta}\frac{\partial\beta_\Delta}{\partial J_\Delta}N_\Delta+\beta_\Delta\right),$$
$$\frac{\partial S_n(A)}{\partial N_\Delta}=\frac{1}{1-n}\left(\frac{\partial\beta_\Delta^{(n)}}{\partial N_\Delta}J_\Delta+\frac{\partial\mu_\Delta^{(n)}}{\partial\beta_\Delta^{(n)}}\frac{\partial\beta_\Delta^{(n)}}{\partial N_\Delta}N_\Delta+\mu_\Delta^{(n)}\right)-\frac{n}{1-n}\left(\frac{\partial\beta_\Delta}{\partial N_\Delta}J_\Delta+\frac{\partial\mu_\Delta}{\partial\beta_\Delta}\frac{\partial\beta_\Delta}{\partial N_\Delta}N_\Delta+\mu_\Delta\right). \tag{8.3}$$

The definitions $\mu_\Delta^{(n)}=\ln[\sum_{j=1/2}^\infty e^{-\beta_\Delta^{(n)}j}g_\Delta^{(n)}(j)]$ and $\mu_\Delta=\ln[\sum_{j=1/2}^\infty e^{-\beta_\Delta j}g_\Delta(j)]$ imply

$$\frac{\partial\mu_\Delta^{(n)}}{\partial\beta_\Delta^{(n)}}=\frac{\sum_{j=1/2}^\infty(-j)e^{-\beta_\Delta^{(n)}j}g_\Delta^{(n)}(j)}{\sum_{j=1/2}^\infty e^{-\beta_\Delta^{(n)}j}g_\Delta^{(n)}(j)}=-\frac{J_\Delta}{N_\Delta},\qquad \frac{\partial\mu_\Delta}{\partial\beta_\Delta}=\frac{\sum_{j=1/2}^\infty(-j)e^{-\beta_\Delta j}g_\Delta(j)}{\sum_{j=1/2}^\infty e^{-\beta_\Delta j}g_\Delta(j)}=-\frac{J_\Delta}{N_\Delta}. \tag{8.4}$$

Inserting in Eq. (8.3), we obtain

$$\frac{\partial S_n(A)}{\partial J_\Delta}=\frac{\beta_\Delta^{(n)}-\beta_\Delta n}{1-n}=\lambda_\Delta(n),$$
$$\frac{\partial S_n(A)}{\partial N_\Delta}=\frac{\mu_\Delta^{(n)}-\mu_\Delta n}{1-n}=\sigma_\Delta(n). \tag{8.5}$$

■

Equation (8.2) suggests the analog between $\lambda(n)^{-1}$ and the temperature, as well as between $-\sigma(n)/\lambda(n)$ and the chemical potential. In the most general situation, Eq. (8.1), the temperature and chemical potential are not constants over the surface $\mathcal{S}$. So $\mathcal{S}$ are in a nonequilibrium state, although every Δ is in equilibrium.

Interestingly Eq. (8.2) shares similarities with the thermodynamical first law of the LQG black hole proposed in [11]. There the authors propose that the quantum isolated horizon is a statistical ensemble of identical spin-network punctures (quantum hairs) on the horizon, and the quasi-local energy of the horizon observed by the near-horizon Unruh observer is proportional to the total area $\mathbf{a}_{\rm BH}$ of the horizon. Then a thermodynamical first law is derived by statistics on the quantum isolated horizon

$$\delta S_{\rm BH}=\lambda\delta J_{\rm BH}+\sigma\delta N_{\rm BH},\qquad J_{\rm BH}\equiv\frac{\mathbf{a}_{\rm BH}}{8\pi\gamma\ell_P^2}, \tag{8.6}$$

where $S_{\rm BH}$ is the black hole entropy, $N_{\rm BH}$ is the total number of punctures on the horizon, λ relates to the Unruh temperature of the observer, and σ relates to the chemical potential. We immediately see the similarity between Eq. (8.2) and the above $\delta S_{\rm BH}$ by relating the entangling surface $\mathcal{S}$ to the black hole horizon, $S_n(A)$ to $S_{\rm BH}$, $J_{\mathcal{S}}$ to $J_{\rm BH}$, and $N_{\mathcal{S}}$ to $N_{\rm BH}$.

IX. REMOVING THE PARALLEL RESTRICTION

Most of the above discussions rely on the parallel restriction on ξ_{af} in spinfoam amplitude. In this section, we relax parallel restrictions to internal ξ_{af}'s and compute the spinfoam amplitude

$$A(\mathcal{K})=\sum_{\{j_f\}}\prod_f A_\Delta(j_f)\int[\mathrm{d}\xi_{\alpha f}\mathrm{d}g^\pm_{v\alpha}]\prod_{f,v,\pm}\langle\xi_{\alpha f}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta f}\rangle^{2j^\pm_f}. \tag{9.1}$$

Instead of imposing the potential $V_{\alpha\Delta}(\xi_{\alpha f})$ to suppress the nonparallel $\xi_{\alpha f}$'s, we are going to integrate out democratically all nonparallel $\xi_{\alpha f}$'s in the following analysis.

We again assume all $j_f\neq 0$, at a polyhedron α and among the facets $f\in\Delta$ (Δ is internal), and we choose one f_0 and set

$$|\xi_{\alpha f_0}\rangle\equiv|\xi_{\alpha\Delta}\rangle, \tag{9.2}$$

for all α containing f_0.

For any other $f\in\Delta$ and $f\neq f_0$, we write

$$|\xi_{\alpha f}\rangle=a_{\alpha f}|\xi_{\alpha\Delta}\rangle+b_{\alpha f}|J\xi_{\alpha\Delta}\rangle,\quad a_{\alpha f}=\cos\left(\frac{\theta_{\alpha f}}{2}\right)e^{i\phi_{\alpha f}/2},\qquad b_{\alpha f}=i\sin\left(\frac{\theta_{\alpha f}}{2}\right)e^{-i\phi_{\alpha f}/2}, \tag{9.3}$$

since $|\xi_{\alpha f}\rangle\in\mathbb{C}^2$ where $|\xi_{\alpha\Delta}\rangle,|J\xi_{\alpha\Delta}\rangle$ is a basis, $\phi_{\alpha f}\in[0,2\pi)$ and $\theta_{\alpha f}\in[0,\pi)$, and we have the gauge equivalence $|\xi_{\alpha f}\rangle\sim e^{i\varphi}|\xi_{\alpha f}\rangle$. We insert the above relation into the following building block of the integrand in $A(\mathcal{K})$:

$$\begin{aligned}\langle\xi_{\alpha f}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta f}\rangle^{2j^\pm_f}=(&\bar a_{\alpha f}a_{\beta f}\langle\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle+\bar b_{\alpha f}b_{\beta f}\langle J\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle\\&+\bar a_{\alpha f}b_{\beta f}\langle\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle+\bar b_{\alpha f}a_{\beta f}\langle J\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle)^{2j^\pm_f}.\end{aligned} \tag{9.4}$$

Applying the multinomial expansion to $\langle\xi_{\alpha f}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta f}\rangle^{2j^\pm_f}$ gives

$$\begin{aligned}=&\sum_{k^\pm_f(v)+l^\pm_f(v)+m^\pm_f(v)+n^\pm_f(v)=2j^\pm_f}\frac{2j^\pm_f!}{k^\pm_f(v)!l^\pm_f(v)!m^\pm_f(v)!n^\pm_f(v)!}\bar a_{\alpha f}^{k^\pm_f(v)+m^\pm_f(v)}\bar b_{\alpha f}^{l^\pm_f(v)+n^\pm_f(v)}a_{\beta f}^{k^\pm_f(v)+n^\pm_f(v)}b_{\beta f}^{l^\pm_f(v)+m^\pm_f(v)}\\&\langle\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle^{k^\pm_f(v)}\langle J\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle^{l^\pm_f(v)}\langle\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle^{m^\pm_f(v)}\langle J\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle^{n^\pm_f(v)},\end{aligned} \tag{9.5}$$

where $k^\pm_f(v),l^\pm_f(v),m^\pm_f(v),n^\pm_f(v)\in\mathbb{Z}_+\cup\{0\}$. Applying the product over $\pm$ and all $f\neq f_0\in\Delta$,

$$\begin{aligned}&\prod_{f\neq f_0,\pm}\langle\xi_{\alpha f}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta f}\rangle^{2j^\pm_f}\\&=\sum_{\substack{\{k^\pm_f(v)\}_f,\{l^\pm_f(v)\}_f,\{m^\pm_f(v)\}_f,\{n^\pm_f(v)\}_f\\k^\pm_f(v)+l^\pm_f(v)+m^\pm_f(v)+n^\pm_f(v)=2j^\pm_f}}\prod_{f\neq f_0;\pm}\frac{2j^\pm_f!}{k^\pm_f(v)!l^\pm_f(v)!m^\pm_f(v)!n^\pm_f(v)!}\\&\quad\times\prod_{f\neq f_0}\bar a_{\alpha f}^{\sum_\pm k^\pm_f(v)+\sum_\pm m^\pm_f(v)}\bar b_{\alpha f}^{\sum_\pm l^\pm_f(v)+\sum_\pm n^\pm_f(v)}a_{\beta f}^{\sum_\pm k^\pm_f(v)+\sum_\pm n^\pm_f(v)}b_{\beta f}^{\sum_\pm l^\pm_f(v)+\sum_\pm m^\pm_f(v)}\\&\quad\times\langle\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle^{K^\pm_\Delta(v)}\langle J\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle^{L^\pm_\Delta(v)}\langle\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle^{M^\pm_\Delta(v)}\langle J\xi_{\alpha\Delta}|g^{\pm-1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle^{N^\pm_\Delta(v)},\end{aligned} \tag{9.6}$$

where

$$K^\pm_\Delta(v)=\sum_{f\neq f_0}k^\pm_f(v),\qquad L^\pm_\Delta(v)=\sum_{f\neq f_0}l^\pm_f(v),\qquad M^\pm_\Delta(v)=\sum_{f\neq f_0}m^\pm_f(v),\qquad N^\pm_\Delta(v)=\sum_{f\neq f_0}n^\pm_f(v), \tag{9.7}$$

satisfying

$$K^\pm_\Delta(v)+L^\pm_\Delta(v)+M^\pm_\Delta(v)+N^\pm_\Delta(v)=2(J^\pm_\Delta-j^\pm_{f_0})\gg 1. \tag{9.8}$$

Therefore at least one of $K^\pm_\Delta(v),L^\pm_\Delta(v),M^\pm_\Delta(v),N^\pm_\Delta(v)$ has to be large.

We integrate nonparallel $\xi_{\alpha f}$ $(f \neq f_0)$ by integrating $\theta_{\alpha f}$ and $\phi_{\alpha f}$ with the standard unit-sphere measure. Explicitly,

$$\begin{aligned}
&\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi_{\alpha f}\int_0^{\pi}\mathrm{d}\theta_{\alpha f}\sin(\theta_{\alpha f})\bar{a}_{\alpha f}^{\sum_\pm k_f^\pm(v)+\sum_\pm m_f^\pm(v)}\bar{b}_{\alpha f}^{\sum_\pm l_f^\pm(v)+\sum_\pm n_f^\pm(v)}a_{\alpha f}^{\sum_\pm k_f^\pm(v')+\sum_\pm n_f^\pm(v')}b_{\alpha f}^{\sum_\pm l_f^\pm(v')+\sum_\pm m_f^\pm(v')}\\
&= e^{i\frac{\pi}{2}[\sum_\pm l_f^\pm(v')+\sum_\pm m_f^\pm(v')-(\sum_\pm l_f^\pm(v)+\sum_\pm n_f^\pm(v))]}\\
&\quad\times\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi_{\alpha f}e^{i\frac{\phi_{\alpha f}}{2}[\sum_\pm k_f^\pm(v')+\sum_\pm n_f^\pm(v')+\sum_\pm l_f^\pm(v)+\sum_\pm n_f^\pm(v)-(\sum_\pm k_f^\pm(v)+\sum_\pm m_f^\pm(v)+\sum_\pm l_f^\pm(v')+\sum_\pm m_f^\pm(v'))]}\\
&\quad\times\int_0^{\pi}\mathrm{d}\theta_{\alpha f}\sin(\theta_{\alpha f})\left[\cos\left(\frac{\theta_{\alpha f}}{2}\right)\right]^{\sum_\pm k_f^\pm(v)+\sum_\pm m_f^\pm(v)+\sum_\pm k_f^\pm(v')+\sum_\pm n_f^\pm(v')}\left[\sin\left(\frac{\theta_{\alpha f}}{2}\right)\right]^{\sum_\pm l_f^\pm(v)+\sum_\pm n_f^\pm(v)+\sum_\pm l_f^\pm(v')+\sum_\pm m_f^\pm(v')}.
\end{aligned} \tag{9.9}$$

Recall that $\sum_\pm k_f^\pm(v')+\sum_\pm l_f^\pm(v)+(\sum_\pm k_f^\pm(v)+\sum_\pm l_f^\pm(v'))=4\sum_\pm j_f^\pm=4j_f$ is even; thus $\sum_\pm k_f^\pm(v')+\sum_\pm l_f^\pm(v)-(\sum_\pm k_f^\pm(v)+\sum_\pm l_f^\pm(v'))$ is also even. Therefore the $\phi_{\alpha f}$-integral constraints

$$\sum_\pm k_f^\pm(v')+\sum_\pm n_f^\pm(v')-\left[\sum_\pm l_f^\pm(v')+\sum_\pm m_f^\pm(v')\right]=\sum_\pm k_f^\pm(v)+\sum_\pm m_f^\pm(v)-\left[\sum_\pm l_f^\pm(v)+\sum_\pm n_f^\pm(v)\right]. \tag{9.10}$$

Recall that $\sum_\pm k_f^\pm(v)+\sum_\pm n_f^\pm(v)+\sum_\pm l_f^\pm(v)+\sum_\pm m_f^\pm(v)=2j_f$ independent of v, and we obtain

$$\sum_\pm k_f^\pm(v')+\sum_\pm n_f^\pm(v')=\sum_\pm k_f^\pm(v)+\sum_\pm m_f^\pm(v)\equiv k_f,\qquad \sum_\pm l_f^\pm(v')+\sum_\pm m_f^\pm(v')=\sum_\pm l_f^\pm(v)+\sum_\pm n_f^\pm(v)\equiv l_f$$

with $k_f+l_f=2j_f$. The integral (9.9) reduces to

$$\frac{1}{2}\int_0^{\pi}\mathrm{d}\theta_{\alpha f}\sin(\theta_{\alpha f})\left[\cos\left(\frac{\theta_{\alpha f}}{2}\right)\right]^{2k_f}\left[\sin\left(\frac{\theta_{\alpha f}}{2}\right)\right]^{2l_f}=\frac{k_f!l_f!}{(k_f+l_f+1)!}. \tag{9.11}$$

Inserting the results into Eqs. (9.12) and (9.1), we write the integral as a sum of partial amplitudes,

$$\begin{aligned}
&\int[\mathrm{d}g_{v\alpha}^\pm\mathrm{d}\xi_{\alpha f}]\prod_{f,v,\pm}\langle\xi_{\alpha f}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|\xi_{\beta f}\rangle^{2j_f^\pm}\\
&=\sum_{\substack{\{k_f^\pm(v)\},\{l_f^\pm(v)\}\\ \{m_f^\pm(v)\},\{n_f^\pm(v)\}}}\int[\mathrm{d}g_{v\alpha}^\pm\mathrm{d}\xi_{\alpha\Delta}]\prod_{\Delta,v,\pm}\prod_{\substack{f\in\Delta\\ f\neq f_0}}\frac{2j_f^\pm!}{k_f^\pm(v)!l_f^\pm(v)!m_f^\pm(v)!n_f^\pm(v)!}\frac{k_f!l_f!}{(2j_f+1)!}\langle\xi_{\alpha\Delta}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|\xi_{\beta\Delta}\rangle^{\tilde{K}_\Delta^\pm(v)}\\
&\quad\times\langle J\xi_{\alpha\Delta}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|J\xi_{\beta\Delta}\rangle^{L_\Delta^\pm(v)}\langle\xi_{\alpha\Delta}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|J\xi_{\beta\Delta}\rangle^{M_\Delta^\pm(v)}\langle J\xi_{\alpha\Delta}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|\xi_{\beta\Delta}\rangle^{N_\Delta^\pm(v)},
\end{aligned} \tag{9.12}$$

where

$$\tilde{K}_\Delta^\pm(v)=K_\Delta^\pm(v)+2j_{f_0}^\pm. \tag{9.13}$$

We introduce shorthand notations to write

$$\int[\mathrm{d}g_{v\alpha}^\pm\mathrm{d}\xi_{\alpha f}]\prod_{f,v,\pm}\langle\xi_{\alpha f}|g_{v\alpha}^{\pm-1}g_{v\beta}^\pm|\xi_{\beta f}\rangle^{2j_f^\pm}\equiv\sideset{}{'}\sum_{\substack{\{\tilde{K}_\Delta^\pm(v)\},\{L_\Delta^\pm(v)\}\\ \{M_\Delta^\pm(v)\},\{N_f^\pm(v)\}}}\prod_\Delta w_\Delta\int[\mathrm{d}g_{v\alpha}^\pm\mathrm{d}\xi_{\alpha\Delta}]e^{S_{KLMN}}, \tag{9.14}$$

where the above sum is constrained by $\sum_\pm\tilde{K}_\Delta^\pm(v')+\sum_\pm N_\Delta^\pm(v')=\sum_\pm\tilde{K}_\Delta^\pm(v)+\sum_\pm M_\Delta^\pm(v)\equiv\tilde{K}_\Delta$, $\sum_\pm L_\Delta^\pm(v')+\sum_\pm M_\Delta^\pm(v')=\sum_\pm L_\Delta^\pm(v)+\sum_\pm N_\Delta^\pm(v)\equiv L_\Delta$, and $\tilde{K}_\Delta^\pm(v)+L_\Delta^\pm(v)+M_\Delta^\pm(v)+N_\Delta^\pm(v)=2J_\Delta^\pm$,

$$S_{KLMN} = \sum_{\Delta,v,\pm} [\tilde{K}^\pm_\Delta(v) \ln \langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | \xi_{\beta\Delta}\rangle + L^\pm_\Delta(v) \ln \langle J\xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle + M^\pm_\Delta(v) \ln \langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle + N^\pm_\Delta(v) \ln \langle J\xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | \xi_{\beta\Delta}\rangle], \tag{9.15}$$

$$w_\Delta = \sum_{\substack{\{k^\pm_f(v)\},\{l^\pm_f(v)\} \\ \{m^\pm_f(v)\},\{n^\pm_f(v)\}}}' \prod_{\substack{f\in\Delta \\ f\neq f_0}} \prod_v \left[\frac{2j^\pm_f!}{k^\pm_f(v)! l^\pm_f(v)! m^\pm_f(v)! n^\pm_f(v)!} \cdot \frac{k_f! l_f!}{(2j_f+1)!} \right]. \tag{9.16}$$

The sum in w_Δ is constrained by $\sum_\pm k^\pm_f(v') + \sum_\pm n^\pm_f(v') = \sum_\pm k^\pm_f(v) + \sum_\pm m^\pm_f(v) \equiv k_f$, $\sum_\pm l^\pm_f(v') + \sum_\pm m^\pm_f(v') = \sum_\pm l^\pm_f(v) + \sum_\pm n^\pm_f(v) \equiv l_f$, $K^\pm_\Delta(v) = \sum_{f\neq f_0} k^\pm_f(v)$, $L^\pm_\Delta(v) = \sum_{f\neq f_0} l^\pm_f(v)$, $M^\pm_\Delta(v) = \sum_{f\neq f_0} m^\pm_f(v)$, $N^\pm_\Delta(v) = \sum_{f\neq f_0} n^\pm_f(v)$.

The new action S_{KLMN} is the old action S in Eq. (4.1) with $\xi_{\alpha f}$ $(f\in\Delta)$ becoming either parallel $\xi_{\alpha f} = \xi_{\alpha\Delta}$ or antiparallel $\xi_{\alpha f} = J\xi_{\alpha\Delta}$. Configurations with some $\xi_{\alpha f}$'s being parallel and others being antiparallel have been discussed in Theorem 4.2 for critical points of S. These critical points also appear in the new action. In contrast to S, here at least one of $K^\pm_\Delta(v), L^\pm_\Delta(v), M^\pm_\Delta(v), N^\pm_\Delta(v)$ has to be large, so it allows us to apply the stationary phase approximation to the integral with the new action S_{KLMN}. The critical points in Theorem 4.2 becomes useful here for computing integrals.

The integral $\int [\mathrm{d}g^\pm_{v\alpha} \mathrm{d}\xi_{\alpha\Delta}] e^{S_{KLMN}}$ has the following feature:

Lemma 9.1: $\int [\mathrm{d}g^\pm_{v\alpha} \mathrm{d}\xi_{\alpha\Delta}] e^{S_{KLMN}}$ prefers large $K^\pm_\Delta(v)$ or $L^\pm_\Delta(v)$ and zero $M^\pm_\Delta(v), N^\pm_\Delta(v)$. $\int [\mathrm{d}g^\pm_{v\alpha} \mathrm{d}\xi_{\alpha\Delta}] e^{S_{KLMN}}$ with nonzero $M^\pm_\Delta(v), N^\pm_\Delta(v)$ is of $O(1/N)$ comparing to the integral with zero $M^\pm_\Delta(v), N^\pm_\Delta(v)$.

Proof: Suppose $M^\pm_\Delta(v)$ is large [the argument of large $N^\pm_\Delta(v)$ is similar],

$$\langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle^{M^\pm_\Delta(v)} = e^{M^\pm_\Delta(v) \ln \langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle} \tag{9.17}$$

participates the integral over $\xi_{\alpha\Delta}$ [we interchange the integral of $\xi_{\alpha\Delta}$ and the finite sum in Eq. (9.12)]. By the stationary phase analysis, this factor in the integrand leads to that critical point the integral must satisfy,

$$g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle = e^{i\varphi^\pm_{\alpha v\beta}} g^\pm_{v\alpha} |\xi_{\alpha\Delta}\rangle, \quad \text{i.e.,} \quad \langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | \xi_{\beta\Delta}\rangle = 0, \tag{9.18}$$

in order that the integrand is not suppressed exponentially. But the integral contains a factor contributed by f_0: $\langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | \xi_{\beta\Delta}\rangle^{2j^\pm_{f_0}}$ which vanishes at the above critical points. Therefore the integral is of $O(1/N)$ by stationary phase analysis and in a neighborhood D containing a single critical point x_c,

$$\int_D \mathrm{d}^n x\, a(x) e^{NS(x)} = \left(\frac{2\pi}{N}\right)^{n/2} \frac{1}{\sqrt{\det(-H)}} \times e^{NS(x_c)} \left[a(x_c) + O\left(\frac{1}{N}\right) \right], \tag{9.19}$$

which is of $O(1/N)$ if $a(x_c) = 0$. The same argument with critical equation, Eq. (9.18), also applies to large $N^\pm_\Delta(v)$.

We cannot have, e.g., both $K^\pm_\Delta(v)$ [or $L^\pm_\Delta(v)$] and $M^\pm_\Delta(v)$ [or $N^\pm_\Delta(v)$] large; otherwise the integral is suppressed exponentially. Indeed Eq. (9.18) is contradicting the first equation in Eq. (4.5), which is a critical equation from large $K^\pm_\Delta(v)$. The integrand is always suppressed exponentially if both $K^\pm_\Delta(v)$ [or $L^\pm_\Delta(v)$] and $M^\pm_\Delta(v)$ [or $N^\pm_\Delta(v)$] are large.

Therefore either $K^\pm_\Delta(v)$ or $L^\pm_\Delta(v)$ has to be large, and then the critical points must satisfy

$$g^\pm_{v\beta} |\xi_{\beta\Delta}\rangle = e^{i\varphi^\pm_{\alpha v\beta}} g^\pm_{v\alpha} |\xi_{\alpha\Delta}\rangle \quad \text{or} \quad g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle = e^{-i\varphi^\pm_{\alpha v\beta}} g^\pm_{v\alpha} | J\xi_{\alpha\Delta}\rangle. \tag{9.20}$$

There is no contradiction between the two equations since J commutes with $g \in \mathrm{SU}(2)$. Either one of them gives

$$\langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle = \langle J\xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | \xi_{\beta\Delta}\rangle = 0. \tag{9.21}$$

Then if $M^\pm_\Delta(v)$ or $N^\pm_\Delta(v)$ is nonzero, the integral is of $O(1/N)$ by the same reason as the above. ■

We set $M^\pm_\Delta(v) = N^\pm_\Delta(v) = 0$ and define

$$S_{KL} = \sum_{v,\Delta,\pm} \tilde{K}^\pm_\Delta(v) \ln \langle \xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | \xi_{\beta\Delta}\rangle + \sum_{v,\Delta,\pm} L^\pm_\Delta(v) \ln \langle J\xi_{\alpha\Delta} | g^{\pm -1}_{v\alpha} g^\pm_{v\beta} | J\xi_{\beta\Delta}\rangle. \tag{9.22}$$

$\tilde{K}^\pm_\Delta(v)$ and $L^\pm_\Delta(v)$ satisfy $\sum_\pm \tilde{K}^\pm_\Delta(v') = \sum_\pm \tilde{K}^\pm_\Delta(v) \equiv \tilde{K}_\Delta$, $\sum_\pm L^\pm_\Delta(v') = \sum_\pm L^\pm_\Delta(v) \equiv L_\Delta$, and $\tilde{K}^\pm_\Delta(v) + L^\pm_\Delta(v) = 2J^\pm_\Delta$.

Since $\mathrm{Re}(S_{KL}) \leq 0$, the condition for preventing the integrand from being exponentially suppressed, $\mathrm{Re}(S_{KL}) = 0$, is equivalent to

$$g^\pm_{v\beta} |\xi_{\beta\Delta}\rangle = e^{i\varphi^\pm_{\alpha v\beta}} g^\pm_{v\alpha} |\xi_{\alpha\Delta}\rangle. \tag{9.23}$$

The action S_{KL} has several scaling parameters $\tilde{K}^\pm_\Delta(v)$, $\tilde{L}^\pm_\Delta(v)$ which may not all be large. But Eq. (9.23) is for all cases.

When we compute $\delta_\xi S_{KL}$, we write $\delta\xi_{\alpha\Delta} = \varepsilon_{\alpha\Delta} J\xi_{\alpha\Delta} + i\eta_{\alpha\Delta}\xi_{\alpha\Delta}$ and $\delta J\xi_{\alpha\Delta} = -\bar{\varepsilon}_{\alpha\Delta}\xi_{\alpha\Delta} - i\eta_{\alpha\Delta}J\xi_{\alpha\Delta}$ where $\varepsilon_{\alpha\Delta} \in \mathbb{C}$ and $\eta_{\alpha\Delta} \in \mathbb{R}$. The coefficient in front of $\xi_{\alpha\Delta}$ is purely imaginary because $\xi_{\alpha\Delta}$ is normalized. Since every $\xi_{\alpha\Delta}$ is shared by two terms with neighboring v's,

$$\delta_{\xi_{\alpha\Delta}} S_{KL} = \sum_{\pm}\left[\tilde{K}^\pm_\Delta(v')\varepsilon_{\alpha f}\frac{\langle\xi_{\beta' f}|(g^\pm_{v'\beta'})^{-1}g^\pm_{v'\alpha}|J\xi_{\alpha f}\rangle}{\langle\xi_{\beta' f}|(g^\pm_{v'\beta'})^{-1}g^\pm_{v'\alpha}|\xi_{\alpha f}\rangle} + \tilde{K}^\pm_\Delta(v)\bar{\varepsilon}_{\alpha f}\frac{\langle J\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}g^\pm_{v\beta}|\xi_{\beta f}\rangle}{\langle\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}g^\pm_{v\beta}|\xi_{\beta f}\rangle} + i(\tilde{K}^\pm_\Delta(v') - \tilde{K}^\pm_\Delta(v))\eta_{\alpha\Delta}\right]$$
$$- \sum_{\pm}\left[L^\pm_\Delta(v')\bar{\varepsilon}_{\alpha f}\frac{\langle J\xi_{\beta' f}|(g^\pm_{v'\beta'})^{-1}g^\pm_{v'\alpha}|\xi_{\alpha f}\rangle}{\langle J\xi_{\beta' f}|(g^\pm_{v'\beta'})^{-1}g^\pm_{v'\alpha}|J\xi_{\alpha f}\rangle} + L^\pm_\Delta(v)\varepsilon_{\alpha f}\frac{\langle \xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}g^\pm_{v\beta}|J\xi_{\beta f}\rangle}{\langle J\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}g^\pm_{v\beta}|J\xi_{\beta f}\rangle} + i(L^\pm_\Delta(v') - L^\pm_\Delta(v))\eta_{\alpha\Delta}\right] = 0 \tag{9.24}$$

by Eqs. (9.23) and the orthogonality between $\xi, J\xi$.

For the derivative in $g^\pm_{v\alpha}$, we use $\delta g^\pm_{v\alpha} = \frac{i}{2}\theta^\pm_{v\alpha}\vec{\sigma}g^\pm_{v\alpha}$ ($\theta_{v\alpha} \in \mathbb{R}$). At the critical point and by Eq. (4.5),

$$\delta_{g^\pm_{v\alpha}} S = \frac{i}{2}\theta^\pm_{v\alpha}\sum_{\Delta\subset\alpha}\kappa_{\alpha\Delta}(v)\left(\tilde{K}^\pm_\Delta(v)\frac{\langle\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}\vec{\sigma}g^\pm_{v\alpha}|\xi_{\alpha f}\rangle}{\langle\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}g^\pm_{v\alpha}|\xi_{\alpha f}\rangle} + \tilde{L}^\pm_\Delta(v)\frac{\langle J\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}\vec{\sigma}g^\pm_{v\alpha}|J\xi_{\alpha f}\rangle}{\langle J\xi_{\alpha f}|(g^\pm_{v\alpha})^{-1}g^\pm_{v\alpha}|J\xi_{\alpha f}\rangle}\right)$$
$$= \frac{i}{2}\theta^\pm_{v\alpha}(1\pm\gamma)g^\pm_{v\alpha}\cdot\sum_{\Delta\subset\alpha}\kappa_{\alpha\Delta}(v)[\tilde{K}^\pm_\Delta(v) - \tilde{L}^\pm_\Delta(v)]\vec{n}_{\alpha\Delta}, \tag{9.25}$$

where $\kappa_{\alpha\Delta}(v) = \pm 1$ satisfying $\kappa_{\alpha\Delta}(v) = -\kappa_{\alpha\Delta}(v')$ appears when $\partial_{g^\pm_{v\alpha}}$ acts on $g^\pm_{v\alpha}$ or $g^{\pm -1}_{v\alpha}$. $\delta_{g^\pm_{v\alpha}} S = 0$ is equivalent to

$$\sum_{\Delta\subset\alpha}\kappa_{\alpha\Delta}(v)[\tilde{K}^\pm_\Delta(v) - L^\pm_\Delta(v)]\vec{n}_{\alpha\Delta} = 0. \tag{9.26}$$

However, there is a subtlety when $|\tilde{K}^\pm_\Delta(v) - L^\pm_\Delta(v)|$ is small. Notice that $\langle J\xi_{\alpha\Delta}|g^{\pm -1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle$ is the complex conjugate of $\langle\xi_{\alpha\Delta}|g^{\pm -1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle$,

$$S_{KL} = \sum_{v,\Delta,\pm}[\tilde{K}^\pm_\Delta(v) - L^\pm_\Delta(v)]\ln\langle\xi_{\alpha\Delta}|g^{\pm -1}_{v\alpha}g^\pm_{v\beta}|\xi_{\beta\Delta}\rangle + 2\sum_{v,\Delta,\pm}L^\pm_\Delta(v)\mathrm{Re}[\ln\langle J\xi_{\alpha\Delta}|g^{\pm -1}_{v\alpha}g^\pm_{v\beta}|J\xi_{\beta\Delta}\rangle]. \tag{9.27}$$

We assume $\tilde{K}^\pm_\Delta(v) \geq \tilde{L}^\pm_\Delta(v)$, while other cases can be worked out analogously. If all $\tilde{K}^\pm_\Delta(v), \tilde{L}^\pm_\Delta(v)$ are large at v, Δ but both $\tilde{K}^\pm_\Delta(v) - \tilde{L}^\pm_\Delta(v)$ are small, then the first term in Eq. (9.27) is subleading, and the contribution from this Δ is negligible in Eq. (9.25). Equation (9.26) with one or more Δ absent corresponds to a semiclassically degenerate tetrahedron.

Equation (9.25) is valid when $\tilde{K}^+_\Delta(v) - \tilde{L}^+_\Delta(v)$ or/and $\tilde{K}^-_\Delta(v) - \tilde{L}^-_\Delta(v)$ is/are large for all involved Δ's. The number of parallel $\xi_{\alpha f} = \xi_{\alpha\Delta}$ is much greater than the number of antiparallel $\xi_{\alpha f} = J\xi_{\alpha\Delta}$. In this case, $\tilde{L}^+_\Delta(v) \ll J^+_\Delta$ and $\tilde{K}^+_\Delta(v) \simeq J^+_\Delta$ [or/and $\tilde{L}^-_\Delta(v) \ll J^-_\Delta$ and $\tilde{K}^-_\Delta(v) \simeq J^-_\Delta$], we obtain the standard tetrahedron closure condition

$$\sum_{\Delta\subset\alpha} J_\Delta\kappa_{\alpha\Delta}(v)\vec{n}_{\alpha\Delta} = 0 \tag{9.28}$$

and recover the critical equations as Eq. (4.5). The solutions of critical equations, Eqs. (9.23) and (9.28), are the same as the situation with the parallel restriction imposed and have been discussed in Sec. IV. This result shows that critical points $(g^\pm_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$, used extensively in Secs. IV, V, and VII, indeed have nontrivial contributions in the stationary approximation of the amplitude $A(\mathcal{K})$ without the parallel restriction.

Depending on the choice of J_Δ, degenerate tetrahedra may still appear even when $\tilde{K}^\pm_\Delta(v) \gg L^\pm_\Delta(v)$, similar to the simplical EPRL/FK amplitude. But the discussion below Eq. (9.27) shows that degenerate tetrahedra become generic in the present situation. The origin of these degenerate tetrahedra is the antiparallel $\xi_{\alpha f} = J\xi_{\alpha\Delta}$ coming from integrating nonparallel $\xi_{\alpha f}$'s. The study of critical points with degenerate tetrahedra is beyond the scope of the present paper, so it is postponed to future research.

Although the integrals with nonzero $M^\pm_\Delta(v), N^\pm_\Delta(v)$ are of $O(1/N)$ comparing to the integrals with $M^\pm_\Delta(v) = N^\pm_\Delta(v) = 0$, we can still perform the same stationary phase analysis to these integrals with small $M^\pm_\Delta(v), N^\pm_\Delta(v)$ by using Eq. (9.19), where critical equations, Eqs. (9.23) and (9.28), still apply. The dual situation with large $M^\pm_\Delta(v)$, $N^\pm_\Delta(v)$ and small $\tilde{K}^\pm_\Delta(v), L^\pm_\Delta(v)$ can be analyzed in a similar way, by simply interchanging the roles $M^\pm_\Delta(v), N^\pm_\Delta(v) \leftrightarrow \tilde{K}^\pm_\Delta(v), L^\pm_\Delta(v)$, and $\xi_{\alpha\Delta} \leftrightarrow J\xi_{\alpha\Delta}$ for some α. The integral with all $M^\pm_\Delta(v), N^\pm_\Delta(v), \tilde{K}^\pm_\Delta(v), L^\pm_\Delta(v)$ large is suppressed exponentially as discussed in Lemma 9.1.

X. DISCUSSION AND OUTLOOK

This paper explores the semiclassical behavior of LQG in small spins and obtains promising results such as the entanglement entropy with thermodynamical analog and Regge geometries emerging from critical points in the stationary phase analysis. There are more interesting perspectives which should be investigated in the future.

In our work, we have seen the small-j semiclassicality always relates to coarse graining; e.g., a semiclassical Regge geometry with J_Δ as a macrostate is a collection of microstates $\{j_f\}$, and the entanglement entropy coarse grains the microstates and gives an analog thermodynamical first law. Moreover, the EPRL-FK model with J_Δ as d.o.f. may be viewed as a coarse-grained effective theory whose fundamental fine-grained theory is the generalized spinfoam model with j_f as d.o.f.. This result opens up a possibility that spinfoam models such as EPRL-FK might not be fundamental but rather coarse-grained effective theories emergent from some fine-grained theories which are more fundamental. In our work, we only consider to coarse grain the face d.o.f. such as spins j_f, but do not consider to coarse grain bulk d.o.f. such as intertwiners or spinfoam vertices in the fine-grained theory. It would be more interesting to coarse grain/fine grain these bulk d.o.f. (there have been some attempts in the literature, e.g., [39–46]). It might be possible that there exists a fine-grained fundamental theory such that the EPRL-FK model emerges from coarse graining both face and bulk d.o.f. This anticipated fine-grained theory might closely relate to the continuum limit of spinfoam formulation.

As is mentioned in Sec. VIII, the analog thermodynamical first law from the entanglement entropy is similar to the first law of the LQG black hole in [11]. This similarity may orient us toward an explanation of black hole entropy from the entanglement entropy in spinfoam formulation. Understanding quantum black hole in spinfoam formulation or other full LQG framework is a long-standing open issue. Our work suggests a new routine toward formulating a black hole in spinfoam. The idea is to consider spinfoam amplitude on a 4-manifold as a subregion in a black hole spacetime such as the Kruskal spacetime, and the spatial boundary Σ to be the spatial slice at the moment $T = 0$ of time reflection symmetry. We may set the critical point $(g^{\pm}_{v\alpha}, \xi_{\alpha\Delta})_c[J_\Delta]$ to correspond to a discrete Kruskal geometry (in this subregion). Σ can be subdivided by the horizon (bifurcate sphere) into A and $\bar{A}$. So we can compute the entanglement Rényi entropy $S_n(A)$ similar to this work. This computation has to be carried out in the Lorentzian spinfoam model, but the derivation and result should be carried over. Then the thermodynamical first law from $S_n(A)$ should be directly related to the black hole thermodynamics.

It would be interesting to relate the entanglement entropy from spinfoam to Jacobson's proposal [47]: The semiclassical Einstein equation can be derived from $\delta S(A) = 0$ where $S(A)$ is the entanglement entropy and satisfies the area law. We hope to relate the entanglement entropy derived here to recent works [9,35] which relate spinfoam amplitude to the Einstein equation.

There are other interesting questions on the semiclassical analysis of the fine-grained spinfoam model $A(\mathcal{K})$, e.g., how to understand the critical points with degenerate tetrahedra and their 4D geometrical interpretation. It would also be interesting if a semiclassical state ψ could be defined with the fine-grained spinfoam model without imposing the parallel restriction and still could be applied to computing entanglement entropy.

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APPENDIX: FACE AMPLITUDE

We follow the choice of face amplitude in [33]. The spinfoam amplitude in holonomy representation gives

$$\psi(\vec{U}) = \sum_{\vec{j},\vec{i}} \prod_f \dim(j_f) \prod_v A_v(j_f, i_\alpha) T_{\vec{j},\vec{i}}(\vec{U}) \qquad (A1)$$

in terms of normalized intertwiners $\langle i_\alpha, i'_\alpha \rangle = \delta_{i,i'}$. $\vec{U}$ are boundary SU(2) holonomies. All face amplitudes are $\dim(j_f) = 2j_f + 1$ at internal and boundary f. The boundary state (neglecting the contracted indices)

$$T_{\vec{j},\vec{i}}(\vec{U}) = \prod_{\text{boundary } f} R^{j_f}(U_f) \prod_{\text{boundary } \alpha} i_\alpha \qquad (A2)$$

is the boundary spin-network basis whose normalization is given by

$$\langle R^j_{mn}, R^{j'}_{m'n'} \rangle = \frac{1}{\dim(j)} \delta_{j,j'} \delta_{mm'} \delta_{nn'}. \qquad (A3)$$

In terms of coherent intertwiners,

$$\psi(\vec{U}) = \sum_{\vec{j}} \prod_f \dim(j_f) \int \underline{\mathrm{d}}\vec{\xi} \prod_v A_v(\vec{j}, \vec{\xi}) T_{\vec{j},\vec{\xi}}(\vec{U}), \qquad (A4)$$

where $T_{\vec{j},\vec{\xi}}(\vec{U})$ is given by replacing i_α in $T_{\vec{j},\vec{i}}(\vec{U})$ with coherent intertwiners. But every integral $\int \underline{\mathrm{d}}\xi_{\alpha f} = \dim(j_f) \int \mathrm{d}\xi_{\alpha f}$ by the resolution of identity for coherent states $\dim(j)\int d\xi|j,\xi\rangle\langle j,\xi| = 1$ where $d\xi$ is the normalized measure on the unit sphere. $A(\mathcal{K})$ in Eq. (4.1) computes the coefficients in front of $T_{\vec{j},\vec{\xi}}(\vec{U})$, and so gives

$$A_f(j_f) = A_\Delta(j_f) = (2j_f+1)^{n_v(\Delta)+1} \quad \text{for internal } f,$$
$$A_f(j_f) = A_\Delta(j_f) = (2j_f+1)^{n_v(\Delta)+2} \quad \text{for boundary } f. \tag{A5}$$

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