Two-form gauge theory dual to scalar-tensor theory

Daisuke Yoshida*

Department of Physics, Kobe University, Kobe 657-8501, Japan

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We generalize the electromagnetic duality between a massless free scalar field and a two-form gauge field in a four-dimensional spacetime to scalar-tensor theories. We derive the action of a two-form gauge field that is dual to two kinds of scalar-tensor theories: the shift symmetric K-essence theory and the shift symmetric Horndeski theory up to quadratic in a scalar field. The former case, the dual two-form, has a nonlinear kinetic term. The latter case, the dual two-form, has nontrivial interactions with gravity through the Einstein tensor. In both cases, the duality relation is modified from the usual case, that is, the dual two-form field is not simply given by the Hodge dual of the gradient of the scalar field.

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I. INTRODUCTION

A scalar field interacting with gravity through nontrivial couplings enables us to construct divergent scenarios of both the early- and late-time Universe. As the first guideline to look for possible forms of interactions, it is important to classify interactions based on the presence/absence of Ostrogradsky's ghost instability (see [1] for a review). An understanding of ghost-free scalar interactions has been deepened, especially after the refinding of Horndeski theory [2] as a generalized Galileon theory [3,4] (see also Ref. [5] for a recent review), which is the most general scalar-tensor theory with second-order derivatives in the Euler-Lagrange equations. It was found that Horndeski theory is not the most general theory avoiding Ostrogradsky's ghost [6,7]. It is possible to construct a theory which has higher derivative terms in the Euler-Lagrange equations apparently but these equations can be rewritten as a set of the equations up to second-order time derivatives. At quadratic order in $\nabla_\mu\nabla_\nu\phi$, where ϕ is a scalar field and ∇_μ is the covariant derivative, the general scalar tensor theory is found in Refs. [8–11], which is called extended scalar-tensor theory [10] or degenerate higher order derivative scalar tensor (DHOST) theory [11]. A cubic order version of DHOST theory is investigated in Ref. [12].

Not only interactions with a scalar field, but also those with other types of fields have been actively investigated. For a $U(1)$ gauge field, Horndeski also found the most general theory with second-order Euler-Lagrange equations called vector-Horndeski theory [13]. If one relaxes $U(1)$ gauge symmetry and considers a massive vector field, a study of nontrivial interactions with gravity has been developed by an analogy of the Horndeski theory for a scalar field. This is called generalized Proca theory [14]. As is similar to scalar-tensor theories, the generalized Proca theory has its further generalization without exciting extra degrees of freedom. The most general theory up to quadratic in $\nabla_\mu A_\nu$, where A_μ is a vector field, is called extended vector-tensor theory [15], which is an analogy of the extended scalar-tensor (or DHOST) theory. Even though a structure of the extended vector-tensor theory is much similar to the extended scalar-tensor theory, the vector theory has an important property: It has been shown that a nontrivial branch of the extended vector-tensor theory, which does not include the generalized Proca theory, has stable cosmological solutions [16], though a similar branch of the DHOST theory does not [17].

Our interest here is interactions between gravity and a two-form gauge field because a two-form gauge field is an essential ingredient of a low energy limit of the string theory as well as scalar and one-form vector fields. Especially, in type-I and heterotic string theory, a two-form field has interactions through the Chern-Simons term of gravity. Then it was shown that, in a four-dimensional spacetime, such a two-form can be regarded as a canonical scalar field with Chern-Simons coupling, which is axion, through the electromagnetic duality [18–20]: As is well known, there is a duality between p and the $D-p-2$ form field in the D dimension at least when there is no interaction. See Ref. [21] for a review of the Chern-Simons coupling of a scalar field. The electromagnetic duality between a scalar and a two-form field is useful even in the context of a black hole hair. It was discussed that a trivial black hole solution with a vanishing scalar field has an

*dyoshida@hawk.kobe-u.ac.jp

"axionic charge" in the two-form description [22]. In the context of cosmology, the two-form and higher form fields also have been studied because they enable us to construct a nontrivial anisotropic universe and are expected to produce statistical anisotropy [23–27]. See also [28] for an application to dark energy.

The purpose of this paper is to develop the way to extend such duality of a canonical scalar field to a subclass of Horndeski theory. As well as the resultant two-form, dual theory will provide a new example of healthy interactions between a two-form field and gravity in four-dimensional spacetime, and it might be a good candidate for healthy theory even in the higher dimension as in the Chern-Simons case. Note that the two-form and scalar field duality in mimetic theory was discussed in [29].

This paper is organized as follows. In the next section, we review the derivation of the electromagnetic duality between a canonical scalar field and a two-form field including the Chern-Simons coupling. After that, we generalize this procedure to the K-essence theory and the shift symmetric Horndeski theory up to quadratic in ϕ in Secs. III and IV, respectively. In Sec. V, we briefly see that our method cannot be applied directly to a more general class of the Horndeski theory. The final section is devoted to summary and discussions.

II. DUALITY BETWEEN CANONICAL SCALAR FIELD AND TWO-FORM FIELD

Let us begin with reviewing the well-known duality between a massless scalar field with the Chern-Simons coupling and a two-form field [18–20]. Let us consider the following action [30]:

$$S^{\phi} = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{\alpha}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \kappa\phi I^{\rm CS}\right], \quad (1)$$

where $I^{\rm CS}$ is the Chern-Simons term for gravity given by

$$I^{\rm CS} = \frac{1}{4}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}. \quad (2)$$

Here $\tilde{R}_{\mu\nu\rho\sigma}$ are components of the Hodge dual of the Riemann two-form,

$$\tilde{R}_{\mu\nu\rho\sigma} \coloneqq \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}. \quad (3)$$

$\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor associated with $g_{\mu\nu}$,

$$\epsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}[\mu\nu\rho\sigma], \qquad \epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}[\mu\nu\rho\sigma], \quad (4)$$

where $[\mu\nu\rho\sigma]$ is the complete antisymmetric tensor with $[0123] = 1$. Here α is a dimensionless constant. Note that our analysis can be applied even when α is a function of other fields; $\alpha = \alpha(\chi^I)$ with some fields χ^I. κ is a constant of mass dimension -1. The simplest example of a duality, that is, a duality between a free scalar and a free two-form field, can be obtained by setting $\kappa = 0$ in the following analysis. To treat a two-form field, it would be useful to write the action by the differential form language. However, since our purpose is to extend the duality relation to a more general scalar-tensor theory and a scalar-tensor theory is usually written in terms of component notations, we will examine the electromagnetic duality in the component notation.

As is well known, the Chern-Simons term can be written as a total derivative form,

$$I^{\rm CS} = -\nabla_\mu J^\mu_{\rm CS}, \quad (5)$$

where $J^\mu_{\rm CS}$ is the Chern-Simons current.[1]

Then, after integrating by parts, the action can be written as

$$S^{\phi} = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{\alpha}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \kappa\partial_\mu\phi J^\mu_{\rm CS}\right]. \quad (6)$$

A technically important step to derive the dual action is to introduce the intermediate action S^{AB} which is defined as follows:

$$S^{AB} = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{\alpha}{2}A_\mu A^\mu + \kappa A_\mu J^\mu_{\rm CS} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\nabla_\rho A_\sigma\right]. \quad (7)$$

The first line is the original action S^ϕ where $\partial_\mu\phi$ is replaced with A_μ. S^{AB} is equivalent with the original scalar action S^ϕ because the equation

$$\frac{1}{\sqrt{-g}}\frac{\delta S^{AB}}{\delta B^{\mu\nu}} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\nabla_\rho A_\sigma = 0 \quad (\Leftrightarrow dA = 0) \quad (8)$$

guarantees the presence of a scalar potential for A_μ. Thus, the Euler-Lagrange equation (8) ensures that A_μ can be written as $A_\mu = \partial_\mu\phi$ at least locally and, by plugging this relation into S^{AB}, one can recover the original action S^ϕ.

[1] A concrete expression for ${J_{\rm CS}}^\mu$ can be written in terms of spin connection $\omega^a{}_b$,

$$J^{\rm CS}_\mu dx^\mu = *\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right).$$

Note that the vierbein which defines this spin connection has nothing to do with our coordinate basis dx^μ and one can freely choose the basis to represent the Chern-Simons current.

The dual *two*-form action can be obtained by eliminating A_μ from the intermediate action S^{AB}. A_μ can be eliminated by using the equation,

$$\frac{1}{\sqrt{-g}}\frac{\delta S^{AB}}{\delta A_\sigma} = -\alpha A^\sigma + \kappa J^{\rm CS}_\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\nabla_\rho B_{\mu\nu} = 0, \quad (9)$$

which means

$$A = \frac{1}{\alpha}((*H) + \kappa J^{\rm CS}), \quad (10)$$

where we introduced the field strength three-form H of the two-form field B which is given by $H = dB$ and $*$ represents the Hodge dual. In component notation, it can be expressed as

$$H_{\mu\nu\rho} = 3\nabla_{[\mu}B_{\nu\rho]}, \qquad (*H)_\mu = \frac{1}{3!}\epsilon_\mu{}^{\nu\rho\sigma}H_{\nu\rho\sigma}. \quad (11)$$

By plugging the above expressions into S^{AB}, we obtain the action of a two-form field which is dual to a canonical free scalar field,

$$S^B = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{12\alpha}\hat{H}_{\mu\nu\rho}\hat{H}^{\mu\nu\rho}\right], \quad (12)$$

where

$$\hat{H}_{\mu\nu\rho} = H_{\mu\nu\rho} + \kappa(*J^{\rm CS})_{\mu\nu\rho}. \quad (13)$$

The action (12) is nothing but a four-dimensional analogy of the interaction between gravity and a two-form gauge field in the type I or heterotic supergravity. To summarize, we have examined the well-known equivalence between a scalar action S^ϕ (1) and a two-form gauge theory S^B (12). The important step of the derivation is to introduce the intermediate action S^{AB}. We will apply this method to more general scalar-tensor theories in the following sections. Note that remembering the relation of A to ϕ, Eq. (10), can be regarded as the duality relation between the original scalar field and the two-form gauge field,

$$*d\phi = \frac{1}{\alpha}\hat{H}. \quad (14)$$

III. DUAL OF SHIFT SYMMETRIC *K*-ESSENCE THEORY

A. Two-form dual action of *K*-essence theory

Here we would like to extend the above analysis to the shift symmetric subclass of K-essence theory [31–33] given by

$$S^\phi = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R + K(X)\right], \quad (15)$$

where $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. We note that a dual description of the K-essence system without gravity is studied in Refs. [34,35] in the context of an effective field theory of superfluid.

We would like to introduce an intermediate action S^{AB} by

$$S^{AB} = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R + K(Y) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\nabla_\rho A_\sigma\right], \quad (16)$$

where $Y = g^{\mu\nu}A_\mu A_\nu$. The original action is recovered by plugging $A_\mu = \partial_\mu\phi$. This replacement is guaranteed by $dA = 0$, which can be derived from the $B_{\mu\nu}$ variation of S^{AB}.

Similar to the analysis in the previous section, the duality relation can be obtained from the A_μ variation of the intermediate action,

$$\frac{1}{\sqrt{-g}}\frac{\delta S^{AB}}{\delta A_\sigma} = 2K'(Y)A^\sigma + \frac{1}{3!}\epsilon^{\sigma\mu\nu\rho}H_{\mu\nu\rho} = 0. \quad (17)$$

By using the relation $A = d\phi$, the duality relation between the original scalar and the two-form field can be understood as

$$-2K'(X)d\phi = *H, \quad (18)$$

which now includes a nonlinear dependence of X;

In order to obtain a two-form action, we need to solve Eq. (17) for A_μ. We can obtain the equation written by Y and $|H|^2 \coloneqq H_{\mu\nu\rho}H^{\mu\nu\rho}$ by squaring Eq. (17),

$$4K'(Y)^2Y = -\frac{1}{3!}|H|^2. \quad (19)$$

By solving this equation for Y, Y can be written as a function of $|H|^2$ implicitly; $Y = Y(|H|^2)$.[2]

Now (17) can be regarded as

$$A_\sigma = -\frac{1}{2K'(Y(|H|^2))}\frac{1}{3!}\epsilon_\sigma{}^{\mu\nu\rho}H_{\mu\nu\rho}, \quad (20)$$

and one can safely eliminate A^μ from the intermediate action S^{AB}. Finally, we obtain

[2]Here we assume

$$(2K''(Y)Y + K'(Y))K'(Y) \neq 0,$$

so that (19) has inverse at least locally. Our assumption is not valid for cuscuton theory [36], $K(X) \propto \sqrt{\pm X}$.

$$S^B \coloneqq S^{AB}|_{(20)} = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R + F(|H|^2)\right], \quad (21)$$

with

$$F(|H|^2) = K(Y) - 2K'(Y)Y, \quad (22)$$

where $Y = Y(|H|^2)$ is a function of $|H|^2$ defined by (19). Thus, the dual of K-essence theory is described by a two-form field with a nonlinear kinetic term. We would like to emphasize that our analysis holds even if K depends on other fields χ^I, namely, $K = K(\chi^I, X)$.

B. Derivation of the original equations of motion

For a complementary check, let us see the equivalence at the level of equations of motion. First, the full set of equations of motion in the original system S^ϕ is given by

$$\frac{2M_{pl}^{-2}}{\sqrt{-g}}\frac{\delta S^\phi}{\delta g^{\mu\nu}} = G_{\mu\nu} - \frac{2}{M_{pl}^2}T^\phi_{\mu\nu} = 0, \quad (23a)$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S^\phi}{\delta\phi} = -2\nabla^\mu(K'(X)\partial_\mu\phi) = 0, \quad (23b)$$

where the energy-momentum tensor $T^\phi_{\mu\nu}$ is given by

$$T^\phi_{\mu\nu} = -K'(X)\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}K(X). \quad (24)$$

We will check if we can derive the same equations from the dual action S^B. The equations of motion from the two-form action S^B can be derived as

$$\frac{2M_{pl}^{-2}}{\sqrt{-g}}\frac{\delta S^B}{\delta g^{\mu\nu}} = G_{\mu\nu} - \frac{2}{M_{pl}^2}T^B_{\mu\nu} = 0, \quad (25a)$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S^B}{\delta B_{\mu\nu}} = -6\nabla^\rho(F'H_\rho{}^{\mu\nu}) = 0, \quad (25b)$$

where $T^B_{\mu\nu}$ is now given by

$$T^B_{\mu\nu} = -3F'H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \frac{1}{2}g_{\mu\nu}F. \quad (26)$$

First, let us focus on Eq. (25b). By introducing the three-form

$$F'H \coloneqq \frac{1}{3!}F'(|H|^2)H_{\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho, \quad (27)$$

it can be written as

$$*d*(F'H) = \frac{1}{2}\nabla^\rho(F'H_\rho{}^{\mu\nu})dx^\mu\wedge dx^\nu = 0. \quad (28)$$

Thus, Eq. (25b) ensures that we can introduce the scalar potential ϕ for $*F'H$,

$$*F'H = -\frac{1}{12}d\phi, \quad (29)$$

where a coefficient is determined so that the above relation reproduces $H = *d\phi$ for a free two-form field $F = -|H|^2/12$. Let us define a function $K(Y)$ and $Y(|H|^2)$ by (22) and (19). Then we can show the relation,

$$\begin{aligned} F'(|H|^2) &= -(K'(Y) + 2YK''(Y))\frac{dY}{d|H|^2} \\ &= -\frac{1}{4K'(Y)}\frac{d}{d|H|^2}(4K'(Y)^2Y) \\ &= \frac{1}{4!K'(Y)}. \end{aligned} \quad (30)$$

Thus, we can rewrite the duality relation (29) as

$$\begin{aligned} H &= -\frac{1}{12F'(|H|^2)}*d\phi \\ &= -2K'(Y(|H|^2))*d\phi, \end{aligned} \quad (31)$$

or in the component notation, by

$$H_{\mu\nu\rho} = -2K'\epsilon_{\mu\nu\rho}{}^\sigma\partial_\sigma\phi. \quad (32)$$

By squaring Eq. (32), we obtain

$$\frac{1}{3!}|H|^2 = -4(K'(Y(|H|^2)))^2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \quad (33)$$

By comparing it with Eq. (19), we obtain

$$Y(|H|^2) = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \eqqcolon X. \quad (34)$$

Hence, the duality relation can be written as

$$H = -2K'(X)*d\phi, \quad (35)$$

which is equivalent with (18). The equations of motion for ϕ (23b) can be obtained from the Bianchi identity $dH = 0$ as

$$\begin{aligned} 0 &= *dH = *d(-2K'(X)*d\phi) \\ &= -2\nabla^\mu(K'(X)\nabla_\mu\phi). \end{aligned} \quad (36)$$

Finally, the equivalence of the energy-momentum tensor can be confirmed just by substituting (18) into (26) as follows:

$$\begin{aligned} T^B_{\mu\nu} &= -3F' H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \frac{1}{2}g_{\mu\nu}F \\ &= -\frac{3}{12^2 F'}\epsilon_{\mu\rho\sigma}{}^{\alpha}\epsilon_\nu{}^{\rho\sigma\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}g_{\mu\nu}F \\ &= K'(g_{\mu\nu}X - \partial_\mu\phi\partial_\nu\phi) + \frac{1}{2}g_{\mu\nu}F \\ &= -K'\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}(2K'X + F) \\ &= -K'\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}K = T^\phi_{\mu\nu}. \end{aligned} \tag{37}$$

IV. DUAL OF A SUBCLASS OF SHIFT SYMMETRIC HORNDESKI THEORY

A. Two-form dual action of Horndeski theory

Next let us include a nontrivial interaction through the Einstein tensor,

$$S^\phi = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}(R-2\Lambda) - \frac{1}{2}\mathcal{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{38}$$

where the effective metric $\mathcal{G}_{\mu\nu}$ is defined by

$$\mathcal{G}_{\mu\nu} \coloneqq \alpha g_{\mu\nu} + \beta G_{\mu\nu}. \tag{39}$$

Here α is a dimension less constant and β has mass dimension -2. We note that the following analysis can apply even when α and β depend on other fields. This action is free of ghost instability due to higher derivatives because it is included by the Horndeski theory. The action of Horndeski theory is given by

$$S^{\rm Horn} = \int d^4x\sqrt{-g}\sum_{n=2}^{5}\mathcal{L}_n^{\rm Horn}, \tag{40}$$

where $\mathcal{L}_n$ are defined by

$$\mathcal{L}_2^{\rm Horn} = G_2(\phi, X), \tag{41a}$$

$$\mathcal{L}_3^{\rm Horn} = G_3(\phi, X)\Box\phi, \tag{41b}$$

$$\mathcal{L}_4^{\rm Horn} = G_4(\phi, X)R - 2G_{4,X}(X)(\Box\phi^2 - (\nabla\nabla\phi)^2), \tag{41c}$$

$$\begin{aligned}\mathcal{L}_5^{\rm Horn} = {}& G_5(\phi,X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\ &+\frac{1}{3}G_{5,X}(\Box\phi^3 - 3\Box\phi(\nabla\nabla\phi)^2 + 2(\nabla\nabla\phi)^3).\end{aligned} \tag{41d}$$

Then our action S^ϕ corresponds to the following choice of the arbitrary functions,

$$G_2(\phi, X) = M_{pl}^2\Lambda - \frac{\alpha + c_1}{2}X, \tag{42a}$$

$$G_3(\phi, X) = c_1\phi, \tag{42b}$$

$$G_4(\phi, X) = \frac{M_{pl}^2}{2} - \frac{\beta + c_2}{2}X, \tag{42c}$$

$$G_5(\phi, X) = c_2\phi. \tag{42d}$$

Here c_1 and c_2 describe redundancy of arbitrary functions and we can set $c_1 = c_2 = 0$ without loss of generality. Note also that (38) is the most general subclass of Horndeski theory that contains terms up to quadratic in ϕ and has the shift symmetry.

The intermediate action S^{AB} can be obtained as

$$\begin{aligned} S^{AB} = \int d^4x\sqrt{-g}\bigg[&\frac{M_{pl}^2}{2}(R-2\Lambda) - \frac{1}{2}\mathcal{G}^{\mu\nu}A_\mu A_\nu \\ &+ \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\nabla_\rho A_\sigma\bigg]. \end{aligned} \tag{43}$$

To remove auxiliary field A_μ, we vary S^{AB} by A_μ and obtain an equation

$$\begin{aligned} \frac{1}{\sqrt{-g}}\frac{\delta S^{AB}}{\delta A_\sigma} &= -\mathcal{G}^{\sigma\mu}A_\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\nabla_\rho B_{\mu\nu} \\ &= -\mathcal{G}^{\sigma\mu}A_\mu + \frac{1}{3!}\epsilon^{\sigma\mu\nu\rho}H_{\mu\nu\rho} = 0. \end{aligned} \tag{44}$$

Contrary to the case of K-essence theory, this can be easily solved for A_μ, provided that $\mathcal{G}^{\mu\nu}$ has the inverse matrix,

$$A_\alpha = \frac{1}{3!}\mathcal{G}^{-1}_{\alpha\sigma}\epsilon^{\sigma\mu\nu\rho}H_{\mu\nu\rho} = \mathcal{G}^{-1}{}_\alpha{}^\sigma(*H)_\sigma. \tag{45}$$

Since A can be understood as $d\phi$, Eq. (45) means the duality relation between a scalar field and a two-form field depends on the Einstein tensor $G_{\mu\nu}$,

$$d\phi = \mathcal{G}^{-1}{}_\alpha{}^\sigma(*H)_\sigma dx^\alpha. \tag{46}$$

The dual two-form action S^B can be obtained by eliminating A_μ from the intermediate action S^{AB} by using Eq. (45):

$$\begin{aligned} S^B &\coloneqq S^{AB}|_{(45)} \\ &= \int d^4x\sqrt{-g}\bigg[\frac{M_{pl}^2}{2}(R-2\Lambda) \\ &\quad + \frac{1}{2\cdot 3!^2}\mathcal{G}^{-1}_{\mu\nu}\epsilon^{\mu\rho\sigma\tau}\epsilon^{\nu\alpha\beta\gamma}H_{\rho\sigma\tau}H_{\alpha\beta\gamma}\bigg]. \end{aligned} \tag{47}$$

Thus, the two-form fields interact with gravity through not only metric but also the Einstein tensor $G_{\mu\nu}$ which is similar to the original scalar-tensor theory. Though we assumed that $\mathcal{G}_\mu{}^\nu$ has the inverse matrix $\mathcal{G}^{-1}{}_\mu{}^\nu$, this condition generally depends on the spacetime metric. For example, if we consider the theory with $\alpha = 0$, then a two-form description can be defined only for a spacetime with $R_{\mu\nu} \neq 0$.

It is useful to derive an expression without inverse matrix $\mathcal{G}^{-1}$. It can be achieved by using a relation,

$$\mathcal{G}^{-1}{}_\mu{}^\nu \mathcal{G}^{-1}{}_\rho{}^\iota \mathcal{G}^{-1}{}_\sigma{}^\kappa \mathcal{G}^{-1}{}_\tau{}^\lambda \epsilon^{\mu\rho\sigma\tau} = \frac{1}{\det(\mathcal{G}^\cdot_\cdot)} \epsilon^{\nu\iota\kappa\lambda},$$

where $\det \mathcal{G}^\cdot_\cdot$ represents the determinant of the matrix $\mathcal{G}_\mu{}^\nu$. Actually, a term in the two-form action can be evaluated as

$$\begin{aligned} &\mathcal{G}^{-1}{}_\mu{}^\nu \epsilon^{\mu\rho\sigma\tau} \epsilon_{\nu\alpha\beta\gamma} H_{\rho\sigma\tau} H^{\alpha\beta\gamma} \\ &= \frac{1}{\det(\mathcal{G}^\cdot_\cdot)} \epsilon^{\nu\iota\kappa\lambda} \epsilon_{\nu\alpha\beta\gamma} \mathcal{G}_\iota{}^\rho \mathcal{G}_\kappa{}^\sigma \mathcal{G}_\lambda{}^\tau H_{\rho\sigma\tau} H^{\alpha\beta\gamma} \\ &= -\frac{3!}{\det(\mathcal{G}^\cdot_\cdot)} \mathcal{G}_\alpha{}^\rho \mathcal{G}_\beta{}^\sigma \mathcal{G}_\gamma{}^\tau H_{\rho\sigma\tau} H^{\alpha\beta\gamma}. \end{aligned} \tag{48}$$

Finally we obtain the two-form action which is dual to (38) as

$$\begin{aligned} S^B = \int d^4x \sqrt{-g} \Big[&\frac{M_{pl}^2}{2} (R - 2\Lambda) \\ &- \frac{1}{12} \frac{1}{\det(\mathcal{G}^\cdot_\cdot)} \mathcal{G}^{\rho\alpha} \mathcal{G}^{\sigma\beta} \mathcal{G}^{\lambda\gamma} H_{\rho\sigma\lambda} H_{\alpha\beta\gamma} \Big]. \end{aligned} \tag{49}$$

This is the main result of this paper. Since the original scalar theory is free of ghost instabilities, this result can also mean that we found a new ghost-free nontrivial interaction between a two-form field and a curvature of spacetime through the duality.

B. Derivation of the original equations of motion

As a complementary check of the duality, let us confirm the equivalence between (38) and (49) at the level of equations of motion. First, the original equations of motion from the scalar-tensor theory can be evaluated as

$$\frac{2M_{pl}^{-2}}{\sqrt{-g}} \frac{\delta S^\phi}{\delta g^{\mu\nu}} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{2}{M_{pl}^2} T^\phi_{\mu\nu} = 0, \tag{50}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S^\phi}{\delta \phi} = \nabla_\mu (\mathcal{G}^{\mu\nu} \nabla_\nu \phi) = 0. \tag{51}$$

Here $T^\phi_{\mu\nu}$ is given by

$$T^\phi_{\mu\nu} = \frac{1}{2} \mathcal{O}_{\mu\nu}{}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + \frac{1}{2} g_{\mu\nu} \left(-\frac{1}{2} \mathcal{G}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \right), \tag{52}$$

where the differential operator $\mathcal{O}$ is defined as

$$(\mathcal{O}_{\mu\nu}{}^{\rho\sigma} f_{\rho\sigma})(x) \coloneqq \frac{1}{\sqrt{-g(x)}} \int d^4y \sqrt{-g(y)} f_{\rho\sigma}(y) \frac{\delta \mathcal{G}^{\rho\sigma}(y)}{\delta g^{\mu\nu}(x)}, \tag{53}$$

and we do not need the concrete expression for $\mathcal{O}^{\rho\sigma}_{\mu\nu}$.

The equations of motion of the two-form field are given by

$$\frac{2M_{pl}^{-2}}{\sqrt{-g}} \frac{\delta S^B}{\delta g^{\mu\nu}} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{2}{M_{pl}^2} T^B_{\mu\nu} = 0, \tag{54}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S^B}{\delta B_{\beta\gamma}} = -\frac{1}{3!} \nabla^\alpha (\mathcal{G}^{-1}{}_\mu{}^\nu \epsilon^{\mu\rho\sigma\tau} \epsilon_{\nu\alpha\beta\gamma} H_{\rho\sigma\tau}) = 0, \tag{55}$$

where $T^B_{\mu\nu}$ is now given by

$$\begin{aligned} T^B_{\kappa\lambda} = &\frac{1}{2 \cdot 3!^2} \mathcal{O}_{\kappa\lambda}{}^{\kappa'\lambda'} \mathcal{G}^{-1}_{\kappa'\mu} \mathcal{G}^{-1}_{\lambda'\nu} \epsilon^{\mu\rho\sigma\tau} \epsilon^{\nu\alpha\beta\gamma} H_{\rho\sigma\tau} H_{\alpha\beta\gamma} \\ &+ \frac{1}{2} g_{\kappa\lambda} \left[-\frac{1}{2 \cdot 3!^2} \mathcal{G}^{-1}_{\mu\nu} \epsilon^{\mu\rho\sigma\tau} \epsilon^{\nu\alpha\beta\gamma} H_{\rho\sigma\tau} H_{\alpha\beta\gamma} \right]. \end{aligned} \tag{56}$$

Let us define a one-form

$$\mathcal{G}^{-1} * H \coloneqq \frac{1}{3!} \mathcal{G}^{-1}_\mu{}^\nu \epsilon_\nu{}^{\rho\sigma\lambda} H_{\rho\sigma\lambda} dx^\mu. \tag{57}$$

Then from Eq. (55), we obtain

$$(*d(\mathcal{G}^{-1} * H))_{\beta\gamma} = -\frac{1}{3!} \nabla^\alpha (\mathcal{G}^{-1}{}_\mu{}^\nu \epsilon^{\mu\rho\sigma\tau} \epsilon_{\nu\alpha\beta\gamma} H_{\rho\sigma\tau}) = 0. \tag{58}$$

That guarantees the presence of a scalar potential,

$$\mathcal{G}^{-1} * H = d\phi. \tag{59}$$

This relation is equivalent with (46).

Now it is clear that the Bianchi identity for H reduces to the equations of motion (51),

$$0 = *dH = \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} \nabla_\mu H_{\nu\rho\sigma} = \nabla_\mu (\mathcal{G}^{\mu\nu} \nabla_\nu \phi), \tag{60}$$

and the energy-momentum tensor of $B_{\mu\nu}$ coincides with that of ϕ,

$$\begin{aligned} T^B_{\kappa\lambda} &= \frac{1}{2} \mathcal{O}_{\kappa\lambda}{}^{\kappa'\lambda'} \mathcal{G}^{-1}_{\kappa'\mu} \mathcal{G}^{-1}_{\lambda'\nu} \left(\frac{1}{3!} \epsilon^{\mu\rho\sigma\tau} H_{\rho\sigma\tau} \right) \left(\frac{1}{3!} \epsilon^{\nu\alpha\beta\gamma} H_{\alpha\beta\gamma} \right) \\ &\quad + \frac{1}{2} g_{\kappa\lambda} \left[-\frac{1}{2} \mathcal{G}^{-1}_{\mu\nu} \left(\frac{1}{3!} \epsilon^{\mu\rho\sigma\tau} H_{\alpha\beta\gamma} \right) \left(\frac{1}{3!} \epsilon^{\nu\alpha\beta\gamma} H_{\rho\sigma\tau} \right) \right] \\ &= \frac{1}{2} \mathcal{O}_{\kappa\lambda}{}^{\kappa'\lambda'} \partial_{\kappa'} \phi \partial_{\lambda'} \phi + \frac{1}{2} g_{\kappa\lambda} \left[-\frac{1}{2} \mathcal{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] \\ &= T^\phi_{\kappa\lambda}. \end{aligned} \tag{61}$$

This result provides a complementary check of the local number of degrees of freedom in the two-form theory. It matches with that of the scalar theory because, for given $g_{\mu\nu}$ and ϕ, the configuration of the two-form field $B_{\mu\nu}$ can be determined, up to gauge choice, by the duality relation (59) without introducing additional integration constants.

V. ON FURTHER GENERALIZATION

We have derived two-form theories dual to the K-essence theory and the shift symmetric Horndeski theory up to quadratic order in ϕ. Before closing our discussion, let us investigate a more general class of the shift symmetric Horndeski theory,

$$G_n(\phi, X) \coloneqq G_n(X). \tag{62}$$

Even for this class of theory, we can construct an intermediate action S^{AB} by the analogy of the previous discussion,

$$S^{AB} = \int d^4x\sqrt{-g}\left[\sum_{n=2}^{5}\mathcal{L}_n^{\rm Horn}|_{\nabla\phi\to A} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\nabla_\rho A_\sigma\right]. \tag{63}$$

In the two examples that we have investigated in this paper, it was possible to eliminate A_μ from the intermediate action S^{AB} by using the equation from the A_μ variation. This fact enabled us to construct the dual two-form action in these cases. However, this property does not hold for the general class of the shift symmetric Horndeski theory because the equation of A_μ generally includes $\nabla_\mu A_\nu$. Actually we can derive the equation

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta S^{AB}}{\delta A_\mu} = {}& 2G_{2,Y}A^\mu + 2G_{3,Y}(A^\mu\nabla_\nu A^\nu - A_\nu\nabla^\mu A^\nu) \\ &+ 2G_{4,Y}A^\mu R - 4G_{4,YY}A^\mu((\nabla_\nu A^\nu)^2 - (\nabla_\nu A_\rho)^2) \\ &+ \cdots + 2G_{5,Y}\nabla_\rho A_\nu(A^\mu G^{\nu\rho} - A^\nu G^{\rho\mu}) + \cdots.\end{aligned} \tag{64}$$

Thus, $\nabla_\mu A_\nu$ terms can be avoided when

$$G_{3,Y} = 0, \quad G_{4,YY} = 0, \quad G_{5,Y} = 0. \tag{65}$$

Since, when G_3 or G_5 is constant, the corresponding $\mathcal{L}_n^{\rm Horn}$ term becomes total derivative, we can set $G_3 = G_5 = 0$. Then our discussion can be applied only for the subclass of the Horndeski theory with

$$G_2(X) = K(X), \tag{66a}$$

$$G_3(X) = 0, \tag{66b}$$

$$G_4(X) = \frac{M_{\rm pl}^2}{2} - \frac{\beta}{2}X, \tag{66c}$$

$$G_5(X) = 0, \tag{66d}$$

where we introduce arbitrary function K and two integration constants of differential equations (65), which are $M_{\rm pl}^2$ and β. This results shows that two-form dual action can be obtained only for the theory we have addressed in the previous two sections or combinations of these theories. Note that when both $K(X)$ and $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ exist at the same time, it is generally difficult to solve the equation

$$2K'(Y)A_\mu - \beta G_\mu{}^\nu A_\nu + (*H)_\mu = 0, \tag{67}$$

for A_μ explicitly.

We would like to emphasize that this result does not mean there is no two-form dual of a scalar-tensor theory beyond (66). Actually we can consider a coupling through the Gauss-Bonnet term,

$$S^{\rm GB} = \int d^4x\sqrt{-g}\lambda\phi I^{\rm GB}, \tag{68}$$

where $I^{\rm GB}$ is the Gauss-Bonnet invariant given by

$$I^{\rm GB} = -\frac{1}{4}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{69}$$

Since the integration of $I^{\rm GB}$ becomes a surface term in four-dimensional spacetime, this coupling respects the shift symmetry $\phi \to \phi + c$. It was shown that the Gauss-Bonnet coupling corresponds to a shift symmetric Horndeski theory with the choice of a function [4,37]

$$G_5 = -4\lambda\log|X|. \tag{70}$$

Clearly this is not included by (66). Nonetheless, we can construct a two-form dual because there is the Gauss-Bonnet current

$$I^{\rm GB} = -\nabla_\mu J^\mu_{\rm GB}, \tag{71}$$

where a concrete expression can be written by connection one-form $\omega_a{}^b$ with respect to some vierbein as [38]

$$J_\mu^{\rm GB}dx^\mu = \left(\omega^{ab}\wedge d\omega^{cd} + \frac{2}{3}\omega^{ab}\wedge\omega^{cf}\wedge\omega_f{}^d\right)\epsilon_{abcd}. \tag{72}$$

Since the derivation of two-form dual action with the Chern-Simons coupling does not need the detail of $J_\mu^{\rm CS}$, one can also derive a two-form dual action with the Gauss-Bonnet coupling by the same manner.

To summarize, our method can directly apply to, in principle, the following action of the scalar-tensor theory:

$$S^\phi = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R + K(X) - \frac{\beta}{2}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \kappa\phi I^{\rm CS} + \lambda\phi I^{\rm GB}\right], \quad (73)$$

though it might be difficult to write down a dual two-form action analytically because of the nonlinear dependence of the kinetic term $K(X)$. Note that here we include the Chern-Simons interaction though it is not included by the Horndeski theory. In the presence of other fields χ^I, our analysis can directly apply even when $K = K(\chi^I, X)$ and $\beta = \beta(\chi^I)$.

VI. SUMMARY AND DISCUSSION

We derived two-form theories which are dual to shift symmetric scalar-tensor theories included by the Horndeski theory. We explicitly showed the equivalence between (15) and (21), and between (38) and (49) at the level of both an action and equations of motion. A two-form field has a nonlinear kinetic term in (21) and has couplings through the Einstein tensor in (49). In both cases, duality relations are modified from the standard relation $d\phi = *dB$, as obtained in (18) and (46). These actions provide nontrivial examples of interaction between a two-form field and gravity. We found that a direct application of our method to the general shift symmetric scalar-tensor theory beyond (73) does not work well.

Since our new two-form theory with interactions through the Einstein tensor is equivalent to the Horndeski theory, it is definitely free of the Ostrogradsky's instability in four-dimensional spacetime. Though we found that it is difficult to generalize our analysis to a more general class of the shift symmetric Horndeski theory, it might be possible to construct a dual two-form theory in the framework of the DHOST theory. Contrary, it is also interesting to explore the most general ghost-free interaction between the two-form and gravity, which might include two-form theories which do not have the dual scalar description. Our duality holds only in a four-dimensional spacetime. Then it is also interesting to study whether the ghost freeness holds in an arbitrary dimension.

Our new two-form interaction would be interesting for applications to inflation and black hole physics. For inflation, the two-form field, which is coupled with inflaton, helps to construct nontrivial anisotropic background solutions and it possibly produces a statistical anisotropy [23,24,24,25,27]. It is not clear how much our new interaction affects it. For black hole physics, we would like to emphasize that the axionic charge [22] of the black hole can be defined only in the two-form view point. This is because the axionic charge depends on the global topology of spacetime while the electromagnetic duality holds locally. Then, it is interesting to clarify the relation of this axionic scalar hair to that of the shift symmetric Horndeski theory studied in Ref. [39]. As another application, we can expect that a quantum effect in the scalar frame and two-form frame can be different because our duality holds only on shell. In the case of the free field, it is known that the two-form theory contains an instanton solution which describes a tunneling process which creates a closed universe from the Minkowski space. We address this application in a subsequent work [40].

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