Static compact objects in Einstein-Cartan theory

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We generalize the Tolman-Oppenheimer-Volkoff equations for space-times endowed with a Weyssenhof-like torsion field in the Einstein-Cartan theory. The new set of structure equations clearly show how the presence of torsion affects the geometry of the space-time. We obtain new exact solutions for compact objects with non-null intrinsic spin surrounded by vacuum, explore their properties, and discuss how these solutions should be smoothly matched to an exterior space-time. We study how the intrinsic spin of matter changes the Buchdahl limit for the maximum compactness of stars. Moreover, under rather generic conditions, we prove that in the context of a Weyssenhof-like torsion, no static, spherically symmetric compact objects supported only by the intrinsic spin can exist. We also provide some algorithms to generate new solutions.

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I. INTRODUCTION

Compact objects, in particular neutron stars, represent one of richest environments to probe fundamental physics due their extreme gravitational fields, densities, and the state of the matter that composes them, especially at the core. The recent detection of gravitational waves due to the coalescing of two orbiting neutron stars [1] opened a new window to study their tidal deformations, allowing the study of the properties of the matter fields that compose this kind of object. Nonetheless, the usage of neutron stars as a physics laboratory is only possible if we have a deep knowledge of their properties. In particular, it is important to understand how the intrinsic spin[1] of the fermionic matter particles affects the behavior of such bodies.

In an astrophysical context, the effects of spin were first considered when Chandrasekhar [2] established that the maximum mass of an ideal white dwarf could hold due to the electron degeneracy pressure, before it underwent continuous gravitational collapse (see also Ref. [3] for the rotating case). In the subsequent years, similar limits relying on the Pauli exclusion principle were proposed for other types of compact objects, namely neutron stars, showing that the spin of matter particles markedly influences astrophysical objects (cf. e.g., Ref. [4]). Nevertheless, the way in which the presence of intrinsic spin affects the properties of astrophysical bodies remains largely unknown.

In an affine theory of gravity, the gravitational field is represented by the geometry of the space-time which is, in turn, determined by the energy and momentum of the matter fields. Mathematically, all classical matter properties are described by an energy-momentum tensor that acts as a source in the field equations. Since spin can be considered as an intrinsic angular momentum of the matter particles, one would expect that this property could also be encoded in an energy-momentum tensor. However, in the theory of general relativity (GR) it appears immediately clear that there is no obvious way to introduce the spin in a way that is consistent with the conservation laws for the total angular momentum. A way around this problem is to endow the space-time with additional geometrical structure, providing extra degrees of freedom to model spin and its relation with the gravitational field. This is the fundamental idea behind the so-called Einstein-Cartan-Sciama-Kible (ECSK) theory of gravity. In this theory the connection is not imposed to be symmetric so that the antisymmetric part of the connection defines an extra tensor field: torsion. In this way, it is

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[1]We should remark that here, and in the following, the word "spin" will be used exclusively to represent the quantum spin of the particles that source the gravitational field equations. In no case will the word spin be associated to any form of rotation of the compact objects we will analyze.

possible to impose a local Poincaré gauge symmetry on the tangent space of each point of the manifold such that the matter intrinsic spin can be related with the torsion tensor field. Indeed, theories of gravity with a nonsymmetric connection (generically called Einstein-Cartan theories) were proposed even before the discovery of spin. Sciama and Kibble [5,6] introduced the idea of connecting the torsion tensor with the matter intrinsic spin, paving the way to a geometrized treatment of spin.

Early works on the ECSK theory focused on the effects of spin on the evolution of gravitational collapse and the possibility of avoidance of singularities [7–11]. Only by the end of the decade, were solutions for spherically symmetric space-times found [12,13]. The solutions in Ref. [12] were obtained by directly solving the field equations for the ECSK theory. Such an approach, though, leads to great difficulties in searching for exact solutions. In this article we will adopt a different method and consider the formalism provided by the $1+1+2$ space-time decomposition [14–17]. Covariant space-time decomposition approaches were initially devised as a powerful tool to explore the properties of cosmological models and their perturbations (see, e.g., [14,18–20]) and only recently have they been employed to deal with space-times of astrophysical interest. In Refs. [21,22], this approach was used to construct—in the context of GR—a covariant version of the Tolman-Oppenheimer-Volkoff equations. The new equations allowed us to pinpoint the mathematical nature of the problem of determining interior solutions for compact objects and, for instance, the treatment of stars with anisotropic pressure. Moreover, in the covariant language it was possible to define algorithms to generate a number of new exact solutions, and to easily obtain general theorems (like the ones [23,24] in GR) which link apparently unrelated solutions.

In this article, we aim to study static compact objects in the context of the ECSK theory, in particular, to study how the presence of spin affects the possible solutions. Moreover, we will also examine how the boundary conditions imposed by the smooth junction of two space-times, with possible non-null torsion, constraint the solutions.

The article is organized as follows: in Sec. II we define the $1+1+2$ formalism and consider the decomposition of some tensorial quantities; in Sec. III we describe the setup that we propose to study and provide the structure equations; in Sec. IV we derive the Tolman-Oppenheimer-Volkoff (TOV) equations for static, locally rotationally symmetric space-times of class I and II in the presence of a non-null torsion field; in Sec. V we generalize the conditions for the smooth junction of two space-times with general torsion tensor fields and apply the results to the particular considered setup; in Sec. VI exact solutions are derived and studied; in Sec. VII we provide a set of algorithms to generate new exact solutions from previously known ones; and in Sec. VIII we summarize the results and conclude.

In this article we shall assume the metric signature $(-+++)$ and work in the geometrized units system where $G=c=1$.

II. THE 1+1+2 DECOMPOSITION

Consider a Lorentzian manifold of dimension 4 and a congruence of timelike curves with tangent vector u. Without loss of generality we can foliate the manifold in 3-hypersurfaces, V, orthogonal at each point to the curves of the congruence, such that all quantities are defined by their behavior along the direction of u and in V. This procedure is usually called "$1+3$ space-time decomposition." Such a decomposition of the space-time manifold relies on the existence of a projector to the hypersurface V which can be naturally defined as

$$h_{\alpha\beta}=g_{\alpha\beta}+u_\alpha u_\beta, \tag{1}$$

where $g_{\alpha\beta}$ represents the space-time metric and $u_\alpha u^\alpha=-1$. The projector $h_{\alpha\beta}$ has the following properties:

$$\begin{aligned} &h_{\alpha\beta}=h_{\beta\alpha}, \qquad h_{\alpha\beta}h^{\beta\gamma}=h^\gamma_\alpha,\\ &h_{\alpha\beta}u^\alpha=0, \qquad h^\alpha_\alpha=3. \end{aligned} \tag{2}$$

The $1+1+2$ decomposition [14–17] builds from the $1+3$ decomposition by defining a congruence of spatial curves with tangent vector field e such that any quantity defined in the submanifold V is defined by its behavior along e and in the 2-surfaces W. We shall refer to W as "the sheet." As before, we can then define a projector onto W by

$$N_{\alpha\beta}=h_{\alpha\beta}-e_\alpha e_\beta, \tag{3}$$

where $e_\alpha e^\alpha=1$, and such that

$$\begin{aligned} &N_{\alpha\beta}=N_{\beta\alpha}, \qquad N_{\alpha\beta}N^{\beta\gamma}=N^\gamma_\alpha,\\ &N_{\alpha\beta}u^\alpha=N_{\alpha\beta}e^\alpha=0, \qquad N^\alpha_\alpha=2. \end{aligned} \tag{4}$$

It is useful to introduce the following tensors:

$$\varepsilon_{\alpha\beta\gamma}=\varepsilon_{\alpha\beta\gamma\sigma}u^\sigma, \qquad \varepsilon_{\alpha\beta}=\varepsilon_{\alpha\beta\gamma}e^\gamma, \tag{5}$$

derived from the covariant Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\sigma}$, with the following properties:

$$\begin{aligned} &\varepsilon_{\alpha\beta\gamma}=\varepsilon_{[\alpha\beta\gamma]}, && \varepsilon_{\alpha\beta}=\varepsilon_{[\alpha\beta]},\\ &\varepsilon_{\alpha\beta\gamma}u^\gamma=0, && \varepsilon_{\alpha\beta}u^\alpha=\varepsilon_{\alpha\beta}e^\alpha=0,\\ &\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\gamma}=h^\mu_\alpha h^\nu_\beta-h^\mu_\beta h^\nu_\alpha, && \varepsilon_\alpha{}^\gamma\varepsilon_{\beta\gamma}=N_{\alpha\beta},\\ &\varepsilon_{\mu\nu\alpha}\varepsilon^{\mu\nu\beta}=2h^\beta_\alpha, && \varepsilon_{\alpha\beta\gamma}=e_\alpha\varepsilon_{\beta\gamma}-e_\beta\varepsilon_{\alpha\gamma}+e_\gamma\varepsilon_{\alpha\beta}. \end{aligned} \tag{6}$$

Using the results in the Appendix, the covariant derivatives of tangent vectors u and e can be written as

$$\delta_\alpha u_\beta = N_\alpha^\sigma N_\beta^\gamma \nabla_\sigma u_\gamma = N_{\alpha\beta}\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega,$$
$$D_\alpha u_\beta = h_\alpha^\sigma h_\beta^\gamma \nabla_\sigma u_\gamma = \delta_\alpha u_\beta + \left(\frac{1}{3}\theta + \Sigma\right)e_\alpha e_\beta + 2\Sigma_{(\alpha}e_{\beta)} - \varepsilon_{\alpha\lambda}\Omega^\lambda e_\beta + e_\alpha \varepsilon_{\beta\lambda}\Omega^\lambda,$$
$$\nabla_\alpha u_\beta = D_\alpha u_\beta - u_\alpha(\mathcal{A}e_\beta + \mathcal{A}_\beta), \tag{7}$$

and

$$\delta_\alpha e_\beta = \frac{1}{2}N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi,$$
$$D_\alpha e_\beta = \delta_\alpha e_\beta + e_\alpha a_\beta,$$
$$\nabla_\alpha e_\beta = D_\alpha e_\beta - u_\alpha \alpha_\beta - \mathcal{A}u_\alpha u_\beta + \left(\frac{1}{3}\theta + \Sigma\right)e_\alpha u_\beta + (\Sigma_\alpha - \varepsilon_{\alpha\sigma}\Omega^\sigma)u_\beta. \tag{8}$$

We shall also need to find the various contributions along u, e and on W of the energy-momentum tensor $\mathcal{T}_{\alpha\beta}$. At this point we shall not assume $\mathcal{T}_{\alpha\beta}$ to have any symmetry. Hence,

$$\begin{aligned}\mathcal{T}_{\alpha\beta} &= \mu u_\alpha u_\beta + p h_{\alpha\beta} + q_{1\alpha}u_\beta + u_\alpha q_{2\beta} + \pi_{\alpha\beta} + m_{\alpha\beta}\\ &= \mu u_\alpha u_\beta + Q_{1\alpha}u_\beta + u_\alpha Q_{2\beta} + Q_1 e_\alpha u_\beta\\ &\quad + Q_2 u_\alpha e_\beta + p_r e_\alpha e_\beta + \Pi_{1\alpha}e_\beta + e_\alpha \Pi_{2\beta}\\ &\quad + p_\perp N_{\alpha\beta} + \Pi_{\alpha\beta} + \varepsilon_{\alpha\beta}\mathbb{M},\end{aligned} \tag{9}$$

with

$$\begin{aligned}
q_{1\alpha} &= -h_\alpha^\sigma u^\gamma \mathcal{T}_{\sigma\gamma}, & \mu &= u^\sigma u^\gamma \mathcal{T}_{\sigma\gamma},\\
q_{2\alpha} &= -u^\sigma h_\alpha^\gamma \mathcal{T}_{\sigma\gamma}, & p &= \frac{1}{3}h^{\alpha\beta}\mathcal{T}_{\alpha\beta},\\
Q_{1\alpha} &= -N_\alpha^\sigma u^\gamma \mathcal{T}_{\sigma\gamma}, & p_r &= p + \Pi = e^\sigma e^\gamma \mathcal{T}_{\sigma\gamma},\\
Q_{2\alpha} &= -u^\sigma N_\alpha^\gamma \mathcal{T}_{\sigma\gamma}, & p_\perp &= p - \frac{1}{2}\Pi = \frac{1}{2}N^{\sigma\gamma}\mathcal{T}_{\sigma\gamma},\\
\Pi_{1\alpha} &= N_\alpha^\sigma e^\gamma \mathcal{T}_{\sigma\gamma}, & Q_1 &= -e^\sigma u^\gamma \mathcal{T}_{\sigma\gamma},\\
\Pi_{2\alpha} &= e^\sigma N_\alpha^\gamma \mathcal{T}_{\sigma\gamma}, & Q_2 &= -u^\sigma e^\gamma \mathcal{T}_{\sigma\gamma},\\
\pi_{\alpha\beta} &= h^\sigma_{\langle\alpha}h_{\beta\rangle}{}^\gamma \mathcal{T}_{\sigma\gamma}, & \Pi &= \frac{1}{3}\mathcal{T}_{\alpha\beta}(2e^\alpha e^\beta - N^{\alpha\beta}),\\
m_{\alpha\beta} &= h^\sigma_{[\alpha}h_{\beta]}{}^\gamma \mathcal{T}_{\sigma\gamma}, & \mathbb{M} &= \frac{1}{2}\varepsilon^{\mu\nu}\mathcal{T}_{\mu\nu},\\
\Pi_{\alpha\beta} &= \mathcal{T}_{\{\alpha\beta\}}, &&
\end{aligned} \tag{10}$$

where the angular and curly parentheses notation is defined in Eq. (A5). Moreover, the following relations are useful:

$$\begin{aligned}q_{1,2\alpha} &= Q_{1,2\alpha} + Q_{1,2}e_\alpha,\\ \pi_{\alpha\beta} &= \Pi_{\alpha\beta} + \Pi\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right)\\ &\quad + \Pi_{1(\alpha}e_{\beta)} + \Pi_{2(\alpha}e_{\beta)}.\end{aligned} \tag{11}$$

In this paper we will assume that the space-time is endowed with a linear, metric compatible connection $C_{\alpha\beta}{}^\gamma$. Such a connection is characterized by the metric connection—the Christoffel symbols—and the torsion tensor field

$$S_{\alpha\beta}{}^\gamma = C^\gamma_{[\alpha\beta]}.$$

Using Eq. (1) we can write the torsion tensor field as

$$S_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta}{}^\mu \bar{S}_{\mu\gamma} + W_{[\alpha|\gamma}u_{|\beta]} + S_{\alpha\beta}u_\gamma + u_{[\alpha}X_{\beta]}u_\gamma, \tag{12}$$

with

$$\begin{aligned}\bar{S}_{\alpha\beta} &= \frac{1}{2}\varepsilon_{\alpha\mu\nu}h_\beta^\sigma S^{\mu\nu}{}_\sigma, & W_{\alpha\beta} &= 2u^\mu h_\alpha^\nu h_\beta^\sigma S_{\mu\nu\sigma},\\ S_{\alpha\beta} &= -h_\alpha^\mu h_\beta^\nu u^\sigma S_{\mu\nu\sigma}, & X_\alpha &= 2u^\mu h_\alpha^\nu u^\sigma S_{\mu\nu\sigma}.\end{aligned} \tag{13}$$

Notice that the tensors defined in Eq. (13) are orthogonal to the tangent vector u.

Now, from the definition of the Riemann tensor, $R_{\alpha\beta\gamma}{}^\delta$:

$$R_{\alpha\beta\gamma}{}^\delta w_\delta = \nabla_\alpha \nabla_\beta w_\gamma - \nabla_\beta \nabla_\alpha w_\gamma + 2S_{\alpha\beta}{}^\delta \nabla_\delta w_\gamma, \tag{14}$$

where w_γ is an arbitrary 1-form; in the case of a Lorentzian manifold with non-null torsion, the Riemann curvature tensor does not possess the same symmetries as the torsion-free case. This is discussed very thoroughly, for instance, in Ref. [25] to which we redirect the reader for further details. In theories with torsion, the Riemann tensor has the following properties:

$$\begin{aligned}R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta},\\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma},\\ R_{[\alpha\beta\gamma]}{}^\delta &= -2\nabla_{[\alpha}S_{\beta\gamma]}{}^\delta + 4S_{[\alpha\beta|}{}^\rho S_{|\gamma]\rho}{}^\delta,\end{aligned} \tag{15}$$

and the modified second Bianchi identity

$$\nabla_{[\alpha}R_{\beta\gamma]\delta}{}^\rho = Q_{\alpha\beta\gamma\delta}{}^\rho, \tag{16}$$

where

$$Q_{\alpha\beta\gamma\delta\rho} = 2S_{[\alpha\beta|}{}^\sigma R_{|\gamma]\sigma\delta\rho}. \tag{17}$$

The previous properties of the Riemann tensor are completely general; in particular, they are valid for space-times of any dimension. Let us now consider the

particular case of a space-time of dimension 4 with torsion. In this case, $R_{\alpha\beta\gamma\delta}$ can be written as the following sum:

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + R_{\alpha[\gamma}g_{\delta]\beta} - R_{\beta[\gamma}g_{\delta]\alpha} - \frac{1}{3}Rg_{\alpha[\gamma}g_{\delta]\beta}, \quad (18)$$

where $C_{\alpha\beta\gamma\delta}$ represents the Weyl tensor, $R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu}$ the Ricci tensor, and R the Ricci scalar. In the presence of torsion, the Weyl tensor is still defined as the trace-free part of the curvature tensor, but it does not retain all the other usual symmetries:

$$\begin{aligned} C_{\alpha\beta\gamma\delta} &= -C_{\beta\alpha\gamma\delta}, \\ C_{\alpha\beta\gamma\delta} &= -C_{\alpha\beta\delta\gamma}, \\ C_{[\alpha\beta\gamma]\delta} &= R_{[\alpha\beta\gamma]\delta} + R_{[\alpha\beta}g_{\gamma]\delta}. \end{aligned} \quad (19)$$

It will be useful for our purposes to give the relation between the derivative of the Weyl tensor and the Riemann tensor. The generalization of the formula given in Ref. [26] (see also Ref. [19]) is

$$\begin{aligned} \nabla_\alpha C^{\gamma\delta\beta\alpha} = &\frac{1}{4}\varepsilon^{\mu\nu\lambda\beta}Q_{\mu\nu\lambda\sigma\rho}\varepsilon^{\sigma\rho\gamma\delta} \\ &+ \frac{3}{2}g^{\beta[\delta}Q^{\gamma]\mu\nu}{}_{\mu\nu} + \nabla^{[\delta}R^{\gamma]\beta} - \frac{1}{6}g^{\beta[\gamma}\nabla^{\delta]}R, \end{aligned} \quad (20)$$

where $Q_{\alpha\beta\gamma\delta\rho}$ is given by Eq. (17).

In what follows, we will need to find the $1+1+2$ decomposition of the Weyl tensor. We start by decomposing $C_{\alpha\beta\gamma\delta}$ into its components along u and V as

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = &-\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}u_{\delta]} + 2u_\beta E_{\alpha[\gamma}u_{\delta]} \\ &- 2\varepsilon_{\alpha\beta\mu}H^{\mu}{}_{[\gamma}u_{\delta]} - 2\varepsilon_{\mu\gamma\delta}\bar{H}^{\mu}{}_{[\alpha}u_{\beta]}, \end{aligned} \quad (21)$$

where

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^\mu u^\nu, \quad (22)$$

$$H_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\mu\nu\beta\delta}u^\delta, \quad (23)$$

$$\bar{H}_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\beta\delta\mu\nu}u^\delta, \quad (24)$$

are the "electric" part and "magnetic" parts of the Weyl tensor, respectively. Equation (21) generalizes the results in Ref. [27] to the case of non-null torsion. Note, however, that differently from the torsionless case, there are two separate tensor quantities associated to the magnetic part of the Weyl tensor: $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$.

From the results in Eq. (19), we see that in the presence of torsion the tensors $E_{\alpha\beta}$, $H_{\alpha\beta}$, and $\bar{H}_{\alpha\beta}$ have the following properties:

$$\begin{aligned} H_{\alpha\beta} &= h^\mu_\alpha h^\nu_\beta H_{\mu\nu}, & H_{\alpha\beta} &= H_{(\alpha\beta)}, \\ \bar{H}_{\alpha\beta} &= h^\mu_\alpha h^\nu_\beta \bar{H}_{\mu\nu}, & \bar{H}_{\alpha\beta} &= \bar{H}_{(\alpha\beta)}, \\ E_{\alpha\beta} &= h^\mu_\alpha h^\nu_\beta E_{\mu\nu}, & E^\alpha{}_\alpha &= 0. \end{aligned} \quad (25)$$

Therefore, $E_{\alpha\beta}$, may not be a symmetric tensor and $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$ do not have to be trace-free. On the other hand, due to the properties of the Levi-Civita tensor, even in the presence of torsion, the magnetic parts of the Weyl tensor are symmetric under the exchange of indexes. These properties allow us to decompose the tensors $E_{\alpha\beta}$, $H_{\alpha\beta}$, and $\bar{H}_{\alpha\beta}$ as

$$E_{\alpha\beta} = \mathcal{E}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{E}_\alpha e_\beta + e_\alpha\bar{\mathcal{E}}_\beta + \mathcal{E}_{\alpha\beta} + \varepsilon_{\alpha\beta}\mathbb{E}, \quad (26)$$

$$H_{\alpha\beta} = \frac{1}{2}N_{\alpha\beta}\mathrm{H} + e_\alpha e_\beta\mathcal{H} + \mathcal{H}_\alpha e_\beta + e_\alpha\mathcal{H}_\beta + \mathcal{H}_{\alpha\beta}, \quad (27)$$

$$\bar{H}_{\alpha\beta} = \frac{1}{2}N_{\alpha\beta}\bar{\mathrm{H}} + e_\alpha e_\beta\bar{\mathcal{H}} + \bar{\mathcal{H}}_\alpha e_\beta + e_\alpha\bar{\mathcal{H}}_\beta + \bar{\mathcal{H}}_{\alpha\beta}, \quad (28)$$

with

$$\begin{aligned} \mathcal{E} &= E_{\mu\nu}e^\mu e^\nu = -N^{\mu\nu}E_{\mu\nu}, & \mathcal{E}_\alpha &= N^\mu_\alpha e^\nu E_{\mu\nu}, \\ \mathbb{E} &= \frac{1}{2}\varepsilon^{\mu\nu}E_{\mu\nu}, & \bar{\mathcal{E}}_\alpha &= e^\mu N^\nu_\alpha E_{\mu\nu}, \\ \mathrm{H} &= N^{\mu\nu}H_{\mu\nu}, & \mathcal{E}_{\alpha\beta} &= E_{\{\alpha\beta\}}, \\ \mathcal{H} &= e^\mu e^\nu H_{\mu\nu}, & \mathcal{H}_\alpha &= N^\mu_\alpha e^\nu H_{\mu\nu}, \\ \bar{\mathrm{H}} &= N^{\mu\nu}\bar{H}_{\mu\nu}, & \bar{\mathcal{H}}_\alpha &= N^\mu_\alpha e^\nu \bar{H}_{\mu\nu}, \\ \bar{\mathcal{H}} &= e^\mu e^\nu \bar{H}_{\mu\nu}, & \mathcal{H}_{\alpha\beta} &= H_{\{\alpha\beta\}}, \\ \bar{\mathcal{H}}_{\alpha\beta} &= \bar{H}_{\{\alpha\beta\}}, \end{aligned} \quad (29)$$

where the curly parentheses notation is defined in Eq. (A5).

III. DECOMPOSITION OF THE FIELD EQUATIONS

We are now in position to apply that framework to study solutions of the Einstein-Cartan theory, characterized by the following field equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi\mathcal{T}_{\alpha\beta}, \quad (30)$$

$$S^{\alpha\beta\gamma} + 2g^{\gamma[\alpha}S^{\beta]}{}_\mu{}^\mu = -8\pi\Delta^{\alpha\beta\gamma}, \quad (31)$$

where $\mathcal{T}_{\alpha\beta}$ represents the canonical energy-momentum tensor and $\Delta^{\alpha\beta\mu}$ the intrinsic hypermomentum, found by varying, independently, the Einstein-Hilbert action with respect to the metric and to the connection. We assume a null cosmological constant.

From the second Bianchi identity (16) and the field equations (30) and (31) we find the conservation laws for the canonical energy-momentum tensor:

$$\nabla_\beta \mathcal{T}_\alpha{}^\beta = 2S_{\alpha\mu\nu}\mathcal{T}^{\nu\mu} + \frac{1}{8\pi}(S_{\alpha\mu}{}^\mu R - S^{\mu\nu\sigma}R_{\alpha\sigma\mu\nu}). \quad (32)$$

Using Eq. (31) we can introduce the intrinsic hypermomentum tensor in the above conservation laws and recover the result in Ref. [28] (see also Refs. [29,30] for similar results derived in a more general context).

We will assume that the source for the above field equations is an uncharged Weyssenhoff fluid [31]. The Weyssenhoff fluid provides a semiclassical description of a perfect fluid composed of fermions, such that the fluid is characterized by its energy density, pressure, and spin density. Its canonical energy-momentum tensor is given by

$$\mathcal{T}_{\alpha\beta} = \mu u_\alpha u_\beta + p h_{\alpha\beta} - (\mathcal{A}e^\mu + \mathcal{A}^\mu)S_{\mu\alpha}u_\beta, \quad (33)$$

where μ and p represent the energy density and pressure of the fluid, respectively.

Following Refs. [32,33], the hypermomentum tensor for the Weyssenhoff spin fluid can be written, in our conventions, as

$$\Delta^{\alpha\beta\gamma} = -\frac{1}{8\pi}\Delta^{\alpha\beta}u^\gamma, \quad (34)$$

where u represents the proper 4-velocity of an element of volume of the fluid and the antisymmetric spin density tensor, $\Delta^{\alpha\beta}$, verifies $\Delta^{\alpha\beta}u_\beta = 0$. From Eq. (31) we find

$$S^{\alpha\beta\gamma} = \Delta^{\alpha\beta}u^\gamma, \quad (35)$$

hence, comparing Eq. (35) with Eq. (12) we see that the Weyssenhoff fluid model implies that the tensors $\bar{S}^{\alpha\beta}$, $W^{\alpha\beta}$, and X^α, Eq. (13), are null and $S^{\alpha\beta} = \Delta^{\alpha\beta}$. In this way, the decomposition of the torsion tensor will coincide with the decomposition of $S^{\alpha\beta}$. Taking into account that $S_{\alpha\beta} \equiv S_{[\alpha\beta]}$ we find

$$S_{\alpha\beta} = \varepsilon_{\alpha\beta}\tau + 2s_{[\alpha}e_{\beta]}, \quad (36)$$

with

$$\tau = \frac{1}{2}\varepsilon^{\mu\nu}S_{\mu\nu}, \qquad s_\alpha = N^\sigma_\alpha e^\gamma S_{\sigma\gamma}. \quad (37)$$

A. The symmetries of the problem

We are interested in solutions of the Einstein-Cartan theory that are static and locally rotationally symmetric (LRS). Following Ref. [34], a space-time is said to be locally rotationally symmetric in neighborhood $B(q)$ of a point q, if there exists a nondiscrete subgroup G of the Lorentz group in the tangent space of each $q' \in B(q)$ which leaves u, the curvature tensor and their derivatives (up to third order) invariant. Assuming G to be one dimensional, we can set at each point the vector field e to have the same direction as an axis of symmetry. Then, LRS implies that all covariantly defined spacelike vectors must have the same direction of e—otherwise they would not be invariant under G. Thus, the vector quantities $\{a_\beta, \alpha_\beta, \Sigma_\beta, \Omega_\beta, A_\beta\}$ are null in such space-times. Also the shear tensors of the congruences of curves associated with u and e projected onto the sheet, $\Sigma_{\alpha\beta}$ and $\zeta_{\alpha\beta}$, must be null since there cannot be any preferred direction at the sheet.[2]

From the definition of LRS space-times, the Riemann curvature tensor must also be invariant under G; therefore, the vector components of the Weyl tensor $\{\mathcal{E}_{1\alpha}, \mathcal{E}_{2\alpha}, (\mathcal{H}_{,2})_\alpha\}$ must be null. Since the Riemann tensor also depends on the torsion tensor, the latter must also be invariant under the action of G. Therefore, from Eqs. (35) and (36), the tensor field s^α, Eq. (37), must be null. In light of this result and taking into account Eq. (35), we also conclude that for a LRS space-time the intrinsic hypermomentum tensor is simply given by

$$\Delta_{\alpha\beta} = \varepsilon_{\alpha\beta}\delta, \quad (38)$$

where $\delta = \frac{1}{2}\varepsilon^{\alpha\beta}\Delta_{\alpha\beta}$, with the constraint $\tau = \delta$.

Now, a LRS space-time is said to be of class I (LRSI) if the congruence of the curves associated with vector field e—defined to have the same direction as the axis of symmetry—is hypersurface orthogonal.[3] If the congruence of curves associated with the vector field u is also hypersurface orthogonal, the space-time is said to be a LRS of class II

[2]It should be remarked here that the presence of a generic torsion tensor field affects the definition of the kinematical quantities [35–38]. See Appendix A 4 for further details. As such, in the presence of a general torsion, LRS implies that the geometric shear vector fields $\Sigma_{g\alpha\beta}$ and $\zeta_{g\alpha\beta}$ must be null and not the quantities $\Sigma_{\alpha\beta} = \sigma_{\{\alpha\beta\}}$ and $\zeta_{\alpha\beta} \equiv \delta_{\{\alpha}e_{\beta\}}$. However, as discussed in Appendix A 4, for a Weyssenhoff fluid those are equal hence, from here on out we shall refer to $\Sigma_{\alpha\beta}$ and $\zeta_{\alpha\beta}$ as the shear tensors, onto the sheet, of the congruences associated with u and e, being implicit that we assume the Weyssenhoff model.

[3]Following Ref. [34], a space-time is said to be a LRSII when it has locally rotational symmetry and the vector fields u and e are hypersurface orthogonal. It just so happens, in space-times with null-torsion, a hypersurface orthogonal congruence has null vorticity. As such, in literature, LRSII space-times are characterized and usually referred to as space-times with locally rotational symmetry and vorticity-free u and e vector fields. As was shown in Ref. [39], this is not the case for space-times with non-null torsion where a hypersurface orthogonal congruence does not have null vorticity. In this article, we will follow the naming convention of Ref. [34]. This has at least one advantage: when comparing results with the null torsion case, we simply have to compare with the same named class; for instance, static spherically symmetric space-times, with or without torsion, always fall in the category of static LRSII space-times.

(LRSII). From the results in Ref. [39], for a torsion tensor given by Eq. (35) we have that e will be hypersurface orthogonal if and only if

$$\xi = 0, \tag{39}$$

and that u will be hypersurface orthogonal if and only if

$$\Omega = \tau, \qquad s_\alpha = 0, \tag{40}$$

where we opted to highlight that s_α will also be null from the imposition that the congruence of u is hypersurface orthogonal.

Before proceeding we should point out the fact that $s_\alpha = 0$ has an interesting effect on the nature of the Weyssenhoff fluid. Comparing Eq. (33) with Eq. (9) we conclude that for a Weyssenhoff fluid the only non-null covariantly defined quantities in Eq. (10) are μ, p, and $q_{1\alpha} = -(\mathcal{A}e^\mu + \mathcal{A}^\mu)S_{\mu\alpha}$. Now, since in a LRS space-time both $\mathcal{A}_\alpha$ and s_α are null, it implies that $q_{1\alpha} = -(\mathcal{A}e^\mu + \mathcal{A}^\mu)S_{\mu\alpha} = 0$, that is, the contributions of spin in the Weyssenhoff fluid model for a LRS space-time, will not appear in the canonical energy-momentum tensor. From this result, one might (wrongly) conclude that torsion has no role in the dynamics of the setup. In reality, torsion will still markedly influence the behavior of the matter fields. Indeed, for instance, when comparing to space-times with null torsion, where LRSII space-times are necessarily irrotational (cf. e.g., Ref. [34]), the presence of a non-null torsion of the form of Eq. (35) will induce a non-null vorticity of the congruence of curves associated with u, Eq. (40). Thus, although in the considered setup spin does not appear in the canonical energy-momentum tensor, it will still markedly change the geometry of the space-time.

An additional assumption we will consider is that the space-time is static. Now, a space-time is said to be stationary if it admits the existence of a timelike Killing vector field Ψ. If the congruence of timelike curves associated with Ψ are also hypersurface orthogonal, the space-time is said to be static. Given that the choice of the vector field u is arbitrary, we can write, at each point, $\Psi = Cu$, where $C = C(x^\alpha)$ is a generic non-null smooth function of the coordinates. The Killing equation $\mathcal{L}_\Psi g_{\alpha\beta} = 0$ in the presence of torsion can be written as

$$\nabla_{(\alpha}\Psi_{\beta)} + 2S_{\sigma(\alpha\beta)}\Psi^\sigma = 0, \tag{41}$$

for any metric compatible connection. Assuming Eq. (35), contracting Eq. (41) with $h^\alpha_\mu h^\beta_\nu$ and $h^{\alpha\beta}$ we have

$$\{\theta, \Sigma, \Sigma_\alpha, \Sigma_{\alpha\beta}\} = 0, \tag{42}$$

and $u^a\partial_a C(x^\alpha) = 0$. All is left now is to impose the condition that Ψ is hypersurface orthogonal. However, if u is hypersurface orthogonal, so is any $\Psi = Cu$. Hence, for the space-time to be static, Eq. (40) must hold.

Lastly, computing the quantities, $N^\alpha_\mu N^\gamma_\nu v^\beta R_{\alpha\beta\gamma\delta}v^\delta$, $\varepsilon_\mu{}^{\alpha\beta}R_{\alpha\beta\gamma\delta}v^\delta$ and $\varepsilon^{\mu\gamma}v^\beta R_{\alpha\beta\gamma\delta}e^\delta$, we also find that in the considered setup

$$\{\mathbb{E}, \mathcal{E}_{\alpha\beta}, \mathcal{H}_{\alpha\beta}, \bar{\mathcal{H}}_{\alpha\beta}\} = 0. \tag{43}$$

Therefore, gathering the previous results, we find that stationary LRSI or LRSII space-times permeated by an uncharged Weyssenhoff fluid are characterized by the following set of quantities: $\{\mu, p, \phi, \Omega, \mathcal{A}, \tau, \mathcal{E}, \mathrm{H}, \bar{\mathrm{H}}, \mathcal{H}, \bar{\mathcal{H}}\}$.

B. Structure equations

We are now in position to find the structure equations for stationary, locally rotationally symmetric space-time filled by a Weyssenhoff fluid in the case where the congruence of spacelike curves associated with e are hypersurface orthogonal; that is, in the case when $\xi = 0$. The nontrivial, independent propagation equations are

$$\hat{p} + \mathcal{A}(\mu + p) = -\frac{1}{4\pi}\tau\bar{\mathcal{H}}, \tag{44}$$

$$\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi) + 2\Omega^2 = 4\pi(\mu + 3p), \tag{45}$$

$$\hat{\phi} + \frac{1}{2}\phi^2 + \mathcal{E} = -\frac{16\pi}{3}\mu, \tag{46}$$

$$\hat{\mathcal{E}} + \frac{3}{2}\mathcal{E}\phi + \Omega\mathrm{H} + 2(\tau - \Omega)\bar{\mathcal{H}} = \frac{8\pi}{3}\hat{\mu}, \tag{47}$$

$$\hat{\mathcal{H}} - \frac{1}{2}\phi(\mathrm{H} - 2\mathcal{H}) + \mathcal{E}(3\Omega - 2\tau) = -8\pi\Omega(\mu + p) + \frac{8\pi}{3}\tau(\mu + 3p), \tag{48}$$

$$2\hat{\Omega} + \Omega\phi = \mathrm{H}, \tag{49}$$

and the constraint equations

$$\mathcal{E} + \mathcal{A}\phi + 2\Omega^2 = \frac{8\pi}{3}(\mu + 3p), \tag{50}$$

$$2\mathcal{A}(\Omega - \tau) - \Omega\phi + \mathcal{H} = 0, \tag{51}$$

$$\Omega(\phi - 2\mathcal{A}) + \bar{\mathrm{H}} = 0, \tag{52}$$

$$\mathrm{H} + \bar{\mathrm{H}} + 2\bar{\mathcal{H}} = 0. \tag{53}$$

Let us discuss the cases when the congruence associated with u is either hypersurface orthogonal or not, separately. Consider the cases when $\Omega \neq \tau$. In such cases we find from Eqs. (44)–(53) the following relation between Ω and τ,

$$(\phi - \mathcal{A})(\tau - \Omega) + \hat{\tau} - \hat{\Omega} = 0, \tag{54}$$

leading us to conclude that the difference between τ and Ω can be uniquely described by the behavior of the variables ϕ and $\mathcal{A}$. Notice that if at an initial instant Ω and τ are different then, unless the term $\phi - A$ diverges, there will be no point in which they are equal. Conversely, if $\Omega = \tau$ at a point these two quantities will be equal at any point.

The relations $\Omega = \tau$ or Eq. (54), for stationary LRSII or LRSI space-times, respectively, have the advantage of not depending directly on the magnetic components of the Weyl tensor and they can replace one of Eqs. (44)–(53). As we shall see, it is useful to remove Eq. (48).

Finally, to close the system, we will need an equation of state that relates the pressure of the fluid with its energy density: $p = p(\mu)$; and an equation that relates the energy density of the fluid with the intrinsic hypermomentum: $\delta = \delta(p(\mu), \mu)$.

IV. GENERALIZED TOV EQUATION FOR STATIONARY LRSI AND LRSII SPACE-TIMES

With the full set of structure equations we are finally in position to make the derivation of the generalized TOV equations. Let us start by introducing the scalar function

$$K = \frac{8\pi}{3}\mu - \mathcal{E} + \frac{1}{4}\phi^2 - 3\Omega^2 + 2\Omega\tau, \tag{55}$$

with the following property:

$$\hat{K} = -\phi K, \tag{56}$$

found from the structure equations. Equation (55) generalizes the expressions in Refs. [16,40].[4] Moreover, since the Gauss equation is unchanged by the presence of torsion, it is possible to prove that, in the cases when the vector fields u and e are hypersurface orthogonal, the quantity K represents the Gaussian curvature of the 2-sheet orthogonal to both u and e.

Now, following the treatment in Refs. [21,22], without loss of generality, let us reparametrize the integral curves of e using, in general, a nonaffine parameter ρ, such that for an arbitrary scalar function F

$$\hat{F} = \phi F_{,\rho}. \tag{57}$$

In particular we have

$$K_{,\rho} = -K. \tag{58}$$

[4]Notice that in Ref. [40] there is a small typographic error.

Introducing the following set of variables:

$$\begin{aligned}
&\mathbb{X} = \frac{\phi_{,\rho}}{\phi}, && \mathbb{B}_1 = \frac{\mathrm{H}}{\phi^2}, && \mathcal{M} = 8\pi\frac{\mu}{\phi^2},\\
&\mathbb{Y} = \frac{\mathcal{A}}{\phi}, && \mathbb{B}_2 = \frac{\bar{\mathrm{H}}}{\phi^2}, && \mathcal{P} = 8\pi\frac{p}{\phi^2},\\
&\mathbb{E} = \frac{\mathcal{E}}{\phi^2}, && \mathbb{D}_1 = \frac{\mathcal{H}}{\phi^2}, && \Delta = \frac{\delta}{\phi},\\
&\mathbb{T} = \frac{\tau}{\phi}, && \mathbb{D}_2 = \frac{\bar{\mathcal{H}}}{\phi^2}, &&\\
&\mathbb{W} = \frac{\Omega}{\phi}, && \mathcal{K} = \frac{K}{\phi^2}, &&
\end{aligned} \tag{59}$$

we can rewrite Eqs. (44)–(53) as

$$2\mathbb{Y}_{,\rho} + 2\mathbb{Y}(\mathbb{X} + \mathbb{Y} + 1) = \mathcal{M} + 3\mathcal{P} - 4\mathbb{W}^2, \tag{60}$$

$$\mathcal{K}_{,\rho} + \mathcal{K}(2\mathbb{X} + 1) = 0, \tag{61}$$

$$\mathcal{P}_{,\rho} + \mathcal{P}(2\mathbb{X} + \mathbb{Y}) + \mathbb{Y}\mathcal{M} = 2\mathbb{T}\mathbb{W}(\mathbb{X} + \mathbb{Y}) + 2\mathbb{T}\mathbb{W}_{,\rho}, \tag{62}$$

$$2\mathbb{W}_{,\rho} + \mathbb{W}(2\mathbb{X} + 1) = \mathbb{B}_1, \tag{63}$$

with the constraints

$$\mathcal{M} + 3\mathcal{P} - 3\mathbb{Y} - 3\mathbb{E} - 6\mathbb{W}^2 = 0, \tag{64}$$

$$2\mathcal{M} + 2\mathbb{X} + 2\mathcal{P} - 2\mathbb{Y} - 4\mathbb{W}^2 + 1 = 0, \tag{65}$$

$$4\mathbb{Y} + 4\mathbb{W}(2\mathbb{T} - \mathbb{W}) - 4\mathcal{P} - 4\mathcal{K} + 1 = 0, \tag{66}$$

$$\mathbb{D}_1 + \mathbb{W}(2\mathbb{Y} - 1) - 2\mathbb{Y}\mathbb{T} = 0, \tag{67}$$

$$\mathbb{B}_2 + \mathbb{W}(1 - 2\mathbb{Y}) = 0, \tag{68}$$

$$\mathbb{B}_1 + \mathbb{B}_2 + 2\mathbb{D}_2 = 0, \tag{69}$$

$$\mathbb{T} = \Delta, \tag{70}$$

and, depending on whether we are considering stationary LRSI or LRSII space-times, we have the extra equation

$$\begin{cases} \mathbb{W}_{,\rho} - \mathbb{T}_{,\rho} = (1 - \mathbb{Y} + \mathbb{X})(\mathbb{T} - \mathbb{W}), & \text{if LRSI} \\ \mathbb{W} = \mathbb{T} & \text{if LRSII.} \end{cases} \tag{71}$$

The system is closed provided and equation of state such that $\mathcal{P} = \mathcal{P}(\mathcal{M})$ and a relation such that $\Delta = \Delta(\mathcal{P}(\mathcal{M}), \mathcal{M})$.

Now, using Eqs. (65) and (66) to eliminate $\mathbb{X}$ and $\mathbb{Y}$ in Eqs. (62) and (71) we find

$$\begin{aligned}\mathcal{P}_{,\rho} &= -\mathcal{P}^2 + \mathcal{P}\left[\frac{7}{4} - 3\mathcal{K} + \mathbb{W}(8\Delta - 7\mathbb{W})\right] \\ &\quad + 2\Delta[(2\mathcal{K}-1)\mathbb{W} - 4\mathbb{W}^2(\Delta - \mathbb{W}) + \mathbb{W}_{,\rho}] \\ &\quad + \mathcal{M}\left(\frac{1}{4} - \mathcal{K} + \mathcal{P} - \mathbb{W}^2\right), \\ \mathcal{K}_{,\rho} &= -2\mathcal{K}\left(3\mathbb{W}^2 - 2\Delta\mathbb{W} + \mathcal{K} - \mathcal{M} - \frac{1}{4}\right), \\ \mathcal{P} &= \mathcal{P}(\mathcal{M}), \\ \Delta &= \Delta(\mathcal{P}(\mathcal{M}), \mathcal{M}),\end{aligned} \tag{72}$$

and

$$\begin{cases} \mathbb{W}_{,\rho} - \Delta_{,\rho} = \frac{1}{2}(\Delta - \mathbb{W}) \times (1 - 2\mathcal{M} - 2\mathcal{P} + 4\mathbb{W}^2), & \text{if LRSI} \\ \mathbb{W} = \Delta, & \text{if LRSII} \end{cases} \tag{73}$$

which represents the covariant TOV equations. The system is completed by the extra relations:

$$\mathcal{K} - \frac{1}{4} + \mathcal{P} + \mathbb{W}(\mathbb{W} - 2\Delta) = \mathbb{Y}, \tag{74}$$

$$\mathcal{K} - \frac{3}{4} - \mathcal{M} + \mathbb{W}(3\mathbb{W} - 2\Delta) = \mathbb{X}, \tag{75}$$

$$\mathcal{M} + \mathbb{W}(6\Delta - 9\mathbb{W}) - 3\left(\mathcal{K} - \frac{1}{4}\right) = 3\mathbb{E}, \tag{76}$$

$$2\mathbb{W}_{,\rho} + \mathbb{W}\left(2\mathcal{K} - 2\mathcal{M} - 4\Delta\mathbb{W} + 6\mathbb{W}^2 - \frac{1}{2}\right) = \mathbb{B}_1, \tag{77}$$

$$\mathbb{W}\left(2\mathcal{K} + 2\mathcal{P} - 4\Delta\mathbb{W} + 2\mathbb{W}^2 - \frac{3}{2}\right) = \mathbb{B}_2, \tag{78}$$

$$\mathbb{B}_1 + \mathbb{B}_2 + 2\mathbb{D}_2 = 0, \tag{79}$$

$$\begin{aligned}&\mathbb{W}(6\Delta\mathbb{W} - 2\mathbb{W}^2 - 4\Delta^2 + 1) \\ &\quad + 2(\Delta - \mathbb{W})\left(\mathcal{K} + \mathcal{P} - \frac{1}{4}\right) = \mathbb{D}_1.\end{aligned} \tag{80}$$

A. The static case

The full set of Eqs. (72)–(80) completely describe the geometry of a stationary LRSI or LRSII space-time filled by an Weyssenhoff fluid. Let us now consider the particular cases when the space-time is static; that is, u is such that $\mathbb{W} = \mathbb{T} = \Delta$.

Introducing the following quantities:

$$\mathcal{M} = \mathcal{M} - \Delta^2, \qquad \mathcal{P} = \mathcal{P} - \Delta^2, \qquad \mathcal{E} = \mathbb{E} + \frac{2}{3}\Delta^2, \tag{81}$$

Eq. (72) is given by

$$\mathcal{P}_{,\rho} = -\mathcal{P}^2 + \mathcal{P}\left[\mathcal{M} + 1 - 3\left(\mathcal{K} - \frac{1}{4}\right)\right] - \mathcal{M}\left(\mathcal{K} - \frac{1}{4}\right), \tag{82}$$

$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathcal{M}\right). \tag{83}$$

$$\mathbb{Y} = \mathcal{K} - \frac{1}{4} + \mathcal{P}, \tag{84}$$

$$\mathbb{X} = \mathcal{K} - \frac{3}{4} - \mathcal{M}, \tag{85}$$

$$3\mathcal{E} = \mathcal{M} - 3\left(\mathcal{K} - \frac{1}{4}\right), \tag{86}$$

which match exactly the expressions found in the theory of general relativity (cf. Ref. [21]) for an effective energy density and pressure and the corrected electric part of the Weyl tensor: $\mathcal{M}$, $\mathcal{P}$, and $\mathcal{E}$. Note that the extra constraints for the magnetic components of the Weyl tensor

$$2\Delta_{,\rho} + 2\Delta\left(\mathcal{K} - \mathcal{M} - \frac{1}{4}\right) = \mathbb{B}_1, \tag{87}$$

$$\mathbb{B}_2 + \Delta\left[1 - 2\mathcal{P} - 2\left(\mathcal{K} - \frac{1}{4}\right)\right] = 0, \tag{88}$$

$$\mathbb{B}_1 + \mathbb{B}_2 + 2\mathbb{D}_2 = 0, \tag{89}$$

$$\mathbb{D}_1 = \Delta, \tag{90}$$

imply that the geometry of the space-time is fundamentally different from the corresponding one in general relativity. Nonetheless, the fact that Eqs. (82)–(86) have the same form for the corrected quantities in Eq. (81) leads us to the notable result:

Proposition 1: At the level of the metric, all static, locally rotationally symmetrics of class II solutions of the theory of general relativity for a perfect fluid with energy momentum $(\mathcal{T}^{GR})_{\alpha\beta} = \mu u_\alpha u_\beta + p h_{\alpha\beta}$ are also solutions of Einstein-Cartan theory sourced by a Weyssenhoff fluid with energy-momentum tensor $(\mathcal{T}^{EC})_{\alpha\beta} = (\mu + \frac{\delta^2}{8\pi})u_\alpha u_\beta + (p + \frac{\delta^2}{8\pi})h_{\alpha\beta}$.

It is important to stress that, because of the nature of the corrections in Eq. (81), solutions which are unacceptable in general relativity due, for example, to negative energy densities or pressure, might still correspond to physically acceptable ones in the Einstein-Cartan case.

In the rest of the article we will consider the case of static spherically symmetric space-times; hence, we will study the solutions of Eqs. (81)–(90).

V. JUNCTION CONDITIONS

In the analysis of compact objects in a geometric theory of gravity it is often necessary to model the space-time as

two distinct manifolds glued together at a common boundary. Such an operation is usually performed using Israel junction conditions [41]. The Israel procedure was initially developed for space-times in the absence of torsion. More recently, a number of works have been published in which the junction conditions are generalized to space-times with torsion in different contexts [42,43]. Here we will summarize the results and extend them in light of the structure equations we have just obtained.

Consider two Lorentzian manifolds with boundary, $\mathcal{V}^-$ and $\mathcal{V}^+$, matched at an hypersurface $\mathcal{N}$, forming a new manifold $\mathcal{V}$. Let n represent the unit normal to $\mathcal{N}$, pointing from $\mathcal{V}^-$ to $\mathcal{V}^+$, and e_a be the tangent vectors to $\mathcal{N}$. Here $\mathcal{N}$ can be either timelike or spacelike. Now, following Ref. [43], for the total space-time to be a valid solution of the field equations and to guarantee that at $\mathcal{N}$ there is no surface layer, the following conditions must be met:

(i) the induced metric at $\mathcal{N}$, as seen from each space-time $\mathcal{V}^-$ and $\mathcal{V}^+$, $h^\pm_{ab} \coloneqq g^\pm_{\alpha\beta} e^\alpha_a e^\beta_b$, must be the same,

$$[h_{ab}]_\pm = 0; \tag{91}$$

(ii) the extrinsic curvature tensor of $\mathcal{N}$ as seen from $\mathcal{V}^-$ and $\mathcal{V}^+$, $Q^\pm_{ab} \coloneqq e^\alpha_a e^\beta_b \nabla^\pm_\alpha n_\beta$, is such that

$$[Q_{ab}]_\pm = 0; \tag{92}$$

(iii) the torsion tensor verifies

$$h^{\alpha\beta}[S_{\mu\beta}{}^\mu]_\pm + \epsilon h^{\alpha\beta}[n^\mu n^\nu S_{\beta\mu\nu}]_\pm = 0. \tag{93}$$

For simplicity we labeled a field Υ defined on the submanifold $\mathcal{V}^+$ or $\mathcal{V}^-$ as $\Upsilon_+ \equiv \Upsilon(\mathcal{V}^+)$ or $\Upsilon_- \equiv \Upsilon(\mathcal{V}^-)$, respectively, and used the notation $[\Upsilon]_\pm$ to represent the difference of a field as measured from each submanifold at the matching surface, i.e., $[\Upsilon]_\pm \equiv \Upsilon(\mathcal{V}^+)|_\mathcal{N} - \Upsilon(\mathcal{V}^-)|_\mathcal{N}$.

Clearly, conditions (91)–(93) reduce to the Israel conditions [41] in the limit of null torsion. In that case, the junction conditions not only guarantee that at the matching surface there is no thin shell but also are necessary and sufficient to guarantee that the singular part of the Riemann tensor is null. However, this is not the case for torsional space-times. In the presence of torsion, assuming only compatibility with the metric, the Riemann tensor of the total space-time is given by[5]

$$\begin{aligned} R_{\alpha\beta\gamma}{}^\rho = {} & \theta(\lambda) R^+_{\alpha\beta\gamma}{}^\rho + \theta(-\lambda) R^-_{\alpha\beta\gamma}{}^\rho \\ & + \delta(\lambda)(A_{\alpha\beta\gamma}{}^\rho + B_{\alpha\beta\gamma}{}^\rho), \end{aligned} \tag{94}$$

[5]Here we mix the distribution associated to a tensor and the tensor itself which is, strictly speaking, an abuse of language. Our conclusions, however, are not influence by this issue. See, e.g., Ref. [44] and references therein for more details.

where $\theta(\lambda)$ represents the Heaviside distribution, $\delta(\lambda)$ the Dirac distribution, λ is the parameter of the integral curves of n, adjusted such that the matching surface is located at $\lambda = 0$, $R^\pm_{\alpha\beta\gamma}{}^\rho$ represent the Riemann tensors of the $\mathcal{V}^-$ and $\mathcal{V}^+$ submanifolds and

$$A_{\alpha\beta\gamma}{}^\rho = \epsilon(n_\beta [\Gamma^\rho_{\alpha\gamma}]_\pm - n_\alpha [\Gamma^\rho_{\beta\gamma}]_\pm), \tag{95}$$

$$B_{\alpha\beta\gamma}{}^\rho = \epsilon(n_\beta [K_{\alpha\gamma}{}^\rho]_\pm - n_\alpha [K_{\beta\gamma}{}^\rho]_\pm), \tag{96}$$

are the singular parts of the—total—Riemann tensor, with $\Gamma^\rho_{\alpha\gamma}$ being the Christoffel symbols, $K_{\alpha\beta}{}^\gamma \equiv S_{\alpha\beta}{}^\gamma + S^\gamma{}_{\alpha\beta} - S_\beta{}^\gamma{}_\alpha$ the contorsion tensor and $\epsilon = n_\mu n^\mu$. We see that, in general, conditions (91)–(93) do not guarantee that both the tensors in Eqs. (95) and (96) are null. A smooth junction of two space-times has to imply that the discontinuities of all curvature tensors across the matching surface have to be at most finite, otherwise, the space-time will be singular at $\mathcal{N}$. In the torsion-free case, no condition on the tensor (96) is required as it is identically zero and therefore does not appear in the Einstein equations. In the Eistein-Cartan case, however, even imposing Eqs. (91)–(93), the remaining singular part of the Riemann tensor will appear in the structure equations, leading to a singularity in $\mathcal{N}$. Different from the standard violation of Israel's condition, such singularity cannot be attributed to the presence of a thin shell since Eqs. (91)–(93) prevent the existence of a surface layer at $\mathcal{N}$. For this reason, in the following we will require a completely smooth matching of the Riemann tensor on the boundary. It is a known result (see, e.g., [44,45] for a clear derivation) that the tensor $A_{\alpha\beta\gamma}{}^\rho$, Eq. (95), is null if and only if $[K_{ab}]_\pm = 0$, where $K_{ab} \coloneqq e^\alpha_a e^\beta_b \tilde{\nabla}_\alpha n_\beta$ represents the extrinsic curvature computed from the metric connection. On the other hand, a necessary and sufficient condition for $B_{\alpha\beta\gamma}{}^\rho$ to be null is given by

$$[K_{\alpha\beta}{}^\rho]_\pm = \epsilon n_\alpha [n^\mu K_{\mu\beta}{}^\rho]_\pm. \tag{97}$$

Thus, we the arrive to the following proposition:

Proposition 2: Let $\mathcal{V}^-$ and $\mathcal{V}^+$ be two Lorentzian manifolds with boundary, endowed with a metric compatible, affine connection. $\mathcal{V}^-$ and $\mathcal{V}^+$ can be smoothly matched at a common, non-null, hypersurface $\mathcal{N}$ when the following three conditions are verified:

(i) the induced metric at $\mathcal{N}$ is such that

$$[h_{ab}]_\pm = 0; \tag{98}$$

(ii) the jump of the extrinsic curvature of $\mathcal{N}$ is null, that is

$$[Q_{ab}]_\pm = 0; \tag{99}$$

(iii) the jump of the contorsion tensor at $\mathcal{N}$ verifies

$$[K_{\alpha\beta}{}^{\rho}]_{\pm} = \epsilon n_{\alpha}[n^{\mu}K_{\mu\beta}{}^{\rho}]_{\pm}, \tag{100}$$

where the vector field n represents the unit normal to $\mathcal{N}$.

Notice that writing $Q_{ab} = K_{ab} - e^{\alpha}_{a}e^{\beta}_{b}K_{\alpha\beta}{}^{\gamma}n_{\gamma}$ and using Eq. (100), Eq. (99) is the same as imposing $[K_{ab}]_{\pm} = 0$.

A. 1 + 1 + 2 junction of static LRSII space-times with torsion

Let us now express conditions (98)–(100) covariantly in the specific case of two static LRSII space-times endowed with a torsion tensor field of the form $S_{\alpha\beta}{}^{\gamma} = \varepsilon_{\alpha\beta}u^{\gamma}\tau$, where τ is a generic function of the space-time coordinates.

In what follows we will be interested in the case when the interior and exterior space-times are to be matched at a timelike hypersurface, orthogonal to the vector field e. Then, condition (98) reads

$$[N_{\alpha\beta} - u_{\alpha}u_{\beta}]_{\pm} = 0, \tag{101}$$

where $N_{\alpha\beta}$ verifies Eq. (98). Using Eq. (8), in the considered setup, Eq. (99) is simply

$$\left[\frac{1}{2}\phi N_{\alpha\beta} - \mathcal{A}u_{\alpha}u_{\beta}\right]_{\pm} = 0, \tag{102}$$

which, contracting with the induced metric at $\mathcal{N}$ and using Eq. (101), gives

$$[\phi + \mathcal{A}]_{\pm} = 0. \tag{103}$$

From Eqs. (101)–(103), we find that at the matching surface the following constraints have to be met:

$$[\phi]_{\pm} = 0, \tag{104}$$

$$[\mathcal{A}]_{\pm} = 0, \tag{105}$$

implying, for $\phi \neq 0$,

$$[\mathbb{Y}]_{\pm} = 0. \tag{106}$$

Given that e is continuous across $\mathcal{N}$, we can integrate Eq. (58), finding $K = k_0 e^{-\rho}$. Using Eqs. (59) and (104) we have

$$[\mathcal{K}]_{\pm} = 0. \tag{107}$$

Using the previous results in Eq. (84) we arrive at

$$[8\pi p - \delta^2]_{\pm} = 0. \tag{108}$$

Finally, for the specific type of torsion that we consider in this article, condition (100) imposes

$$[\delta]_{\pm} = 0, \tag{109}$$

then, from Eq. (108),

$$[p]_{\pm} = 0. \tag{110}$$

We have then found that for a smooth matching between two static LRSII space-times endowed with a torsion tensor field of the form $S_{\alpha\beta}{}^{\gamma} = \varepsilon_{\alpha\beta}u^{\gamma}\tau$, both the pressure of the fluid and the spin density, as seen from each space-time, must match at $\mathcal{N}$.

VI. EXACT SOLUTIONS FOR STATIC LRSII SPACE-TIMES

Given the set of structure equations (81)–(90) that describe the behavior of a static, LRSII space-time filled by a Weyssenhoff fluid, let us now find and study some exact solutions.

As was stated before, the system of structure equations is not closed until an equation of state and an expression for the spin density are provided. Let us then consider some particular relations for the pressure, energy, and spin densities of the fluid in order to gain some insight into the behavior of compact objects in a fully relativistic theory with non-null spin.

For the remaining of the article we will consider only the particular case of spherically symmetric space-times. Moreover, in what follows we will refer to static, spherically symmetric compact objects as "stars." Although this is an abuse of language, it is also a trend in the literature since such systems are expected to be a good model for slowly varying astrophysical bodies.

A. Effective constant energy-density and the Buchdahl limit

We start by considering the case of a system where the effective energy density is assumed to be constant, that is,

$$8\pi\mu - \delta^2 = \tilde{\mu}_0, \tag{111}$$

where $\tilde{\mu}_0 \in \mathbb{R}$. Notice that, contrary to the case of null torsion, the above assumption does not have to imply that the energy density, μ, is constant.

Using Eqs. (56) and (57) we have

$$K(\rho) = \frac{e^{-\rho}}{r_0^2}, \tag{112}$$

where r_0 is an integration constant. Equation (112) then yields

$$\mathcal{M}(\rho) = \tilde{\mu}_0 r_0^2 e^{\rho}\mathcal{K}(\rho). \tag{113}$$

Equation (113) allows us to solve Eq. (83), finding

$$\mathcal{K}(\rho) = \frac{3}{12 - 4\tilde{\mu}_0 r_0^2 e^{\rho} + 3\mathcal{K}_0 e^{-\frac{\rho}{2}}}, \tag{114}$$

where $\mathcal{K}_0$ is yet another integration constant. Setting $\mathcal{K}_0 = 0$ to avoid a conical singularity at $\rho \to -\infty$ [46], the structure equations yield

$$p(\rho) - \frac{\delta(\rho)^2}{8\pi} = -\frac{\tilde{\mu}_0(P_0 + 3\sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}})}{24\pi(P_0 + \sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}})}, \tag{115}$$

$$\mathcal{A}(\rho) = -\frac{\tilde{\mu}_0 r_0 e^{\frac{\rho}{2}}}{\sqrt{3}(P_0 + \sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}})}, \tag{116}$$

$$\phi(\rho) = \frac{2}{r_0\sqrt{3}} e^{-\frac{\rho}{2}} \sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}}, \tag{117}$$

$$\mathcal{E} = -\frac{2}{3}\delta^2, \tag{118}$$

where we have chosen the direction of e so that ϕ is positive, and the value of the integration constant P_0 is to be determined by the boundary conditions.

Let us now assume that relations (111)–(118) describe the interior of a compact object matched at a boundary $\mathcal{N}$ to an exterior space-time modeled by the Schwarzschild vacuum solution. From Eqs. (109) and (110) we find that the quantity in Eq. (115) must be zero at the boundary. Setting, without loss of generality, the matching hypersurface to be at $\rho = 0$, we find

$$P_0 = -3\sqrt{3 - \tilde{\mu}_0 r_0^2}. \tag{119}$$

The matching conditions, Eqs. (104) and (107), imply that interior and exterior observers agree on the value of the circumferential radius of $\mathcal{N}$, say r_0, and the Schwarzschild parameter, M, is given by

$$M = \frac{\tilde{\mu}_0 r_0^3}{6}. \tag{120}$$

Moreover, from condition (109) we find that the spin density must go to zero at the matching surface, that is, $\delta(\rho = 0) = 0$.

Given the previous results, we are now in position to study some effects arising from the presence of spin in compact objects. In the remainder of this subsection, for clarity, we shall write the results in terms of the circumferential radius r. Using the fact that, in the considered setup, the quantity K, Eq. (55), represents the Gaussian curvature of the 2-sheet, we have that the parameter ρ and r are related by

$$\rho = 2\ln\left(\frac{r}{r_0}\right), \tag{121}$$

where we have set the value of the arbitrary scaling factor to be r_0.

Now, defining the central pressure $p_c := p(\rho \to -\infty)$, from Eqs. (115) and (119), we have

$$p_c = -\frac{\mu_c\left(1 - \sqrt{1 - \frac{2M}{r_0}}\right)}{1 - 3\sqrt{1 - \frac{2M}{r_0}}} + \frac{\delta_c^2\left(1 - 2\sqrt{1 - \frac{2M}{r_0}}\right)}{4\pi\left(1 - 3\sqrt{1 - \frac{2M}{r_0}}\right)}, \tag{122}$$

where $\mu_c \equiv \mu(\rho \to -\infty)$ and $\delta_c \equiv \delta(\rho \to -\infty)$. If we compared directly the above expression to a similar system in GR, we would see that the second term on the right-hand side of Eq. (122) represents an explicit contribution due to the presence of spin. However, there is a subtlety: Eq. (120) indicates that the presence of spin also modifies the matching radius r_0 and the value of the Schwarzschild parameter M, making it difficult to draw conclusions only on (122).

A clearer idea of the differences between our case and GR can be obtained by computing the maximum mass that can be held by a star with constant radius. Considering Eq. (111) and if neither the densities μ_c and δ_c diverge, the central pressure in Eq. (122) will go to infinity when $r_0 = \frac{9}{4}M$ or, using Eq. (120), when

$$M_{\max} = \frac{4}{9\sqrt{3\pi}}\left(\mu - \frac{\delta^2}{8\pi}\right)^{-\frac{1}{2}}. \tag{123}$$

This result makes it clear that, when compared to a system with the same energy density μ in GR, the presence of spin increases the maximum allowed mass.

In analogy with the calculation of the Buchdahl limit in GR we can generalize this discussion to nonconstant $\tilde{\mu}$. Consider the quantity $\tilde{\mu} := 8\pi\mu - \delta^2$ and assume it to be non-negative and $d\tilde{\mu}/dr \leq 0$, for $r \in [0, r_0]$. Following the same reasoning of Ref. [47] (see also [46]) we can find an upper limit for the amount of mass a star with constant radius can hold:

$$\frac{m(r_0)}{r_0} \leq \frac{4}{9}, \tag{124}$$

with

$$m(r_0) = \frac{1}{2}\int_0^{r_0} \tilde{\mu}(r) r^2 dr. \tag{125}$$

At first sight, the expression in Eq. (124) matches the one found by Buchdahl [47] for GR. However, there is a correction due to the presence of spin in the function $m(r_0)$, Eq. (125), leading us to conclude that for the same value of the circumferential radius, r_0, a star can hold more matter in the presence of spin than in the null-spin case. It is also worth mentioning that the quantity $m(r_0)$ agrees with the value of the Schwarzschild parameter of the exterior space-time; therefore, the gravitational mass of such objects is determined not only by the energy density, μ, but also by the spin density, δ, which was expected because of

the specific way in which spin gravitates in our specific Einstein-Cartan setup.

As a final comment, although *a priori* there is nothing that forces δ^2 to be smaller than $8\pi\mu$, it is expected that in stars—even neutron stars—$\delta^2 \ll \mu$ (see Refs. [10,48]), hence, $\tilde{\mu} \geq 0$, as was assumed in the derivation of Eq. (124). On the other hand, the requirement that $d\tilde{\mu}/dr \leq 0$ might not be as physically reasonable as in the case of GR since, as we will see below, the presence of spin allows for a richer possible behavior for the matter variables.

B. Spin held stars

In the previous subsection we have considered a classical model for a relativistic star which is similar to the simplest model for these types of objects in general relativity. However, the presence of spin allows solutions which are not contemplated in Einstein's theory. The prototype of such objects is a star which is supported only by the gravitation of the spin of the Weyssenhoff fluid. In the remainder of the subsection, we will analyze this case and prove the following result:

Proposition 3: There are no static, spherically symmetric solutions of the Einstein-Cartan theory sourced by a Weyssenhoff fluid with null isotropic pressure that have all the following properties:

(1) $\delta(r)$ is non-null for $r \in [0, r_0[$ and $\delta(r_0) = 0$, for some $r_0 > 0$;
(2) $\delta^2(r)$ is a monotonically decreasing function for all $r \in [0, r_0]$;
(3) the spin and energy density functions: $\delta(r)$ and $\mu(r)$, are at least of class C^1 and the function $\mathcal{A}(r)$ is differentiable for all $r \in [0, r_0]$;
(4) the function $M(r) \coloneqq \frac{1}{2}\int_0^r [8\pi\mu(r) - \delta^2(r)]x^2 dx$ is such that $2M(r) < r$, for all $r \in]0, r_0]$.

To prove Proposition 3 we will consider first the behavior of the quantities of interest in a neighborhood of the center, $r = 0$, and then on the boundary of the star. In doing so, in order to make the reasoning more intuitive, we shall consider here that the integral curves of the vector field e are parametrized by the circumferential radius r.

Defining the quantities

$$\tilde{\mu}(r) = 8\pi\mu(r) - \delta^2(r),$$
$$\tilde{p}(r) = 8\pi p(r) - \delta^2(r), \tag{126}$$

we find from the structure equations

$$\frac{r}{2}\phi(r)\tilde{p}_{,r} = -\mathcal{A}(\tilde{\mu} + \tilde{p}), \tag{127}$$

$$\frac{r}{2}\phi(r)\mathcal{A}_{,r} + \mathcal{A}^2 + \mathcal{A}\phi = \frac{1}{2}(\tilde{\mu} + 3\tilde{p}), \tag{128}$$

$$\tilde{p} = \mathcal{A}\phi - K + \frac{1}{4}\phi^2, \tag{129}$$

with

$$K(r) = \frac{1}{r^2}, \tag{130}$$

$$\phi(r) = \frac{2}{r}\sqrt{1 - \frac{2M(r)}{r}}, \tag{131}$$

and

$$M(r) = \frac{1}{2}\int_0^r \tilde{\mu}(x)x^2 dx, \tag{132}$$

where, without loss of generality, we chose the direction of e so that $\phi(r)$ is non-negative. Moreover, from Eqs. (129) and (131) we find the useful relation

$$\mathcal{A}\phi = \frac{2M(r)}{r^3} + \tilde{p}. \tag{133}$$

We will consider now the case of a static, spherically symmetric compact object held entirely by spin, that is, the case when $p(r) = 0$ and $\tilde{p}(r) = -\delta^2(r)$, smoothly matched to an exterior space-time modeled by a vacuum solution of the Einstein-Cartan field equations. Moreover, we will assume that for $r > 0$, $2M(r) < r$, otherwise the scalar $\phi(r)$ would take complex values.

1. Behavior at the center

Assuming that the functions $\mu(r)$, $\delta^2(r) \in C^1$ we can write in a small enough neighborhood of $r = 0$:

$$\mu(r) = \mu(0) + \mu_{,r}(0)r,$$
$$\delta^2(r) = \delta^2(0) + (\delta^2)_{,r}(0)r, \tag{134}$$

where the comma represents partial—or total—derivative with respect to the variable in front. From Eq. (134), we find that in a small enough neighborhood of $r = 0$, the mass function (132) is described by

$$2M(r) = \frac{\tilde{\mu}(0)}{3}r^3 + \frac{1}{4}(\tilde{\mu}_{,r}(0))r^4. \tag{135}$$

In particular, we find that $M(r)$ goes to zero at least as fast as r^3.

Now, Eqs. (127) and (133) yield

$$\frac{2}{r}\left(1 - \frac{2M(r)}{r}\right)\frac{d\delta^2}{dr} = \left(\frac{2M(r)}{r^3} - \delta^2\right)(\tilde{\mu} - \delta^2). \tag{136}$$

In a region where $r \in [0, \epsilon[$, with $\epsilon \ll 1$, the rhs of this equation takes values in $\mathbb{R}$, therefore,

$$(\delta^2)_{,r}(0) = 0. \tag{137}$$

Repeating the same reasoning in Eq. (133) we find that

$$\mathcal{A}(0) = 0. \tag{138}$$

Let us now assume that there exists an $r_a > 0$ where for $r \in]0, r_a[$, $\mathcal{A}(r) > 0$. From Eq. (127) we will find that in this region $\tilde{\mu} + \tilde{p} \leq 0$ which implies that $\tilde{\mu} + 3\tilde{p} < 0$. Then, from Eq. (128) we find that $\mathcal{A}_{,r}(r) < 0$, for all $r \in]0, r_a[$. This, however, violates the initial hypothesis since, $\mathcal{A}(0) = 0$ and we assume that $\mathcal{A}(r) > 0$, for $r \in]0, r_a[$, that is, $\mathcal{A}_{,r}(r)$ would have to be positive for some $r \in [0, r_a[$.

Another possibility is that for a region $r \in [0, r_a]$, $\mathcal{A}(r) = 0$ and for $r \in]r_a, r_c]$ with $r_c > r_a$, $\mathcal{A}(r) > 0$. If this were the case, since for $r \in]r_a, r_c]$, $\mathcal{A}(r) > 0$, there would exist a value $r_b \in]r_a, r_c]$ such that $\mathcal{A}_{,r}(r_b) > 0$. Using this in Eq. (128), at $r = r_b$ we find

$$\tilde{\mu} + 3\tilde{p} > 0 \Rightarrow \tilde{\mu} + \tilde{p} > 0, \tag{139}$$

but from Eq. (127) and imposing that $(\delta^2)_{,r} \leq 0$ we find that $\tilde{\mu} + \tilde{p}|_{r=r_b} \leq 0$, contradicting (139).

Another possibility is that $\mathcal{A}(r) = 0$, for all $r \in [0, r_0]$. From Eqs. (127) and (128) this simply represents a vacuum solution as such it does not represent a solution for a compact object.

Gathering these results, we conclude that there exists an $r_d > 0$ such that in the region $[0, r_d[$, $\mathcal{A}(r) \leq 0$ and it must take negative values in some subregion.

2. Behavior at the boundary

Let us now define the boundary of the compact object as the hypersurface at which the spin density goes to zero, that is, $\delta^2(r_0) = 0$. In such a hypersurface we have three possible behaviors for the function $\mathcal{A}$:

(1) $\mathcal{A}(r_0) < 0$;
(2) $\mathcal{A}(r_0) > 0$;
(3) $\mathcal{A}(r_0) = 0$.

Let us consider each case separately.

(1) The case $\mathcal{A}(r_0) < 0$

From Eq. (133) we have that at $r = r_0$

$$\mathcal{A}\phi|_{r=r_0} = \frac{2M(r_0)}{r_0^3} < 0. \tag{140}$$

Therefore, from Eq. (132) there exists a region $]r_f, r_g[$ where

$$\tilde{\mu}(r) < 0, \tag{141}$$

then $\tilde{\mu} + \tilde{p} < 0$, in that region. From Eq. (127), to guarantee that the spin density is a monotonically decreasing function of r, we find that $\mathcal{A}(r) \geq 0$, for $r \in]r_f, r_g[$. So, either $\mathcal{A}(r) = 0 \wedge \mathcal{A}_{,r}(r) = 0$ for all $r \in]r_f, r_g[$, that is, the function $\mathcal{A}(r)$ takes the value zero and stays zero for all $r \in]r_f, r_g[$; or $\mathcal{A}(r) > 0$ for some $r \in]r_f, r_g[$. The former case is not possible: from Eq. (128), $\tilde{\mu}(r) + 3\tilde{p}(r) = 0$, hence, $\tilde{\mu}(r) \geq 0$, for all $r \in]r_f, r_g[$, which contradicts the inequality (141). As for the latter—the case when $\mathcal{A}(r) > 0$, for some $r \in]r_f, r_g[$—in the previous subsection it was shown that for some subregion of $[0, r_d[$, $\mathcal{A}(r) \leq 0$, therefore the region $]r_f, r_g[$ cannot be a subregion of $[0, r_d[$. With this said, since $\mathcal{A}(r)$ is a differentiable function, there exists a region with, say, $r = r_e < r_0$, where $\mathcal{A}(r_e) > 0 \wedge \mathcal{A}_{,r}(r_e) > 0$. Then, from Eq. (128)

$$\tilde{\mu} + 3\tilde{p}|_{r=r_e} > 0 \Rightarrow \tilde{\mu} + \tilde{p}|_{r=r_e} > 0. \tag{142}$$

However, substituting this result in Eq. (127) we find $(\delta^2)_{,r}(r_e) > 0$, which contradicts the assumption that the spin density is a monotonically decreasing function.

(2) The case $\mathcal{A}(r_0) > 0$

For the case when $\mathcal{A}(r_0) > 0$, we can simply repeat the proof in the previous subsubsection and conclude in the same way that the assumptions are violated in a region. We just remark that the point with radial coordinate $r = r_e$, in the proof, can always be chosen such that $r_e < r_0$ since, for whatever the value of $\mathcal{A}(r_0) > 0$, there is a point where $0 < \mathcal{A}(r < r_0) < \mathcal{A}(r_0)$.

(3) The case $\mathcal{A}(r_0) = 0$

In this the case when $\mathcal{A}(r_0) = 0$ we have, from Eq. (133), that

$$M(r_0) = 0. \tag{143}$$

From this we have three possibilities:

(a) $\tilde{\mu}(r) = 0$, for $r \in [0, r_a]$;
(b) $\tilde{\mu}(r) < 0$, for $r \in]0, r_a]$;
(c) $\tilde{\mu}(r) > 0$, for $r \in]0, r_a]$;

for some $r_a > 0$.

Let us consider each case individually.

(a) In the case when $\tilde{\mu}(r) = 0$, for $r \in [0, r_a]$ we have from Eq. (133) that $\mathcal{A}(r) < 0$. However, using this result in (127), we see that it implies that the spin density is an increasing function of r, violating the hypothesis.

(b) In the case when the corrected energy density is such that $\tilde{\mu}(r) < 0$, for $r \in]0, r_a]$, from (132) we have that the mass function is negative, in this region. From Eq. (133) we than conclude that $\mathcal{A}(r) < 0$, $r \in]0, r_a]$. However, going back to Eq. (127), we find that the spin density is an increasing function of r, violating the hypothesis.

(c) Finally, consider the case when $\tilde{\mu}(r) > 0$, for $r \in]0, r_a]$. From (132), this implies that the mass function is positive in this region. Since Eq. (143) must be verified, there must be a region where $\tilde{\mu}(r) < 0$. We can then repeat the arguments of the case $\mathcal{A}(r_0) < 0$, which lead to the conclusion that the hypothesis would be violated in some region inside the star.

Gathering the previous results we have proven the result in Proposition 3.

We end this section by remarking that if instead of imposing $\tilde{p} = -\delta^2$ we only imposed that $p < \delta^2$, that is, the thermodynamical pressure is always smaller than correction due to the spin density, then all the previous results are valid if $\tilde{p}_{,r} \geq 0$. Notice, however, that in this scenario, this condition simply measures the gradient of the quantity $8\pi p - \delta^2$.

C. Reconstructing exact solutions

As in the case of the theory of general relativity, when torsion is present it is possible to generate exact solutions via reconstruction algorithms [21,22]. The idea is to assign a given metric tensor and deduce the corresponding behavior of the energy density, pressure, and spin density.

Analyzing Eqs. (60) and (61) and using Eqs. (64) and (66) shows that different from the case of anisotropic compact objects in general relativity [22], the structure equations cannot be solved for the spin density. This implies that the reconstruction algorithm can only be used if an additional relation is provided, either relating the spin density to the other matter variables or an equation of state for matter.

In the following we will show some applications of these algorithms which return some interesting solutions from a physical point of view.

1. Connecting the spin density to the energy density: "Buchdhal stars"

A natural additional relation is to have the spin density be proportional to the energy density of the Weissenhoff fluid. In this case, however, the junction conditions that we have seen in Sec. V A pose the problem to have both the energy density and the pressure to be zero at the boundary. A class of solutions which are devised to have exactly this property was given by Buchdhal [49]. We will now reconstruct this solution in the case of Eqs. (82)–(86).

Consider a spherically symmetric space-time characterized by the line element

$$ds^2 = -A(w)dt^2 + B(w)dw^2 + C(w)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{144}$$

where

$$\eta(w) = \frac{(a-1)\sin(Rw)}{Rw}, \qquad A(w) = \frac{a(1+a-\eta)}{1+a+\eta},$$
$$B(w) = \frac{(1+a+\eta)}{a(1+a-\eta)} \qquad C(w) = \frac{w^2(1+a+\eta)^2}{4a^2}, \tag{145}$$

and w is connected to ρ by the relation

$$e^{\rho} = \frac{w^2}{4a^2}(1+a+\eta)^2. \tag{146}$$

Notice that the circumferential radius, r, vanishes when $w = 0$.

From Eqs. (60), (61), (83), and (84), assuming $\mathbb{W} = \Delta$ and $\Delta^2 = \gamma\mathcal{M}$, we find

$$\mathcal{M} = \frac{2\mathcal{K}_{,\rho} + 4\mathcal{K}^2 - \mathcal{K}}{4(1-\gamma)\mathcal{K}}, \tag{147}$$

$$\mathcal{P} = Y - \mathcal{K} + \frac{1}{4} - \frac{2\mathcal{K}_{,\rho} + 4\mathcal{K}^2 - \mathcal{K}}{4(1-\gamma)\mathcal{K}}, \tag{148}$$

$$0 = (2Y+1)\mathcal{K}_{,\rho} - 4\mathcal{K}^2 - \mathcal{K}[4Y_{,\rho} + 4(Y-1)Y - 1]. \tag{149}$$

The form of Y and $\mathcal{K}$ that satisfies the constraint (149) can be found directly from their definition in a general coordinate system (see Refs. [21,22])

$$Y = \frac{1}{2}\frac{CA_{,w}}{AC_{,w}} = \frac{(1+a)w\eta_{,w}}{2(\eta - a - 1)(1+a+\eta+w\eta_{,w})},$$
$$\mathcal{K} = \frac{BC}{(C_{,w})^2} = \frac{a(1+a+\eta)}{(1+a-\eta)(1+a+\eta+w\eta_{,w})^2}. \tag{150}$$

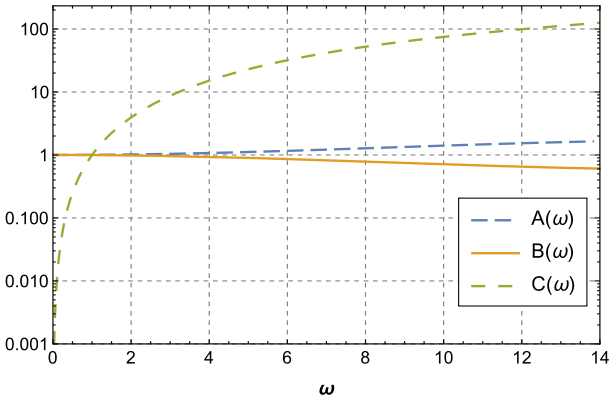
Using Eqs. (147), (148) and (150), the energy density and the pressure are then given by

$$\mu = \frac{aR^2\eta(3\eta - 2 - 2a)}{8\pi(\gamma-1)(1+a+\eta)^2},$$
$$p = \frac{aR^2\eta[2\gamma(2\eta - a - 1) - \eta]}{8\pi(\gamma-1)(1+a+\eta)^2}. \tag{151}$$

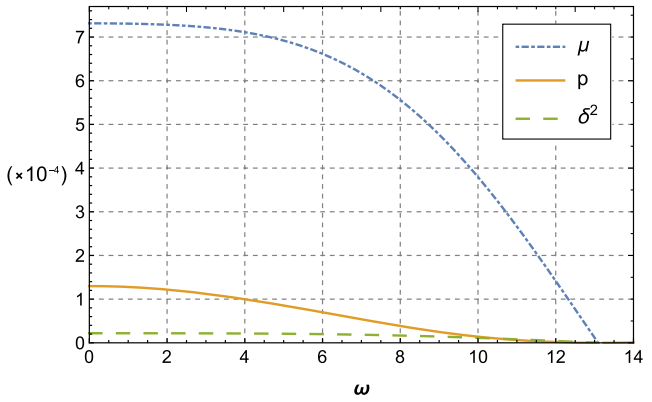
As said, this family of solutions has, by construction, the property that the pressure, energy, and spin densities all vanish at a particular hypersurface. In Figs. 1–3 we present the behavior of these quantities for a few combinations of the parameters, showing that the values of the parameters a and γ have a direct impact in the profile of the densities, whereas, the parameter R defines the value when the matter variables go to zero. Moreover, from the plots it is clear that the presence of spin markedly changes the type of behavior the matter may have. In particular, for certain values of the parameters a and γ the functions μ, p, or δ might not be monotonically decreasing functions of the coordinate w.

2. Connecting the spin density to the pressure

Another option that reduces the number of conditions related to the junction is to associate the spin density to the pressure. This choice, which at first might appear unnatural, corresponds to the case in which the spin depends on the equation of state. We can imagine that particles with spin will create different structures not unlike the ones that

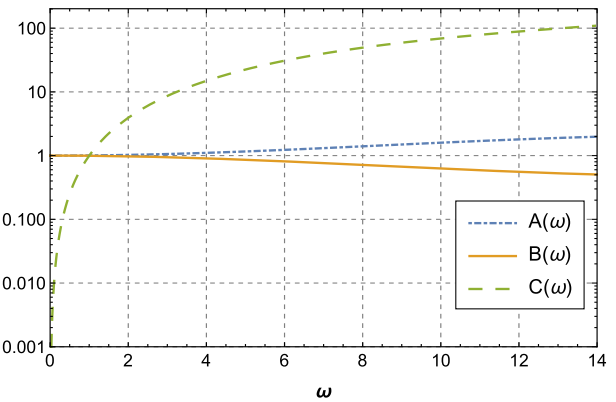


(a) Coefficients of the metric in Eqs. (144) and (145).

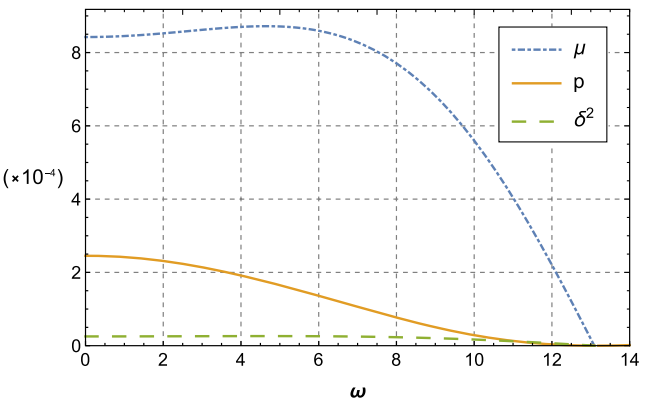


(b) Thermodynamic quantities in Eqs. (151) and the spin density.

FIG. 1. Plots of the behavior of the metric components (a) and matter variables (b) associated with the solution in Eqs. (144) and (145) for $a = 1.6$, $\gamma = 0.03/(8\pi)$, and $R = 0.24$.



(a) Coefficients of the metric in Eqs. (144) and (145).



(b) Thermodynamic quantities in Eqs. (151) and the spin density.

FIG. 2. Plots of the behavior of the metric components (a) and matter variables (b) associated with the solution in Eqs. (144) and (145) for $a = 1.9$, $\gamma = 0.03/(8\pi)$, and $R = 0.24$.

characterize the crystalline phases of water ice (see, e.g., Ref. [50]). Our ansatz refers to this kind of effect.

The reconstruction equations in this case, setting $\Delta^2 = \gamma\mathcal{P}$, read

$$\mathcal{M} = \frac{4\mathcal{K}^2 + 2\mathcal{K}_{,\rho} - \mathcal{K}}{4(1-\gamma)\mathcal{K}} - \frac{\gamma[\mathcal{K}_{,\rho} + \mathcal{K}(4\mathcal{K} - 2Y - 1)]}{2(1-\gamma)\mathcal{K}}, \tag{152}$$

$$\mathcal{P} = \frac{1 - 4\mathcal{K} + 4Y}{4(1-\gamma)}, \tag{153}$$

$$0 = (2Y+1)\mathcal{K}_{,\rho} - 4\mathcal{K}^2 - \mathcal{K}[4Y_{,\rho} + 4(Y-1)Y - 1]. \tag{154}$$

Let us now consider a metric in which the (0, 0) coefficient, A, is given by

$$A = A_0(a + br_0^2 e^{\rho})^2, \tag{155}$$

where a, b and r_0 are arbitrary constants. From the definition of Y one obtains

$$Y = \frac{1}{2}\frac{A_{,\rho}}{A} = \frac{br_0^2 e^{\rho}}{a + br_0^2 e^{\rho}}, \tag{156}$$

and from Eq. (154) it follows that

$$\mathcal{K} = \frac{(a + 3br_0^2 e^{\rho})^{2/3}}{\mathcal{K}_0 e^{\rho} + 4(a + 3br_0^2 e^{\rho})^{2/3}}, \tag{157}$$

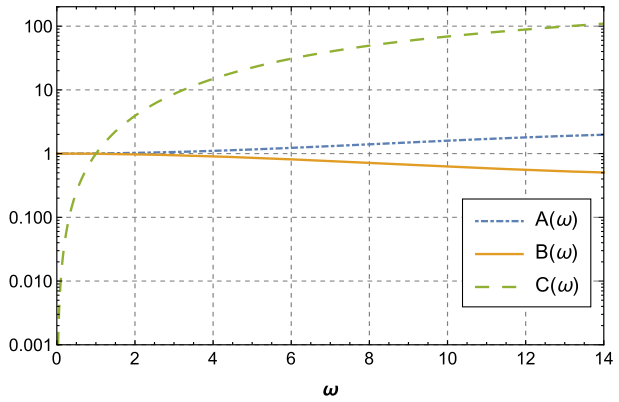
where $\mathcal{K}_0$ is an integration constant. In terms of the area radius r, this result corresponds to the line element

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{158}$$

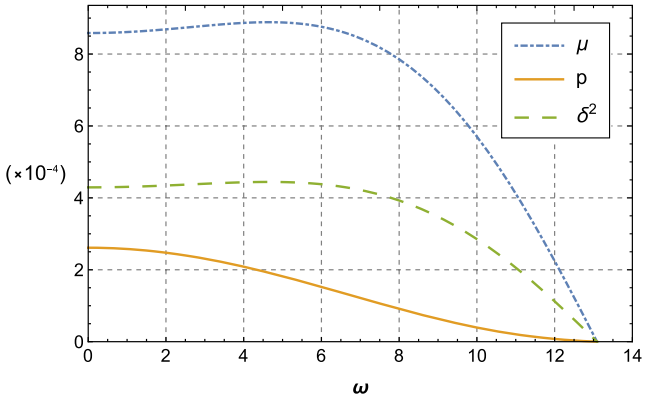
with

$$A(r) = A_0(a + br^2)^2,$$
$$B(r) = \left(1 + \frac{cr^2}{(a + 3br^2)^{2/3}}\right)^{-1}. \tag{159}$$

The energy density and the pressure are given by



(a) Coefficients of the metric in Eqs. (144) and (145).



(b) Thermodynamic quantities in Eqs. (151) and the spin density.

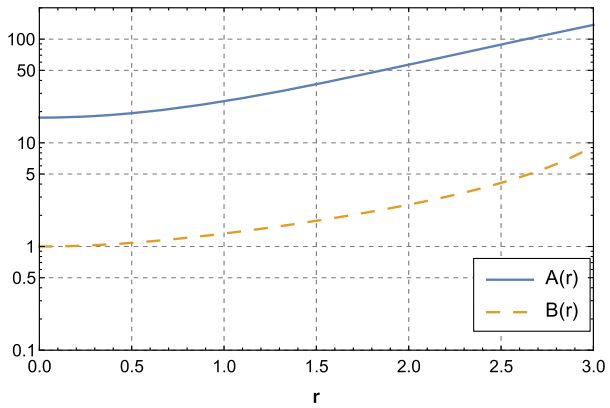
FIG. 3. Plots of the behavior of the metric components (a) and matter variables (b) associated with the solution in Eqs. (144) and (145) for $a = 1.9$, $\gamma = 0.5/(8\pi)$, and $R = 0.24$.

$$\mu = \frac{b^2r^2[5c(1-4\gamma)r^2 - 12\gamma(a+3br^2)^{2/3}]}{8\pi(\gamma-1)(a+br^2)(a+3br^2)^{5/3}} - \frac{4ab[\gamma(a+3br^2)^{2/3} + 2c(2\gamma-1)r^2]}{8\pi(\gamma-1)(a+br^2)(a+3br^2)^{5/3}} + \frac{a^2c(3-4\gamma)}{8\pi(\gamma-1)(a+br^2)(a+3br^2)^{5/3}},$$
$$p = \frac{4b(a+3br^2)^{2/3} + ac + 5bcr^2}{8\pi(1-\gamma)(a+br^2)(a+3br^2)^{2/3}}. \quad (160)$$

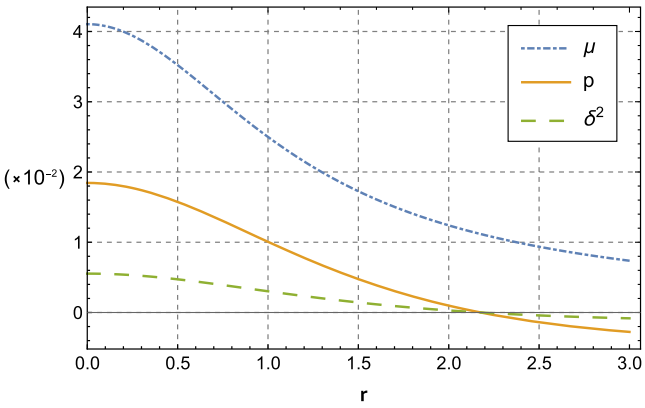
We give in Fig. 4 the behavior of this solution for specific values of the parameters, showing the existence of a hypersurface where both p and δ^2 vanish, so that we can smoothly match such solution with a vacuum exterior space-time.

Another example, based on the same assumptions, can be given considering

$$A = A_0(a + \sqrt{c - be^{\rho}})^2, \quad (161)$$



(a) Coefficients of the metric in Eqs. (158) and (159).



(b) The thermodynamic quantities in Eqs. (160).

FIG. 4. Plots of the behavior of the metric components (a) and matter variables (b) associated with the solution in Eqs. (158)–(160) in the case $a=5$, $b = 1$, $c = -1$, $\gamma = 0.3/(8\pi)$, and $A_0 = 0.7$.

which corresponds to

$$Y = -\frac{be^{\rho}}{2\sqrt{c - be^{\rho}}(a + \sqrt{c - be^{\rho}})}, \quad (162)$$

Eq. (154) then gives

$$\mathcal{K} = \frac{c\psi(a\sqrt{c-be^{\rho}} - 2be^{\rho} + c)}{(c-be^{\rho})[4\psi(a\sqrt{c-be^{\rho}} - 2be^{\rho} + c) - bde^{\rho}]}, \quad (163)$$

with

$$\psi = \left(\frac{\sqrt{a^2+8c} + a + 4\sqrt{c-be^{\rho}}}{\sqrt{a^2+8c} - a - 4\sqrt{c-be^{\rho}}}\right)^{\frac{a}{\sqrt{a^2+8c}}}. \quad (164)$$

Using the area radius r, we find the following solution for the metric (158):

$$A = A_0[a + y(r)]^2,$$
$$B = \frac{4c[ay(r) + 2y(r)^2 - c]}{y(r)^2[4ay(r) + 8y(r)^2 + d\psi(r)(y(r)^2 - c) - 4c]}, \tag{165}$$

where

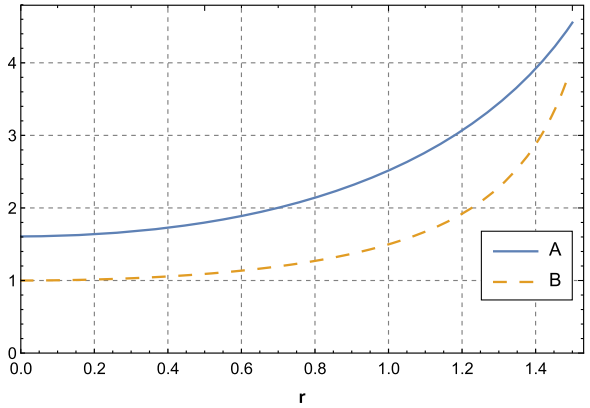
$$y(r) = \sqrt{c - \frac{br^2}{r_0^2}},$$
$$\psi(r) = \left(\frac{\sqrt{a^2+8c} + a + 4y(r)}{\sqrt{a^2+8c} - a - 4y(r)}\right)^{\frac{a}{\sqrt{a^2+8c}}}, \tag{166}$$

with the following expressions for the energy density and pressure of the fluid:

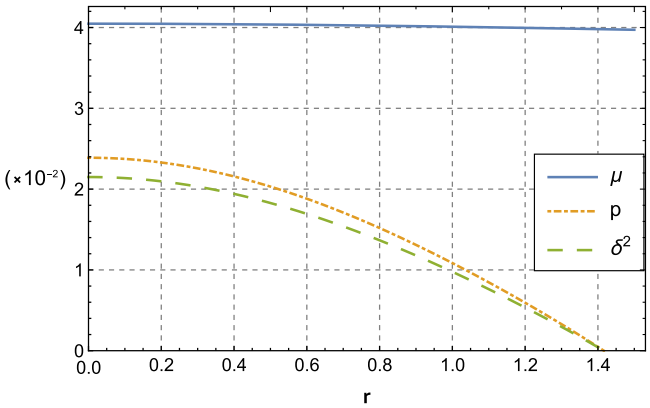
$$\mu = \frac{bd\gamma[6y^5 + 7ay^4 + 2y^3(a^2 - 3c) - 4acy^2]\psi}{16\pi c(\gamma-1)r_0^2(a+y)[c - y(a+2y)]^2} + \frac{bd\gamma c^2(2y+a)\psi}{16\pi c(\gamma-1)r_0^2(a+y)[c - y(a+2y)]^2} - \frac{bd[6y^4 + 3ay^3 - 5cy^2 + 2c^2]\psi}{32\pi c(\gamma-1)r_0^2[c - y(a+2y)]^2} + \frac{b[2\gamma(2a+3y) - 3(a+y)]}{8\pi c r_0^2(\gamma-1)(a+y)},$$
$$p = \frac{bdy(ay - 2c + 3y^2)\psi}{32\pi c(\gamma-1)r_0^2(a+y)(ay - c + 2y^2)} + \frac{b(a+3y)}{8\pi c r_0^2(\gamma-1)(a+y)}. \tag{167}$$

In Fig. 5 we show the behavior of this solution for specific values of the parameters. Notice that this solution also admits the existence of a common hypersurface where both p and δ vanish.

Before finishing this section we remark that, as shown by Figs. 1–5, in all considered cases it is possible to find values of the parameters for which all the thermodynamical quantities and spin density are positive, hence, all the classical energy conditions are valid.



(a) The coefficients of the metric in Eq. (158) and (165).



(b) The thermodynamic quantities in Eqs. (167).

FIG. 5. Plots of the behavior of the metric components (a) and matter variables (b) associated with the solution in Eqs. (158), (165), and (160) in the case $A_0 = 1$, $a = -3$, $b = 1$, $c = 3$, $d = 0.03$, $r_0 = 1$, and $\gamma = 0.9/(8\pi)$.

VII. GENERATING THEOREMS

As discussed in Ref. [21], the form of the structure equations (82)–(90) is especially useful to find algorithms for generating new solutions from previous known ones.

Consider a solution for the structure equations (82)–(90) characterized by the functions

$$\{\mathcal{P}_0, \mathcal{M}_0, \Delta_0, K_0, \mathcal{E}_0, \mathbb{X}_0, \mathbb{Y}_0, (\mathbb{B}_1)_0, (\mathbb{B}_2)_0, (\mathbb{D}_1)_0, (\mathbb{D}_2)_0\}. \tag{168}$$

Given the quantities

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1, \qquad \mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1,$$
$$\Delta^2 = {\Delta_0}^2 + {\Delta_1}^2, \qquad \mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1, \tag{169}$$

where $\{\mathcal{P}_1, \mathcal{M}_1, \Delta_1, \mathcal{K}_1\}$ are sufficiently smooth arbitrary functions, let us search conditions on the deforming functions so that the set $\{\mathcal{P}, \mathcal{M}, \Delta, \mathcal{K}, \mathcal{E}, \mathbb{X}, \mathbb{Y}, \mathbb{B}_1, \mathbb{B}_2, \mathbb{D}_1, \mathbb{D}_2\}$ is a solution of the structure equations.

Substituting Eq. (169) in Eq. (83) we find

$$\partial_\rho \mathcal{K}_1 + 2{\mathcal{K}_1}^2 + 2\mathcal{K}_0({\Delta_1}^2 - \mathcal{M}_1) - \mathcal{K}_1\left(2\mathcal{M}_0 - 4\mathcal{K}_0 - 2{\Delta_0}^2 + 2{\Delta_1}^2 - 2\mathcal{M}_1 + \frac{1}{2}\right) = 0. \tag{170}$$

This equation has the form of a Riccati differential equation to which, in general, there are no known closed form solutions. We can, nonetheless, consider particular cases so that the previous equation reduces to a Bernoulli differential equation, where general closed form solutions exist.

A. Case 1

Let us first consider that

$$\mathcal{M}_1 = \Delta_1^2. \tag{171}$$

In this case, Eq. (170) can be readily integrated for $\mathcal{K}_1$, such that

$$\mathcal{K}_1 = 0 \vee \mathcal{K}_1(\rho) = \frac{\mathrm{Exp}[\int_{\rho_0}^{\rho} \Lambda dx]}{\mathcal{K}_\star + 2\int_{\rho_0}^{\rho} \mathrm{Exp}[\int_{y_0}^{y} \Lambda dx] dy}, \tag{172}$$

where $\mathcal{K}_\star$ is an integration constant and

$$\Lambda = 2\mathcal{M}_0 - 2{\Delta_0}^2 - 4\mathcal{K}_0 + \frac{1}{2}. \tag{173}$$

Using Eq. (171) in Eq. (82) we find

$$\partial_\rho \mathcal{P}_1 + \mathcal{P}_1\left(3\mathcal{K}_0 + 3\mathcal{K}_1 - \mathcal{M}_0 + 2\mathcal{P}_0 - {\Delta_0}^2 - \frac{7}{4}\right) + {\mathcal{P}_1}^2 + \frac{1}{4}\mathcal{F} = 0, \tag{174}$$

with

$$\begin{aligned}\mathcal{F} = {} & 4\mathcal{M}_0({\Delta_1}^2 + \mathcal{K}_1) - 8(\mathcal{P}_0 + \mathcal{P}_1){\Delta_1}^2 + 12\mathcal{P}_0\mathcal{K}_1 \\ & + 4{\Delta_0}^2{\Delta_1}^2 - 4\partial_\rho{\Delta_1}^2 + 4{\Delta_1}^4 + 7{\Delta_1}^2 \\ & - 12{\Delta_1}^2\mathcal{K}_0 - 16{\Delta_0}^2\mathcal{K}_1 - 12{\Delta_1}^2\mathcal{K}_1.\end{aligned} \tag{175}$$

For Eq. (174) to reduce to a Bernoulli-like differential equation we will require $\mathcal{F}(\rho) = 0$, that is,

$$\partial_\rho{\Delta_1}^2 - {\Delta_1}^4 - \mathcal{K}_1(\mathcal{M}_0 + 3\mathcal{P}_0 - 4{\Delta_0}^2) - {\Delta_1}^2\left(\frac{7}{4} + \mathcal{M}_0 - 3\mathcal{K}_0 - 3\mathcal{K}_1 - 2\mathcal{P}_0 - 2\mathcal{P}_1 + {\Delta_0}^2\right) = 0, \tag{176}$$

which, by setting $\mathcal{K}_1 = 0$ or $\mathcal{M}_0 + 3\mathcal{P}_0 - 4{\Delta_0}^2 = 0$, can be formally solved, such that

$$\Delta_1^2(\rho) = 0 \vee {\Delta_1}^2(\rho) = \frac{\mathrm{Exp}[\int_{\rho_0}^{\rho} \Phi dx]}{\Delta_\star - \int_{\rho_0}^{\rho} \mathrm{Exp}[\int_{y_0}^{y} \Phi dx] dy}, \tag{177}$$

where $\Delta_\star$ is an integration constant and

$$\Phi = \mathcal{M}_0 - 3\mathcal{K}_0 - 3\mathcal{K}_1 - 2\mathcal{P}_0 - 2\mathcal{P}_1 + {\Delta_0}^2 + \frac{7}{4}. \tag{178}$$

Consequently, from Eq. (174), we find

$$\mathcal{P}_1(\rho) = 0 \vee \mathcal{P}_1(\rho) = \frac{\mathrm{Exp}[\int_{\rho_0}^{\rho} \Gamma dx]}{\mathcal{P}_\star + \int_{\rho_0}^{\rho} \mathrm{Exp}[\int_{y_0}^{y} \Gamma dx] dy}, \tag{179}$$

with

$$\Gamma = \mathcal{M}_0 - 2\mathcal{P}_0 + {\Delta_0}^2 - 3\mathcal{K}_0 - 3\mathcal{K}_1 + \frac{7}{4}, \tag{180}$$

and $\mathcal{P}_\star$ is an integration constant.

Before we conclude this section, we should stress that Eqs. (172), (177), and (179) present two possible solutions for the considered functions and all combinations of those solutions verify the structure equations with $\mathcal{M}_1 = \Delta_1^2$, leading, *a priori*, to distinct solutions.

B. Case 2

Another possibility to solve Eq. (170) is the case when

$$(\mathcal{K}_0 + \mathcal{K}_1)(2\Delta_1^2 - 2\mathcal{M}_1) = G(\rho)\mathcal{K}_1 + Q(\rho){\mathcal{K}_1}^2, \tag{181}$$

where $G(\rho)$ and $Q(\rho)$ are sufficiently smooth, arbitrary functions. Setting

$$2\Delta_1^2 - 2\mathcal{M}_1 = \mathcal{K}_1 Q(\rho), \qquad G(\rho) = \mathcal{K}_0 Q(\rho), \tag{182}$$

and substituting Eqs. (181) and (182) in Eq. (170) we find

$$\partial_\rho \mathcal{K}_1 + \mathcal{K}_1\left[2{\Delta_0}^2 - 2\mathcal{M}_0 + 4\mathcal{K}_0 + \mathcal{K}_0 Q(\rho) - \frac{1}{2}\right] + [2 + Q(\rho)]{\mathcal{K}_1}^2 = 0, \tag{183}$$

which, providing an expression for $Q(\rho)$ can be solved for $\mathcal{K}_1$, or vice-versa.

Now, to solve the remaining equations for the functions Q, $\mathcal{P}_1$, and Δ_1, we will consider that the original solution is such that $\mathcal{M}_0 = \mathcal{P}_0 = \Delta_0 = 0$, that is, the original space-time is described by a vacuum solution of the field equations. From Eq. (82) we then find

$$\partial_\rho \mathcal{P}_1 + {\mathcal{P}_1}^2 + \mathcal{P}_1\left[3\mathcal{K}_0 + 3\mathcal{K}_1 - \frac{7}{4}\right] + \mathcal{J}(\rho) = 0, \tag{184}$$

where

$$\begin{aligned}\mathcal{J}(\rho) = {} & -\partial_\rho{\Delta_1}^2 + {\Delta_1}^4 + \frac{1}{2}Q(\rho)\mathcal{K}_1\left[\mathcal{P}_1 - \mathcal{K}_0 - \mathcal{K}_1 + \frac{1}{4}\right] \\ & + {\Delta_1}^2\left(-2\mathcal{P}_1 - \frac{1}{2}Q(x)\mathcal{K}_1 - 3\mathcal{K}_0 - 3\mathcal{K}_1 + \frac{7}{4}\right).\end{aligned} \tag{185}$$

As before, to reduce Eq. (184) to a Bernoulli differential equation we will impose $\mathcal{J}(\rho) = 0$. Unfortunately, this equation itself is also not possible to solve in general since it has the form of a Riccati differential equation. Let us then further impose the last term in the first line of the previous equation to be zero. Solving for $\mathcal{K}_1$, we have

$$\mathcal{K}_1 = \mathcal{P}_1 - \mathcal{K}_0 + \frac{1}{4}, \tag{186}$$

where we have ignored the solutions where $Q = 0 \vee \mathcal{K}_1 = 0$ since they lead to a particular case of subsection VII A.

Considering the constraint that originally we have a vacuum solution, substituting Eq. (186) in Eq. (183) we find,

$$Q = -\frac{8(2\partial_\rho \mathcal{P}_1 + 4{\mathcal{P}_1}^2 + \mathcal{P}_1)}{(4\mathcal{P}_1 + 1)(4\mathcal{P}_1 - 4\mathcal{K}_0 + 1)}. \tag{187}$$

Gathering the previous results we find the following expressions for the remaining perturbations

$$\begin{aligned}
\mathcal{P}_1 &= \frac{e^\rho}{\mathcal{P}_\star + 4e^\rho}, \\
\Delta_1^2 &= 0 \vee {\Delta_1}^2 = \frac{\mathrm{Exp}[-\int_{\rho_0}^{\rho} \Phi dx]}{\Delta_\star - \int_{\rho_0}^{\rho} \mathrm{Exp}[-\int_{y_0}^{y} \Phi dx] dy}, \\
\mathcal{M}_1 &= \Delta_1^2 + \frac{2\partial_\rho \mathcal{P}_1 + 4{\mathcal{P}_1}^2 + \mathcal{P}_1}{4\mathcal{P}_1 + 1},
\end{aligned} \tag{188}$$

where $\mathcal{P}_\star$ and $\Delta_\star$ are integrating constants and

$$\Phi = 2\mathcal{P}_1 + \frac{1}{2}Q(x)\mathcal{K}_1 + 3\mathcal{K}_0 + 3\mathcal{K}_1 - \frac{7}{4}. \tag{189}$$

Notice that we did not consider the case when $\mathcal{P}_1 = 0$ since it would lead to the case when $Q(\rho) = 0$, which, as mentioned before, represents a particular case of Sec. VII A. Let us also remark that, for solutions generated using the above equations, the functional form of the pressure, $\mathcal{P} \equiv \mathcal{P}_1$, is independent of the original solution and completely determined up to a constant. Moreover, notice that the pressure—in such solutions—is only null when $\rho \to -\infty$.

C. Case 3

Let us now consider the deformations in Eq. (169) with the extra constraint

$$\mathbb{Y} = \mathbb{Y}_0, \tag{190}$$

that is, we will impose that the function $\mathbb{Y}$ is unchanged between the original and the perturbed space-time. This is a generalization of the deformations considered in Refs. [21,23], for non-null-spin density. Substituting Eqs. (169) and (190) in Eq. (84) we find that

$$\mathcal{P}_1 = \Delta_1^2 - \mathcal{K}_1. \tag{191}$$

Using Eqs. (169), (190), and (191) in Eqs. (83) and (82) we find the following relations for $\mathcal{M}_1$ and $\mathcal{K}_1$:

$$\mathcal{M}_1 = \frac{\mathcal{K}_1(2\mathbb{Y}_0 + 3)}{2\mathbb{Y}_0 + 1} + \Delta_1^2, \tag{192}$$

$$\mathcal{K}_1 = \frac{\mathrm{Exp}[-\int_{\rho_0}^{\rho} \Phi dx]}{\mathcal{K}_\star - \int_{\rho_0}^{\rho} \frac{4}{2\mathbb{Y}_0+1} \mathrm{Exp}[-\int_{y_0}^{y} \Phi dx] dy} - \mathcal{K}_0, \tag{193}$$

where

$$\Phi = \frac{\mathcal{K}_0(6 + 4\mathbb{Y}_0)}{2\mathbb{Y}_0 + 1} + 2\Delta_0^2 - 2\mathcal{M}_0 - \frac{1}{2}, \tag{194}$$

and $\mathcal{K}_\star$ is an integration constant. Equations (191)–(194) generalize the results in Ref. [21] in the presence of a non-null-spin density.[6]

Contrary to the previous cases, Eqs. (191)–(194) do not completely determine the system since the function Δ_1^2 is unconstrained. Notice that $\mathcal{K} \equiv \mathcal{K}_1$ is determined uniquely by the unperturbed solution and Δ_1^2 will only affect $\mathcal{M}_1$ and $\mathcal{P}_1$. Therefore, provided an unperturbed solution, the metric of the perturbed space-time is completely determined by Eqs. (190) and (193). As already pointed out, Δ_1^2 will not only affect the energy density and the pressure of the fluid but also the Weyl tensor components. Thus, although the metric of the space-time is independent of Δ_1^2, the geometry is profoundly influenced by the presence of spin.

VIII. CONCLUSIONS

In this paper we have used the $1+1+2$ formalism to derive the structure equations for LRSI and LRSII, stationary space-times with a Weyssenhof-like torsion field in the context the ECSK theory of gravity. The structure of the covariant equations show in detail how the spin interacts with the space-time via the torsion tensor. In particular, the presence of a torsion tensor field separates the magnetic part of the Weyl tensor in two distinct tensors, which behave differently. Even in the case of static LRSII space-times, the magnetic parts of the Weyl tensor do not vanish and some of its components depend on both the value and spatial derivative of the spin density. This suggests, in particular, that the effects of spin on the matter fluid, even in the regimes expected to be found in neutron stars, may not be negligible, as it was previously thought (see, e.g., [13]), even in the case in which the contribution to the spin is very small.

The $1+1+2$ equations were then used to derive the covariant Tolman-Oppenheimer-Volkoff equations for ECSK gravity for LRSI and LRSII space-times. In the case of LRSII space-times, the equations are structurally very similar to the ones of GR. Indeed, this similarity allows us to recast them into the same form of the GR TOV

[6]Notice that there is an error in the expression for $\mathcal{M}_1$ in Ref. [21]. The correct expression is found by setting $\Delta_0^2 = \Delta_1^2 = 0$ in Eq. (192).

equations via a redefinition of the matter variables and the electric part of the Weyl tensor. As a consequence we found that at the level of the metric it is possible to map static, locally rotationally symmetric solutions of class II from the Einstein-Cartan theory to the ones of the theory of general relativity. Moreover, due to this mapping and the rescale in the matter variables, some GR solutions which are physically irrelevant become, in the context of ECSK gravity, interesting.

When we examine in detail the physical properties of physically relevant solutions, the differences between the Einstein-Cartan theory and GR become once more evident. This is particularly true looking at junction conditions. We found that the requirement that all of the components of the Riemann tensor have finite discontinuities across the separation surface $\mathcal{N}$ leads to additional constraints with respect to the tornsionless case. This is especially evident when looking at the structure equations for stationary LRSI and LRSII space-times sourced by a Weyssenhoff fluid. In these equations the magnetic parts of the Weyl tensor depend explicitly on the derivatives of the torsion tensor and the classical Israel junction conditions of GR do not guarantee these terms to be finite across $\mathcal{N}$. As consequence of the generalized junction conditions, in the considered setup, observers at the interior and exterior space-times must measure the same value for the spin density at $\mathcal{N}$, turning the task of finding physically relevant solutions even more daunting.

Using the full set of structure equations and boundary conditions provided by the junction formalism, we were able to study various properties of possible solutions. We started by analyzing how the presence of spin changes the Buchdahl limit for the maximum compactness of a star. We concluded that the spin-geometry coupling allows stars with a given circumferential radius to hold more matter than the corresponding GR ones. Next we considered the case of static, spherically symmetric compact objects entirely held by the matter spin, smoothly matched to a vacuum exterior. This scenario was expected to represent a good model for cold neutron stars, where the thermodynamical pressure is negligible when compared to the spin density. We found, surprisingly, that such objects cannot be simultaneously static, spherically symmetric and smoothly matched to a vacuum exterior. This is a strong result and it is necessary to discuss in detail the hypothesis that led to such conclusion. More specifically our conclusion may not be valid if:

(i) the spin density is not a monotonically decreasing function of the radial coordinate inside the star;
(ii) we consider a nonvacuum exterior space-time;
(iii) we replace the uncharged Weyssenhoff fluid model;
(iv) we allow the presence of a thin shell.

The first possibility might lead to a total energy density and a pressure density which is not monotonically decreasing. While this is not a strong enough reason to discard this case, we expect these oscillation to make the solution unstable under small perturbations. The second case suggests that if ECSK theory had a nontrivial vacuum (vortical) solution, one could smoothly match the interior to it, bypassing the requirement of the spin density to vanish at a hypersurface. At present there is no evidence that such a solution might/should exist. Indeed the theory is expected to reduce to GR in vacuum. For what concerns hypothesis (iii), the Weyssenhoff fluid can be advocated to be a good model for the matter fluids that might constitute cold neutron stars. However, in this work we made the simplifying assumption that the fluid is electrically neutral. If, instead, a charged Weyssenhoff fluid model is considered, we expect that other effects will appear—such as anisotropic pressure—which may drastically change the behavior of the fluid. As for the last possibility, although a smooth junction with a vacuum exterior might represent a more reasonable scenario, it might be argued that neutron stars may have a well-defined surface; therefore, it is not completely unreasonable to consider the presence of a thin shell of matter at the matching surface.

On top of the zero pressure solution considered above, we have also considered solutions in which pressure is nonzero. Using reconstruction algorithms, we have been able to obtain various classes of solutions for the interior of static, spherically symmetric compact objects that can be smoothly matched to a Schwarzschild exterior. One family of those solutions, which we dubbed Buchdahl stars, represent a very interesting scenario: they admit the existence of a common hypersurface where the pressure, spin density, and energy density all vanish. This model, studied for the first time by Buchdahl for gaseous stars in GR [49], represents the scenario where the fluid that composes a star will smoothly dissipate away from a denser core and transition to vacuum. These solutions also provided a key example for the effects that spin may have on the behavior of the fluid. Figures 1–3 clearly exemplify that even if the spin density is much smaller than the other matter variables, it allows for a much richer behavior for the fluid.

The natural question that emerges is about the stability of these solutions. Because of the nontrivial role of the magnetic part of the Weyl tensor, no standard "zeroth order" (and Newtonian based) criterion is necessarily valid in our case. In cases in which the spin density is decreasing, one very heuristic criterium of stability of our solutions is to guarantee (as we have done) that both energy density and pressure of the fluid are decreasing functions of the radial coordinate when the spin is small at least in a nonempty set of values of the parameters. However, a complete study of the stability of the solutions we have found requires a more careful study, which will be the topic of a series of future works.

Finally, as in the case of GR, in ECSK theory it is also possible to derive generating theorems. In this work we

have presented several algorithms to generate new exact solutions from previously known ones. We should stress here that the results we obtained followed from the simple idea of finding conditions so that the Riccati differential equations would reduce to Bernoulli equations. Although this scheme allowed us to find various generating algorithms, we make no claim that we have exhausted all possibilities for finding new ones. On this note, the integrability conditions for Riccati-type equations in Refs. [51–53] were also considered. However, these did not lead to useful results in the considered context.

ACKNOWLEDGMENTS

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APPENDIX: COVARIANTLY DEFINED QUANTITIES FOR THE DERIVATIVES OF THE TANGENT VECTORS

Using the definitions of the projector operators on the hypersurfaces V and W, let us show how the covariant derivatives of the tangent vector fields v and e can be uniquely decomposed in their components along u, e, and W.

1. Decomposition on the sheet W

Let us first consider the projected covariant derivatives of the tensors u and e on the sheet. These can be uniquely decomposed as

$$\delta_\alpha u_\beta \equiv N_\alpha{}^\sigma N_\beta{}^\gamma \nabla_\sigma u_\gamma = \frac{1}{2} N_{\alpha\beta}\tilde{\theta} + \Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega, \tag{A1}$$

where

$$\tilde{\theta} = \delta_\alpha u^\alpha, \qquad \Sigma_{\alpha\beta} = \delta_{\{\alpha} u_{\beta\}}, \qquad \Omega = \frac{1}{2}\varepsilon^{\sigma\gamma}\delta_\sigma u_\gamma, \tag{A2}$$

and

$$\delta_\alpha e_\beta = \frac{1}{2} N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi, \tag{A3}$$

with

$$\phi = \delta_\alpha e^\alpha, \qquad \zeta_{\alpha\beta} = \delta_{\{\alpha} e_{\beta\}}, \qquad \xi = \frac{1}{2}\varepsilon^{\sigma\gamma}\delta_\sigma e_\gamma, \tag{A4}$$

where the curly brackets represent the projected symmetric part without the trace of a tensor in W, that is, for a tensor $\psi_{\alpha\beta}$,

$$\psi_{\{\alpha\beta\}} = \left[N^\mu{}_{(\alpha}N_{\beta)}{}^\nu - \frac{N_{\alpha\beta}}{2}N^{\mu\nu}\right]\psi_{\mu\nu}. \tag{A5}$$

Using the 2-form volume $\varepsilon_{\alpha\beta}$ a completely antisymmetric tensor defined on the sheet, $\psi_{[\alpha\beta]}$, can be written as

$$\psi_{[\alpha\beta]} = \varepsilon_{\alpha\beta}\left(\frac{1}{2}\epsilon^{\gamma\sigma}\psi_{\gamma\sigma}\right). \tag{A6}$$

This property was used in Eq. (5).

2. Decomposition on V

The decomposition of the projected covariant derivatives of u^α onto V is given by

$$D_\alpha u_\beta = h_\alpha^\sigma h_\beta^\gamma \nabla_\sigma u_\gamma = \frac{1}{3}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \tag{A7}$$

with

$$\theta = h^{\alpha\beta}D_\alpha u_\beta, \tag{A8}$$

$$\sigma_{\alpha\beta} = D_{\langle\alpha}u_{\beta\rangle}, \tag{A9}$$

$$\omega_{\alpha\beta} = h^\sigma_{[\alpha}h_{\beta]}{}^\gamma D_\sigma u_\gamma, \tag{A10}$$

where we used the angular brackets to represent the projected symmetric part without trace of a tensor on V, that is, for a tensor, $\psi_{\alpha\beta}$,

$$\psi_{\langle\alpha\beta\rangle} = \left[h^\mu{}_{(\alpha}h_{\beta)}{}^\nu - \frac{h_{\alpha\beta}}{3}h^{\mu\nu}\right]\psi_{\mu\nu}. \tag{A11}$$

The scalar and tensor quantities in Eqs. (A8)–(A10) can themselves be further decomposed in their contributions exclusively on W and along e, such that

$$\theta = \tilde{\theta} + \bar{\theta}, \tag{A12}$$

where $\tilde{\theta}$ is defined in Eq. (A2) and

$$\bar{\theta} = -u_\beta(e^\alpha D_\alpha e^\beta) = -u_\beta\hat{e}^\beta; \tag{A13}$$

$$\sigma_{\alpha\beta} = \Sigma_{\alpha\beta} + 2\Sigma_{(\alpha}e_{\beta)} + \Sigma\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right), \tag{A14}$$

with

$$\Sigma_{\alpha\beta} = \sigma_{\{\alpha\beta\}}, \qquad \Sigma_\alpha = N^\gamma_\alpha e^\beta \sigma_{\gamma\beta}, \qquad \Sigma = e^\alpha e^\beta \sigma_{\alpha\beta} = -N^{\alpha\beta}\sigma_{\alpha\beta}, \tag{A15}$$

and

$$\omega_{\alpha\beta} = \varepsilon_{\alpha\beta}\Omega - \varepsilon_{\alpha\lambda}\omega^{\lambda}e_{\beta} + e_{\alpha}\varepsilon_{\beta\lambda}\omega^{\lambda}, \tag{A16}$$

where Ω is given in Eq. (A2) and

$$\omega^{\lambda} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}D_{\mu}u_{\nu}, \tag{A17}$$

which can be itself decomposed as

$$\omega^{\lambda} = \Omega e^{\lambda} + \Omega^{\lambda}, \quad \text{with} \quad \Omega^{\lambda} = N^{\lambda}_{\alpha}\omega^{\alpha} = \frac{1}{2}N^{\lambda}_{\alpha}\varepsilon^{\mu\nu\alpha}D_{\mu}u_{\nu}, \tag{A18}$$

therefore, equivalently,

$$\omega_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}(\Omega e^{\gamma} + \Omega^{\gamma}). \tag{A19}$$

The quantities θ, Σ, $\tilde{\theta}$, and $\bar{\theta}$ are not independent, in fact:

$$\bar{\theta} = \frac{1}{3}\theta + \Sigma, \tag{A20}$$

$$\tilde{\theta} = \frac{2}{3}\theta - \Sigma; \tag{A21}$$

as such, when setting up the $1+1+2$ formalism only two are chosen. The convention followed here uses the variables θ and Σ.

For the projected covariant derivative of the vector field e on V we have

$$D_{\alpha}e_{\beta} = h_{\alpha}{}^{\sigma}h_{\beta}{}^{\gamma}\nabla_{\sigma}e_{\gamma} = \delta_{\alpha}e_{\beta} + e_{\alpha}a_{\beta}, \tag{A22}$$

where $\delta_{\alpha}e_{\beta}$ is given by Eq. (A3) and

$$a_{\alpha} = e^{\mu}D_{\mu}e_{\alpha} = \hat{e}_{\alpha}. \tag{A23}$$

3. Decomposition on the full manifold

Finally, we can decompose the total covariant derivatives of u^{α} and e^{α}, such that

$$\nabla_{\alpha}u_{\beta} = -u_{\alpha}(\mathcal{A}e_{\beta} + \mathcal{A}_{\beta}) + D_{\alpha}u_{\beta}, \tag{A24}$$

with

$$\mathcal{A} = -u_{\gamma}u^{\mu}\nabla_{\mu}e^{\gamma} = -u_{\gamma}\dot{e}^{\gamma}, \qquad \mathcal{A}_{\alpha} = N_{\alpha\beta}\dot{u}^{\beta}, \tag{A25}$$

and

$$\nabla_{\alpha}e_{\beta} = D_{\alpha}e_{\beta} - u_{\alpha}\alpha_{\beta} - \mathcal{A}u_{\alpha}u_{\beta} + \left[\frac{1}{3}\theta + \Sigma\right]e_{\alpha}u_{\beta} + [\Sigma_{\alpha} - \varepsilon_{\alpha\sigma}\Omega^{\sigma}]u_{\beta}, \tag{A26}$$

where

$$\alpha_{\mu} = h^{\sigma}_{\mu}\dot{e}_{\sigma}. \tag{A27}$$

4. The actual physical kinematical variables

As discussed in Refs. [35–38], the presence of a generic torsion field will affect the definition of the kinematical quantities that characterize a congruence of curves, such that, θ, $\sigma_{\alpha\beta}$, and $\omega_{\alpha\beta}$, Eqs. (A8)–(A10), in general, do not represent the actual geometric—physical—expansion, shear, and vorticity of the timelike congruence to which u is tangent. These, however, are related with the actual kinematical quantities by

$$\theta_g = \theta + W^{\sigma}_{\sigma}, \quad \sigma_{g\alpha\beta} = \sigma_{\alpha\beta} + W_{\langle\alpha\beta\rangle}, \quad \omega_{g\alpha\beta} = \omega_{\alpha\beta} + W_{[\alpha\beta]}, \tag{A28}$$

where we have used the index g to represent the physical—geometric—kinematical quantities and the definition of angular brackets is given in Eq. (A11). In the same way the presence of the torsion field will modify the kinematical quantities ϕ, $\zeta_{\alpha\beta}$, and ξ, Eq. (A4). In particular we have the following relations:

$$\phi_g = \phi + 2S_{\gamma\mu\nu}e^{\gamma}N^{\mu\nu}, \qquad \zeta_{g\alpha\beta} = \zeta_{\alpha\beta} + 2S_{\gamma\mu\nu}e^{\gamma}N^{\mu}_{\langle\alpha|}N^{\nu}_{|\beta\rangle}, \qquad \xi_g = \xi + S_{\gamma\mu\nu}e^{\gamma}\varepsilon^{\mu\nu}. \tag{A29}$$

In the particular setup that we propose to study—Weyssenhoff-like torsion—the extra terms in the rhs of Eqs. (A28) and (A29) that depend explicitly of the torsion tensor will be null; therefore, in our case, the indicated quantities will correspond to the actual geometric kinematical quantities.

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