Two-vierbein gravity action from the gauge theory of the conformal group

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(Received 2 April 2019; published 7 October 2019)

We study the gravity action built from two gauge fields corresponding to the generators of the conformal group. Starting with the action from which one can obtain Einstein gravity and conformal gravity upon imposing suitable constraints, we keep two independent gauge fields and integrate out the field corresponding to the generator of Lorentz transformations. We identify the two gauge fields with two vierbeins and perturb them around anti–de Sitter space. This gives the linearized equations that differ from both Einstein gravity and conformal gravity linearized equations. We also study the linearized equations for one gauge field perturbed around the flat space and one around zero, and the case in which the gauge fields are proportional to each other.

DOI: 10.1103/PhysRevD.100.084012

I. INTRODUCTION

Conformal gravity was interpreted as a gauge theory of conformal group O(4,2) by Kaku *et al.* [1] in 1977. The motivation to study it was the fact that Einstein gravity has been viewed as a gauge theory of the de Sitter group O(3,2) [2], which upon contraction reduces to the Poincaré group. Squaring the curvatures of the de Sitter group, one obtains Einstein gravity [2], while the Poincaré group and the de Sitter group are subgroups of the conformal group O(4,2). It is natural to look at the square of the curvature of O(4,2). To achieve the invariance of a constructed action under proper conformal gauge transformations, the authors had to require that the gauge generator of the translations vanishes. The resulting action is invariant under conformal transformations, and it is a gauge theory of the conformal group. It is built out of three independent gauge fields. Upon integrating out the gauge fields, we are left with the remaining two. This situation where one encounters two different fields appears in bimetric gravity models, which contain two dynamical metrics. These models [3–5] originated from the de Rham–Gabadadze–Tolley (dRGT) massive gravity model [6–9]. It has been shown that other higher derivative theories, one of them being conformal gravity, can be rewritten and obtained from bimetric and partially massless bimetric theory [10]. This has further motivated a study of bimetric gravity [3], whose action takes the form [3]

$$S = M_g^2 \int d^4x \sqrt{-\det g} R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f} R^{(f)} + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n\left(\sqrt{g^{-1} f}\right). \quad (1)$$

$R^{(g)}$ and $R^{(f)}$ are Ricci scalars with respect to metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, M_g and M_f are two different Planck masses, and M_{eff} is an effective Planck mass. The e_n are elementary symmetric polynomials in eigenvalues of $\sqrt{g^{-1} f}$, and β_n are four combinations of the mass of the graviton, the cosmological constant, and the free parameters. The graviton mass and cosmological constants for $g_{\mu\nu}$ and $f_{\mu\nu}$ are among five free parameters of the theory. Four-dimensional spin-2 theories have recently been studied within the different dimensional reduction schemes coming from five-dimensional Chern-Simons gauge theories. The resulting actions were four-dimensional generalizations of Einstein-Cartan theory, conformal gravity, and bimetric gravity [11].

Here, we study linearized gravity, perturbed around maximally symmetric space, as a gauge theory of the conformal group while keeping two dynamical gauge fields. We find that perturbing the equations around anti–de Sitter (AdS) space gives degeneracy in the fields. The reason for this comes from the symmetric appearance of the gauge fields in the initial action and perturbation around maximally symmetric space. The linearized theory is different from the sum of linearized Einstein gravities for the two metrics since the equations of motion do not come from the corresponding Einstein actions, where the linearized MacDowell-Mansouri action has been studied in Ref. [12]. It also differs from linearized conformal gravity since we do not require invariance under the proper

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conformal gauge transformations, and a vanishing of the generator of translations which has in Ref. [1] been imposed "by hand."

Comparison with linearized Einstein gravity (EG) and conformal gravity (CG) further shows that the original action should consist out of the two Ricci scalars, one for each metric, and an additional potential. Just like CG, the action has one dimensionless parameter α, but two dynamical gauge fields, as one would expect from gauge theory for bimetric gravity. We also compare the linearized equations to the linearized equations of bimetric gravity. One could remove the degeneracy between the fields by introducing a parameter multiplying one of the gauge fields; however, the fields would still be linearly dependent. In order for them not to be linearly dependent, one would need to have the kinetic part modified. Another possibility for removing the degeneracy would be to perturb the fields around different backgrounds; for example, one of the fields could be perturbed around the AdS background, and another around a black hole. For now, we focus on the perturbations of both of the fields around AdS space, perturbation of one field around AdS space and the other around flat space, and in the nonperturbative case where gauge fields depend linearly on each other. The content of the article is as follows. Section II describes the action and corresponding equations of motion, while Sec. III analyzes them as a perturbation around the maximally symmetric spaces. In Sec. IV we obtain the linearized equations of motion for the two gauge field fluctuations, perturbed around the AdS space. In Sec. V we show an example of linearization around Minkowski space, while in Sec. VI we consider the case in which the gauge fields are proportional to each other. In Sec. VII we discuss the results and possible future prospects.

II. ACTION

The most general parity conserving quadratic action that can be constructed using the curvatures of a conformal group with no dimensional constants is [1]

$$I=\frac{\alpha}{8}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd}R_{\mu\nu ab}(J)R_{\rho\sigma cd}(J) \tag{2}$$

for α as a dimensionless constant,

$$\begin{aligned}R_{\mu\nu ab}(J)=\mathcal{R}_{\mu\nu ab}-2(e_{a\mu}f_{b\nu}-e_{b\mu}f_{a\nu})\\ +2(e_{a\nu}f_{b\mu}-e_{b\nu}f_{a\mu}),\end{aligned} \tag{3}$$

and

$$\mathcal{R}_{\mu\nu ab}=-\partial_\mu\omega_{\nu ab}+\partial_\nu\omega_{\mu ab}+\omega^c_{\mu a}\omega_{\nu cb}-\omega^c_{\nu a}\omega_{\mu cb}. \tag{4}$$

It consists of the gauge fields $e_{a\mu}$ and $f_{a\mu}$, which appear symmetrically in the action, and spin connection $\omega_{\mu ab}$. If we rewrite the action using Eq. (3) and omit the topologically invariant, Gauss-Bonnet term $(\mathcal{R}_{\mu\nu ab}(\omega))^2$, the action becomes

$$\begin{aligned}I&=\frac{\alpha}{8}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd}(-16\mathcal{R}_{\mu\nu ab}e_{c\rho}f_{d\sigma}+64e_{a\mu}f_{b\nu}e_{c\rho}f_{d\sigma})\\ &=\frac{\alpha}{8}\int d^4xL,\end{aligned} \tag{5}$$

which contains three independent fields $\omega_{\mu ab}$, $e_{a\mu}$, and $f_{a\mu}$. The fields $e_{a\mu}$ and $f_{a\mu}$ appear symmetrically in action, so we treat them on equal footing. If one imposes the requirement that the action is invariant under proper conformal gauge transformations, one needs to require that the gauge generator of translations

$$\begin{aligned}R_{\mu\nu a}(P)=-(\partial_\mu e_{a\nu}-\omega_\mu{}^b{}_a e_{b\nu})+(\partial_\nu e_{a\mu}-\omega_\nu{}^b{}_a e_{b\mu})\\ +(e_{a\mu}b_\nu-e_{a\nu}b_\mu)\end{aligned} \tag{6}$$

vanishes. This constraint on the generator determines the gauge field $\omega_{\mu ab}$ identified with spin connection. The gauge field b_ν is a generator of dilatations, and it does not appear in the action. Action (5) is scale and proper conformal invariant for $\omega=\omega(e)$. Keeping this spin connection, one can also integrate out the nonpropagating field $f_{a\mu}$ to obtain the

$$I=\frac{\alpha}{8}\int d^4xC_{\mu\nu ab}C_{\rho\sigma cd}\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd} \tag{7}$$

conformal gravity action; here, $C_{\mu\nu ab}$ is a Weyl tensor.

One more approach to consider action is without background expectation value for the field $f_{a\mu}$. One can integrate out $f_{a\mu}$ to obtain an action that depends on $\omega_{\mu ab}$ and $e_{a\mu}$. The action would be nonunitary and similar to the Weyl squared action but different from it since $\omega_{\mu ab}$ would be an independent field and not a function of $e_{a\mu}$.

A. Equations of motion

Varying the Lagrangian under action (5) with respect to $\omega_{\mu ab}$, one obtains its equation of motion,

$$\begin{aligned}\delta_\omega L=(-2e_{c\nu}\partial_\rho f_{d\sigma}+2e_{c\nu}\omega^k_{\rho d}f_{k\sigma}\\ -2f_{c\nu}\partial_\rho e_{d\sigma}+2f_{c\nu}\omega^k_{\rho d}e_{k\sigma})\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd}=0,\end{aligned} \tag{8}$$

in terms of the $e_{a\mu}$ and $f_{a\mu}$ gauge fields. Since the fields $e_{a\mu}$ and $f_{a\mu}$ appear symmetrically, we can compute the equation of motion for one gauge field and know it for the other gauge field as well. If we assume that $e_{a\mu}$ is invertible and has a nonzero determinant, we can determine its equation of motion from variation with respect to $e^{i\kappa}$,

$$\delta_e L = \epsilon^{\mu\nu\kappa\sigma}\epsilon^{abid}[-\mathcal{R}_{\mu\nu ab} f_{d\sigma} + 8 f_{b\nu} f_{d\sigma} e_{a\mu}] = 0, \quad (9)$$

while for the analogous equation for $f^{i\kappa}$ we have to take analogous assumptions for $f_{a\mu}$,

$$\delta_f L = \epsilon^{\mu\nu\kappa\sigma}\epsilon^{abid}[-\mathcal{R}_{\mu\nu ab} e_{d\sigma} + 8 e_{b\nu} e_{d\sigma} f_{a\mu}] = 0, \quad (10)$$

which corresponds to [1]

$$f_{a\mu} = -\frac{1}{4}\left(R_{a\mu} - \frac{1}{6} R e_{a\mu}\right). \quad (11)$$

Here, we have used the contractions

$$R_{b\mu} = \mathcal{R}_{\mu\nu ab} e^{a\nu}, \qquad R = R_{a\mu} e^{a\mu}, \quad (12)$$

and $f_{\mu\nu} = e^a{}_\mu f_{a\nu}$. Equation (10), inserted back, is known to give conformal gravity action for a vanishing of the translation generator [1,13,14]. However, we keep both of the gauge fields dynamical and perturbatively solve Eq. (8) for $\omega_{\mu ab}$.

We introduce perturbations of the gauge fields

$$e^a_\mu = v^a_\mu + \eta \chi^a_\mu + \eta^2 \zeta^a_\mu + \cdots, \quad (13)$$

$$f^a_\mu = f^{(0)a}_\mu + \eta \theta^a_\mu + \eta^2 \psi^a_\mu + \cdots, \quad (14)$$

and the perturbation of spin connection $\omega_{\mu ab}$,

$$\omega_{\mu ab} = \omega^{(0)}_{\mu ab} + \eta \omega^{(1)}_{\mu ab} + \eta^2 \omega^{(2)}_{\mu ab} + \cdots, \quad (15)$$

with an η small perturbation parameter. In the expansion of curvatures in Eq. (10),

$$R_{b\mu} = R^{(0))}_{b\mu} + \eta R^{(1)}_{b\mu} + \cdots \quad (16)$$

for $R^{(0)}_{b\mu} = \mathcal{R}^{(0)}_{\mu\nu ab} v^{a\nu}$, one needs to take into account the contractions $R^{(1)}_{b\mu} = \mathcal{R}^{(1)}_{\mu\nu ab} v^{a\nu} + \mathcal{R}^{(0)}_{\mu\nu ab} \tilde{\chi}^{a\nu}$ from Eq. (12). Analogously, the expansion of the Ricci scalar is

$$R = R^{(0)}_{b\mu} v^{b\mu} + \eta(R^{(1)}_{b\mu} v^{b\mu} + R^{(0)}_{b\mu} \tilde{\chi}^{b\mu}) + \cdots. \quad (17)$$

The allowed vacuum points around which we can perturb the action and equations of motion need to be backgrounds with curvature. One would naively perturb the fields around the flat background; however, the choice of $e_{a\mu} = f_{a\mu}$ would not satisfy the equation of motion for $f_{a\mu}$ or $e_{a\mu}$ if both of them are flat. If one of them were flat, the other one would have to be zero. One could further analyze around which backgrounds is it allowed to perturb the solution by studying the allowed solutions, as was done for Einstein theory in Ref. [15].

III. PERTURBATION AROUND $v_{a\mu} = f^{(0)}_{a\mu}$

We choose the background with $v^a_\mu = f^{(0)a}_\mu$. In the leading order the solution for equation of motion (8) is

$$\omega^{(0)}_{\nu ab} = -\frac{1}{2}(v_b{}^\beta \partial_\beta v_{a\nu} + v_a{}^\alpha v^\beta_b v^c_\nu(-\partial_\alpha v_{c\beta} + \partial_\beta v_{c\alpha}) - v_a{}^\beta \partial_\beta v_{b\nu} - v_b{}^\beta \partial_\nu v_{a\beta} + v_a{}^\beta \partial_\nu v_{b\beta}), \quad (18)$$

which agrees with the well-known spin connection for Einstein gravity. Leading order equations (9) and (10) will expectedly give an equal solution, which is an Einstein action with the cosmological constant

$$R^{(0)}_{\mu\nu} - 4 v_{\mu\nu} = 0. \quad (19)$$

Here, we have defined $v_{\mu\nu} = v_{b\mu} v^b{}_\nu$. For the analysis of the linear order, it is convenient to introduce the tensor

$$e_{a\mu} f_{b\nu} = Q_{ab\mu\nu}, \quad (20)$$

whose subleading order reads

$$Q^{(1)}_{ab\mu\nu} = v_{b\nu}\chi_{a\mu} + v_{a\mu}\theta_{b\nu}, \quad (21)$$

and we rewrite the subleading order of Eq. (8) in terms of it:

$$v^d{}_{[\nu} v^k{}_\sigma \omega^{(1)c}_{\rho]}{}_k - v^c{}_{[\nu} v^k{}_\sigma \omega^{(1)d}_{\rho]}{}_k - \partial_{[\rho} Q^{(1)[cd]}{}_{\nu\sigma]} = 0. \quad (22)$$

The combinations of the $Q_{ab\mu\nu}$ tensor which appear in Eq. (22) allow us to rewrite the partial derivatives in terms of the general covariant derivative defined on the background space because the Christoffels and spin connections added and subtracted to form the covariant derivative exactly cancel. One obtains

$$\omega^{(1)}_{\kappa ab} = \frac{1}{2} v_c{}^\alpha v_d{}^\beta (v_{b\kappa} v_a{}^\gamma - v_b{}^\gamma v_{a\kappa}) \nabla_{[\alpha} Q^{(1)[cd]}{}_{\beta\gamma]} + v_d{}^\beta (v_a{}^\alpha \eta_{bc} - v_b{}^\alpha \eta_{ac}) \nabla_{[\alpha} Q^{(1)[cd]}{}_{\delta\kappa]}. \quad (23)$$

The subleading order of the spin connection consists of the background vielbeins which are a solution of Eq. (19), Einstein spaces, and fluctuations $\chi_{a\mu}$, $\theta_{a\mu}$, which will be defined through Eqs. (9) and (10). The subleading order of Eq. (9),

$$\theta_{b\mu} = -\frac{1}{4}\left(R^{(1)}_{b\mu} - \frac{1}{6} R^{(0)} \chi_{b\mu} - \frac{1}{6} R^{(1)} v_{b\mu}\right), \quad (24)$$

consists of

$$R^{(1)}_{b\mu} = (-\partial_\mu \omega^{(1)}_{\nu ab} + \partial_\nu \omega^{(1)}_{\mu ab} + \omega^{c(0)}_{\mu a}\omega^{(1)}_{\nu cb} - \omega^{c(0)}_{\nu a}\omega^{(1)}_{\mu cb} + \omega^{c(1)}_{\mu a}\omega^{(0)}_{\nu cb} - \omega^{c(1)}_{\nu a}\omega^{(0)}_{\mu cb}) v^{a\nu} + (-\partial_\mu \omega^{(0)}_{\nu ab} + \partial_\nu \omega^{(0)}_{\mu ab} + \omega^{c(0)}_{\mu a}\omega^{(0)}_{\nu cb} - \omega^{c(0)}_{\nu a}\omega^{(0)}_{\mu cb}) \tilde{\chi}^{a\nu} \tag{25}$$

for

$$R^{(1)} = R^{(1)}_{b\mu} v^{b\mu} + R^{(0)}_{b\mu} \tilde{\chi}^{b\mu} \tag{26}$$

and $R^{(0)} = R^{(0)}_{b\mu} e^{b\mu}$, and it gives the dependence of $\chi_{a\mu}$ and $\theta_{a\mu}$.

IV. AdS BACKGROUND

We set the background perturbation to the AdS metric, which is Weyl flat, allowing us to write

$$v_{a\mu} = \rho(x)\delta_{a\mu} \tag{27}$$

and the leading order spin connection

$$\omega^{(0)}_{\nu ab} = -\delta_{[a\nu}\partial_{b]}\rho(x); \tag{28}$$

here, we denote $\partial_b = \delta^\mu_b \partial_\mu$. Equations (9) and (10) reduce to $R^{(1)}_{a\mu} = 4\delta_{\mu\nu}$. The subleading order of Eq. (8), just as Eq. (23) after a few technical manipulations, shows that the linear term in the $\omega_{\mu ab}$ perturbation can be rewritten in terms of the sum of two linear terms of Einstein spin connections,

$$\omega^{(1)}_{\kappa ak} = \omega^{(1)}_{\kappa ak}(\chi) + \omega^{(1)}_{\kappa ak}(\theta). \tag{29}$$

Here,

$$\omega^{(1)}_{\kappa ak}(\chi) = -\frac{1}{4\rho}(\delta^\alpha_a \nabla_\alpha \chi_{k\kappa} + \delta^\alpha_k \nabla_\kappa(\chi_{a\alpha}) + \delta^\alpha_k \delta^b_\kappa \delta^\beta_a \nabla_\beta \chi_{b\alpha}) - a \leftrightarrow k \tag{30}$$

is the linearized spin connection for Einstein gravity, and ∇ denotes the Lorentz covariant derivative. For transparency, we keep the Lorentz covariant derivative, and do not evaluate it for background AdS. The expression for the linearized spin connection evaluated on AdS is given in the Appendix. This form of $\omega^{(1)}_{\mu ab}$ allows us to split the curvatures in parts depending only on $\chi_{a\mu}$ or $\theta_{a\mu}$ fluctuation. Therefore, we can write the subleading order of the Riemann tensor as sum of linearized Riemann tensors for Einstein gravity. The subleading order of the Ricci tensor, however, will not be possible to write in the form of two linearized Ricci tensors for Einstein gravity because of the term $R^{(0)}_{\mu\nu ab}\tilde{\chi}^{a\nu}$ ($R^{(0)}_{\mu\nu ab}\tilde{\theta}^{a\nu}$), which is visible from Eq. (25):

$$R^{(1)}_{b\mu} = (R^{(1)}_{\mu\nu ab}(\chi) + R^{(1)}_{\mu\nu ab}(\theta))v^{a\nu} + R^{(0)}_{\mu\nu ab}\tilde{\chi}^{a\nu}. \tag{31}$$

Here,

$$R^{(1)}_{\mu\nu ab}(\chi) = -\partial_\mu \omega^{(1)}_{\nu ab}(\chi) + \partial_\nu \omega^{(1)}_{\mu ab}(\chi) + \omega^{c(0)}_{\mu a}\omega^{(1)}_{\nu cb}(\chi) - \omega^{c(0)}_{\nu a}\omega^{(1)}_{\mu cb}(\chi) + \omega^{c(1)}_{\mu a}(\chi)\omega^{(0)}_{\nu cb} - \omega^{c(1)}_{\nu a}(\chi)\omega^{(0)}_{\mu cb} \tag{32}$$

is a linearized Riemann tensor for Einstein gravity. We contract Eq. (24) with $v^b{}_\sigma$ and write

$$\theta_{b\mu}v^b{}_\sigma = -\frac{1}{4}\left(R^{(1)}_{b\mu}v^b{}_\sigma - \frac{1}{6}R^{(0)}\chi_{b\mu}v^b{}_\sigma - \frac{1}{6}R^{(1)}v_{b\mu}v^b{}_\sigma\right). \tag{33}$$

In terms of the Einstein gravity perturbations in the fields $\chi_{a\mu}$ and $\theta_{a\mu}$, using Eqs. (31) and (26), this is

$$\theta_{b\mu}v^b{}_\sigma = -\frac{1}{4}\Bigg(\bigg(R^{(1)}_{\mu\nu ab}(\chi) + R^{(1)}{}_{\mu\nu ab}(\theta)v^{a\nu} + R^{(0)}{}_{\mu\nu ab}\tilde{\chi}^{a\nu} - \frac{1}{6}R^{(0)}\chi_{b\mu}\bigg)v^b{}_\sigma - \frac{1}{6}((R^{(1)}_{\alpha\nu ac}(\chi) + R^{(1)}_{\alpha\nu ac}(\theta))v^{a\nu} + R^{(0)}_{\alpha\nu ac}\tilde{\chi}^{a\nu})v^{c\alpha}v_{b\mu}v^b{}_\sigma - \frac{1}{6}R^{(0)}_{c\alpha}\tilde{\chi}^{c\alpha}v_{b\mu}v^b{}_\sigma\Bigg). \tag{34}$$

This way, one obtains the constraint on the $\chi_{\mu\nu}$ related to $\theta_{\mu\nu}$. An analogous appearance of both equations of motions for the $f_{a\mu}$ and $e_{a\mu}$ gauge fields assuming them invertible implies that equation for $\chi_{b\mu}$ is

$$\chi_{b\mu}v^b{}_\sigma = -\frac{1}{4}\Bigg(\bigg(R^{(1)}_{\mu\nu ab}(\theta)v^{a\nu} + R^{(1)}{}_{\mu\nu ab}(\chi)v^{a\nu} + R^{(0)}{}_{\mu\nu ab}\tilde{\theta}^{a\nu} - \frac{1}{6}R^{(0)}\theta_{b\mu}\bigg)v^b{}_\sigma - \frac{1}{6}(R^{(1)}_{\alpha\nu ac}(\theta)v^{a\nu}v^{c\alpha} + R^{(1)}_{\alpha\nu ac}(\chi)v^{a\nu}v^{c\alpha} + 2R^{(0)}_{\alpha\nu ac}\tilde{\theta}^{a\nu}v^{c\alpha})v_{b\mu}v^b{}_\sigma\Bigg). \tag{35}$$

If we subtract Eqs. (35) and (34), we obtain

$$(\theta_{b\mu} - \chi_{b\mu})v^b{}_\sigma = -\frac{1}{4}\Bigg(\bigg(R^{(0)}{}_{\mu\nu ab}(\tilde{\chi}^{a\nu} - \tilde{\theta}^{a\nu}) - \frac{1}{6}R^{(0)}(\chi_{b\mu} - \theta_{b\mu})\bigg)v^b{}_\sigma - \frac{1}{6}(2R^{(0)}_{\alpha\nu ac}v^{c\alpha})v_{b\mu}v^b{}_\sigma(\tilde{\chi}^{a\nu} - \tilde{\theta}^{a\nu})\Bigg). \tag{36}$$

The equation does not contain any linearized curvatures due to their cancellation. The reason for this is that the terms with the linearized Riemann tensor can be written as a sum of the linear Riemann tensor for Einstein gravity and can contain both perturbations, $\chi_{a\mu}$ and $\theta_{a\mu}$, in both Eq. (34) and Eq. (35). Subtracting the equations will cancel these terms. Using the conventions $R^{(0)}_{\mu\nu\alpha\beta}=-\tilde{\lambda}(-v_{\mu\beta}v_{\nu\alpha}+v_{\mu\alpha}v_{\nu\beta})$, $R_{\alpha\beta}=3\tilde{\lambda}v_{\alpha\beta}$, $\tilde{\chi}^{a\nu}=-\chi^{a\nu}$, and $\tilde{\theta}^{a\nu}=-\theta^{a\nu}$, we evaluate Eq. (36) and get

$$\tilde{\lambda}(\theta_{\mu\sigma}-\chi_{\mu\sigma})=2(2+\tilde{\lambda})(\theta_{\sigma\mu}-\chi_{\sigma\mu}) \tag{37}$$

for $\tilde{\lambda}=-1$; this is

$$\chi_{\mu\sigma}-\theta_{\mu\sigma}=2(\theta_{\sigma\mu}-\chi_{\sigma\mu}) \tag{38}$$

or

$$\theta_{\mu\sigma}+2\theta_{\sigma\mu}=2\chi_{\sigma\mu}+\chi_{\mu\sigma}. \tag{39}$$

Owing to Lorentz invariance, we can impose a gauge in which $\chi_{a\mu}$ is a symmetric matrix, $\chi_{a\mu}=\chi_{\mu a}$. This would imply that $\chi_{a\mu}v^a_\nu=\chi_{\mu a}v^a_\nu\rightarrow\chi_{\mu\nu}=\chi_{\nu\mu}$. This condition requires that

$$\theta_{\mu\sigma}+2\theta_{\sigma\mu}=3\chi_{\sigma\mu}. \tag{40}$$

Summing Eqs. (35) and (34) and using the same notation give

$$(\theta_{\sigma\mu}+\chi_{\sigma\mu})=-\frac{1}{4}(k.t.+\tilde{\lambda}(\chi_{\mu\sigma}+\theta_{\mu\sigma})-2\tilde{\lambda}(\chi_{\sigma\mu}+\theta_{\sigma\mu})) \tag{41}$$

for the *k.t.* kinetic term

$$k.t.=\left(2R^{(1)}_{\mu\nu ab}(\chi+\theta)v^{a\nu}v^b{}_\sigma-\frac{1}{3}R^{(1)}_{\alpha\nu ac}(\chi+\theta)v^{a\nu}v^{c\alpha}v_{\mu\sigma}\right). \tag{42}$$

To evaluate the linear term $R^{(1)}_{\mu\nu ab}=\delta R_{\mu\nu ab}$, we linearize the tensor in the metric formulation and use the projection to the tetrad formulation

$$\begin{aligned}\delta R_{\mu\nu cd}(\chi)&\equiv R^{(1)}_{\mu\nu cd}(\chi)\\&=R^{(1)}{}_{\lambda\sigma\mu\nu}v^\lambda_c v^\sigma_d(\chi)-R^{(0)}_{\mu\nu ab}\delta^a_c\chi^b{}_d-R^{(0)}_{\mu\nu ab}\chi^a{}_c\delta^b_d.\end{aligned} \tag{43}$$

We then obtain

$$\begin{aligned}k.t.&=6\tilde{\lambda}(h_{\mu\sigma}+q_{\mu\sigma})-\mathcal{D}_\sigma\mathcal{D}_\mu(h+q)-\mathcal{D}^2(h_{\mu\sigma}+q_{\mu\sigma})\\&\quad+2\mathcal{D}_{(\mu}\mathcal{D}_\alpha(h^\alpha_{\sigma)}+q^\alpha_{\sigma)})-\frac{1}{3}(3\tilde{\lambda}(h+q)-\mathcal{D}^2(h+q)\\&\quad+\mathcal{D}_\alpha\mathcal{D}_\beta(h^{\alpha\beta}+q^{\alpha\beta}))v_{\mu\sigma}-2\tilde{\lambda}(\chi_{\mu\sigma}+\theta_{\mu\sigma}).\end{aligned} \tag{44}$$

Here, we have defined $h_{\mu\nu}=v_{a\mu}\chi^a{}_\nu+v_{a\nu}\chi^a{}_\mu$ and $q_{\mu\nu}=v_{a\mu}\theta^a{}_\nu+v_{a\mu}\theta^a{}_\nu$, their traces h and q, respectively, and we have not used any gauge conditions. The last term in Eq. (44) comes from the two last terms in Eq. (43). For the sum of the constraint equations on the linear term in the perturbation of the gauge field, from Eq. (41) we obtain

$$\begin{aligned}0=&-\mathcal{D}_\sigma\mathcal{D}_\mu(h+q)\\&-\mathcal{D}^2(h_{\mu\sigma}+q_{\mu\sigma})+2\mathcal{D}_{(\mu}\mathcal{D}_\alpha(h^\alpha_{\sigma)}+q^\alpha_{\sigma)})\\&-\frac{1}{3}(-\mathcal{D}^2(h+q)+\mathcal{D}_\alpha\mathcal{D}_\beta(h^{\alpha\beta}+q^{\alpha\beta}))v_{\mu\sigma}\\&+6\tilde{\lambda}(h_{\mu\sigma}+q_{\mu\sigma})-\tilde{\lambda}(h+q)v_{\mu\sigma}\\&-\tilde{\lambda}(\chi_{\mu\sigma}+\theta_{\mu\sigma})-(2\tilde{\lambda}-4)(\chi_{\sigma\mu}+\theta_{\sigma\mu})\end{aligned} \tag{45}$$

for $2\mathcal{D}_{(\mu}\mathcal{D}_\alpha h_{\sigma)}{}^\alpha=\mathcal{D}_\mu\mathcal{D}_\alpha h_\sigma{}^\alpha+\mathcal{D}_\sigma\mathcal{D}_\alpha h_\mu{}^\alpha$. One can also choose the de Donder gauge $\mathcal{D}_\alpha(h^\alpha{}_\beta+q^\alpha{}_\beta)=\frac{1}{2}\mathcal{D}_\beta(h+q)$, which keeps in the equation Laplace operators acting on the sum of the symmetrized linear terms in the expansion of the gauge field, their traces, and the mass terms

$$\begin{aligned}0=&-\mathcal{D}^2(h_{\mu\sigma}+q_{\mu\sigma})+\frac{1}{6}\mathcal{D}^2(h+q)v_{\mu\sigma}+6\tilde{\lambda}(h_{\mu\sigma}+q_{\mu\sigma})\\&-\tilde{\lambda}(h+q)v_{\mu\sigma}-\tilde{\lambda}(\chi_{\mu\sigma}+\theta_{\mu\sigma})-(2\tilde{\lambda}-4)(\chi_{\sigma\mu}+\theta_{\sigma\mu}).\end{aligned} \tag{46}$$

For $\tilde{\lambda}=-1$ Eq. (46) becomes

$$\begin{aligned}v^a{}_\sigma T^{(1)}{}_{a\mu}\equiv&-\mathcal{D}^2(h_{\mu\sigma}+q_{\mu\sigma})+\frac{1}{6}\mathcal{D}^2(h+q)v_{\mu\sigma}\\&-6(h_{\mu\sigma}+q_{\mu\sigma})+(h+q)v_{\mu\sigma}\\&+(\chi_{\mu\sigma}+\theta_{\mu\sigma})+6(\chi_{\sigma\mu}+\theta_{\sigma\mu})\\=&-\mathcal{D}^2(h_{\mu\sigma}+q_{\mu\sigma})+\frac{1}{6}\mathcal{D}^2(h+q)v_{\mu\sigma}-5(h_{\mu\sigma}+q_{\mu\sigma})\\&+5(\chi_{\sigma\mu}+\theta_{\sigma\mu})+(h+q)v_{\mu\sigma};\end{aligned} \tag{47}$$

we call this equation "$v^a{}_\sigma T^{(1)}{}_{a\mu}$." From Eqs. (38) and (47) one can notice that fluctuations cannot be fixed independently; they appear as a sum, which implies that there is an extra symmetry.

Highly symmetric equations (38) and (47) are pointing out the degeneracy of the perturbations around the maximally symmetric background. This becomes obvious when one tries to symmetrize Eq. (38). One obtains the equality $\chi_{\mu\sigma}+\chi_{\sigma\mu}=\theta_{\sigma\mu}+\theta_{\mu\sigma}$, which inserted into symmetrized equation (47) leads to two equal equations for $\chi_{\mu\sigma}+\chi_{\sigma\mu}$

and $\theta_{\mu\sigma} + \theta_{\sigma\mu}$. One could further analyze symmetrized equation (47) as

$$0 = -2\mathcal{D}^2(h_{\mu\sigma} + q_{\mu\sigma}) + \frac{1}{3}\mathcal{D}^2(h+q)v_{\mu\sigma} - 5(h_{\mu\sigma} + q_{\mu\sigma}) + 2(h+q)v_{\mu\sigma}, \tag{48}$$

rewriting the perturbations in the transverse traceless split, and one could consider its one loop partition function; however, one would have to keep in mind the implications of Eq. (38).

Independently, one can antisymmetrize Eq. (47), which will lead to cancellation of the derivatives and $\chi_{\mu\sigma} - \chi_{\sigma\mu} = -\theta_{\mu\sigma} + \theta_{\sigma\mu}$. With the Lorentz invariance requirement that $\chi_{\sigma\mu}$ is symmetric, antisymmetrizing Eq. (40), one obtains that $\theta_{\sigma\mu}$ is also symmetric. Equation (40) will then lead to $\theta_{\mu\sigma} = \chi_{\mu\sigma}$.

Equation (48), however, cannot be compared to the known linearized equations of EG or CG. As shown in subsection C of the Appendix on the example of Einstein gravity, projection of the general perturbed tensor $T_{\mu\nu} = T^{(0)}_{\mu\nu} + \eta T^{(1)}_{\mu\nu}$ is $T^{(1)}_{\mu\nu} = v^a{}_\mu T^{(1)}{}_{a\nu} + \chi^a{}_\nu T^{(0)}_{a\mu}$. We can recognize Eq. (47) as the $v^a{}_\mu T^{(1)}{}_{a\nu}$ part of the equation. To be able to compare the equation with linearized EG and CG from the literature, we have to obtain $T^{(1)}_{\mu\nu}$, i.e., we have to add $\chi^a{}_\nu T^{(0)}_{a\mu}$ to the $v^a{}_\mu T^{(1)}{}_{a\nu}$ tensor. After that, Eq. (47) becomes

$$T^{(1)}_{\sigma\mu} = -8(\chi_{\sigma\mu} + \theta_{\sigma\mu}) - 5(h_{\mu\sigma} + q_{\mu\sigma}) - \mathcal{D}^2(h_{\mu\sigma} + q_{\mu\sigma}) - v_{\mu\sigma}\left(-(h+q) - \frac{1}{6}\mathcal{D}^2(h+q)\right) = 0, \tag{49}$$

which can be symmetrized to give

$$-9(h_{\mu\sigma} + q_{\mu\sigma}) - \mathcal{D}^2(h_{\mu\sigma} + q_{\mu\sigma}) - v_{\mu\sigma}\left(-(h+q) - \frac{1}{6}\mathcal{D}^2(h+q)\right) = 0. \tag{50}$$

One can compare this to the linearized minimal bimetric gravity model where, for the massless spin-2 particle $h_{\mu\nu}$ and a massive spin-2 particle $u_{\mu\nu}$ of mass m, one has [3]

$$S = \int d^4x(h_{\mu\nu}\hat{\epsilon}^{\mu\nu\alpha\beta}h_{\alpha\beta} + u_{\mu\nu}\hat{\epsilon}^{\mu\nu\alpha\beta}u_{\alpha\beta}) - \frac{m^2}{4}\int d^4x(u^{\mu\nu}u_{\mu\nu} - u^\mu{}_\mu u^\nu{}_\nu). \tag{51}$$

Here, $\hat{\epsilon}^{\mu\nu\alpha\beta}$ denotes the Einstein-Hilbert (EH) kinetic operator. One can notice that Eq. (49), as well as linear equations that would come from Eq. (51), has the form of two equal operators acting on two separate fields and a mass term. In Eq. (49) the kinetic operator is not EH. One could think of the equation as consisting of two EH operators and additional mass terms. When Eq. (49) is symmetrized and one obtains Eq. (50), there are two equal kinetic operators for two degenerate fields, which can be thought of as two EH operators and mass terms. Upon lifting the degeneracy between the fields, one should be able to diagonalize the resulting equation such that there are two EH operators, one for each field, and remaining terms which belong only to one massive field, as in Eq. (51).

Analysis of the spin-2 massive graviton has been done in tetrad formulation for the dRGT model using similar methods [16]. A possibly convenient area of further consideration might be in terms of the field $Q_{\mu\nu\alpha\beta}$. If we express the subleading order equation (38) in terms of this tensor, it reads

$$Q^{(1)}_{\beta\mu\nu\sigma} - Q^{(1)}_{\nu\sigma\beta\mu} = 2(Q^{(1)}_{\sigma\nu\mu\beta} - Q^{(1)}_{\mu\beta\sigma\nu}), \tag{52}$$

while symmetrized equation (48) is

$$0 = -2\mathcal{D}^2(Q^{(1)}_{\mu\beta\sigma\nu} + Q^{(1)}_{\beta\mu\nu\sigma} + Q^{(1)}_{\sigma\nu\mu\beta} + Q^{(1)}_{\nu\sigma\beta\mu}) + \frac{1}{3}\mathcal{D}^2 Q^{(1)}v_{\mu\sigma}v_{\beta\nu} + 2Q^{(1)}v_{\mu\sigma}v_{\beta\nu} - 5(Q^{(1)}_{\mu\beta\sigma\nu} + Q^{(1)}_{\beta\mu\nu\sigma} + Q^{(1)}_{\sigma\nu\mu\beta} + Q^{(1)}_{\nu\sigma\beta\mu}). \tag{53}$$

It can be useful to notice the property

$$Q^{(1)}_{\nu\mu\beta\sigma} + Q^{(1)}_{\beta\sigma\nu\mu} = Q^{(1)}_{\beta\mu\nu\sigma} + Q^{(1)}_{\nu\sigma\beta\mu}. \tag{54}$$

V. $e_{a\mu}$ PERTURBED AROUND THE FLAT BACKGROUND AND $f_{a\mu}$ AROUND ZERO

The linearized equations of motion when $e_{a\mu}$ is perturbed around the flat background and $f_{a\mu}$ around zero in Eqs. (13) and (14) imply δ^a_μ and zero, respectively, for leading order terms, and the subleading terms remain to be determined. The equation of motion for $\omega_{\mu ab}$ in the leading order vanishes because it is multiplied by the leading order term in the expansion of $f_{a\mu}$. This naturally makes Eqs. (9) and (10) identically zero.

The subleading order of $\omega_{\mu ab}$,

$$\omega^{(1)}_{\mu ab} = \frac{1}{6}(\delta_b{}^\rho(-\partial_\mu\chi_{a\rho} + \partial_\rho\chi_{a\mu}) + \delta_a{}^\rho(\partial_\mu\chi_{b\rho} - \partial_\rho\chi_{b\mu}) + \delta_a{}^\rho\delta_b{}^\alpha e^{(0)d}{}_\mu(\partial_\alpha\chi_{d\rho} - \partial_\rho\chi_{d\alpha})), \tag{55}$$

agrees with a subleading term of $\omega_{\mu ab}$ in Einstein gravity, while the subleading order of Eq. (10) is

$$\theta^{(1)}_{b\mu} = -\frac{1}{4}\left(R^{(1)}_{b\mu} - \frac{1}{6}R^{(1)}\delta_{b\mu}\right). \tag{56}$$

The curvature terms in expansion are $R^{(1)}_{b\mu} = (-\partial_\mu \omega^{(1)}_{\nu ab} + \partial_\nu \omega^{(1)}_{\mu ab})\delta^{a\nu}$ and $R^{(1)} = R^{(1)}_{b\mu}\delta^{b\mu}$. Following the procedure of the previous chapter,

$$R^{(1)}_{b\beta} = \frac{1}{2}(\partial_\alpha \partial_\gamma h^\gamma_\beta - \partial_\beta \partial_\alpha h + \partial_\beta \partial_\gamma h^\gamma_\alpha - \partial_\gamma \partial^\gamma h_{\alpha\beta})\delta^\alpha_b \tag{57}$$

and

$$R^{(1)} = \partial_\beta \partial_\alpha h^{\alpha\beta} - \partial_\beta \partial^\beta h. \tag{58}$$

Using the de Donder gauge and writing the derivatives with $\mathcal{D}$,

$$\theta_{\sigma\mu} = \frac{1}{8}\left(\mathcal{D}^\alpha \mathcal{D}_\alpha h_{\sigma\mu} - \frac{1}{3}\mathcal{D}_\alpha \mathcal{D}^\alpha h \delta_{\sigma\mu}\right). \tag{59}$$

We can notice that there is dependency only on χ on the right-hand side of Eq. (59), which is a result of the fact that, in $\omega^{(1)}_{\mu ab}$, we have only $\chi_{a\mu}$ appearing. The subleading order of $\omega^{(1)}_{\mu ab}$ does not depend on $\theta_{a\mu}$ because, in the equation of motion that determines $\omega^{(1)}_{\mu ab}$, fields $f_{a\mu}$ appear in pairs, which will make such terms vanish in the subleading order when $f^{(0)}_{a\mu}$ is vanishing.

The leading order of the Eq. (9) will vanish because the perturbation of the $f_{a\mu}$ field is expanded around zero. The subleading order will also vanish because the first term of Eq. (9) is given by $R^{(0)}_{\mu\nu ab}\theta_{a\mu} + R^{(1)}_{\mu\nu ab} f^{(0)}_{a\mu}$, both of which vanish. The second term in Eq. (9) will be multiplied by the vanishing background $f^{(0)}_{a\mu}$.

VI. $e_{a\mu}$ IS PROPORTIONAL TO $f_{a\mu}$

Taking the condition

$$f_{a\mu} = \rho(x) e_{a\mu} \tag{60}$$

in the equation for $\omega_{\mu ab}$, Eq. (8), with

$$f^a_\mu = \rho(x) e^a_\mu, \quad f^\mu_a = \rho(x)^{-1} e^\mu_a, \quad f^{a\mu} = \rho(x)^{-1} e^{a\mu}, \tag{61}$$

one obtains

$$2\rho(x) e^{[c|}{}_{[\nu} e^k{}_\sigma \omega_{\rho]k}{}^{|d]} = 2\rho(x) e^{[c}{}_{[\nu} \partial_\rho e^{d]}{}_{\sigma]} + e^{[c}{}_{[\nu} \partial_\rho \rho(x) e^{d]}{}_{\sigma]}. \tag{62}$$

To find $\omega_{\mu ab}$, we multiply Eq. (62) with $e^\nu{}_k \eta^{di} \eta^{\rho\beta}$, $\delta^\beta{}_\delta e^\nu{}_j e^\rho{}_a \eta^{di}$, and $e^\beta{}_k e^\rho{}_a \eta^{di}$, respectively, and solve the system of equations for $\omega_{\mu ab}$:

$$\begin{aligned}\omega_{\nu ab} = {} & \frac{1}{2\rho(x)}(e_{a\nu} e_b{}^\beta \partial_\beta \rho(x) - e_a{}^\beta e_{b\nu} \partial_\beta \rho(x)) \\ & - \frac{1}{2}(e_b{}^\beta \partial_\beta e_{a\nu} + e^\alpha_a e^\beta_b e^c_\nu(-\partial_\alpha e_{c\beta} + \partial_\beta e_{c\alpha}) \\ & - e_a{}^\beta \partial_\beta e_{b\nu} - e_b{}^\beta \partial_\nu e_{a\beta} + e_a{}^\beta \partial_\nu e_{b\beta}).\end{aligned} \tag{63}$$

This form of $\omega_{\mu ab}$ has been expected based on the known solution from Kaku *et al.* [1], where agreement is obtained by setting $\rho(x)$ as a constant. The condition of proportionality (60) would give the action

$$I = \int d^4x L_s = 8\alpha \int d^4x \rho(x)(R + 24\rho(x))e, \tag{64}$$

which is equal to Einstein gravity for $\rho(x) = 1$. Here, we used contractions

$$R_{b\mu} = R^{(0)}_{\mu\nu ab} e^{a\nu}, \qquad R = R_{\mu a} e^{a\mu}. \tag{65}$$

Obtaining Einstein gravity from Weyl gravity has been studied from different angles [17,18]. In Ref. [18] the relation between the Weyl and Einstein gravities have been studied via breaking conformal gauge symmetries. After imposing the relation between the gauge fields $f_{\mu\nu}$ and $e_{\mu\nu}$ which breaks the conformal gauge symmetries, the obtained Lagrangian agrees with the Lagrangian in Eq. (64) when $\rho(x) \to -\frac{1}{4}\rho(x_0)$; i.e., $\rho(x)$ is taken to be $-\frac{1}{4}\rho(x_0)$ constant.

VII. DISCUSSION

We have studied linearized equations of motion of the parity conserving action constructed from curvatures of conformal group. Since we have not imposed additional constraints by hand, the result is highly symmetric. One can notice that the symmetry which appears between the linearized fields $\chi_{\mu\nu}$ and $\theta_{\mu\nu}$ is a consequence of the symmetry which appears in the action, and one can speculate on whether its origin reaches the relations among the generators of special conformal transformations (SCTs) and translations (Ts) in the conformal group. The difference between the SCTs and Ts in a conformal group is due to a minus sign that, if absorbed in the SCT generator, reemerges in a change of sign of different commutation relation.

We have obtained the constraint equations on the fluctuations in the expansion of the gauge fields $e_{a\mu}$ and $f_{a\mu}$ around the background AdS. When the constraint equations are symmetrized, one obtains two equal linearized expressions for both fields. The reason for this degeneracy, besides the conformal group, is in the perturbation around AdS space. For comparison, EG describes a massless graviton, and CG describes one massless and one partially massless mode. Here, the perturbations are linearly dependent on each other, and the system has degeneracy. In order to count precisely the number of degrees of freedom, one would have to perform a canonical analysis of the theory. Based on current results, one may expect one massless and partially massless or massive mode. Inspecting the linearized equations and comparing them with the linearized equations of EG and CG, it is possible to speculate that the original effective theory consists of two Ricci scalars each for one metric and an additional potential. The exact form of the potential is yet to be studied. The parameter of the theory is an α dimensionless parameter inherited from the starting action. This is similar to the theory with CG, but unlike in CG there are two dynamical gauge fields, which is similar to dRGT theory.

It would be interesting to compute observables such as the one loop partition function for this theory and compare it to Einstein and conformal gravity, and possibly to look for generalizations to higher spins. If the generalization were to arbitrary dimensions, one could consider the general d-dimensional conformal algebra and its implications, which one could relate and motivate with multimetric theories [19]. One could also look into the implications of the gauge (40) and obtain symmetric vielbeins, as was done in Ref. [20].

ACKNOWLEDGMENTS

We are grateful to Arkady Tseytlin for the discussions and comments on the draft. The work was supported by Project No. J 4129-N27 in the framework of the Erwin-Schrödinger Program of the Austrian Science Fund (FWF), and by Grant No. ST/P000762/1 of the Science and Technology Facilities Council (STFC).

APPENDIX A: INVERSE GAUGE FIELDS

To obtain the inverse of the perturbed gauge field f_μ^a, one starts with the general form of the inverse gauge field $\tilde{f}_a^\mu$. The expansion of the latter,

$$\tilde{f}_b^\mu = \tilde{f}_b^{(0)\mu} + \eta\tilde{\theta}_b^{(1)\mu} + \eta^2\tilde{\theta}_b^{(2)\mu} + \eta^3\tilde{\theta}_b^{(3)\mu}, \quad \text{(A1)}$$

in $\mathcal{O}(0)$ order requires one to satisfy $\tilde{f}_b^{(0)\mu} f_\mu^{(0)a} = \delta_b^a$. Multiplication of the two expansions in the leading order gives that $\tilde{f}_b^{(0)\mu} = f_b^{(0)\mu}$. The subleading order $\mathcal{O}(1)$ gives the condition

$$f_\alpha^{(0)a}\tilde{\theta}_b^{(1)\alpha} + \tilde{f}_b^{(0)\mu}\theta_\mu^a = 0,$$

from which it follows that $\tilde{\theta}_b^{(1)\alpha} = -f_b^{(0)\mu}\theta_\mu^{(1)a} f_a^{(0)\alpha}$. The order $\mathcal{O}(2)$ leads to

$$\tilde{\theta}_b^{(2)\alpha} = -f_b^{(0)\mu}\theta_{(2)\mu}^a f_a^{(0)\alpha} + \theta_\gamma^{(1)a} f_a^{(0)\alpha} f_b^{(0)\beta}\theta_\beta^c f_c^{(0)\gamma}. \quad \text{(A2)}$$

APPENDIX B: AdS BACKGROUND

When we consider above computation of the linear $\omega_{\mu ab}$ on the AdS background, it is most convenient to start from the equations of motion for $\omega_{\mu ab}$. We can notice that Eq. (8) can be written as

$$\alpha(e_{c\nu}R_{\rho\sigma d}(K) + f_{c\nu}R_{\rho\sigma d}(P))\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd} = 0 \quad \text{(B1)}$$

for

$$R_{\mu\nu a}(P) = -(\partial_\mu e_{a\nu} - \omega_{\mu a}^b e_{b\nu}) + (\partial_\nu e_{a\mu} - \omega_{\nu a}^b e_{b\mu}), \quad \text{(B2)}$$

$$R_{\mu\nu a}(K) = -(\partial_\mu f_{a\nu} - \omega_{\mu a}^b f_{b\nu}) + (\partial_\nu f_{a\mu} - \omega_{\nu a}^b f_{b\mu}). \quad \text{(B3)}$$

In the leading order Eq. (B1) reads

$$\alpha(v_{c\nu}R_{\rho\sigma d}^{(0)}(K) + f_{c\nu}^{(0)}R_{\rho\sigma d}^{(0)}(P))\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd} = 0, \quad \text{(B4)}$$

where we have used index (0) in $R_{\mu\nu a}^{(0)}$ to accent the order of perturbation. Since we use $f_{c\nu}^{(0)} = v_{c\nu}$, the equation reduces to

$$2\alpha v_{c\nu}R_{\rho\sigma d}^{(0)}(P)\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd} = 0, \quad \text{(B5)}$$

where we can recognize the appearance of the no torsion condition, which corresponds to the requirement that the covariant derivative of the AdS vielbein vanishes. That means that in the subleading order

$$\begin{aligned}\alpha[v_{c\nu}(R_{\rho\sigma d}^{(1)}(K) + R_{\rho\sigma d}^{(1)}(P)) + \chi_{c\nu}R_{\rho\sigma d}^{(0)}(P)\\ + \theta_{c\nu}R_{\rho\sigma d}^{(0)}(P)]\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd} = 0,\end{aligned} \quad \text{(B6)}$$

the second and the third term may be taken to zero due to the no torsion condition, so one obtains

$$\alpha v_{c\nu}(R_{\rho\sigma d}^{(1)}(K) + R_{\rho\sigma d}^{(1)}(P))\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd} = 0 \quad \text{(B7)}$$

for

$$\begin{aligned}R_{\rho\sigma d}^{(1)}(P) = -(\partial_\mu\chi_{a\nu} - \omega_\mu^{(0)b}{}_a\chi_{b\nu} - \omega_\mu^{(1)b}{}_a v_{b\nu})\\ + \partial_\nu\chi_{a\mu} - \omega_\nu^{(0)b}{}_a\chi_{b\mu} - \omega_\nu^{(1)b}{}_a v_{b\mu},\end{aligned} \quad \text{(B8)}$$

and $R_{\rho\sigma d}^{(1)}(K)$ gives the same expression with $\theta_{a\mu}$ on the place of $\chi_{a\mu}$ in Eq. (B8).

Analogous to the procedure for Eq. (8), we can dualize Eq. (B7) to obtain the equation for $\omega_{\mu ab}^{(1)}$:

$$v_{c[\nu}(R_{\rho\sigma]d}^{(1)}(K) + R_{\rho\sigma]d}^{(1)}(P) - v_{d[\nu}(R_{\rho\sigma]c}^{(1)}(K) + R_{\rho\sigma]c}^{(1)}(P)) = 0. \quad \text{(B9)}$$

To solve Eq. (B9) for $\omega^{(1)}_{\mu ab}$, we obtain three tensorial equations whose manipulation leads to the expression for $\omega^{(1)}_{\mu ab}$. The simplification that can be taken for the AdS background is that the AdS background is Weyl flat, and one can define

$$v_{a\mu} = \rho(x)\delta_{a\mu}. \tag{B10}$$

Here, $\rho(x)$ denotes the function of the coordinates on the manifold. The multiplication for obtaining the tensorial equations is therefore also done by using Eq. (B10). To express $\omega^{(1)}_{\mu ab}$, we use the *Mathematica* package xAct [21] and classify the terms as follows:

(1) Terms $\omega^{(1)}_{\mu ab}(\omega,\chi,\theta)$ with $\omega_{\mu ab}$, $\chi^{(1)}_{a\mu}$, and $\theta^{(1)}_{a\mu}$.

(2) Terms $\omega^{(1)}_{\mu ab}(\partial\chi)$ with $\partial_\mu \chi_{a\nu}$.

(3) Terms $\omega^{(1)}_{\mu ab}(\partial\theta)$ with $\partial_\mu \theta_{a\nu}$.

There are no terms that involve the partial derivative acting on the background vielbein. The reason for this becomes clear from Eq. (B8). In the linear order we can have the partial derivative of the background vielbein only from $\omega^{(0)}_{\mu ab}$, while the remaining terms vanished due to the no torsion condition. (Below we omit writing (0) in $\omega^{(0)}$ for simplicity.)

For the terms in 1, we obtain

$$\tilde{\omega}^{(1)}_{\kappa ak}(\omega,\chi,\theta) = -\frac{1}{4\rho}[(\omega_k{}^b{}_\kappa + \omega_\kappa{}^b{}_k)(\theta_{ba}+\chi_{ba}) - \omega_a{}^b{}_k(\theta_{b\kappa}+\chi_{ba})]; \tag{B11}$$

here,

$$\omega^{(1)}_{\kappa ak}(\omega,\chi,\theta) = \tilde{\omega}^{(1)}_{\kappa ak}(\omega,\chi,\theta) - \tilde{\omega}^{(1)}_{\kappa ka}(\omega,\chi,\theta). \tag{B12}$$

The terms in 2 are $\omega^{(1)}_{\mu ab}(\partial\chi) = \tilde{\omega}^{(1)}_{\mu ab}(\partial\chi) - \tilde{\omega}^{(1)}_{\mu ba}(\partial\chi)$ and

$$\tilde{\omega}^{(1)}_{\mu ab}(\partial\chi) = -\frac{1}{4\rho}\delta_k{}^\alpha\delta_\kappa{}^b\partial_a\chi_{b\alpha} - \frac{1}{4\rho}\partial_a\chi_{k\kappa} - \frac{1}{4\rho}\delta_k{}^\alpha\partial_\kappa\chi_{a\alpha}, \tag{B13}$$

and the terms in 3 are equal to the terms in 2, with $\theta_{a\mu}$ in place of $\chi_{a\mu}$: $\omega^{(1)}_{\mu ab}(\partial\theta) = \tilde{\omega}^{(1)}_{\mu ab}(\partial\theta) - \tilde{\omega}^{(1)}_{\mu ba}(\partial\theta)$:

$$\tilde{\omega}^{(1)}_{\mu ab}(\partial\theta) = -\frac{1}{4\rho}\delta_k{}^\alpha\delta_\kappa{}^b\partial_a\theta_{b\alpha} - \frac{1}{4\rho}\partial_a\theta_{k\kappa} - \frac{1}{4\rho}\delta_k{}^\alpha\partial_\kappa\theta_{a\alpha}. \tag{B14}$$

To identify the covariant derivatives, let us rewrite the $\theta_{a\mu}$ part of Eq. (B11) with indices on $\omega_{\mu ab}$ not contracted:

$$-\frac{1}{4\rho}[(\delta^\beta_k\delta^\alpha_a\delta^c_\kappa\omega_\beta{}^b{}_c + \delta^\alpha_a\omega_\kappa{}^b{}_k)\theta_{b\alpha} - \delta^\alpha_a\omega_\alpha{}^b{}_k\theta_{b\kappa}]. \tag{B15}$$

Combining the third term from Eq. (B15) and the appropriate term from Eq. (B11), we have

$$\delta^\alpha_a(\partial_\alpha\theta_{k\kappa} - \omega_\alpha{}^c{}_k\theta_{c\kappa}) = \delta^\alpha_a\nabla_\alpha\theta_{k\kappa}. \tag{B16}$$

The remaining terms from Eq. (B15) analogously combine with the antisymmetric pairs of the terms in Eq. (B11) to form covariant derivatives. Taking into account $\chi_{a\mu}$, $\theta_{a\mu}$, and Eqs. (B11)–(B14), we obtain

$$\omega^{(1)}_{\kappa ak} = -\frac{1}{4\rho}(\delta^\alpha_a\nabla_\alpha(\theta_{k\kappa}+\chi_{k\kappa}) + \delta^\alpha_k\nabla_\kappa(\theta_{a\alpha}+\chi_{a\alpha}) + \delta^\alpha_k\delta^b_\kappa\delta^\beta_a\nabla_\beta(\theta_{b\alpha}+\chi_{b\alpha})) - a\leftrightarrow k. \tag{B17}$$

For the EG spin connection it holds that

$$\omega^{\mathrm{EG}}{}_\mu{}^a{}_b = -e_b{}^\nu\mathcal{D}_\mu e_a{}^\mu, \tag{B18}$$

which is equal to Eq. (18) in the leading order, where we denote the covariant derivative with $\mathcal{D}$. In the linearized order this is

$$\omega^{\mathrm{EG}(1)}{}_\mu{}^a{}_b(\chi) = -\tilde{\chi}_b{}^\nu\mathcal{D}_\mu v_a{}^\nu v_b{}^\nu\mathcal{D}^{(1)}_\mu v_a{}^\nu - v_b{}^\nu\mathcal{D}_\mu\tilde{\chi}_a{}^\nu. \tag{B19}$$

We can write Eq. (29) as

$$\omega^{(1)}_{\mu ab}(\chi+\theta) = \omega^{\mathrm{EG}(1)}_{\mu ab}(\chi) + \omega^{\mathrm{EG}(1)}_{\mu ab}(\theta). \tag{B20}$$

Linearizing Eq. (B20) around AdS, we can write the terms

$$\omega^{(1)\mathrm{EG}}_{\mu ab(\mathrm{AdS})}(\chi) = \frac{1}{2\rho^2}((\chi_{b\mu}-\chi_{\mu b})\partial_a\rho + (-\chi_{a\mu}+\chi_{\mu a})\partial_b\rho + (-\chi_{ab}+\chi_{ba})\partial_\mu\rho \tag{B21}$$

$$+((-\eta_{b\mu}-\eta_{\mu b})\chi^\nu{}_a + (\eta_{a\mu}+\eta_{\mu a})\chi^\nu{}_b)\partial_\nu\rho) \tag{B22}$$

$$+\frac{1}{2\rho}(\partial_a\chi_{b\mu} - \partial_b\chi_{a\mu} + \delta_b{}^\nu\partial_\mu\chi_{a\nu} - \delta_a{}^\nu\partial_\mu\chi_{b\nu} + \delta_\mu{}^c(\delta_b{}^\lambda\partial_a\chi_{c\lambda} - \delta_a{}^\lambda\partial_b\chi_{c\lambda})) \tag{B23}$$

and $\omega^{(1)\mathrm{EG}}_{\mu ab(\mathrm{AdS})}(\theta)$ analogously. We can notice that the choice of symmetric perturbation $\chi_{\mu b} = \chi_{b\mu}$, $\chi_{ab} = \chi_{ba}$ reduces Eq. (B23) to

$$\omega^{(1)\mathrm{EG}}_{\mu ab(\mathrm{AdS})\,\mathrm{symmetric}} = \frac{(-\eta_{\mu b}\chi_a{}^\nu + \eta_{\mu a}\chi_b{}^\nu)\partial_\nu\rho}{\rho^2} \tag{B24}$$

$$+\frac{1}{2\rho}(\partial_a\chi_{b\mu} - \partial_b\chi_{a\mu} + \delta_\mu{}^c(\delta_b{}^\lambda\partial_a\chi_{c\lambda} - \delta_a{}^\lambda\partial_b\chi_{c\lambda})). \tag{B25}$$

Equation (B20) also requires that

$$-\frac{1}{4\rho}(v_a^\alpha\nabla_\alpha(\chi_{k\kappa})+v_k^\alpha\nabla_\kappa(\chi_{a\alpha})+v_k^\alpha v_\kappa^b v_a^\beta\nabla_\beta(\chi_{b\alpha}))-a\leftrightarrow k$$
$$=(-\tilde{\chi}_k{}^\nu\mathcal{D}_\kappa v_\nu{}^d+v^\nu{}_k\Gamma^{(1)\alpha}{}_{\kappa\nu}v_\alpha{}^d-v^\nu{}_k\mathcal{D}_\kappa\chi_d{}^\nu)\eta_{dc},\qquad \text{(B26)}$$

where $\Gamma^{(1)\alpha}_{\kappa\nu}$ is Christoffel $\Gamma^\alpha_{\kappa\nu}=\frac{1}{2}e^{\alpha\beta}(\partial_\kappa e_{\beta\nu}+\partial_\nu e_{\beta\kappa}-\partial_\beta e_{\kappa\nu})$ expanded for $e_{\mu\nu}=e_{a\mu}e^a{}_\nu$, its expansion $e_{\mu\nu}=v_{\mu\nu}+h_{\mu\nu}$, and

$$h_{\mu\nu}=v_{a\mu}\chi^a{}_\nu+\chi_{a\mu}v^a{}_\nu.\qquad \text{(B27)}$$

We have defined $h_{\mu\nu}$ as a symmetric term in the perturbation of $e_{\mu\nu}$. Expansion is analogous for $\theta_{a\mu}$,

$$q_{\mu\nu}=v_{a\mu}\theta^a{}_\nu+\theta_{a\mu}v^a{}_\nu.\qquad \text{(B28)}$$

Proving that Eq. (B20) holds makes it possible to write the perturbation as a sum of perturbations in Einstein gravity. We can consider the linearized projection of the Riemann tensor from the vielbein to metric formulation. For the projection of the Riemann tensor, we know that $R^\lambda{}_{\sigma\mu\nu}=e_a{}^\lambda e^b{}_\sigma R_{\mu\nu}{}^a{}_b$. When we rewrite the definition of $R^a{}_{b\mu\nu}$, Eq. (12), in terms of Eq. (B18), $\omega_\mu{}^a{}_b=e^a{}_\alpha e_b{}^\beta\Gamma^\alpha_{\mu\beta}-e_b{}^\alpha\partial_\mu e^a{}_\alpha$, the projection gives us $R^\lambda_{\sigma\mu\nu}$. The terms in the computation that contain one partial derivation ∂_μ, ∂_ν, and their combination $f(\partial_\mu,\partial_\nu)$ (for the f function in ∂_μ and ∂_ν) at leading order separately cancel. Analogously, we consider them in linearized order.

We write the projection

$$R^\lambda{}_{\sigma\mu\nu}=e_a^\lambda e_\sigma^b(-\partial_\mu(e_\rho^a e_b^\tau\Gamma^\rho_{\nu\tau})+\partial_\nu(e_\rho^a e_b^\tau\Gamma^\rho_{\mu\tau})$$
$$+\partial_\mu e_b^\tau\partial_\nu e_\tau^a-\partial_\nu e_b^\tau\partial_\mu e_\tau^a\qquad \text{(B29)}$$

$$-(e_\rho^a e_c^\tau\Gamma^\rho_{\mu\tau}-e_c^\tau\partial_\mu e_\tau^a)(e_{\rho'}^c e_b^{\tau'}\Gamma^{\rho'}_{\nu\tau'}-e_b^{\tau'}\partial_\nu e_{\tau'}^c)\qquad \text{(B30)}$$

$$+(e_\rho^a e_c^\tau\Gamma^\rho_{\nu\tau}-e_c^\tau\partial_\nu e_\tau^a)(e_{\rho'}^c e_b^{\tau'}\Gamma^{\rho'}_{\mu\tau'}-e_b^{\tau'}\partial_\mu e_{\tau'}^c))\qquad \text{(B31)}$$

and linearize it. The linearized order projection is

$$R^{(1)\lambda}{}_{\sigma\mu\nu}(\chi)=v^{a\lambda}v_\sigma^b R^{(1)}{}_{\mu\nu ab}(\chi)+R^{(0)}{}_{\mu\nu ab}v^{a\lambda}\chi_\sigma^b$$
$$+R^{(0)}{}_{\mu\nu ab}\tilde{\chi}^{a\lambda}v_\sigma^b,\qquad \text{(B32)}$$

the subleading order of $R^\lambda_{\sigma\mu\nu}=-\partial_\mu\Gamma^\lambda_{\nu\sigma}+\partial_\nu\Gamma^\lambda_{\mu\sigma}-\Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\nu\sigma}+\Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\mu\sigma}$.

APPENDIX C: COMPARISON WITH EINSTEIN GRAVITY

An analogous consideration of Einstein gravity would lead to equations of motion in the subleading order

$$G^{(1)}_{a\mu}=R^{(1)}_{a\mu}-\frac{1}{2}R^{(1)}e_{a\mu}-\frac{1}{2}R\chi_{a\mu}=0.\qquad \text{(C1)}$$

Using the above method and the de Donder gauge leads to the constraint on $\chi_{a\mu}$

$$-\tilde{\lambda}\chi_{\mu\nu}-\mathcal{D}^2\chi_{\mu\nu}+\frac{1}{2}(2\tilde{\lambda}\chi+\mathcal{D}^2\chi)v_{\mu\nu}=0.\qquad \text{(C2)}$$

To compare this with the familiar result for the linearized Einstein operator, we have to consider $h_{\mu\nu}=2\chi_{\mu\nu}$, which is symmetric, and

$$G^{(1)}_{\mu\nu}=G^{(1)}_{a\mu}v^a_\nu+G^{(0)}_{a\mu}\chi^a_\nu,\qquad \text{(C3)}$$

where $G^{(0)}_{a\mu}\chi^a_\nu=-3\tilde{\lambda}\chi_{\mu\nu}$. We also need to take into account the cosmological constant, which is $6\tilde{\lambda}\chi_{\mu\nu}$ for four dimensions. Adding this to Eq. (C2), we obtain a familiar result,

$$2\tilde{\lambda}\chi_{\mu\nu}-\mathcal{D}^2\chi_{\mu\nu}+\frac{1}{2}(2\tilde{\lambda}\chi+\mathcal{D}^2\chi)v_{\mu\nu}=0.\qquad \text{(C4)}$$

APPENDIX D: RELATIONS USED IN TEXT

Here we list several equations that were used in text

$$\frac{1}{4}\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma=e,\qquad \text{(D1)}$$

$$\frac{1}{2!}\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}e^c{}_\rho e^d{}_\sigma=e(e^\mu{}_a e^\nu{}_b-e^\nu{}_a e^\mu{}_b),\qquad \text{(D2)}$$

$$\delta e=ee^\mu_a\delta e^a_\mu,\qquad \text{(D3)}$$

where e is a determinant.

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