$P-V$ criticality in the extended phase space of black holes in Einstein-Horndeski gravity

Ya-Peng Hu,[1,2,*] Hong-An Zeng,[1,†] Zhi-Ming Jiang,[1,‡] and Hongsheng Zhang[3,§]

[1]*College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*
[2]*Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China*
[3]*School of Physics and Technology, University of Jinan, 336 West Road of Nan Xinzhuang, Jinan, Shandong 250022, China*

(Received 25 April 2019; published 2 October 2019)

Gravity is believed to have a deep and inherent relation to thermodynamics. We study phase transitions and critical behaviors in the extended phase space of asymptotic anti–de Sitter black holes in Einstein-Horndeski gravity. We demonstrate that the black hole in Einstein-Horndeski gravity undergoes a phase transition and $P-V$ criticality mimicking the van der Waals gas-liquid system. The key approach in our study is to introduce a more reasonable pressure instead of the previous pressure $P=-\Lambda/8\pi$ related to the cosmological constant Λ, and this proper pressure is motivated by the asymptotical behavior of the black hole. Moreover, we also obtain $P-V$ criticality in the two cases with $\Lambda=0$ and $\Lambda>0$ for the first time, which implies that the cosmological constant Λ may not be a necessary pressure candidate for black holes at the microscopic level. We present critical exponents for these phase transition processes.

DOI: 10.1103/PhysRevD.100.084004

I. INTRODUCTION

The relation between thermodynamics and gravity theory is an interesting and profound issue. The four thermodynamic laws of a stationary black hole system (in fact, spacetime thermodynamics, because the physical quantities in black hole thermodynamics should be treated as the quantities of the globally asymptotic flat manifold) were developed in Ref. [1] and confirmed by Hawking radiation [2]. After this pioneering discovery, more analogies were found between ordinary thermodynamics and black hole thermodynamics, including several phase transitions and critical phenomena. A well-known case is the so-called Hawking-Page phase transition [3], where a first-order phase transition was found to occur in the Schwarzschild–anti de Sitter (AdS) black hole spacetime. In addition, the phase transitions between radiation gas and a black hole were further investigated in an isolated box [4,5], where an equilibrium between the absorbed and radiated particles in a black hole system can be reached. Moreover, a phase transition similar to the Van der Waals gas/liquid transition was found for Reissner-Nordström AdS (RN-AdS) black holes [7]. These studies revealed significant properties of gravitational systems that may be helpful for solving some extremely difficult problems beyond gravity through investigations of phase transitions in gravitational systems. For example, the Hawking-Page phase transition maps to the quark confinement/deconfinement transition in the AdS/CFT scenario [6], which has gained much more physical significance.

In recent years, a similar phase transition mimicking the Van der Waals gas/liquid transition in the RN-AdS black hole system was further investigated in Ref. [8]. No pressure P or volume V were defined in the previous work [7], while we use the $P-V$ diagram to characterize the van der Waals liquid-gas phase transition. In Ref. [8], a crucial step was to introduce a thermodynamic pressure P, which is related to the negative cosmological constant Λ in the Reissner-Nordström AdS black hole system as $P=-\Lambda/8\pi$. Hence, this phase transition is also called the $P-V$ criticality of black holes, and is similar to the $P-V$ diagrams of a black hole system and the Van der Waals–Maxwell gas/liquid system. Note that this behavior of $P-V$ criticality was already found in other black hole systems with asymptotical AdS behavior [9–17]. Moreover, in this situation where the pressure P is proportional to the negative cosmological constant, the mass of a black hole becomes enthalpy rather than energy, which is required by thermodynamic laws [18]. In these laws, the term dP appears, which implies that the cosmological "constant" may be a variable, and there has been a large amount of discussion about a variable vacuum energy in the context of cosmology since the discovery of cosmic acceleration. Theoretically, we have several motivations to invoke the

*huyp@nuaa.edu.cn
†hazeng@nuaa.edu.cn
‡jzm931106lj@gmail.com
§sps_zhanghs@ujn.edu.cn

assumption of a variable cosmological "constant." If the present theory is not a "fundamental theory," the constants in the theory may be variables. For example, the cosmological constant becomes a variable in gauged supergravity. And furthermore, the cosmological constant, Newton's constant, and other coupling constants may become variables in quantum theory, whose vacuum expectation values are the observed values [19]. Thus, it may be reasonable to identify the cosmological constant as the pressure in $P-V$ criticality in gravity and allow the pressure to be a variable. Such an identity leads to several sensible results [8–17]. However, it requires further study, since the $P-V$ criticality of a gravitational system compared to a gas/liquid system is fundamentally an analogy. There is no sound statistical-mechanical mechanism behind gravitational thermodynamics, and thus we are unable to prove that the pressure P should be identified as the cosmological constant Λ at the microscopic level, such as $P=-\Lambda/8\pi$.

Scalar-tensor gravity has a fairly long history and several different extensions, while Horndeski scalar-tensor theory is the most generic scalar-tensor theory. In Horndeski scalar-tensor theory, the equations of motion consist of terms which have at most second-order derivatives acting on each field, though it permits higher-order derivatives in the action [20]. This property is very similar to Lovelock gravity. From different considerations, Horndeski theory was rediscovered in a different form in cosmological studies [21,22]. Black hole solutions in a special Einstein-Horndeski gravity were derived in Refs. [23,24], while the thermodynamics of black holes in Einstein-Horndeski gravity were investigated in Refs. [25,26]. In this paper, we show that there is no $P-V$ criticality in black holes in Einstein-Horndeski theory if one treats the cosmological constant, as usual as the pressure. We demonstrate that the proper pressure is different from the cosmological constant. With this new pressure, the $P-V$ criticality appears. Moreover, we find $P-V$ criticality in the two cases with $\Lambda=0$ and $\Lambda>0$ for the first time, which implies that the cosmological constant Λ may not be a necessary pressure candidate for black holes at the microscopic level.

This article is organized as follows. In the next section, we briefly review an exact solution in the Einstein-Horndeski theory. In Sec. III, we demonstrate that the $P-V$ phase transition does occur with a new definition of the pressure that is different from the cosmological constant. In Sec. IV, we conclude this paper.

II. THERMODYNAMICS OF STATIC BLACK HOLE SOLUTIONS IN HORNDESKI GRAVITY

The most general action of Horndeski gravity can be seen in Ref. [22]. In this paper, we investigate a special case in Horndeski gravity with a nonminimal kinetic coupling, and the corresponding action is written as [24–26]

$$S=\int\sqrt{-g}d^4x\left[(R-2\Lambda)-\frac{1}{2}(\zeta g_{\mu\nu}-\eta G_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{2.1}$$

where ζ is a coupling constant, and η stands for the coupling strength with the dimension of length squared. Besides the electromagnetic field $F_{\mu\nu}$, a real scalar field ϕ also exists, which has a nonminimal kinetic coupling with the metric and Einstein tensor. From this action, the equations of motion are

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\frac{1}{2}(\zeta T_{\mu\nu}+\eta\Xi_{\mu\nu}+E_{\mu\nu}), \tag{2.2}$$

$$\nabla_\mu[(\zeta g^{\mu\nu}-\eta G^{\mu\nu})\nabla_\nu\phi]=0, \tag{2.3}$$

$$\nabla_\mu F^{\mu\nu}=0, \tag{2.4}$$

where $T_{\mu\nu}$, $\Xi_{\mu\nu}$, and $E_{\mu\nu}$ are defined as

$$T_{\mu\nu}\equiv\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi, \tag{2.5}$$

$$\begin{aligned}\Xi_{\mu\nu}\equiv{}&\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi R-2\nabla_\rho\phi\nabla_{(\mu}\phi R^\rho_{\nu)}-\nabla^\rho\phi\nabla^\lambda\phi R_{\mu\rho\nu\lambda}\\&-(\nabla_\mu\nabla^\rho\phi)(\nabla_\nu\nabla_\rho\phi)+(\nabla_\mu\nabla_\nu\phi)\Box\phi+\frac{1}{2}G_{\mu\nu}(\nabla\phi)^2\\&-g_{\mu\nu}\left[-\frac{1}{2}(\nabla^\rho\nabla^\lambda\phi)(\nabla_\rho\nabla_\lambda\phi)+\frac{1}{2}(\Box\phi)^2-\nabla_\rho\phi\nabla_\lambda\phi R^{\rho\lambda}\right],\end{aligned} \tag{2.6}$$

$$E_{\mu\nu}\equiv F^\rho_\mu F_{\nu\rho}-\frac{1}{4}g_{\mu\nu}F^2. \tag{2.7}$$

In this paper, we focus on the static black hole solutions of Eqs. (2.2)–(2.4) with spherical symmetry, while the metric, scalar field, and Maxwell field are simplified as

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+g(r)\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2), \tag{2.8}$$

$$\phi=\phi(r),\qquad A=\Psi(r)\mathrm{d}t. \tag{2.9}$$

After this simplification, an analytical solution can be obtained [24–26],

$$
\begin{aligned}
f(r) &= \frac{\zeta r^2}{3\eta} - \frac{2M}{r} + \frac{3\zeta+\Lambda\eta}{\zeta-\Lambda\eta} + \left(\frac{\zeta+\Lambda\eta+\frac{\zeta^2 q^2}{4\eta}}{\zeta-\Lambda\eta}\right)^2 \frac{\tan^{-1}\left(\sqrt{\frac{\zeta}{\eta}}r\right)}{\sqrt{\frac{\zeta}{\eta}}r} + \frac{\zeta^2 q^2}{(\zeta-\Lambda\eta)^2 r^2} - \frac{\zeta^2 q^4}{48(\zeta-\Lambda\eta)^2 r^4} + \frac{\zeta^3 q^4}{16\eta(\zeta-\Lambda\eta)^2 r^2},\\
g(r) &= \frac{\zeta^2[4(\zeta-\Lambda\eta)r^4 + 8\eta r^2 - \eta q^2]^2}{16 r^4 (\zeta-\Lambda\eta)^2(\zeta r^2+\eta)^2 f(r)},\\
\psi^2(r) &= -\frac{\zeta^2[4(\zeta+\Lambda\eta)r^4+\eta q^2][4(\zeta-\Lambda\eta)r^4+8\eta r^2-\eta q^2]^2}{32\eta r^6(\zeta-\Lambda\eta)^2(\zeta r^2+\eta)^3 f(r)},\\
\Psi(r) &= \Psi_0 + \frac{1}{4}\frac{q\sqrt{\zeta}}{\eta^{\frac{3}{2}}}\left[\frac{4\eta(\zeta+\Lambda\eta)+\zeta^2 q^2}{\zeta-\Lambda\eta}\right]\tan^{-1}\left(\frac{1}{\sqrt{\frac{\zeta}{\eta}}r}\right) - \frac{\zeta q(\zeta q^2+8\eta)}{4\eta r(\zeta-\Lambda\eta)} + \frac{\zeta q^3}{12 r^3(\zeta-\Lambda\eta)},
\end{aligned}
\tag{2.10}
$$

where $\psi(r) \equiv \phi'(r)$ and Ψ_0 is an integration constant. Note that the solution (2.10) is same as that in Ref. [25] after taking the identity $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1}(1/x)$ into account, and both ζ and η can not be zero while obtaining this solution.[1] Therefore, we cannot switch off the scalar field within this solution family, which implies that this solution is usually not continuously connected with the maximally symmetric background, and hence does not contain the Reissner-Nordström-AdS solution. Despite this, the case with $M=0$ and $q=0$ is also a regular spacetime, which can describe an asymptotically AdS gravitational soliton solution [24]. In particular, if we further require $\zeta=-\Lambda\eta$, this asymptotically AdS gravitational soliton solution becomes the pure AdS solution. Hence, in this particular case, we can find that the Schwarzschild-AdS solution is recovered with $\zeta=-\Lambda\eta$ and $q=0$. However, the Reissner-Nordström-AdS solution still cannot be recovered in this particular case. In this paper, we only consider the solution (2.10) with asymptotic AdS behavior, and hence we further require that $\zeta/\eta>0$ and $\zeta\neq\Lambda\eta$. The temperature of this black hole in Horndeski gravity is easily obtained as

$$
\begin{aligned}
T &= \frac{1}{4\pi}\sqrt{g'(r_h)f'(r_h)}\\
&= \frac{1}{4\pi r_h}\left(\frac{2\zeta}{\zeta-\Lambda\eta} + \frac{\zeta r_h^2}{\eta} - \frac{\zeta q^2}{4 r_h^2(\zeta-\Lambda\eta)}\right),
\end{aligned}
\tag{2.11}
$$

where r_h is the location of the outer horizon satisfying $f(r_h)=0$, and the first law of thermodynamics of the black hole solution (2.10) was carefully investigated in Ref. [25],

$$
dE = TdS + \Phi_e dQ_e + \Phi_\chi dQ_\chi, \tag{2.12}
$$

where the energy E, entropy S, charge potential Φ_e, charge Q_e, scalar charge Q_χ, and its conjugate potential Φ_χ are calculated as

$$
\begin{aligned}
E &= \frac{M(\zeta-\eta\Lambda)}{2\zeta} - \frac{\pi[q^2\zeta^2+4\eta(\zeta+\eta\Lambda)]^2}{128\zeta^{3/2}\eta^{3/2}(\zeta-\eta\Lambda)},\\
S &= \frac{2\pi^2 r_h^3(\zeta-\eta\Lambda)}{\zeta(1+r_h^2\frac{\zeta}{\eta})}T, \qquad \Phi_e = \Psi_0,\\
Q_e &= \frac{q}{4}, \qquad Q_\chi = 2\sqrt{2}\pi\sqrt{-\frac{q^2+4r_h^4(\frac{\zeta}{\eta}+\Lambda)}{r_h^2(\eta+r_h^2\zeta)}},\\
\Phi_\chi &= -\frac{r_h^2 T\eta}{32\pi}Q_\chi.
\end{aligned}
\tag{2.13}
$$

Interestingly, due to the scalar charge Q_χ on the horizon, the entropy in Eq. (2.13) disagrees with the Bekenstein-Hawking entropy, where the entropy was explicitly calculated in Ref. [25] using the Wald formalism as $S=(1+\frac{\eta}{4}(\frac{\phi'(r)^2}{g(r)})|_{r_h})\frac{A}{4}$, and $A=4\pi r_h^2$ is the horizon area of a black hole.

Note that, during investigations of the $P-V$ criticality of asymptotical AdS black hole systems, the cosmological constant Λ is usually considered as a dynamical variable. In our case, the parameters ζ and η in the Lagrangian of the action (2.1) can be also considered as dynamical variables. However, due to the absence of a scalar field potential in the action, we just have one independent parameter for these two parameters ζ and η, i.e., ζ or η can be set to unity by the redefinition of ϕ. Therefore, for simplicity and without loss of generality, we just consider ζ as the one independent parameter and fix η as a constant in the following.[2] In this case, the first law of thermodynamics of the black hole solution (2.12) can be written as

[1] For the equation of motion for the scalar field, one can obtain Eq. (10) from Ref. [24] or Eq. (3.1) from Ref. [25]. From this equation, obviously η cannot be zero. For the case $\zeta\neq 0$, one obtains the solution (2.10). Reference [24] also discussed the particular case $\zeta=0$. However, this case cannot be contained in the solution (2.10).

[2] Note that, for the case where η is the one independent parameter with constant ζ, the volume V_B will be changed. However, changes in V_B do not affect the detailed pressure function $P(r_h,T)$ [Eq. (3.4)] in the following. Hence, the $P-V$ criticality of these two cases is treated similarly.

$$dE = TdS + \Phi_e dQ_e + \Phi_\chi dQ_\chi + V_A d\Lambda + V_B dP_B, \tag{2.14}$$

where V_A is the corresponding conjugate quantity to Λ and V_B is the corresponding conjugate quantity to $P_B \equiv \zeta/(8\pi\eta)$, which is explicitly shown in Appendix A.

In addition, from the formula for the temperature in Eq. (2.11), it can be further rewritten as

$$4r_h{}^4\Pi^{*2} + (8r_h{}^2 - 4\Lambda r_h{}^4 - 16\pi T r_h{}^3 - q^2)\Pi^* + 16\pi T r_h{}^3\Lambda = 0, \tag{2.15}$$

where a new parameter $\Pi^* \equiv \frac{\zeta}{\eta}$ has been defined for convenience of later investigations.

III. $P-V$ CRITICALITY IN THE EXTENDED PHASE SPACE OF BLACK HOLES IN HORNDESKI GRAVITY

Many works have investigated $P-V$ criticality in the extended phase spaces of asymptotical AdS black holes [8–17]. Since the black hole solution in Eq. (2.10) also has asymptotical AdS behavior [24–26], we will investigate the $P-V$ criticality in its extended phase space in Horndeski gravity.

A. No $P-V$ criticality with the pressure defined as $P = -\frac{1}{8\pi}\Lambda$

In many previous investigations of $P-V$ criticality in extended phase spaces of black holes with asymptotical AdS behavior [8–17], the thermodynamic pressure P of a black hole system was usually defined as

$$P = -\frac{1}{8\pi}\Lambda. \tag{3.1}$$

Therefore, as the corresponding conjugate quantity to Λ, V_A is easily seen to be proportional to the thermodynamic volume of this black hole system from Eq. (2.14). In this subsection, we will check the possibility of $P-V_A$ criticality by using this definition of pressure.

Note that discussions of $P-V$ criticality are usually based on investigations of the pressure function $P(r_h, T)$ through the $P-r_h$ diagram [8–17]. Since V_A is complicated in our case, we also investigate $P-V$ criticality using the corresponding function $P(r_h, T)$ through the $P-r_h$ phase diagram. In Appendix B, we also give a simple proof of the equivalence between investigations of $P-V$ criticality using either $P-V$ or $P-r_h$ phase diagrams. For the pressure function $P(r_h, T)$, after inserting Eq. (3.1) into Eq. (2.15), we easily obtain

$$P(r_h, T) = \frac{4r_h{}^4\Pi^{*2} + (8r_h{}^2 - 16\pi T r_h{}^3 - q^2)\Pi^*}{8\pi(16\pi T r_h{}^3 - 4r_h{}^4\Pi^*)}, \tag{3.2}$$

and it is difficult to analytically investigate whether there is $P-V$ criticality like in the van der Waals liquid-gas system. Therefore, we check the criticality by plotting the $P-r_h$ phase diagram for different cases, and we find that there is no evidence for the existence of $P-V$ criticality in this case, since the $P-r_h$ phase diagrams are all similar to the following diagram in Fig. 1.[3]

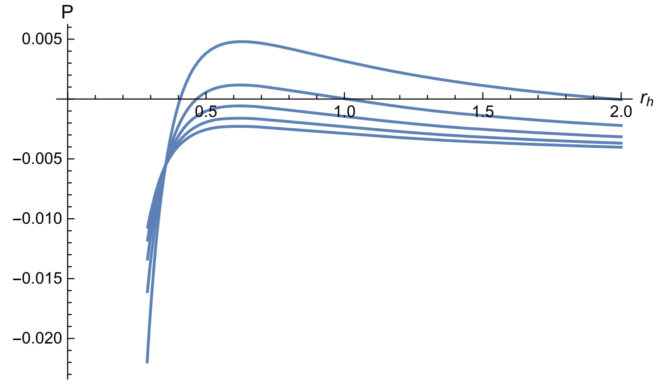

FIG. 1. The case with $q = 1$ and $\Pi^* = 0.14$, for $T = 0.15$ to 0.35 (top to bottom) with an interval of 0.05.

B. $P-V$ criticality with the more reasonable pressure $P = \frac{\zeta}{8\pi\eta}$

In the above subsection, we found that there is no $P-V$ criticality with a pressure defined as $P = -\frac{1}{8\pi}\Lambda$. However, the solution in Eq. (2.10) is an asymptotical AdS solution with an effective AdS radius l_{eff} satisfying $l_{\mathrm{eff}}^2 = \frac{3\eta}{\zeta}$, which is independent of the cosmological constant Λ. Therefore, a more reasonable pressure is defined as

$$P = -\frac{\Lambda_{\mathrm{eff}}}{8\pi} = \frac{\zeta}{8\pi\eta} = \frac{\Pi^*}{8\pi} = P_B, \tag{3.3}$$

where Λ_{eff} is an effective cosmological constant related to l_{eff} as $\Lambda_{\mathrm{eff}} = -\frac{3}{l_{\mathrm{eff}}^2} = -\frac{\zeta}{\eta}$. In this subsection, we determine the possibility of $P-V$ criticality by using this new reasonable pressure in Eq. (3.3). Note that the pressure P is a dynamical variable in the $P-V$ criticality, while there are two parameters ζ and η in the new pressure (3.3). For the sake of simplicity, we just consider the parameter ζ as a dynamical variable in the following, while the parameter η is fixed as a constant. In this case, similar to the above subsection, V_B is also easily seen to be a new thermodynamic volume of this black hole system from Eq. (2.14), and this new thermodynamic volume is the conjugate quantity to this new pressure. In the following, we will also use the pressure function $P(r_h, T)$ and plot $P-r_h$ phase diagrams to investigate the $P-V$ criticality. A simple proof of the equivalence between investigations of $P-V$ criticality using either $P-V_B$ or $P-r_h$ phase diagrams is presented in Appendix B.

[3]Strictly speaking, here regions with positive pressure in the diagram are physical.

As the new pressure function $P(r_h, T)$ corresponds to the newly defined pressure (3.3), it can also be solved using Eq. (2.15), and one obtains two branches,

$$P(r_h, T) = \frac{-(8{r_h}^2 - 4\Lambda {r_h}^4 - 16\pi T {r_h}^3 - q^2) \pm \sqrt{(8{r_h}^2 - 4\Lambda r_h^4 - 16\pi T {r_h}^3 - q^2)^2 - 256\pi\Lambda r_h^7 T}}{64\pi r_h^4}. \quad (3.4)$$

For the case with zero cosmological constant $\Lambda = 0$, one branch is $P(r_h, T) = 0$, while the other is

$$P(r_h, T) = -\frac{8{r_h}^2 - 16\pi T {r_h}^3 - q^2}{32\pi r_h^4} = \frac{\omega_1}{r_h} + \frac{\omega_2}{{r_h}^2} + \frac{\omega_4}{{r_h}^4}, \quad (3.5)$$

where

$$\omega_1 = \frac{T}{2}, \qquad \omega_2 = -\frac{1}{4\pi}, \qquad \omega_4 = \frac{q^2}{32\pi}. \quad (3.6)$$

Note that this nonzero pressure function $P(r_h, T)$ [Eq. (3.5)] is similar to the ones in an RN-AdS black hole [8] and massive gravity [16]. Moreover, the specific volume v of this black hole system is related to the horizon radius as $v = 2r_h$ [8]. Therefore, an analytical investigation of the $P - V$ criticality of this black hole system is similar to those in Refs. [8,16], and hence the critical point is obtained as [16]

$$r_{hc} = \sqrt{-\frac{6w_4}{w_2}}, \quad w_{1c} = -\frac{4}{3} w_2 \sqrt{-\frac{w_2}{6w_4}}, \quad P_c = \frac{w_2^2}{12w_4}, \quad (3.7)$$

while the corresponding critical exponents around the critical point are $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, and $\delta = 3$. These critical exponents are the same as those in Refs. [8,16].

For the case with a negative cosmological constant $\Lambda < 0$, we should choose the $+$ branch of the function $P(r_h, T)$ in Eq. (3.4), since the pressure in the $-$ branch is negative. In this $+$ branch, the analytical investigation of $P - V$ criticality is complicated, because the pressure function $P(r_h, T)$ in Eq. (3.4) is complicated. For simplicity, we just plot the function $P(r_h, T)$ to find its $P - V$ criticality in this $+$ branch. Indeed, we find that there is $P - V$ criticality for some fixed parameters, e.g., $q = 0.3$ and $\Lambda = -0.3$ in Fig. 2. However, it should be pointed out that $P - V$ criticality does not occur for all cases with fixed parameters. For example, in Fig. 3 there is no evidence that the case with $q = 5$ and $\Lambda = -0.3$ exhibits $P - V$ criticality.

Now we calculate the critical exponents in the case with fixed parameters $q = 0.3$ and $\Lambda = -0.3$. First, the critical point is determined as the inflection point in the $P - r_h$ diagram, i.e.,

$$\left.\frac{\partial P}{\partial r_h}\right|_{r_h = r_{hc}, T = T_c} = \left.\frac{\partial^2 P}{\partial r_h^2}\right|_{r_h = r_{hc}, T = T_c} = 0. \quad (3.8)$$

From these two equations, we numerically obtain the critical point as $r_{hc} \approx 0.2598$, $T_c \approx 0.8002$, and $P_c \approx 0.5775$.

On the other hand, as in typical thermodynamic systems, the critical exponents α, β, γ, and δ are defined as follows:

$$\begin{aligned} C_v &\sim |t|^{-\alpha}, \\ \Delta_v &\sim v_l - v_s \sim |t|^{\beta}, \\ K_T &= -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T \sim |t|^{-\gamma}, \\ P - P_c &\sim |v - v_c|^{\delta}, \end{aligned} \quad (3.9)$$

where $t = T/T_c - 1$, C_v is the specific heat at constant volume, v stands for the specific volume of a black hole

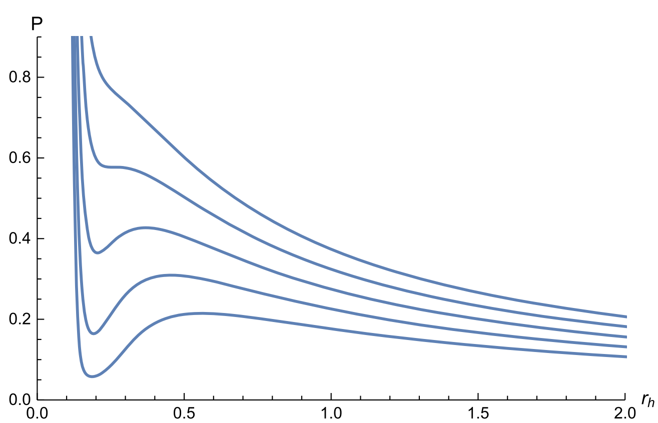

FIG. 2. The case with $q = 0.3$ and $\Lambda = -0.3$, for $T = 0.6$ to 1.0 (bottom to top) with an interval of 0.1.

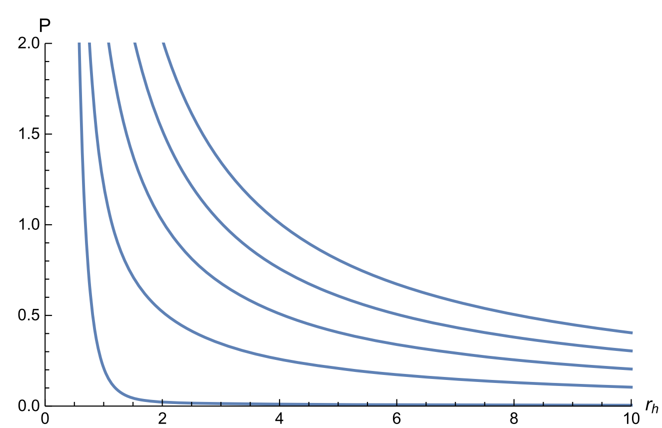

FIG. 3. The case with $q = 5$ and $\Lambda = -0.3$, for $T = 1.1$ to 9.1 (bottom to top) with an interval of 2.0.

with $v = 2r_h$, and κ_T is the isothermal compressibility. The subscript c stands for the critical point, while l and s refer to large and small black hole phases, similar to the gas and liquid phases of typical thermodynamic systems.

For the critical exponent α, we need to calculate the isopyknic heat capacity C_v at the critical point. The entropy of the black hole (2.10) was obtained in Ref. [25] as

$$S = \frac{\pi(8r_h^2 - 4r_h^4(\Lambda - \Pi^*) - q^2)}{8(r_h^2\Pi^* + 1)}, \tag{3.10}$$

and hence the isopyknic heat capacity $C_v = T(\frac{\partial S}{\partial T})_v$ at the critical point is a constant. Therefore, we have $\alpha = 0$.

For the critical exponent β, we first Taylor expand the pressure function $P(r_h, T)$ [Eq. (3.4)] near the critical point,

$$P = P_c + Rt + Bt\omega + D\omega^3 + Kt^2 + \cdots, \tag{3.11}$$

where

$$t = \frac{T}{T_c} - 1, \qquad \omega = \frac{r_h}{r_{hc}} - 1, \tag{3.12}$$

and the expansion coefficients are numerically obtained as

$$\begin{aligned} R &= T_c\left[\left(\frac{\partial P}{\partial T}\right)_{r_h}\right]_c \approx 1.4898, \\ B &= T_c r_c\left[\left(\frac{\partial^2 P}{\partial T \partial r_h}\right)\right]_c \approx -1.4120, \\ D &= r_c^3\frac{1}{3!}\left[\left(\frac{\partial^3 P}{\partial {r_h}^3}\right)_T\right]_c \approx -0.7449, \\ K &= T_c^2\frac{1}{2!}\left[\left(\frac{\partial^2 P}{\partial T^2}\right)_{r_h}\right]_c \approx 0.1230. \end{aligned} \tag{3.13}$$

For constant t, one finds $dP = (Bt + 3D\omega^2)d\omega$; therefore, after using Maxwell's area law [8,16,17], we obtain the following equation:

$$\int_{\omega_l}^{\omega_s} \omega(Bt + 3D\omega^2)d\omega + \int_{\omega_l}^{\omega_s} (Bt + 3D\omega^2)d\omega = 0, \tag{3.14}$$

where ω_l and ω_s correspond to volumes of the large and small black hole in two different phases. We further consider the condition $P_l = P_s$, i.e., the end point of vapor and the staring point of liquid during the coexistence phase have the same pressure as in typical thermodynamic liquid-gas systems, and hence from Eq. (3.11) this condition implies

$$Bt(\omega_l - \omega_s) + D(\omega_l^3 - \omega_s^3) = 0, \tag{3.15}$$

which means that the second integral in Eq. (3.14) vanishes. Therefore, Eq. (3.14) is further reduced as

$$Bt(\omega_l^2 - \omega_s^2) + \frac{3D}{2}(\omega_l^4 - \omega_s^4) = 0. \tag{3.16}$$

The nontrivial solutions of Eqs. (3.14) and (3.16) are $\omega_l = (-Bt/D)^{1/2}$ and $\omega_s = -(-Bt/D)^{1/2}$. It is easy to find that

$$\Delta_v \sim v_l - v_s = (\omega_l - \omega_s)v_c \sim |t|^{1/2}, \tag{3.17}$$

which determines the critical exponent $\beta = 1/2$.

For the critical exponent γ, the isothermal compressibility K_T can be calculated using $(\partial P/\partial r_h)_T$ from Eq. (3.11), and it is given by

$$\left(\frac{\partial P}{\partial r_h}\right)_T \simeq \frac{B}{r_{hc}}t, \tag{3.18}$$

where $\partial\omega/\partial r_h = 1/r_{hc}$ has been used. Therefore, near the critical point, the value of K_T is

$$K_T \simeq \frac{1}{Bt} \sim t^{-1}, \tag{3.19}$$

which implies $\gamma = 1$. Finally, when $T = T_c$, from Eq. (3.11) we find that

$$P - P_c \sim \omega^3 \sim (r_h - r_{hc})^3, \tag{3.20}$$

which implies that the value of the critical exponent δ is $\delta = 3$.

For the case with a positive cosmological constant $\Lambda > 0$, both of the branches of the pressure function $P(r_h, T)$ in Eq. (3.4) can be chosen, while the constraint $8{r_h}^2 - 4\Lambda {r_h}^4 - 16\pi T{r_h}^3 - q^2 < 0$ should be satisfied to keep the pressure $P(r_h, T)$ positive. For the + branch, we find that there is $P-V$ criticality and we plot the phase diagram with $q = 0.3$ and $\Lambda = 0.3$ in Fig. 4. There is no evidence for the existence of $P-V$ criticality for the − branch. In Fig. 5, we plot some phase diagrams as an illustration. Therefore, we just focus on the + branch in the following, and calculate its corresponding critical exponents. For fixed parameters $q = 0.3$ and $\Lambda = 0.3$, the critical point is numerically obtained as $r_{hc} \approx 0.2598$, $T_c \approx 0.8333$, and $P_c \approx 0.6014$. As in the

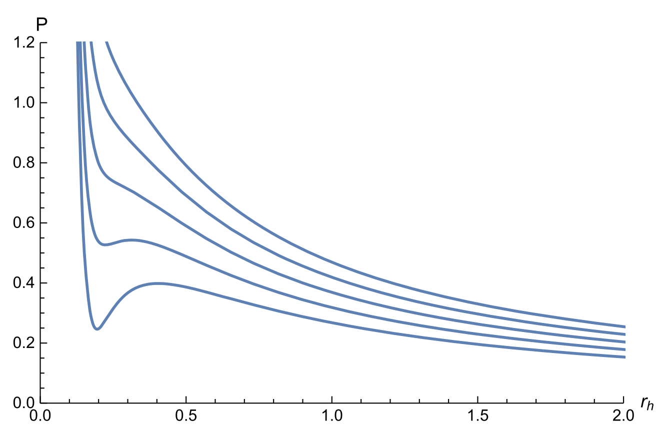

FIG. 4. The case with $q = 0.3$ and $\Lambda = 0.3$, for $T = 0.8$ to 1.2 (bottom to top) with an interval of 0.1.

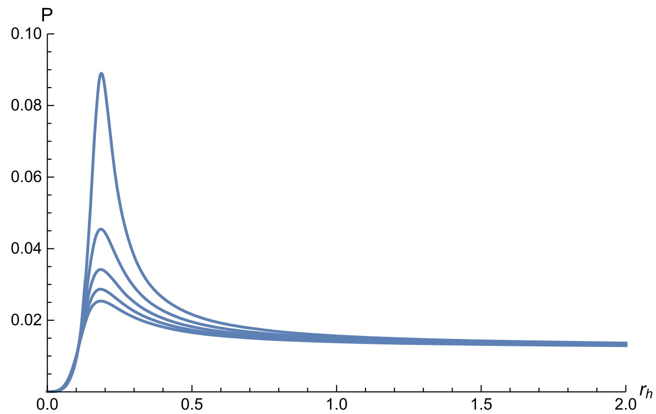

FIG. 5. The case with $q = 0.3$ and $\Lambda = 0.3$, for $T = 0.8$ to 1.2 (top to bottom) with an interval of 0.1.

above procedure, we Taylor expand the pressure function (3.4) near the critical point, while the corresponding expansion coefficients are

$$R = T_c\left[\left(\frac{\partial P}{\partial T}\right)_{r_h}\right]_c \approx 1.6598,$$
$$B = T_c r_c\left[\left(\frac{\partial^2 P}{\partial T\partial r_h}\right)\right]_c \approx -1.7525,$$
$$D = r_c^3 \frac{1}{3!}\left[\left(\frac{\partial^3 P}{\partial {r_h}^3}\right)_T\right]_c \approx -0.8299,$$
$$K = T_c^2 \frac{1}{2!}\left[\left(\frac{\partial^2 P}{\partial T^2}\right)_{r_h}\right]_c \approx -0.1632. \qquad (3.21)$$

These four coefficients are also nonzero, and it is obvious that this case is similar to the above case with $q = 0.3$ and $\Lambda = -0.3$. Therefore, the critical exponents are the same as in the above case, i.e., $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, and $\delta = 3$.

Interestingly (and obviously), all of these derived critical exponents satisfy the following thermodynamic scaling laws:

$$\alpha + 2\beta + \gamma = 2, \qquad \alpha + \beta(1+\delta) = 2,$$
$$\gamma(1+\delta) = (2-\alpha)(\delta - 1), \qquad \gamma = \beta(\delta - 1), \qquad (3.22)$$

which are same as those in the van der Waals liquid-gas system.

IV. CONCLUSION AND DISCUSSION

Einstein-Horndeski theory is the most general covariant scalar-tensor theory. Strangely enough, it was shown that $P-V$ criticality mimicking the van der Waals liquid-gas phase transition never occurs in Einstein-Horndeski theory, while such a criticality has been widely found in many gravity theories. In this paper, we investigated the $P-V$ criticality behavior in the extended phase space of a black hole system in Einstein-Horndeski theory. Through cautious analysis of the structure of this black hole system in this theory, we demonstrated that the pressure should be $P = \frac{\zeta}{8\pi\eta}$ rather than the cosmological constant $P = -\frac{1}{8\pi}\Lambda$. Physically, $\frac{\zeta}{\eta}$ uniquely corresponds to the AdS radius, and thus works as an effective cosmological constant. With this new definition of pressure, we have obtained that $P-V$ criticality indeed occurs in Einstein-Horndeski theory. Furthermore, we also calculated the critical exponents in this process and checked the scaling laws between these exponents. The exponents follow those in mean field theory, which agrees with previous studies of $P-V$ criticality in many gravity theories. It should be pointed out that $\Lambda_{\rm eff}$ is an effective cosmological constant satisfying $\Lambda_{\rm eff} = -\frac{3}{l_{\rm eff}^2} = -\frac{\zeta}{\eta}$, which is a parameter that is independent of the cosmological constant Λ. Hence, the pressure $P = \frac{\zeta}{8\pi\eta}$ can be totally independent of the cosmological constant Λ, which is a significant difference from previous works. Therefore, this paper is the first to obtain $P-V$ criticality in the two cases with $\Lambda = 0$ and $\Lambda > 0$, which implies that the cosmological constant Λ may be not a necessary pressure candidate for black holes at the microscopic level.

For the cases investigated in this paper, a subtlety related to the parameter region should be clarified, since a real scalar field ψ with $\psi^2(r) > 0$ should exist to make the solution (2.10) represent a black hole. More precisely, the corresponding constraint from Eq. (2.10) is

$$-\frac{4(\zeta + \Lambda\eta)r^4 + \eta q^2}{\eta(\zeta r^2 + \eta)} = -\frac{4(\frac{\zeta}{\eta} + \Lambda)r^4 + q^2}{\zeta r^2 + \eta}$$
$$= -\frac{4(\frac{\zeta}{\eta} + \Lambda)r^4 + q^2}{\eta(\frac{\zeta}{\eta} r^2 + 1)} > 0. \qquad (4.1)$$

In the case where the pressure is defined as $P = -\frac{1}{8\pi}\Lambda$, we have chosen $q = 1$ and $\Pi^* = \frac{\zeta}{\eta} = 0.14$ in Fig. 1. From this diagram, we see that $P = -\frac{1}{8\pi}\Lambda < 0.005$, and hence $\frac{\zeta}{\eta} + \Lambda > 0$. Therefore, the constraint (4.1) can be satisfied by setting both ζ and η to be negative. On the other hand, for the cases where the pressure is defined as $P = \frac{\zeta}{8\pi\eta}$, we easily obtain $\frac{\zeta}{\eta} + \Lambda > 0$ in cases with $\Lambda = 0$ and $\Lambda = 0.3 > 0$, while $\frac{\zeta}{\eta} + \Lambda$ is also positive in our case with $\Lambda = -0.3 < 0$ since $P = \frac{\zeta}{8\pi\eta} > 0.05$ in Fig. 2. Therefore, for these cases with the newly defined pressure, the constraint (4.1) can also be satisfied by setting both ζ and η to be negative.

Note that most of our calculations are numerical, and whether an analytical investigation of the critical point and exponents is possible is still an open question. On the other hand, there is no uniqueness theorem in Einstein-Horndeski theory, and hence further investigations of the critical behavior and critical exponents of more solutions in Einstein-Horndeski theory will also be an interesting issue. In addition, the critical exponents of $P-V$ criticality obtained in this paper and many cases all follow those in mean field theory. Critical exponents beyond mean field theory in black hole systems need to be studied further to address the underlying question of whether critical exponents beyond mean field theory depend on a special black hole solution or special gravity theory. As we know, some quantities like the temperature of a black hole only depend on the black hole solution, while quantities like the relation between the entropy of a black hole and the area of its event horizon depend on the gravitational theory [27]. Finally, another interesting issue involves the AdS/CFT correspondence. In this scenario, gravitational theory in the bulk spacetime corresponds to a conformal field theory dwelling on the asymptotical AdS conformal boundary. Since investigations of $P-V$ criticality also have an asymptotic AdS behavior in the bulk spacetime, the corresponding physical processes on the boundary should be studied further.

ACKNOWLEDGMENTS

Y. P. H. is thankful for the discussions with Profs. Hong-Bao Zhang, Hai-Qing Zhang, and Hai-Shan Liu, and Drs. Run-Qiu Yang and Yihao Yin, and also thanks the anonymous referee for constructive and helpful comments. This work is supported by the National Natural Science Foundation of China (NSFC) under Grants No. 11575083 and No. 11565017.

APPENDIX A: FIRST LAW OF THERMODYNAMICS OF BLACK HOLES REINVESTIGATED

There are three parameters Λ, ζ, and η in the Lagrangian of the action (2.1). In previous investigations of the $P-V$ criticality of asymptotical AdS black hole systems, the cosmological constant Λ was usually considered as a dynamical variable. In this paper, the parameter ζ is also considered as a dynamical variable, while η is fixed as a constant for simplicity. Therefore, the first law of thermodynamics of the black hole solution in Eq. (2.12) can be rewritten as

$$dE = TdS + \Phi_e dQ_e + \Phi_\chi dQ_\chi + V_A d\Lambda + V_B dP_B, \quad \text{(A1)}$$

where the energy E, temperature T, entropy S, charge potential Φ_e, charge Q_e, scalar charge Q_χ, and its conjugate potential Φ_χ are explicitly shown in Eqs. (2.11) and (2.13). In addition, V_A is the corresponding conjugate quantity to Λ, while V_B is the corresponding conjugate quantity to $P_B \equiv \zeta/(8\pi\eta)$, and their complicated expressions are

$$\begin{aligned} V_A = \frac{1}{768 r_h^4}\Bigg(&-\frac{24 r_h^5\zeta(-q^2\eta+8r_h^2\eta+4r_h^4(\zeta-\eta\Lambda))}{(r_h^2\zeta+\eta)(\zeta-\eta\Lambda)} - \frac{24 r_h^5\zeta(-q^2\eta+8r_h^2\eta+4r_h^4(\zeta-\eta\Lambda))}{(r_h^2\zeta+\eta)(-\zeta+\eta\Lambda)} \\ &+\frac{1}{\zeta^3\eta^{3/2}(r_h^2\zeta+\eta)(\zeta-\eta\Lambda)^2}\Bigg(-2r_h\zeta^{3/2}\eta(r_h^2\zeta+\eta)(2\sqrt{\zeta}\sqrt{\eta}(-48q^2r_h^2\zeta^2\eta+q^4\zeta^2(-3r_h^2\zeta+\eta) \\ &+16r_h^4(r_h^2\zeta-3\eta)(\zeta-\eta\Lambda)^2)+3\pi r_h^3(q^4\zeta^4+16q^2\zeta^3\eta+16\eta^2(3\zeta^2+2\zeta\eta\Lambda-\eta^2\Lambda^2))) \\ &+12r_h^4\zeta\sqrt{\frac{\zeta}{\eta}}\eta^{3/2}(r_h^2\zeta+\eta)(q^2\zeta^2+12\zeta\eta-4\eta^2\Lambda)(q^2\zeta^2+4\eta(\zeta+\eta\Lambda))\mathrm{ArcTan}\left[r_h\sqrt{\frac{\zeta}{\eta}}\right]\Bigg)\Bigg), \end{aligned} \quad \text{(A2)}$$

$$\begin{aligned} V_B = \frac{\pi\eta}{96 r_h^4\zeta^3(r_h^2\zeta+\eta)^2}\Bigg(&\frac{1}{\zeta-\eta\Lambda}6r_h^3\zeta^4(q^2-4r_h^2+4r_h^4\Lambda)(-q^2\eta+8r_h^2\eta+4r_h^4(\zeta-\eta\Lambda)) \\ &+\frac{1}{-\zeta+\eta\Lambda}6r_h^3\zeta^4(q^2-4r_h^2+4r_h^4\Lambda)(-q^2\eta+8r_h^2\eta+4r_h^4(\zeta-\eta\Lambda)) \\ &+\frac{1}{\eta^{3/2}(\zeta-\eta\Lambda)^2}(r_h^2\zeta+\eta)\Bigg(-3\pi r_h^4\sqrt{\zeta}(r_h^2\zeta+\eta)(q^4\zeta^4(3\zeta-5\eta\Lambda)+8q^2\zeta^2\eta(\zeta^2-4\zeta\eta\Lambda-\eta^2\Lambda^2) \\ &-16\eta^2(\zeta^3+7\zeta^2\eta\Lambda+3\zeta\eta^2\Lambda^2-3\eta^3\Lambda^3))+2r_h\zeta\sqrt{\eta}(q^4\zeta^2(2\eta^3\Lambda+3r_h^4\zeta^2(3\zeta-5\eta\Lambda)+2r_h^2\zeta\eta(3\zeta-5\eta\Lambda)) \\ &+24q^2r_h^2\zeta^2\eta(-4\eta^2\Lambda+r_h^2(\zeta^2-4\zeta\eta\Lambda-\eta^2\Lambda^2))+16r_h^4(2r_h^4\zeta^3(\zeta-\eta\Lambda)^2+2r_h^2\zeta\eta(\zeta-3\eta\Lambda)(\zeta-\eta\Lambda)^2 \\ &+3\eta^2(\zeta^3-\zeta^2\eta\Lambda+3\zeta\eta^2\Lambda^2-3\eta^3\Lambda^3)))+6r_h^4\sqrt{\frac{\zeta}{\eta}}\sqrt{\eta}(r_h^2\zeta+\eta)(q^2\zeta^2+4\eta(\zeta+\eta\Lambda))(q^2\zeta^2(3\zeta-5\eta\Lambda) \\ &-4\eta(\zeta^2+6\zeta\eta\Lambda-3\eta^2\Lambda^2))\mathrm{ArcTan}\left[r_h\sqrt{\frac{\zeta}{\eta}}\right]\Bigg)\Bigg). \end{aligned} \quad \text{(A3)}$$

APPENDIX B: PROOFS OF THE EQUIVALENCE BETWEEN INVESTIGATIONS OF $P-V$ CRITICALITY WITH $P-V$ OR $P-r_h$ DIAGRAMS

Note that discussions of $P-V$ criticality usually involve investigations of the pressure function $P(r_h, T)$ instead of the function $P(V, T)$ [8–17]. In those previous works, the thermodynamic volume V was usually $V = \frac{4}{3}\pi r_h^3$, and thus the equivalence between investigations of $P-V$ criticality using either $P(V, T)$ or $P(r_h, T)$ was easily obtained. In our case, the functions V_A and V_B are complicated. Therefore, we will give a simple proof in the following to show the equivalence between investigations of $P-V$ criticality using either $P-V$ or $P-r_h$ phase diagrams in our cases, i.e., we obtain the same critical point and critical exponents when using either $P-V$ or $P-r_h$ phase diagrams. For simplicity and consistency with the constraint from the real scalar field in Eq. (2.10), we fix the constant parameter η as $\eta = -1$.

1. Same critical point

Besides the three parameters Λ, ζ, and η in the action, there are in fact two other parameters M and q that come from the black hole solution (2.10). For simplicity, η is fixed as $\eta = -1$, and therefore the only parameters are Λ, ζ, M, and q. In this subsection, we give a simple proof that the critical points are same when using either $P(V, T)$ or $P(r_h, T)$. Without loss of generality, we take into account the pressure function $P(V_B, T)$ with the newly defined pressure. In this case, ζ and M are dynamical variables, while q and Λ are fixed parameters.

The critical point from the pressure function $P(V_B, T)$ is

$$\left(\frac{\partial P}{\partial V_B}\right)_T = 0, \qquad \left(\frac{\partial^2 P}{\partial {V_B}^2}\right)_T = 0, \tag{B1}$$

while the critical point from the pressure function $P(r_h, T)$ is

$$\left(\frac{\partial P}{\partial r_h}\right)_T = 0, \qquad \left(\frac{\partial^2 P}{\partial {r_h}^2}\right)_T = 0. \tag{B2}$$

In the following, we will prove that Eqs. (B1) and (B2) give the same critical point. A key point is that we use V_B and T as the two dynamical variables, and hence $P(r_h, T) = P(r_h(V_B, T), T) = P(V_B, T)$; thus, we can obtain

$$\left(\frac{\partial P}{\partial V_B}\right)_T = \left(\frac{\partial P}{\partial r_h}\right)_T \left(\frac{\partial r_h}{\partial V_B}\right)_T, \tag{B3}$$

$$\left(\frac{\partial^2 P}{\partial {V_B}^2}\right)_T = \left[\left(\frac{\partial^2 P}{\partial {r_h}^2}\right)_T \left(\frac{\partial r_h}{\partial V_B}\right)_T + \left(\frac{\partial P}{\partial r_h}\right)_T \left(\frac{\partial}{\partial r_h}\right)_T \left(\frac{\partial r_h}{\partial V_B}\right)_T\right] \left(\frac{\partial r_h}{\partial V_B}\right)_T. \tag{B4}$$

We will prove that $(\frac{\partial r_h}{\partial V_B})_T$ is nonzero around the critical point in the following subsection. Therefore, Eqs. (B1) and (B2) are easily found to be equivalent, and hence they obtain the same critical point.

2. Same critical exponents

In this paper, $P-V$ criticality only exists in the case with the newly defined pressure. Therefore, we focus on this case in the following discussions. The critical exponent α is deduced from either

$$C_{V_B} = \left(\frac{\mathrm{TdS}}{\mathrm{dT}}\right)_{V_B} \tag{B5}$$

or

$$C_v = \left(\frac{\mathrm{TdS}}{\mathrm{dT}}\right)_v, \tag{B6}$$

while v is the specific volume $v = 2r_h$, and the entropy S is obtained in Eq. (3.10) as the function $S(P, r_h)$. In our cases, we can prove that the isopyknic heat capacity C_{V_B} and C_v at the critical point are both constant. Therefore, we have $\alpha = 0$.

For the critical exponents β, γ, and δ, we need to expand the pressure function $P(V_B, T)$ around the critical point like $P(r_h, T)$ in Eq. (3.11). In the case with the pressure function $P(V_B, T)$, the expansion around the critical point is

$$\begin{aligned} P_B(V_B, T) = P_{\mathrm{Bc}} &+ \left[\left(\frac{\partial P_B}{\partial T}\right)_{V_B}\right]_c (T - T_c) \\ &+ \frac{1}{2}\left[\left(\frac{\partial^2 P_B}{\partial T^2}\right)_{V_B}\right]_c (T - T_c)^2 \\ &+ \left[\frac{\partial^2 P_B}{\partial T \partial V_B}\right]_c (V_B - V_{\mathrm{Bc}})(T - T_c) \\ &+ \frac{1}{6}\left[\left(\frac{\partial^3 P_B}{\partial {V_B}^3}\right)_T\right]_c (V_B - V_{\mathrm{Bc}})^3 + \cdots, \end{aligned} \tag{B7}$$

where B labels the pressure P corresponding to V_B. Hereafter, i.e., $P_B(V_B, T) = P(V_B, T)$. We can obtain

$$\left(\frac{\partial P_B}{\partial T}\right)_{V_B} = \left(\frac{\partial P_B}{\partial r_h}\right)_T \left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \left(\frac{\partial P_B}{\partial T}\right)_{r_h}, \tag{B8}$$

where $P_B(r_h, T) = P_B(r_h(V_B, T), T) = P_B(V_B, T)$ has been used, while the other coefficients are as follows:

$$\begin{aligned}
\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial P_B}{\partial V_B}\right)_T &= \left(\frac{\partial}{\partial T}\right)_{V_B}\left[\left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial r_h}{\partial V_B}\right)_T\right] \\
&= \left(\frac{\partial r_h}{\partial V_B}\right)_T\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial P_B}{\partial r_h}\right)_T + \left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial r_h}{\partial V_B}\right)_T \\
&= \left[\left(\frac{\partial^2 P_B}{\partial {r_h}^2}\right)_T\left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \frac{\partial^2 P_B}{\partial T \partial r_h}\right]\left(\frac{\partial r_h}{\partial V_B}\right)_T + \left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial r_h}{\partial V_B}\right)_T,
\end{aligned} \tag{B9}$$

and

$$\begin{aligned}
\left(\frac{\partial^2 P_B}{\partial T^2}\right)_{V_B} &= \left(\frac{\partial}{\partial T}\right)_{V_B}\left[\left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \left(\frac{\partial P_B}{\partial T}\right)_{r_h}\right] \\
&= \left(\frac{\partial r_h}{\partial T}\right)_{V_B}\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial P_B}{\partial r_h}\right)_T + \left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial r_h}{\partial V_B}\right)_T + \left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial P_B}{\partial T}\right)_{r_h} \\
&= \left[\left(\frac{\partial^2 P_B}{\partial {r_h}^2}\right)_T\left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \frac{\partial^2 P_B}{\partial T \partial r_h}\right]\left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial r_h}{\partial V_B}\right)_T + \left(\frac{\partial}{\partial T}\right)_{V_B}\left(\frac{\partial P_B}{\partial T}\right)_{r_h}, \\
\left(\frac{\partial^3 P_B}{\partial {V_B}^3}\right)_T &= \left(\frac{\partial r_h}{\partial V_B}\right)_T\left(\frac{\partial}{\partial r_h}\right)_T\left(\frac{\partial^2 P_B}{\partial {V_B}^2}\right)_T \\
&= \left[\left(\frac{\partial}{\partial r_h}\right)_T\left(\left[\left(\frac{\partial^2 P_B}{\partial {r_h}^2}\right)_T\left(\frac{\partial r_h}{\partial V_B}\right)_T + \left(\frac{\partial P_B}{\partial r_h}\right)_T\left(\frac{\partial}{\partial r_h}\right)_T\left(\frac{\partial r_h}{\partial V_B}\right)_T\right]\left(\frac{\partial r_h}{\partial V_B}\right)_T\right)\right]\left(\frac{\partial r_h}{\partial V_B}\right)_T.
\end{aligned} \tag{B10}$$

At the critical point, we can find that these coefficients are nonzero. For example, we obtain these coefficients in the case with $q = 0.3$ and $\Lambda = -0.3$ as

$$\begin{aligned}
&\left[\left(\frac{\partial P_B}{\partial T}\right)_{V_B}\right]_c = \left[\left(\frac{\partial P_B}{\partial T}\right)_{r_h}\right]_c = 1.8617, \qquad \left[\left(\frac{\partial^2 P_B}{\partial T^2}\right)_{V_B}\right]_c = 2\left[\frac{\partial^2 P_B}{\partial T \partial r_h}\left(\frac{\partial r_h}{\partial T}\right)_{V_B}\right]_c + \left[\left(\frac{\partial^2 P_B}{\partial T^2}\right)_{r_h}\right]_c = -9.700, \\
&\left[\frac{\partial^2 P_B}{\partial T \partial V_B}\right]_c = \left[\frac{\partial^2 P_B}{\partial T \partial r_h}\left(\frac{\partial r_h}{\partial V_B}\right)_T\right]_c = -17.1345, \qquad \left[\left(\frac{\partial^3 P_B}{\partial {V_B}^3}\right)_T\right]_c = \left[\left(\frac{\partial^3 P_B}{\partial {r_h}^3}\right)_T\left(\frac{\partial r_h}{\partial V_B}\right)_T^3\right]_c = -4093.1,
\end{aligned} \tag{B11}$$

where several numerical values at the critical point have been used:

$$\begin{aligned}
&\left[\left(\frac{\partial V_B}{\partial r_h}\right)_{P_B}\right]_c \approx 0.3964, \quad \left[\left(\frac{\partial V_B}{\partial P_B}\right)_{r_h}\right]_c \approx -0.1581, \quad \left[\left(\frac{\partial T}{\partial r_h}\right)_{P_B}\right]_c \approx 5.9369 \times 10^{-15}, \quad \left[\left(\frac{\partial T}{\partial P_B}\right)_{r_h}\right]_c \approx 0.5372, \\
&\left[\left(\frac{\partial r_h}{\partial V_B}\right)_T\right]_c \approx 2.5230, \quad \left[\left(\frac{\partial P_B}{\partial V_B}\right)_T\right]_c \approx -2.7886 \times 10^{-14}, \quad \left[\left(\frac{\partial r_h}{\partial T}\right)_{V_B}\right]_c \approx 0.7424, \quad \left[\left(\frac{\partial P_B}{\partial T}\right)_{V_B}\right]_c \approx 1.8617, \\
&\left[\left(\frac{\partial P_B}{\partial T}\right)_{r_h}\right]_c \approx 1.8617, \quad \left[\left(\frac{\partial^2 P_B}{\partial T \partial r_h}\right)\right]_c \approx -6.7913, \quad \left[\left(\frac{\partial^3 P_B}{\partial {r_h}^3}\right)_T\right]_c \approx -254.853, \quad \left[\left(\frac{\partial^2 P_B}{\partial T^2}\right)_{r_h}\right]_c \approx 0.3839.
\end{aligned}$$

Since the coefficients in Eq. (B11) are nonzero, similar to discussions in the main text and Refs. [8,16,17], the three critical exponents β, γ and δ are same.

Note that the quantities in the above expressions are not an easy work to obtain. Therefore, in the following we will give a detailed calculation of $(\frac{\partial r_h}{\partial V_B})_T$ as an example. The starting point is that we have obtained the temperature and thermodynamic volume as functions of r_h and P_B, i.e., $T(r_h, P_B)$ and $V_B(r_h, P_B)$ in Eqs. (2.11) and (A3). Since there are just two dynamical variables in our cases, in principle we can also use T and V_B as the two new dynamical variables. Therefore, from $T(r_h, P_B) \equiv T(r_h(T, V_B), P_B(T, V_B))$ and $V_B(r_h, P_B) \equiv V_B(r_h(T, V_B), P_B(T, V_B))$, we deduce the following key relations:

$$\begin{aligned}
\mathrm{dT} &= \left(\frac{\partial T}{\partial r_h}\right)_{P_B}\mathrm{dr}_h + \left(\frac{\partial T}{\partial P_B}\right)_{r_h}\mathrm{dP}_B \\
&= \left(\frac{\partial T}{\partial r_h}\right)_{P_B}\left[\left(\frac{\partial r_h}{\partial V_B}\right)_T\mathrm{dV}_B + \left(\frac{\partial r_h}{\partial T}\right)_{V_B}\mathrm{dT}\right] + \left(\frac{\partial T}{\partial P_B}\right)_{r_h}\left[\left(\frac{\partial P_B}{\partial V_B}\right)_T\mathrm{dV}_B + \left(\frac{\partial P_B}{\partial T}\right)_{V_B}\mathrm{dT}\right] \\
&= \left[\left(\frac{\partial T}{\partial r_h}\right)_{P_B}\left(\frac{\partial r_h}{\partial V_B}\right)_T + \left(\frac{\partial T}{\partial P_B}\right)_{r_h}\left(\frac{\partial P_B}{\partial V_B}\right)_T\right]\mathrm{dV}_B + \left[\left(\frac{\partial T}{\partial r_h}\right)_{P_B}\left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \left(\frac{\partial T}{\partial P_B}\right)_{r_h}\left(\frac{\partial P_B}{\partial T}\right)_{V_B}\right]\mathrm{dT}, \qquad \text{(B12)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{dV}_B &= \left(\frac{\partial V_B}{\partial r_h}\right)_{P_B}\mathrm{dr}_h + \left(\frac{\partial V_B}{\partial P_B}\right)_{r_h}\mathrm{dP}_B \\
&= \left(\frac{\partial V_B}{\partial r_h}\right)_{P_B}\left[\left(\frac{\partial r_h}{\partial V_B}\right)_T\mathrm{dV}_B + \left(\frac{\partial r_h}{\partial T}\right)_{V_B}\mathrm{dT}\right] + \left(\frac{\partial V_B}{\partial P_B}\right)_{r_h}\left[\left(\frac{\partial P_B}{\partial V_B}\right)_T\mathrm{dV}_B + \left(\frac{\partial P_B}{\partial T}\right)_{V_B}\mathrm{dT}\right] \\
&= \left[\left(\frac{\partial V_B}{\partial r_h}\right)_{P_B}\left(\frac{\partial r_h}{\partial v_B}\right)_T + \left(\frac{\partial V_B}{\partial P_B}\right)_{r_h}\left(\frac{\partial P_B}{\partial V_B}\right)_T\right]\mathrm{dV}_B + \left[\left(\frac{\partial V_B}{\partial r_h}\right)_{P_B}\left(\frac{\partial r_h}{\partial T}\right)_{V_B} + \left(\frac{\partial V_B}{\partial P_B}\right)_{r_h}\left(\frac{\partial P_B}{\partial T}\right)_{V_B}\right]\mathrm{dT}. \qquad \text{(B13)}
\end{aligned}$$

From these relations, we obtain four equations that should be satisfied:

$$\begin{cases}
(\frac{\partial V_B}{\partial r_h})_{P_B}(\frac{\partial r_h}{\partial V_B})_T + (\frac{\partial V_B}{\partial P_B})_{r_h}(\frac{\partial P_B}{\partial V_B})_T = 1, \\
(\frac{\partial T}{\partial r_h})_{P_B}(\frac{\partial r_h}{\partial V_B})_T + (\frac{\partial T}{\partial P_B})_{r_h}(\frac{\partial P_B}{\partial V_B})_T = 0,
\end{cases} \qquad \text{(B14)}$$

$$\begin{cases}
(\frac{\partial V_B}{\partial r_h})_{P_B}(\frac{\partial r_h}{\partial T})_{V_B} + (\frac{\partial V_B}{\partial P_B})_{r_h}(\frac{\partial P_B}{\partial T})_{V_B} = 0, \\
(\frac{\partial T}{\partial r_h})_{P_B}(\frac{\partial r_h}{\partial T})_{V_B} + (\frac{\partial T}{\partial P_B})_{r_h}(\frac{\partial P_B}{\partial T})_{V_B} = 1.
\end{cases} \qquad \text{(B15)}$$

Therefore, we finally obtain

$$\begin{cases}
(\frac{\partial r_h}{\partial V_B})_T = \frac{1}{(\frac{\partial V_B}{\partial r_h})_{P_B} - (\frac{\partial V_B}{\partial P_B})_{r_h}\frac{(\frac{\partial T}{\partial r_h})_{P_B}}{(\frac{\partial T}{\partial P_B})_{r_h}}}, \\
(\frac{\partial P_B}{\partial V_B})_T = \frac{1}{(\frac{\partial V_B}{\partial P_B})_{r_h} - (\frac{\partial V_B}{\partial r_h})_{P_B}\frac{(\frac{\partial T}{\partial P_B})_{r_h}}{(\frac{\partial T}{\partial r_h})_{P_B}}}, \\
(\frac{\partial r_h}{\partial T})_{V_B} = \frac{1}{(\frac{\partial T}{\partial r_h})_{P_B} - (\frac{\partial T}{\partial P_B})_{r_h}\frac{(\frac{\partial V_B}{\partial r_h})_{P_B}}{(\frac{\partial V_B}{\partial P_B})_{r_h}}}, \\
(\frac{\partial P_B}{\partial T})_{V_B} = \frac{1}{(\frac{\partial T}{\partial P_B})_{r_h} - (\frac{\partial T}{\partial r_h})_{P_B}\frac{(\frac{\partial V_B}{\partial P_B})_{r_h}}{(\frac{\partial v_B}{\partial r_h})_{P_B}}}.
\end{cases} \qquad \text{(B16)}$$

Therefore, from the known functions $T(r_h, P_B)$ and $V_B(r_h, P_B)$ in Eqs. (2.11) and (A3), we can easily obtain $(\frac{\partial r_h}{\partial V_B})_T$ and other quantities around the critical point.

[1] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973); J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).

[2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[3] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).

[4] G. W. Gibbons and M. J. Perry, Proc. R. Soc. A **358**, 467 (1978).

[5] Y. P. Hu, F. Pan, and X. M. Wu, Phys. Lett. B **772**, 553 (2017).
[6] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[7] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D **60**, 104026 (1999).
[8] D. Kubiznak and R. B. Mann, J. High Energy Phys. 07 (2012) 033.
[9] S. Gunasekaran, R. B. Mann, and D. Kubiznak, J. High Energy Phys. 11 (2012) 110; N. Altamirano, D. Kubiznak, and R. B. Mann, Phys. Rev. D **88**, 101502 (2013).
[10] S. H. Hendi and M. H. Vahidinia, Phys. Rev. D **88**, 084045 (2013); S. H. Hendi and A. Dehghani, Eur. Phys. J. C **79**, 227 (2019).
[11] R. Zhao, H. H. Zhao, M. S. Ma, and L. C. Zhang, Eur. Phys. J. C **73**, 2645 (2013).
[12] S. W. Wei and Y. X. Liu, Phys. Rev. D **87**, 044014 (2013); Phys. Rev. Lett. **115**, 111302 (2015); **116**, 169903(E) (2016).
[13] E. Spallucci and A. Smailagic, Phys. Lett. B **723**, 436 (2013).
[14] Y. G. Miao and Z. M. Xu, Phys. Rev. D **98**, 084051 (2018).
[15] R. G. Cai, L. M. Cao, L. Li, and R. Q. Yang, J. High Energy Phys. 09 (2013) 005.
[16] J. Xu, L. M. Cao, and Y. P. Hu, Phys. Rev. D **91**, 124033 (2015).
[17] B. R. Majhi and S. Samanta, Phys. Lett. B **773**, 203 (2017); K. Bhattacharya and B. R. Majhi, Phys. Rev. D **95**, 104024 (2017); K. Bhattacharya, B. R. Majhi, and S. Samanta, Phys. Rev. D **96**, 084037 (2017).
[18] D. Kastor, S. Ray, and J. Traschen, Classical Quantum Gravity **26**, 195011 (2009).
[19] G. W. Gibbons, R. Kallosh, and B. Kol, Phys. Rev. Lett. **77**, 4992 (1996); J. D. E. Creighton and R. B. Mann, Phys. Rev. D **52**, 4569 (1995).
[20] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[21] C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, Phys. Rev. D **84**, 064039 (2011).
[22] X. Gao, Phys. Rev. D **90**, 081501 (2014); **90**, 104033 (2014); X. Gao and Z. b. Yao, arXiv:1806.02811.
[23] A. Anabalon, A. Cisterna, and J. Oliva, Phys. Rev. D **89**, 084050 (2014); M. Minamitsuji, Phys. Rev. D **89**, 064017 (2014).
[24] A. Cisterna and C. Erices, Phys. Rev. D **89**, 084038 (2014).
[25] X. H. Feng, H. S. Liu, H. Lü, and C. N. Pope, Phys. Rev. D **93**, 044030 (2016).
[26] Y. G. Miao and Z. M. Xu, Eur. Phys. J. C **76**, 638 (2016).
[27] R. G. Cai, L. M. Cao, and Y. P. Hu, J. High Energy Phys. 08 (2008) 090.