Constant-roll approach to noncanonical inflation

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The scenario of constant-roll inflation in the framework of a noncanonical inflaton model is studied. Both of these modifications lead to the appearance of some differences in the slow-roll parameters besides the Friedmann equations, resulting in a better justification of theoretical predictions compared to observations. Phenomenologically, by assuming a constant η, i.e., a second slow-roll parameter, and recalculating the related perturbation equations, obviously there should appear some modification in the scalar spectral index and amplitude of scalar perturbations. It is shown that finding an exact solution for the Hubble parameter is one of the main advantages and triumphs of this approach. Also, whereas making a connection between subhorizon and super-horizon regions has a crucial role in inflationary studies, the main perturbation parameters are obtained at the horizon crossing time. To examine the accuracy of our results, we consider the Planck 2018 results as a confident criterion. To do so by virtue of the $r - n_s$ diagram, the acceptable ranges of the free parameters of the model are illustrated. As a result, it is found that the second slow-roll parameter should be a positive constant and smaller than unity. By constraining the free parameters of the model, also, an energy scale of inflation is estimated that is of order 10^{-2}. Even more, by investigating the attractor behavior of the model, it is clear that the aforementioned properties can be appropriately satisfied.

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I. INTRODUCTION

The first ideas about the very early Universe to cope with (at least) three fundamental problems of the standard big bang model date back to 1981, and the paper written by Guth [1], called "old inflation." Before this idea, a type of inflationary model was proposed by A. Starobinsky based on the theory of conformal anomaly in quantum gravity [2,3]. The model was complicated and had a different goal than solving the horizon and flatness problem. The old inflation model posed a problem in justifying a smooth exit at the end of inflation. To solve this fatal problem, many models of inflation have been put forward such as new inflation [4,5], chaotic inflation [6], k-inflation [7,8], brane inflation [9,10], G-inflation [11–14], warm inflation [15–21], etc. Inflation is assumed to be a phase of a very rapid accelerated expansion of the super–high energy Universe. In single or multiple field scenarios of inflation, the Universe usually evolved in the presence of scalar fields [22,23]. Nonetheless, there are some models in which instead of scalar fields, for instance, the gauge fields play the main role [24,25].

Most inflationary models are based on the slow-roll assumption, in which the scalar field is present and rolls down slowly from the top of its potential to the bottom of the hill. The flatness of the potential provides a condition for having a quasi–de Sitter expansion, weak dependence on the time of the Hubble parameter, especially at the beginning of the inflation. Besides the aforementioned necessary parts to run inflation, we need small enough slow-roll parameters to cope with the well-known three problems of the hot big bang theory [26]. The definition of the first slow-roll parameter is based on the time evolution of the Hubble parameter divided by its square, i.e., $\epsilon = -\dot{H}/H^2$, and the condition $\epsilon < 1$ is required to have an accelerated expansion phase at the initiation eras ($\ddot{a} > 0$). The second slow-roll parameter is defined as $\eta = \ddot{\phi}/H\dot{\phi}$, in which we hold out the rate of the time derivative of the scalar field during the Hubble time [27]. Smallness of the latter parameter states that $\dot{\phi}$ should vary very slowly, and it therefore guarantees the appropriate behavior of the inflation.

Whereas the single field canonical versions of inflation are not able to cover all results that arise through observation, one

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might seek a modification in the inflationary models [28–30]. One possible extension, for instance, is quasisingle field inflation, or maybe multiple field inflation [31–33], etc. Another proposal addresses the scalar field where the Lagrangian is a function of the scalar field ϕ and the kinetic term $X = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$; i.e., $\mathcal{L}(\phi, X)$ is the k-essence model [7,8,34]. In this model, the sound speed is no longer equal to the light speed, and it could be smaller, which leads to a smaller tensor-to-scalar ratio to meet the observational range. One type of the aforementioned Lagrangian is with a modified kinetic term of the canonical scalar field Lagrangian. In this work, the modification to the kinetic term is restricted to the time derivative of the scalar field so that it is possible to receive the canonical one. This type of Lagrangian has received attention for application to cosmological studies; for instance, one can see [35]. In that reference, De-Santiago *et al.* studied the dynamical system of the Universe when it is filled with a barotropic fluid and a scalar field with a modified kinetic term. Also, in [36], a noncanonical scalar field with a modified Lagrangian has been taken as a candidate for dark energy. It has been claimed that, for a simple choice of the modified kinetic term, this model can be considered a unified model of dark matter and dark energy. Besides, these models, which usually are based on modification of the kinetic portion, have received more attention recently because of their ability to justify the inflation behavior. In [37], Mukhanov and Vikman showed that, for specific choices of free parameters, the tensor-to-scalar ratio, which originates from enhancement of the sound speed, could increase. It has been indicated that other consequences of this model can be enumerated as larger energy scales and higher temperatures for reheating [38]. In the latter reference, one can find some good clues about intermediate inflation for the noncanonical model with a power-law kinetic term. Additionally Unnikrishnan *et al.* [39] determined that, despite the canonical intermediate inflation, the noncanonical model could properly satisfy the observational data in their case for chaotic inflation. The quantities of the model are derived, and the existence of an inflationary attractor is confirmed. Even more, Unnikrishnan *et al.* showed that, for a steep potential, the model is able to properly describe inflation. In addition, the results of [40] have clarified the fact that the power-law inflation in the standard model of inflation leads to a tensor-to-scalar ratio which is out of range compared to observational data. However, by using a noncanonical scalar field, there could be a new power-law inflation for which predictions are consistent with the Planck results. In [41] the scenario of warm inflation is extended to the noncanonical case, where it means that the kinetic term will be modified again. The new, but still scale invariant, curvature spectrum is obtained, and it is demonstrated that the tensor-to-scalar ratio is insignificant for the strong regime and significant for the weak regime.

Such models have a better consistency with observational data compared to canonical scalar field models. Some interesting features of the noncanonical scalar field model can be addressed as follows [42]:

(a) Steep potentials like $v(\phi) \propto \phi^{-n}$, which are known as the dark energy potential, could give a better inflaton potential in a noncanonical scalar field than in canonical ones.
(b) The consistency relation $r = -8n_T$ is violated in the noncanonical scalar field model of inflation.
(c) In the canonical scalar field models of inflation, the exponential potential roughly stands in an acceptable range of data. However, this type of potential in the noncanonical model of inflation could show better agreement with the data.

The slow-roll features of inflation are almost provided by a potential with a flat part. It prompts this question: what happens if the potential is exactly flat? This question was considered for the first time in [43]. From the equation of motion of the scalar field, it is determined that for a flat potential the second slow-roll parameter becomes $\eta = -3$, which is not smaller than unity; actually it is of order 1. After that, in [44], Namjoo *et al.* studied the non-Guassianity of the case, and it was specified that the non-Gaussianity cannot be ignored anymore and that it could be of order 1 [44]. The concept of having flat potential was generalized in [45], where Martin *et al.* assumed that η could be a constant. They found an approximate solution for the model and obtained a scalar perturbation amplitude that could vary even on super-horizon scales, and also, for some choices, it could be scale invariant. In [46], where for the first time the term "constant roll" was addressed, the same model was reconsider by using the Hamilton-Jacobi formalism [47–53] and the authors found an exact solution for the Hubble parameter which possesses the attractor behavior as well. Also, it was concluded that the power spectrum could remain scale invariant for specific choices of the constant. The scenario of constant-roll inflation in modified gravity was studied in [54–58], and the generalized version of this approach, known as smooth-roll inflation could be found in [59–61].

The interesting feature and application of noncanonical scalar field in slow-roll inflationary scenarios motivated us to investigate this model for constant-roll inflation. The perturbation equations will be reconsidered, and we will find the modified amplitude of scalar perturbations, and also, the correction to the scalar spectral index are determined which are of the second order of η. Comparing the result with observational data showed that, for some specific values of η, one could have obtained a scale invariant perturbation on superhorizon scales.

The paper is organized as follows: In Sec. II, the general formulas of the framework of the noncanonical scalar field will be obtained, and they are rewritten for specific choices of the kinetic term. The slow-roll parameters and the differential equation for the Hubble parameter will be addressed in Sec. III, where the constant-roll approach is applied to the equations. The scalar and tensor perturbations will be discussed in Sec. IV, and the power spectrum of the perturbations is derived. And finally we conclude and discuss our results in Sec. V.

II. NONCANONICAL SCALAR FIELD MODEL

The action is assumed to be

$$S = \frac{-1}{16\pi G}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}(\phi, X), \quad (1)$$

where $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$, and where $\mathcal{L}(\phi, X)$ is the Lagrangian of the noncanonical scalar field that in general is an arbitrary function of the scalar field ϕ and X. Variation of the action with respect to the metric comes to the field equation of the model

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left(\frac{\partial\mathcal{L}}{\partial X}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\right), \quad (2)$$

and also, variation of the action with respect to the field come to the following equation of motion:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0. \quad (3)$$

It is assumed that the geometry of the Universe is described by a spatially flat Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \quad (4)$$

Since the term in parentheses on the right-hand side of field equation (1) is the energy-momentum tensor of the scalar field, in a comparison to the energy-momentum of a perfect fluid $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$, it is concluded that the energy density, pressure, and four velocity of the field are

$$\rho = 2X\frac{\partial\mathcal{L}}{\partial X} - \mathcal{L}, \qquad p = \mathcal{L}, \qquad u_\mu = \frac{\partial_\mu\phi}{\sqrt{2X}}. \quad (5)$$

Substituting this metric into field equation (2), one arrives at the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho, \qquad \dot{H} = -4\pi G(\rho + p). \quad (6)$$

Also, the equation of motion (3) for this geometry reads [42]

$$\left(\frac{\partial\mathcal{L}}{\partial X} + 2X\frac{\partial^2\mathcal{L}}{\partial X^2}\right)\ddot{\phi} + \left(3H\frac{\partial\mathcal{L}}{\partial X} + \dot{\phi}\frac{\partial\mathcal{L}}{\partial X\partial\phi}\right)\dot{\phi} - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \quad (7)$$

In this work, we are going to work with a specific type of k-essence Lagrangian which contains a modification of the kinetic term. We notice that this model leads to interesting results working out inflation besides the study of dynamical system of the Universe and dark energy. The Lagrangian is assumed to be

$$\mathcal{L} = X\left(\frac{X}{M^4}\right)^{\alpha-1} - V(\phi), \quad (8)$$

where α is a dimensionless constant and M is a constant with mass dimension. Using this definition for a scalar field Lagrangian, its energy density and pressure are expressed by

$$\rho = (2\alpha - 1)X\left(\frac{X}{M^4}\right)^{\alpha-1} + V(\phi),$$
$$p = X\left(\frac{X}{M^4}\right)^{\alpha-1} - V(\phi). \quad (9)$$

Then, equation of motion (7) is reduced to

$$\ddot{\phi} + \frac{3H}{2\alpha - 1}\dot{\phi} + \left(\frac{2M^4}{\dot{\phi}^2}\right)^{\alpha-1}\frac{V_\phi(\phi)}{\alpha(2\alpha - 1)} = 0. \quad (10)$$

Introducing the Hubble parameters as a function of scalar field, and using Eqs. (6) and (9) and relation $\dot{H} = \dot{\phi}H_\phi(\phi)$, the time derivatives of the scalar field are given by

$$\dot{\phi}^{2\alpha-1} = \frac{-2^\alpha M_p^{2(2\alpha-1)}\mu^{4(\alpha-1)}}{\alpha}H_{,\phi}(\phi), \quad (11)$$

in which $M_p^2 = 1/8\pi G$ and $\mu \equiv M/M_p$. Then, substituting this into the Friedmann equation (6) and using Eq. (9), the potential of the scalar field reads

$$V(\phi) = 3M_p^2H^2(\phi) - \frac{(2\alpha-1)M_p^{4\alpha}}{2^\alpha M^{4(\alpha-1)}}\left[\frac{-2^\alpha\mu^{4(\alpha-1)}}{\alpha}H_{,\phi}(\phi)\right]^{\frac{2\alpha}{2\alpha-1}}. \quad (12)$$

For the rest of the paper, the reduced mass Planck is taken to be $M_p = 1$ for more convenience, and we also define the new parameter $\lambda = M^{4(\alpha-1)}$.

III. NONCANONICAL INFLATION

During inflation, the Universe undergoes an extreme expansion in a short period of time. Here, it is assumed that the inflation is caused by a noncanonical scalar field. Usually, inflation is describe by using slow-roll parameters. The first slow-roll parameter indicates the rate of the Hubble parameter during a Hubble time as [27]

$$\epsilon = -\frac{\dot{H}}{H^2} \quad (13)$$

so that, to have a quasi–de Sitter expansion, this parameter should be much smaller than unity [27]. Using this equation and Eq. (6), one could obtain the time derivative of the scalar field in terms of the Hubble parameter and the slow-roll parameters ϵ as [42]

$$\dot{\phi}^{2\alpha} = \frac{2^\alpha \lambda}{\alpha} \epsilon H^2. \tag{14}$$

Assuming the Hubble parameter to be a function of the scalar field, $H \coloneqq H(\phi)$, one has $\dot{H} = \dot{\phi} H_{,\phi}$. Then, the slow-roll parameter ϵ reads

$$\epsilon = \left(\frac{2^\alpha \lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{H_{,\phi}^{\frac{2\alpha}{2\alpha-1}}}{H^2}. \tag{15}$$

The second slow-roll parameter is defined as $\eta = \ddot{\phi}/\dot{\phi}H$, where in constant-roll inflation it is assumed to be a constant, $\eta = \beta$. Using this definition and also Eqs. (14) and (15), we arrive at the following differential equation for the Hubble parameter,

$$H_{,\phi}^{\frac{2-2\alpha}{2\alpha-1}} H_{,\phi\phi} = C_0 H, \qquad C_0 \equiv (2\alpha - 1)\beta \left(\frac{\alpha}{2^\alpha \lambda}\right)^{\frac{1}{2\alpha-1}}, \tag{16}$$

which comes to the same differential equation for the Hubble parameter as in [46], where Motohashi *et al.* consider the constant-roll inflation for the canonical scalar field. Since $H_{,\phi} = dH/d\phi$, there is $d\phi = dH/H_{,\phi}$. Therefore, for the second derivative of the Hubble parameter in terms of the scalar field, we have $H_{,\phi\phi} = dH_{,\phi}/d\phi = H_{,\phi} dH_{,\phi}/dH$. Substituting this into the above differential equation, one arrives at

$$H_{,\phi}^{\frac{2\alpha}{2\alpha-1}} = \frac{\alpha C_0}{2\alpha - 1} H^2 + C_1, \tag{17}$$

where C_1 is the constant of integration. Taking another integration from Eq. (17), the scalar field could be expressed in terms of the Hubble parameter as

$$\phi = \phi_0 + \frac{H}{C_1}\, {}_2F_1\left[\frac{1}{2}, 1 - \frac{1}{2\alpha}, \frac{3}{2}, \frac{C_0 \alpha}{C_1(1-2\alpha)} H^2\right], \tag{18}$$

in which ϕ_0 is a constant of integration too. From Eqs. (17) and (18), it seems that every quantity could be expressed in terms of the Hubble parameter, such as the slow-roll parameter

$$\epsilon = \left(\frac{2^\alpha \lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{\left(\frac{\alpha C_0}{2\alpha-1} H^2 + C_1\right)}{H^2}, \tag{19}$$

and also, for the number of e-folds, there is

$$N = \int_{t_i}^{t_e} H dt = \int_{H_i}^{H_e} \frac{H}{\dot{H}} dH = \int_{H_i}^{H_e} \frac{-1}{\epsilon H} dH$$
$$= -\left(\frac{\alpha}{2^\alpha \lambda}\right)^{\frac{1}{2\alpha-1}} \int_{H_i}^{H_e} \frac{H dH}{\frac{\alpha C_0}{2\alpha-1} H^2 + C_1}, \tag{20}$$

and by taking the integral, one arrives at

$$N = -\left(\frac{\alpha}{2^\alpha \lambda}\right)^{\frac{1}{2\alpha-1}} \frac{2\alpha - 1}{2\alpha C_0} \ln\left[\frac{\alpha C_0}{2\alpha - 1} H^2 + C_1\right]\Bigg|_{H_i}^{H_e}, \tag{21}$$

where H_e is the Hubble parameter at the end of inflation, and where H_i is the Hubble parameter at the horizon exit time. From Eq. (12), the potential of the scalar field could also be derived only as a function of the Hubble parameter

$$V = 3M_p^2 H^2 \left[1 - \frac{(2\alpha - 1)}{3\alpha} \left(\frac{2^\alpha \lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{\left(\frac{\alpha C_0}{2\alpha-1} H^2 + C_1\right)}{H^2}\right]. \tag{22}$$

IV. PERTURBATIONS OF NONCANONICAL SCALAR FIELD

Assume a small inhomogeneity of the scalar field as $\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x})$. This perturbation with respect to the background $\phi_0(t)$ (from now on, we will omit the subscript "0") induces a perturbation to the metric because, from the field equation, it is clear that geometry and matter are tightly coupled. The metric in the longitudinal gauge is written as

$$ds^2 = (1 + 2\Phi(t, \mathbf{x}))dt^2 - a^2(t)(1 - 2\Psi(t, \mathbf{x}))\delta_{ij} dx^i dx^j, \tag{23}$$

where, by assuming a diagonal tensor for the spatial part of the energy-momentum tensor (i.e., $\delta T^i_j \propto \delta^i_j$), we have $\Psi(t, \mathbf{x}) = \Phi(t, \mathbf{x})$.

The action for linear scalar perturbation is derived as [8,62]

$$S = \frac{1}{2} \int \left(v'^2 + c_s^2 v (\nabla v)^2 + \frac{z''}{z} v\right) d\tau d^3\mathbf{x}, \tag{24}$$

in which $v \equiv z\zeta$, where ζ is the curvature perturbation given by $\zeta = \Phi + H \frac{\delta\phi}{\dot{\phi}}$, and where the prime denotes the derivative with respect to the conformal time τ, $a d\tau = dt$. Also, c_s is the sound speed of the model, which is stated as

$$c_s = \frac{1}{\sqrt{2\alpha - 1}}, \tag{25}$$

which is a constant. To get a physical sound speed, the constant α should always be positive and bigger than 0.5, and also, to not exceed the speed of light, it should be bigger than 1.

The quantity z is known as the Mukhanov variable, which, for our model, is defined as [8,62]

$$z^2 \equiv \frac{2\alpha a X}{c_s^2 H^2} \left(\frac{X}{M^4}\right)^{\alpha - 1}. \tag{26}$$

From Eq. (24), the equation for perturbation quantity v becomes

$$\frac{d^2}{d\tau^2}v(\tau,\mathbf{x}) - c_s^2\nabla^2 v(\tau,\mathbf{x}) - \frac{z''}{z}v(\tau,\mathbf{x}) = 0, \quad (27)$$

and by utilizing the Fourier mode, one arrives at

$$\frac{d^2}{d\tau^2}v_k(\tau) + \left(c_s^2k^2 - \frac{1}{z}\frac{d^2z}{d\tau^2}\right)v_k(\tau) = 0. \quad (28)$$

Before discussing the solution of the above equation, we try to compute the term z''/z. From the definition of z and the slow-roll parameters of the previous section, we get

$$z \equiv \sqrt{\frac{2\alpha}{2^\alpha\lambda}\frac{a\dot{\phi}^2}{c_sH}}, \quad (29)$$

$$\frac{dz}{d\tau} = z(aH)[1+\alpha\eta+\epsilon]. \quad (30)$$

To calculate the second derivative, we first need to obtain the derivative of the slow-roll parameters with respect to the conformal time,

$$\frac{d\epsilon}{d\tau} = (aH)[2\alpha\eta\epsilon+\epsilon^2]. \quad (31)$$

Then, the second derivative of the quantity z with respect to the conformal time reads

$$\frac{1}{z}\frac{d^2z}{d\tau^2} = (aH)^2[2+2\epsilon+3\eta+(\alpha+2\alpha\theta)\eta\epsilon+\alpha^2\eta^2]. \quad (32)$$

Here, $\theta=\pm1$ and appears through the time derivative of the scalar field. [From Eqs. (14) and (15), $\dot{\phi}^2$ are derived in terms of the scalar field. On the other hand, the time derivative of ϵ can be expressed as $\dot{\epsilon}=\dot{\phi}\epsilon_{,\phi}$. Then, we need $\dot{\phi}$, which in general is $\dot{\phi}=\theta\sqrt{\mathcal{O}}$.]

On subhorizon scales, where $c_sk \gg aH$ (i.e., $c_sk \gg z''/z$), the quantity v_k is obtained as

$$\frac{d^2}{d\tau^2}v_k(\tau) + c_s^2k^2v_k(\tau) = 0, \quad \Rightarrow \quad v_k(\tau) = \frac{1}{\sqrt{2c_sk}}e^{ic_sk\tau}. \quad (33)$$

In order to find the general solution of Eq. (28), we make the variable changes $v_k=\sqrt{-\tau}f_k$ and $x\equiv -c_sk\tau$. After some manipulation, the find the following Bessel differential equation,

$$\frac{d^2f_k}{dx^2} + \frac{1}{x}\frac{df_k}{dx} + \left(1-\frac{\nu^2}{x^2}\right)f_k = 0, \qquad \frac{z''}{z} = \frac{\nu^2-\frac{1}{4}}{\tau^2}, \quad (34)$$

where we use $a^2H^2=(1+\epsilon)^2/\tau^2$, and keeping only the first order of the slow-roll parameter ϵ, the parameter ν is acquired as

$$\nu^2 = \frac{9}{4}+6\epsilon+3\alpha\eta+(7\alpha+2\alpha\theta)\epsilon\eta+(1+2\epsilon)\alpha^2\eta^2. \quad (35)$$

The general solution for the above differential equation is the first and second kind of Hankel function. However, to have consistency with the solution on subhorizon scales, the second kind of Hankel function should be ignored; therefore, the final solution is acquired as

$$v_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{2}(\nu+\frac{1}{2})}\sqrt{-\tau}H_\nu^{(1)}(-c_sk\tau). \quad (36)$$

The spectrum of curvature perturbation is defined as

$$\mathcal{P}_s = \frac{k^3}{2\pi^2}|\zeta|^2 = \frac{k^3}{2\pi^2}\left|\frac{v_k}{z}\right|^2. \quad (37)$$

On superhorizon scales, the asymptotic behavior of the Hankel function is

$$\lim_{-k\tau\to\infty}H_\nu^{(1,2)}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2c_sk}}e^{\mp(ic_sk\tau+\delta)},$$
$$\delta = \frac{1}{2}\left(\nu+\frac{1}{2}\right). \quad (38)$$

Then, the spectrum of curvature perturbation on superhorizon scales will be

$$\mathcal{P}_s = \left(\frac{2^{\nu-\frac{3}{2}}\Gamma(\nu)}{\Gamma(3/2)}\right)^2\left(\frac{H^2}{2\pi\sqrt{c_s(\rho+p)}}\right)^2\left(\frac{c_sk}{aH}\right)^{3-2\nu}. \quad (39)$$

V. CONSISTENCY WITH OBSERVATION

An advantage of the inflationary scenario is the prediction for quantum perturbations including the scalar, vector, and tensor ones. Among them, scalar perturbations are considered to be the primary seeds for large structure formation, and tensor perturbation is interpreted as the source of gravitational waves. During inflation, these perturbations are stretched out to the horizon and remain invariant. On the other hand, the observational data almost indicate a scale invariant spectrum for curvature perturbations. This scale invariant feature is described by the scalar spectral index, so it is defined as $\mathcal{P}_s = A_s^2(\frac{c_sk}{aH})^{n_s-1}$, where A_s^2 is the amplitude of scalar perturbation at horizon crossing $c_sk=aH$. For $n_s=1$, the amplitude of scalar perturbation is exactly scale invariant; however, the latest observational data imply that $\ln(A_s\times10^{10})=3.044\pm0.014$ and $n_s=0.9649\pm0.0042$, expressing an almost scale invariant perturbation [63]. To measure the tensor perturbations, the usual procedure is indirectly through the parameter r known as the tensor-to-scalar ratio, $r = \mathcal{P}_t/\mathcal{P}_s$. The data originated by Planck 2018 show that there is only an upper bound on this parameter, $r < 0.064$, and it is still not measured exactly [63].

Consistency of the presented model with observational data is the main goal of this section. At the end of inflation, the first slow-roll parameter reaches unity [64], which happens for $H = H_e$,

$$\epsilon(H_e) = \left(\frac{2^\alpha \lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{\left(\frac{\alpha C_0}{2\alpha-1} H^2 + C_1\right)}{H^2} = 1, \tag{40}$$

which leads to the following value for the Hubble parameter at the end of inflation:

$$H_e^2 = \left(\frac{2^\alpha \lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{-C_1}{\alpha\eta - 1}. \tag{41}$$

To guarantee the positiveness of H_e^2, the term $-C_1\lambda/(\alpha\eta-1)$ should always be positive. At the horizon exit, the slow-roll parameter ϵ is smaller than unity, and the corresponding Hubble parameter could be determine through the number of e-fold equation (20)

$$H_\star^2(N) = \left(\frac{2^\alpha \lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{-C_1}{\alpha\eta - 1} \left(\frac{e^{2\alpha\eta N} + \alpha\eta - 1}{\alpha\eta}\right), \tag{42}$$

which is expressed in terms of the number of e-fold parameters N. The right-hand side should always be positive, so besides the above restriction, the term in the parentheses should always be positive. Note that, in order to overcome the horizon and flatness problems, we need about 60–70 e-folds [34].

From Eqs. (6), (5), and (14), we determine that the time derivatives of the Hubble parameter are negative, so by passing time and approaching the end of inflation, the Hubble parameter decreases. On the other hand, since $\epsilon = 1$ is taken as the end of inflation, it is expected that the first slow-roll parameter smaller than 1 will be for bigger values of the Hubble parameter—namely, the horizon crossing occurs for bigger values of the Hubble parameter. This point is illustrated in Fig. 1, where by passing time and decreasing the Hubble parameter, the parameter ϵ approaches 1, indicating the end of inflation (the selected values for the constant in the figure is based on the results that will be determined below).

Using Eq. (42), the main perturbation parameters such as the scalar spectral index, amplitude of scalar perturbation, and tensor-to-scalar ratio could be obtained in terms of N as

$$\epsilon(N) = \frac{\alpha\beta e^{2\alpha\beta N}}{e^{2\alpha\beta N} + \alpha\beta - 1}, \tag{43}$$

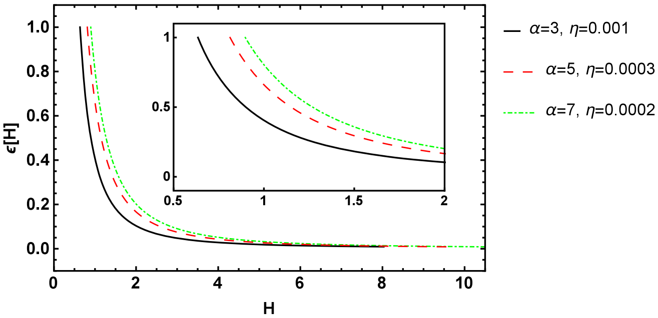

FIG. 1. Behavior of the first slow-roll parameter ϵ in terms of the Hubble parameter for different values of α and η.

$$n_s(N) = 4 - 2\nu(N), \tag{44}$$

$$\nu^2(N) = \frac{9}{4} + 3\alpha\beta + \alpha^2\beta^2 + (6 + 7\alpha + 2\alpha\theta + 2\alpha^2\beta^2)\epsilon(N), \tag{45}$$

$$\mathcal{P}_s(N) = \frac{1}{8\pi^2}\left(\frac{2^{\nu(N)-\frac{3}{2}}\Gamma(\nu(N))}{\Gamma(3/2)}\right)^2 \frac{H^2(N)}{c_s \epsilon(N)}, \tag{46}$$

$$r(N) = 16\left(\frac{\Gamma(3/2)}{2^{\nu(N)-\frac{3}{2}}\Gamma(\nu(N))}\right)^2 c_s \epsilon(N). \tag{47}$$

It is clear that the scalar spectral index and the tensor-to-scalar ratio depend only on the constants α and η. Using the Planck $r - n_s$ diagram, we could obtain a range of these constants that puts the model predictions about n_s and r in the range of data. Figure 2 shows this area, in which the light blue area illustrates the range related to the 95% C.L., and the dark blue section determines values of α and η that put n_s and r in the 68% C.L. It shows that the second slow-roll parameter should be positive, and there is no consistent result for negative η. The second slow-roll parameter could be of order 10^{-3}, but the best results occur for smaller values. By increasing the constant η, the range of α decreases.

The other constants M and C_1 appear in the amplitude of the curvature perturbation and also in the potential of the scalar field through the Hubble parameter. Imposing the observational data for the amplitude of curvature perturbations determines that we should get

$$\frac{-C_1 M^{\frac{4(\alpha-1)}{2\alpha-1}}}{\alpha\beta - 1} = \left(\frac{\alpha}{2^\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{\mathcal{P}_s^\star}{8\pi^2 c_s \epsilon^\star}\left(\frac{\Gamma(3/2)}{2^{\nu^\star-\frac{3}{2}}\Gamma(\nu^\star)}\right)^2 \times \frac{\alpha\eta}{e^{2\alpha\eta N} + \alpha\eta - 1}. \tag{48}$$

The amplitude of the curvature perturbation is clear from the data, so by choosing some values for the constants α and η from Fig. 2, the right-hand side would be clear.

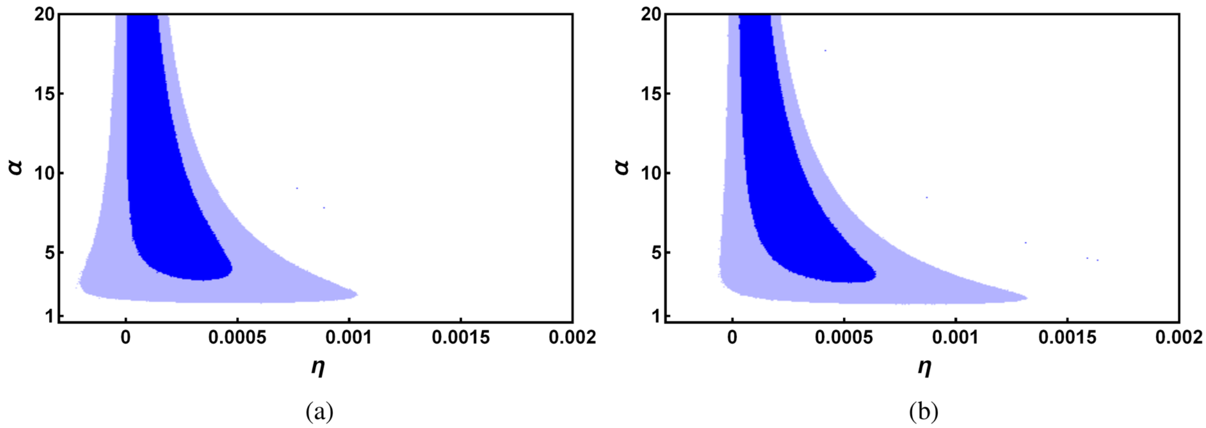

FIG. 2. Parametric plot for the constant η and α such that, for these values of the constants, the scalar spectral index and tensor-to-scalar ratio stands in the observational range. The figure has been plotted for $\theta = -1$ and the number of e-folds: (a) $N = 65$, (b) $N = 70$.

Table I expresses these terms for different choices of α and η.

Also, the term on the left-hand side of Eq. (48) appears in the potential of the scalar field, for which, at the time of horizon crossing, one has

$$V^{\star} = 3\left(\frac{2^{\alpha}}{\alpha}\right)^{\frac{1}{2\alpha-1}} \frac{-C_1 M^{\frac{4(\alpha-1)}{2\alpha-1}}}{\alpha\beta - 1} \frac{e^{2\alpha\eta N}}{\epsilon^{\star}}\left[1 - \frac{(2\alpha-1)}{3\alpha}\epsilon^{\star}\right]. \quad (49)$$

Then, using the determined values of α and η, the inflation energy scale is determined, which is implied in Table I. It is clearly seen that the energy scale of inflation is about 10^{-2}.

VI. ATTRACTOR BEHAVIOR

Consideration of the attractor behavior of the solution will be performed following the same process as mentioned in [59,60,65], which is a simple approach and also an effective one, as one works in the Hamilton-Jacobi formalism. In this formalism, it is assumed that there is a homogeneous perturbation to the Hubble parameter as $H = H_0 + \delta H$. Now by introducing it in the Hamilton-Jacobi equation (12) and keeping the terms up to the first order of δH, one arrives at

$$\frac{\delta H'}{\delta H} = 3\left(\frac{2^{\alpha}\lambda}{\alpha}\right)\frac{H_0}{\dot{\phi}_0^{2\alpha}} H'_0, \quad (50)$$

By taking the integral from the above equation, we have

$$\delta H = \delta H_i \exp\left(-3\int_{H_i}^{H} \frac{H}{\dot{H}} dH\right), \quad (51)$$

where δH_i is the perturbation at the initial time. From the above equation, it is realized that, by approaching the end of inflation, the term within the power of the exponential term grows, and the negative sign brings out the fact that the perturbation δH decreases continuously. This feature is determined in Fig. 3, where the integral has been plotted for different values of α, β, and M, and for all cases, the term decreases during inflation. Therefore, it results that the model satisfies the attractor behavior.

TABLE I. We determine the constant term $C_1 M^{\frac{4(\alpha-1)}{2\alpha-1}}$ by using Fig. 2 and the data for the curvature perturbations. Then, the energy scale of inflation is illustrated for these values of the constants. The results have been derived for the number of e-folds $N = 65$.

α	η	$-C_1 M^{\frac{4(\alpha-1)}{2\alpha-1}}$	$V^{\star}$
3	0.0002	-4.56×10^{-11}	2.26×10^{-8}
3	0.001	-3.30×10^{-11}	1.92×10^{-8}
5	0.0003	-5.73×10^{-11}	2.94×10^{-8}
5	0.0005	-4.69×10^{-11}	2.66×10^{-8}
7	0.0002	-6.73×10^{-11}	3.53×10^{-8}
7	0.0004	-5.58×10^{-11}	3.21×10^{-8}

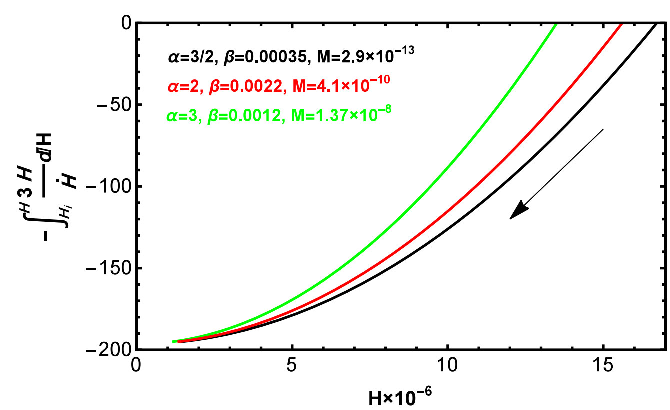

FIG. 3. The power of the exponential term in Eq. (51) during inflation. By passing time and approaching the end of inflation, the power increases with negative sign. Then, the perturbation δH exponentially decreases during inflation, and the solution is attractive.

VII. CONCLUSION

This work has relied on a scenario of constant-roll evolution in the initial Universe. It has been supposed that inflation has been derived by considering a modification in the kinetic term of the Lagrangian called the noncanonical model of the scalar field. In the constant-roll scenario, it has been assumed that the second slow-roll parameter, i.e., η, has to be a constant, and therefore we have had to recalculate the necessary inflationary parameters. By calculating the slow-roll parameters in terms of the Hubble parameter, we could derive a differential equation for the Hubble parameter, which could lead to the corresponding differential equation for the canonical scalar field model by taking $\alpha = 1$. On the other hand, by taking into account the assumption of the constant-roll method, the perturbation equation should be recalculated again. Doing so, obviously there are some modifications in the amplitude of scalar perturbation, and the scalar spectral index leads to the appearance of the second order of η in our investigation.

By finding an exact solution, which is one of the main advantages and triumphs of this approach, it was realized that every parameter of the model can be expressed in terms of the Hubble parameter. Therefore, comparing our approach with the usual Hamilton-Jacobi formalism of inflation, the Hubble parameter here behaves as the scalar field in that approach. The importance of this result goes back to this fact: there is no need to introduce an ansatz for a Hubble parameter based on the scalar field. The condition $\epsilon = 1$ clarified the Hubble parameter at the end of inflation, i.e., H_e, and using the number of e-folds, the Hubble parameter is determined in terms of the number of e-folds and other constant parameters of the model. Utilizing this result, we express the main perturbation parameters of the model in terms of the number of e-folds to obtain better estimates compared to the observations.

It has been shown that the scalar spectral index and the tensor-to-scalar ratio at the horizon exit can be obtained in terms of two free parameters of the model—namely, α and η. Then, by making use of the $r - n_s$ diagram originating from Planck 2018, the proper ranges based on these two parameters are estimated and depicted in Fig. 2. It is noticed for any values of α and η in the best estimated range that both the scalar spectral index and the tensor-to-scalar ration are in good agreement with the observational data. The results show that the second slow-roll parameter η is constant, as it should be, and is of order 10^{-4}, and by increasing the constant α, the range of η becomes smaller. On the other hand, applying the data for the amplitude of curvature perturbation, the constant $C_1 M^{\frac{4(\alpha-1)}{2\alpha-1}}$ is determined, and imposing this result in the potential of the scalar field, it is found that the energy scale of inflation is around $V^{\star 1/4} \propto 10^{-2}$.

Ultimately, the attractor behavior of the model when we assume a homogeneous perturbation to the Hubble parameter is investigated. By introducing into the Hamilton-Jacobi equation, it leads to a differential equation of the homogeneous perturbation. From Fig. 3, it is obvious that the perturbation decreases exponentially by approaching the end of inflation, which indicates the attractor behavior of the solution.

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