Evaluation of entanglement entropy in high energy elastic scattering

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Entanglement of the two scattered particles is expected to occur in elastic collisions, even at high energy where they are in competition with inelastic ones. We study how to evaluate quantitatively the corresponding entanglement entropy S_{EE}. For this sake, we regularize the divergences occurring in the formal derivation of S_{EE} using a regularization procedure acting on the two-particle Hilbert space of final states. A quantitative application is performed in proton-proton collisions at collider energies, comparing the results of S_{EE} with two different cutoffs and with a volume-regularization obtained by a prescription fixing the finite two-body Hilbert space volume. A significant entanglement is found which persists even at the highest available energies.

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I. INTRODUCTION

Entanglement is a significant phenomenon in quantum theories and has been attracting many interests of scientists in various research areas. In this paper we are interested in the entanglement of scattering particles. How much are the particles entangled due to the scattering interaction? This is a simple and fundamental question. A way to answer it is to evaluate the entanglement entropy of the final state of particles. For this sake, two-body elastic scattering appears to be a case study for entanglement in the final state.

In Ref. [1] the entanglement in momentum Hilbert space in the scattering process has been studied, and the entanglement entropy of the final state of two particles has been calculated in weak coupling perturbation by applying the method developed by Ref. [2] for momentum space entanglement. Reference [3] also has considered the entanglement in momentum Hilbert space for the elastically scattering particles, but has formulated nonperturbatively the entanglement entropy by the use of S-matrix theory [4,5]. Reference [3], as a result, has derived an adequate formalism for the entanglement entropy and has suggested an entropy formula of the two-particle final state after the elastic scattering. Additionally the entanglement entropy in this formula includes the influential effect of inelastic processes which are present in the overall set of the possible final states at a given high energy.

However there is a problem of divergence in the entanglement entropy, which is caused by the infinite volume of the momentum Hilbert space in Refs. [1,3]. Indeed the formula in Ref. [3] is written in terms of not only physical observables, i.e., the elastic and total cross sections, but also the cutoff parameter for the infinite volume. One of the subjects in this paper is, starting from Ref. [3], to formulate a finite entanglement entropy formula by identifying the physical origin of the divergence in the entropy formula and using it to appropriately regularize this divergence.

As mentioned above, the entanglement in scattering process is a fundamental issue. For the sake of completion, we quote some works [6–10] related to this issue, while being of a different focus than ours. Reference [6] has computed the variation of entanglement entropy in an elastic scattering of two interacting scalar particles at one-loop perturbative level. Reference [7] has studied the entanglement entropy and mutual information in a fermion-fermion scattering. Reference. [8] are

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concerned with quantum measurement theory and relativistic scattering theory, and has studied the entanglement entropy of an apparatus particle scattered off and a set of system particles. Reference [9] has suggested another derivation of the momentum space entanglement entropy in the scattering at weak coupling. Reference [10] has discussed the entanglement entropy in a deep inelastic scattering.

In our study, having performed the regularization and using the obtained formula, it is interesting to evaluate the entanglement entropy for concrete particle scattering. We thus apply our formalism in high energy proton-proton scattering at Tevatron and LHC energies in order to evaluate the entanglement entropy of two-body elastic final states at the highest available energies.

The plan of our study is as follows: In Sec. II we reformulate the entanglement entropy of scattering particles, starting from Ref. [3], in order to determine the physical origin of the divergences one encounters and to properly regularize them. In Sec. III, by using the entanglement entropy formulas obtained for different regularization procedures, we evaluate the regularized entanglement entropy in proton-proton scattering. We compare two different cutoff methods with the case of a volume-regularization given by an adequate prescription for the regularized Hilbert space volume without explicit cutoff procedure. Section IV is devoted to a discussion of the results, an outlook on further directions and a conclusion.

II. FORMULATION OF ENTANGLEMENT ENTROPY

In this section, we start by recalling the formal derivation (see Ref. [3]) of the entanglement entropy. Then we reformulate the derivation in order to focus on the divergences one encounters. Our goal is to find the physical origin of these divergences, identify the divergent factor and propose the way to obtain a finite formula for the entanglement entropy of the two outgoing particles.

A. Density matrix

Let us consider elastic scattering of two particles A and B which have initial 3-momentum $\vec{k}$ and $\vec{l}$ respectively. Note that in the high energy regime inelastic scattering together with the elastic one have a large contribution. In fact, both types of scattering are related through the unitarity relations. Using the generic entanglement formalism, the statistical entanglement between the particles, A and B, with final 3-momentum respective $\vec{p}$ and $\vec{q}$ is expressed in terms of the entanglement entropy S_{EE} as follows: One starts with the overall density matrix ρ in the Hilbert space spanned by two-body final states $|\vec{p},\vec{q}\rangle \equiv |\vec{p}\rangle_A \otimes |\vec{q}\rangle_B$.[1] One defines a reduced density matrix as $\rho_A = \mathrm{tr}_B\rho$, where one sums over the states of particle B. Then the entanglement entropy is given by $S_{\mathrm{EE}} = -\mathrm{tr}_A\rho_A \ln\rho_A$, or equivalently by $S_{\mathrm{EE}} = \lim_{n\to1} S_{\mathrm{RE}}(n) = -\lim_{n\to1} \frac{\partial}{\partial n}\mathrm{tr}_A(\rho_A)^n$, where $S_{\mathrm{RE}}(n) = \frac{1}{1-n}\ln \mathrm{tr}_A(\rho_A)^n$ is the Rényi entropy.

The overall density matrix reads

$$\rho = \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}\frac{d^3\vec{p}'}{2E_{A\vec{p}'}}\frac{d^3\vec{q}'}{2E_{B\vec{q}'}}|\vec{p},\vec{q}\rangle\langle\vec{p},\vec{q}|\mathcal{S}|\vec{k},\vec{l}\rangle\langle\vec{k},\vec{l}|\mathcal{S}^\dagger|\vec{p}',\vec{q}'\rangle\langle\vec{p}',\vec{q}'|, \tag{2.1}$$

where $\mathcal{S}$ is the S-matrix operator projecting the two-body initial state $|\vec{k},\vec{l}\rangle\langle\vec{k},\vec{l}|$ onto two-body final states. In Eq. (2.1), the integration measure is the Lorentz invariant one $\frac{d^3\vec{p}}{2E_{\vec{p}}}$ for on-shell particles and $\mathcal{N}$ is a normalization ensuring the condition $\mathrm{tr}_A\mathrm{tr}_B\rho = 1$. Tracing out ρ with respect to the Hilbert space of particle B, we obtain the reduced density matrix,

$$\begin{aligned}\rho_A \equiv \mathrm{tr}_B\rho &= \int \frac{d^3\vec{q}''}{2E_{B\vec{q}''}}\langle\vec{q}''|\rho|\vec{q}''\rangle \\ &= \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}\frac{d^3\vec{p}'}{2E_{A\vec{p}'}}(\langle\vec{p},\vec{q}|\mathcal{S}|\vec{k},\vec{l}\rangle\langle\vec{k},\vec{l}|\mathcal{S}^\dagger|\vec{p}',\vec{q}\rangle)|\vec{p}\rangle\langle\vec{p}'|.\end{aligned} \tag{2.2}$$

Taking into account energy-momentum conservation and the kinematics of elastic scattering $|\vec{k},\vec{l}\rangle \to |\vec{p},\vec{q}\rangle$ with $|\vec{k}| = |\vec{l}| = |\vec{p}| = |\vec{q}|$, one obtains

$$\rho_A = \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{\delta(0)\delta(E_{A\vec{p}} + E_{B\vec{k}+\vec{l}-\vec{p}} - E_{A\vec{k}} - E_{B\vec{k}})}{2E_{A\vec{p}}2E_{B\vec{k}+\vec{l}-\vec{p}}}(\langle\vec{p},\vec{k}+\vec{l}-\vec{p}|\mathbf{s}|\vec{k},\vec{l}\rangle\langle\vec{k},\vec{l}|\mathbf{s}^\dagger|\vec{p},\vec{k}+\vec{l}-\vec{p}\rangle)|\vec{p}\rangle\langle\vec{p}|, \tag{2.3}$$

[1] Although the complete relativistic quantum numbers of a particle state are denoted by momentum and spin (or helicity) as $|\vec{p},s\rangle$, we focus only on the momentum Hilbert space in this paper. We will give some comments on the helicity in Sec. IV.

where $\langle\vec{p},\vec{q}|\mathcal{S}|\vec{k},\vec{l}\rangle \equiv \delta^{(4)}(P^{(4)}_{p+q} - P^{(4)}_{k+l})\langle\vec{p},\vec{q}|\mathbf{s}|\vec{k},\vec{l}\rangle$ with the notation $P^{(4)}$ for the center-of-mass energy-momentum vector. The density matrix (2.3) is normalized by its unit trace;

$$1 = \mathrm{tr}_A\mathrm{tr}_B\rho = \mathrm{tr}_A\rho_A = \int \frac{d^3\vec{p}''}{2E_{A\vec{p}''}}\langle\vec{p}''|\rho_A|\vec{p}''\rangle$$
$$= \frac{1}{\mathcal{N}}\int d^3\vec{p}\,\frac{\delta^{(4)}(0)\delta(|\vec{p}|-|\vec{k}|)}{4|\vec{k}|(E_{A\vec{k}}+E_{B\vec{k}})}|\langle\vec{p},-\vec{p}|\mathbf{s}|\vec{k},-\vec{k}\rangle|^2, \tag{2.4}$$

giving

$$\mathcal{N} = \delta^{(4)}(0)\mathcal{N}',$$
$$\mathcal{N}' = \int d^3\vec{p}\,\frac{\delta(|\vec{p}|-|\vec{k}|)}{4|\vec{k}|(E_{A\vec{k}}+E_{B\vec{k}})}|\langle\vec{p},-\vec{p}|\mathbf{s}|\vec{k},-\vec{k}\rangle|^2, \tag{2.5}$$

where Eqs. (2.4) and (2.5) are expressed using the center-of-mass frame. Note that the $\delta(0)$ coming from the energy conservation in Eq. (2.3) cancels the similar one in Eqs. (2.5), leaving an overall $\delta^{(3)}(0)$ due to the normalization. We shall discuss later the potential divergence related to this 3-dimensional δ-function.

One finally gets

$$\rho_A = \frac{1}{\mathcal{N}'\delta^{(3)}(0)}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{\delta(p-k)}{4k(E_{A\vec{k}}+E_{B\vec{k}})}$$
$$\times |\langle\vec{p},-\vec{p}|\mathbf{s}|\vec{k},-\vec{k}\rangle|^2|\vec{p}\rangle\langle\vec{p}|, \tag{2.6}$$

where for further purpose we quote

$$p = |\vec{p}|, \qquad k = |\vec{k}|, \qquad \frac{\vec{p}\cdot\vec{k}}{pk} = \cos\theta, \tag{2.7}$$

and θ is the center-of-mass scattering angle.

B. Entanglement entropy

By performing the product of the n density operators of the form (2.6), one obtains the formal expression for the entanglement entropy through the calculation of $\mathrm{tr}_A(\rho_A)^n$ as

$$\mathrm{tr}_A(\rho_A)^n$$
$$= \int d^3\vec{p}\,\delta^{(3)}(0)\left(\delta(p-k)\frac{|\langle\vec{p},-\vec{p}|s|\vec{k},-\vec{k}\rangle|^2}{\mathcal{N}'\delta^{(3)}(0)4k(E_{A\vec{k}}+E_{B\vec{k}})}\right)^n. \tag{2.8}$$

The overall $\delta^{(3)}(0)$ in the integration comes from taking the trace over the A particle's 3-momentum.

Let us now introduce the partial wave expansion of the reduced S-matrix element [4,5],

$$\langle\vec{p},-\vec{p}|\mathbf{s}|\vec{k},-\vec{k}\rangle$$
$$= \frac{E_{A\vec{k}}+E_{B\vec{k}}}{\pi k}\cdot\sum_{\ell=0}^{\infty}(2\ell+1)(1+2i\tau_\ell)P_\ell(\cos\theta)$$
$$= \frac{E_{A\vec{k}}+E_{B\vec{k}}}{\pi k}\cdot 2\left(\delta(1-\cos\theta)+\frac{i}{16\pi}\mathcal{A}(s,t)\right), \tag{2.9}$$

where one used the known summation formula of Legendre polynomials P_ℓ,

$$\delta(1-\cos\theta) = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta), \tag{2.10}$$

together with the partial wave expansion of the scattering amplitude,

$$\mathcal{A} = 16\pi\sum_{\ell=0}^{\infty}(2\ell+1)\tau_\ell P_\ell(\cos\theta), \tag{2.11}$$

and

$$s_\ell = 1+2i\tau_\ell. \tag{2.12}$$

is the two-body S-matrix ℓ^{th} partial wave. It becomes clear from Eq. (2.9) that the powers of δ-functions in Eqs. (2.5), (2.6) and (2.8) give rise to divergences. In order to exhibit these divergences for further regularization, we introduce the divergent full phase-space "volume,"

$$V \equiv 2\delta(0) = \sum_{\ell=0}^{\infty}(2\ell+1), \tag{2.13}$$

which we now prove that it is the key factor determining all divergences we encounter in the derivation of the entanglement entropy.

Inserting the S-matrix element (2.9) into the expression for $\mathcal{N}'$ in Eqs. (2.5), one obtains

$$\mathcal{N}' = \frac{E_{A\vec{k}}+E_{B\vec{k}}}{\pi k}\sum_{\ell=0}^{\infty}(2\ell+1)|s_\ell|^2. \tag{2.14}$$

With this expression one can reexpress Eq. (2.8) as

$$\begin{aligned} \mathrm{tr}_A(\rho_A)^n &= \int d^3\vec{p}\,\delta^{(3)}(0)\left[\frac{\delta(p-k)}{\delta^{(3)}(0)4\pi k^2}\frac{|\sum_\ell(2\ell+1)s_\ell P_\ell(\cos\theta)|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}\right]^n \\ &= \int_0^\infty dp2\pi p^2\int_{-1}^1 d\cos\theta\frac{\delta(p-k)}{4\pi k^2}\frac{|\sum_\ell(2\ell+1)s_\ell P_\ell(\cos\theta)|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}\left[\frac{\delta(p-k)}{\delta^{(3)}(0)4\pi k^2}\frac{|\sum_\ell(2\ell+1)s_\ell P_\ell(\cos\theta)|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}\right]^{n-1} \\ &= \left(\frac{\delta(0)}{\delta^{(3)}(0)2\pi k^2}\right)^{n-1}\int_{-1}^1 d\cos\theta\left[\frac{1}{2}\frac{|\sum_\ell(2\ell+1)s_\ell P_\ell(\cos\theta)|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}\right]^n, \end{aligned} \tag{2.15}$$

where we reduce the momentum integration to the scattering angle and factorize out a constant prefactor in the last line of (2.15) between parentheses. This prefactor can be expressed in terms of the (infinite) phase-space volume (2.13), using the mathematical identity of δ-functions in spherical coordinates with azimuthal symmetry,

$$\delta^{(3)}(\vec{p}-\vec{k}) = \frac{\delta(p-k)}{4\pi k^2}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta). \tag{2.16}$$

In the $\cos\theta \to 1$ limit we formally obtain for the inverse prefactor in (2.15),

$$2\pi k^2\frac{\delta^{(3)}(0)}{\delta(0)} = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1) = \frac{V}{2}. \tag{2.17}$$

All in all we can rewrite Eq. (2.8) as

$$\mathrm{tr}_A(\rho_A)^n = \left(\frac{V}{2}\right)^{1-n}\int_{-1}^1 d\cos\theta[\mathcal{P}(\theta)]^n, \tag{2.18}$$

$$\mathcal{P}(\theta) = \frac{1}{2}\frac{|\sum_\ell(2\ell+1)s_\ell P_\ell(\cos\theta)|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}, \tag{2.19}$$

where, using the orthogonality property of Legendre polynomials, $\mathcal{P}(\theta)$ is of norm

$$\int_{-1}^1 d\cos\theta\mathcal{P}(\theta) = 1. \tag{2.20}$$

Substituting Eq. (2.12) into Eq. (2.19), one writes

$$\begin{aligned} \mathcal{P}(\theta) &= \delta(1-\cos\theta)\left(1-\frac{2\sum_\ell(2\ell+1)|\tau_\ell|^2}{V/2-\sum_\ell(2\ell+1)f_\ell}\right) \\ &\quad+\frac{|\sum_\ell(2\ell+1)\tau_\ell P_\ell(\cos\theta)|^2}{V/2-\sum_\ell(2\ell+1)f_\ell}, \end{aligned} \tag{2.21}$$

where the f_ℓ are the partial wave components of the inelastic cross section related to the elastic ones τ_ℓ through the unitarity relation, $s_\ell s_\ell^* = 1–2f_\ell$, or equivalently

$$f_\ell = 2(\mathrm{Im}\tau_\ell - |\tau_\ell|^2). \tag{2.22}$$

Indeed, the standard expressions for physical scattering observables in terms of partial wave components τ_ℓ and f_ℓ read

$$\begin{aligned} \sigma_{\mathrm{tot}} &= \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{Im}\tau_\ell, \quad \sigma_{\mathrm{el}} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)|\tau_\ell|^2, \\ \sigma_{\mathrm{inel}} &= \frac{2\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)f_\ell \end{aligned} \tag{2.23}$$

and

$$\frac{d\sigma_{\mathrm{el}}}{dt} = \frac{\pi}{k^4}\left|\sum_\ell(2\ell+1)\tau_\ell P_\ell(\cos\theta)\right|^2 = \frac{|\mathcal{A}|^2}{256\pi k^4}, \tag{2.24}$$

where the Mandelstam variable $t = 2k^2(\cos\theta - 1)$. We finally find the following expression for $\mathcal{P}(\theta)$;

$$\begin{aligned} \mathcal{P}(\theta) &= \delta(1-\cos\theta)\cdot\left(1-\frac{\sigma_{\mathrm{el}}}{\pi V/k^2-\sigma_{\mathrm{inel}}}\right) \\ &\quad+\frac{1}{\sigma_{\mathrm{el}}}\frac{d\sigma_{\mathrm{el}}}{d\cos\theta}\cdot\left(\frac{\sigma_{\mathrm{el}}}{\pi V/k^2-\sigma_{\mathrm{inel}}}\right). \end{aligned} \tag{2.25}$$

Using Eq. (2.18), we write formally the entanglement entropy as

$$S_{\mathrm{EE}} = -\lim_{n\to1}\frac{\partial}{\partial n}\mathrm{tr}_A(\rho_A)^n = \ln\frac{V}{2} - \int_{-1}^1 d\cos\theta\mathcal{P}(\theta)\ln\mathcal{P}(\theta). \tag{2.26}$$

From Eqs. (2.25) and (2.26), we observe that the divergences, in particular those due to the product of the δ-functions contained in $[\mathcal{P}(\theta)]^n$ in the definition of $\mathrm{tr}_A(\rho_A)^n$ in Eq. (2.18), are related to the infinite phase-space "volume" $V(=\infty)$ defined in Eq. (2.13). In this case $\mathcal{P}(\theta)$ reduces to $\delta(1-\cos\theta)$ and the entanglement entropy is zero. However, It is physically obvious that at each center-of-mass energy, only a finite (of order const $\times\, k^2$) number of partial waves contribute to the final interacting states. Indeed, in the formal calculation of the entanglement entropy we performed, all two-body states in the Hilbert space have been included for the summation of final states, whether they come from the interaction or not. Therefore we have to restore a projection of the two-body Hilbert space onto the set of interacting ones. We are thus led to interpret the divergence due to δ-functions and the "volume" V, as due to the infinite number of noninteracting two-body states. Hence an appropriate regularization is required.

C. Volume-regularization

As we pointed out in the previous subsection, the first term in Eq. (2.25) comes from the part of the two-body Hilbert space of the final states which does not correspond to the interacting states at the given energy. In an ideal cutoff independent way to avoid such noninteracting modes, we are led to note that the volume V could be regularized to $\tilde{V}$ so that the first term vanishes, i.e.,

$$\tilde{V} = \frac{k^2\sigma_{\rm tot}}{\pi}, \qquad \tilde{\mathcal{P}}(\theta) = \frac{1}{\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{d\cos\theta} = \frac{2k^2}{\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{dt}. \tag{2.27}$$

We call it the volume-regularization assumption.

From the second equation in Eqs. (2.27), and recalling the normalization condition (2.20), one realizes that $\tilde{\mathcal{P}}(\theta)$ can be interpreted as the physical probability of interaction.

The relations (2.27) lead the formal entanglement entropy (2.26) to the volume-regularized entanglement entropy,

$$\tilde{S}_{\rm EE} = -\int_{-\infty}^{0} dt \frac{1}{\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{dt}\ln\left(\frac{4\pi}{\sigma_{\rm tot}\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{dt}\right). \tag{2.28}$$

However currently we do not know yet which could be an effective regularization of the partial wave components leading to the volume-regularization without modifying the observables. The volume-regularization can thus be called ideal, since it only depends on measurable observables, and not on any cutoff. In the following sections, we try some concrete regularization methods, in order to obtain an approximation of the ideal determination (2.28) of the entanglement entropy and compare it with the one obtained from Eq. (2.28).

III. EVALUATION OF THE REGULARIZED ENTANGLEMENT ENTROPY

A. Cutoff regularization

We shall make use of the impact parameter b and the corresponding representation of observables, which correspond to a description of high-energy scattering observables, appropriate to our goal. The scattering amplitude (2.11) by the partial wave expansion is rewritten in the impact-parameter representation as

$$\begin{aligned}\mathcal{A} &= 16\pi\sum_{\ell=0}^{\infty}(2\ell+1)\tau_\ell P_\ell(\cos\theta) \\ &= 32\pi k^2\int_0^\infty bdb\tau(b)J_0(b\sqrt{-t}),\end{aligned} \tag{3.1}$$

where J_n is the well-known Bessel function of order n. In other words, $\tau(b)$ is defined by this equation.

In actual physics experiments, τ_ℓ for large ℓ, i.e., $\tau(b)$ for large b (because of $bk\sim\ell$), does not contribute to the scattering amplitude. Therefore we are led to a regularization truncating the large b modes by introducing a cutoff function $c(b)$ satisfying $\lim_{b\to\infty}c(b)=0$, so that the amplitude becomes

$$\hat{\mathcal{A}} = 32\pi k^2\int_0^\infty bdbc(b)\tau(b)J_0(b\sqrt{-t}). \tag{3.2}$$

This prescription gives an approximation of physical Hilbert space. Following this scattering amplitude, the differential elastic cross section becomes

$$\frac{d\hat{\sigma}_{\rm el}}{dt} = 4\pi\left|\int_0^\infty bdbc(b)\tau(b)J_0(b\sqrt{-t})\right|^2, \tag{3.3}$$

and the total, elastic and inelastic cross sections become

$$\hat{\sigma}_{\rm tot} = 8\pi\int_0^\infty bdbc^2(b){\rm Im}\tau(b), \tag{3.4}$$

$$\hat{\sigma}_{\rm el} = \int_{-\infty}^{0} dt\frac{d\hat{\sigma}_{\rm el}}{dt} = 8\pi\int_0^\infty bdbc^2(b)|\tau(b)|^2, \tag{3.5}$$

$$\hat{\sigma}_{\rm inel} = 4\pi\int_0^\infty bdbc^2(b)f(b). \tag{3.6}$$

Since the relation $\hat{\sigma}_{\rm tot} = \hat{\sigma}_{\rm el} + \hat{\sigma}_{\rm inel}$ is preserved by the regularization, $f(b)$ is written in terms of $\tau(b)$ as $f(b) = 2({\rm Im}\tau(b) - |\tau(b)|^2)$. This expression in the impact parameter space corresponds to Eq. (2.22).

Under the cutoff approximation, the volume of the regularized Hilbert space $\tilde{V}$ is

$$\tilde{V} \approx \hat{V} = \frac{k^2}{\pi}\hat{\sigma}_{\rm tot}. \tag{3.7}$$

and the entanglement entropy (2.28) is

$$\hat{S}_{\rm EE} = -\int_{-\infty}^{0} dt\frac{1}{\hat{\sigma}_{\rm el}}\frac{d\hat{\sigma}_{\rm el}}{dt}\ln\left(\frac{4\pi}{\hat{\sigma}_{\rm tot}\hat{\sigma}_{\rm el}}\frac{d\hat{\sigma}_{\rm el}}{dt}\right). \tag{3.8}$$

It is important to note that $\tilde{\mathcal{P}}(\theta)$ in Eq. (2.27) keeps to be a finite probability distribution verifying positivity and unit norm even under the cutoff approximation, i.e.,

$$\hat{\mathcal{P}}(\theta) = \frac{2k^2}{\hat{\sigma}_{\rm el}}\frac{d\hat{\sigma}_{\rm el}}{dt}, \tag{3.9}$$

since Eq. (3.5) leads to

$$\int_{-1}^{1} d\cos\theta \hat{\mathcal{P}}(\theta) = \int_{-\infty}^{0} dt \frac{1}{\hat{\sigma}_{\mathrm{el}}} \frac{d\hat{\sigma}_{\mathrm{el}}}{dt} = 1. \quad (3.10)$$

1. Step-function cutoff

In order for a concrete evaluation of the entanglement entropy, we, for instance, employ a step-function as the simplest cutoff function:

$$c(b) = \begin{cases} 1 & (b \leq 2\Lambda) \\ 0 & (b > 2\Lambda) \end{cases}. \quad (3.11)$$

The scattering amplitude (3.2) becomes

$$\hat{\mathcal{A}} = 32\pi k^2 \int_0^{2\Lambda} bdb\tau(b) J_0(b\sqrt{-t}). \quad (3.12)$$

This cutoff truncates the modes whose impact parameter is larger than the maximal impact parameter 2Λ. Since the impact parameter b is related with angular momentum ℓ by $b = \ell/k$, ℓ has an upper bound L defined by $2\Lambda k \equiv L$. Therefore one can also recognize the scattering amplitude as $\hat{\mathcal{A}} = 16\pi \sum_{\ell=0}^{L} (2\ell+1)\tau_\ell P_\ell(\cos\theta)$. Simultaneously the cutoff regularizes the infinite volume V of the full Hilbert space as

$$\hat{V} = 2k^2 \int_0^{2\Lambda} bdb = 4k^2\Lambda^2. \quad (3.13)$$

Then the condition (3.7) determines Λ such that

$$4\pi\Lambda^2 = \hat{\sigma}_{\mathrm{tot}}. \quad (3.14)$$

Under the cutoff (3.11) we write the differential cross section (3.3), the total cross section (3.4) and the elastic cross section (3.5) as

$$\frac{d\hat{\sigma}_{\mathrm{el}}}{dt} = 4\pi \left| \int_0^{2\Lambda} bdb\tau(b) J_0(b\sqrt{-t}) \right|^2, \quad (3.15)$$

$$\hat{\sigma}_{\mathrm{tot}} = 8\pi \int_0^{2\Lambda} bdb \,\mathrm{Im}\tau(b), \quad (3.16)$$

$$\hat{\sigma}_{\mathrm{el}} = 8\pi \int_0^{2\Lambda} bdb |\tau(b)|^2. \quad (3.17)$$

2. Gaussian cutoff

By concrete comparison with the step-function cutoff, let us consider a Gaussian cutoff function;

$$c(b) = \exp\left(-\frac{1}{2} \cdot \frac{b^2}{4\Lambda^2}\right), \quad (3.18)$$

corresponding to an impact-parameter width 2Λ. Then the differential cross section (3.3), the total cross section (3.4) and the elastic cross section (3.5) become

$$\frac{d\hat{\sigma}_{\mathrm{el}}}{dt} = 4\pi \left| \int_0^{\infty} bdb e^{-\frac{b^2}{8\Lambda^2}} \tau(b) J_0(b\sqrt{-t}) \right|^2, \quad (3.19)$$

$$\hat{\sigma}_{\mathrm{tot}} = 8\pi \int_0^{\infty} bdb e^{-\frac{b^2}{4\Lambda^2}} \mathrm{Im}\tau(b), \quad (3.20)$$

$$\hat{\sigma}_{\mathrm{el}} = 8\pi \int_0^{\infty} bdb e^{-\frac{b^2}{4\Lambda^2}} |\tau(b)|^2. \quad (3.21)$$

Since (3.7) shows that the Hilbert space volume is regularized in the same way as the total cross section, the regularized Hilbert space volume under the Gaussian cutoff (3.18) becomes

$$\hat{V} = 2k^2 \int_0^{\infty} bdb c^2(b) = 4k^2\Lambda^2, \quad (3.22)$$

and the condition (3.7) is written as

$$4\pi\Lambda^2 = \hat{\sigma}_{\mathrm{tot}}. \quad (3.23)$$

This condition has the same expression as the one (3.14) in the step function cutoff.

B. Application: The diffraction peak approximation in proton-proton scattering at high energy

We concentrate on the proton-proton scattering, because we can use the experimental data given by the Tevatron (at $\sqrt{s} = 1800$ GeV) and the LHC (at $\sqrt{s} = 7000$, 8000, 13000 GeV), of which data are listed in Table I. Note that the difference between $\bar{p}$-p and p-p scattering at the Tevatron and LHC energies is not expected to be relevant in our study and thus has been neglected.

Since we must know the differential cross section $\frac{d\sigma_{\mathrm{el}}}{dt}$ as a function of t in order to evaluate the entanglement entropy (3.8), here we assume the diffraction peak model, which is described by the following scattering amplitude:

TABLE I. Experimental cross sections by Tevatron and LHC, central values.

$\sqrt{s}$ [GeV]	σ_{tot} [mb]	σ_{el} [mb]	Refs.
1800	72.1	16.6	[11,12]
7000	98.58	25.43	[13]
8000	101.7	27.1	[14,15]
13 000	110.6	31.0	[16]

$$\mathcal{A}(s,t) = is\sigma_{\text{tot}} e^{\frac{1}{2}Bt}, \tag{3.24}$$

where B is the slope parameter. We assume sufficiently high energy, so that $s \approx 4k^2$. The differential elastic cross section is

$$\frac{d\sigma_{\text{el}}}{dt} = \frac{\sigma_{\text{tot}}^2}{16\pi} e^{Bt}, \tag{3.25}$$

and the elastic cross section is

$$\sigma_{\text{el}} = \int_{-\infty}^{0} dt \frac{d\sigma_{\text{el}}}{dt} = \frac{\sigma_{\text{tot}}^2}{16\pi B}. \tag{3.26}$$

Therefore the slope parameter B can be written in terms of σ_{tot} and σ_{el} as

$$B = \frac{\sigma_{\text{tot}}^2}{16\pi\sigma_{\text{el}}}. \tag{3.27}$$

From Eq. (3.1) and (3.24), $\tau(b)$ is calculated,

$$\begin{aligned} \tau(b) &= \frac{1}{32\pi k^2} \int_0^\infty \sqrt{-t} d\sqrt{-t} \mathcal{A}(s,t) J_0(b\sqrt{-t}) \\ &= i \frac{\sigma_{\text{tot}}}{8\pi B} e^{-\frac{b^2}{2B}}. \end{aligned} \tag{3.28}$$

1. Step-function cutoff

In terms of Eq. (3.28) we write down the truncated differential cross section (3.15),

$$\frac{d\hat{\sigma}_{\text{el}}}{dt} = \frac{\sigma_{\text{tot}}^2}{16\pi B^2} \left(\int_0^{2\Lambda} bdbe^{-\frac{b^2}{2B}} J_0(b\sqrt{-t}) \right)^2 \tag{3.29}$$

and compute the truncated cross sections (3.16) and (3.17),

$$\hat{\sigma}_{\text{tot}} = \sigma_{\text{tot}}(1 - e^{-\frac{2}{B}\Lambda^2}), \quad \hat{\sigma}_{\text{el}} = \frac{\sigma_{\text{tot}}^2}{16\pi B}(1 - e^{-\frac{4}{B}\Lambda^2}). \tag{3.30}$$

Then the condition (3.14) determining Λ becomes

$$\frac{4\pi\Lambda^2}{\sigma_{\text{tot}}} = 1 - e^{-\frac{2}{B}\Lambda^2}. \tag{3.31}$$

By using the data in Table I, we numerically calculate the cutoff parameter Λ, the truncated cross sections (3.30) and the entanglement entropy (3.8), and the results are shown in Table II.

TABLE II. The cutoff (Λ), the cross sections ($\hat{\sigma}_{\text{tot}}$, $\hat{\sigma}_{\text{el}}$) and the entanglement entropy ($\hat{S}_{\text{EE}}$) in the step-function regularization. The slope B is calculated by Eq. (3.27) from the experimental data of σ_{tot} and σ_{el}.

$\sqrt{s}$ [GeV]	Λ [fm]	$\hat{\sigma}_{\text{tot}}$ [mb]	$\hat{\sigma}_{\text{el}}$ [mb]	$\hat{S}_{\text{EE}}$	B [GeV^{-2}]
1800	0.6550	53.91	15.54	1.193	16.00
7000	0.7988	80.18	24.54	1.192	19.52
8000	0.8192	84.34	26.31	1.197	19.50
13000	0.8659	94.23	30.32	1.212	20.16

2. Gaussian cutoff

The differential cross section (3.19) truncated by the Gaussian cutoff with Eq. (3.28) is written down as

$$\frac{d\hat{\sigma}_{\text{el}}}{dt} = \frac{\sigma_{\text{tot}}^2}{16\pi} \left(1 + \frac{B}{4\Lambda^2}\right)^{-2} \exp\left(\frac{B}{1 + \frac{B}{4\Lambda^2}} t\right). \tag{3.32}$$

In the same way we calculate the truncated cross sections (3.20) and (3.21), so that

$$\hat{\sigma}_{\text{tot}} = \sigma_{\text{tot}} \left(1 + \frac{B}{2\Lambda^2}\right)^{-1}, \qquad \hat{\sigma}_{\text{el}} = \frac{\sigma_{\text{tot}}^2}{16\pi B} \left(1 + \frac{B}{4\Lambda^2}\right)^{-1}. \tag{3.33}$$

The condition (3.14) fixes Λ as

$$\Lambda = \sqrt{\frac{\sigma_{\text{tot}}}{4\pi} - \frac{B}{2}}. \tag{3.34}$$

Furthermore one can write down the entanglement entropy (3.8) as

$$\hat{S}_{\text{EE}} = 1 - \ln \frac{4\pi B(1 + \frac{B}{2\Lambda^2})}{\sigma_{\text{tot}}(1 + \frac{B}{4\Lambda^2})}. \tag{3.35}$$

The numerical evaluation of the cutoff, the total and elastic cross sections and the entanglement entropy are shown in Table III.

3. Comparison with volume-regularization

In order to compare the cutoff regularizations with the volume-regularization, let us try to evaluate the entanglement entropy $\tilde{S}_{\text{EE}}$ in Eq. (2.28) by the volume-regularization. Although we do not know how to concretely realize the volume-regularization, we compute $\tilde{S}_{\text{EE}}$ by the use of

TABLE III. The cutoff (Λ), the cross sections ($\hat{\sigma}_{\text{tot}}$, $\hat{\sigma}_{\text{el}}$) and the entanglement entropy ($\hat{S}_{\text{EE}}$) in the Gaussian regularization.

$\sqrt{s}$ [GeV]	Λ [fm]	$\hat{\sigma}_{\text{tot}}$ [mb]	$\hat{\sigma}_{\text{el}}$ [mb]	$\hat{S}_{\text{EE}}$
1800	0.5121	32.96	10.41	0.6009
7000	0.6359	50.81	17.30	0.7539
8000	0.6555	53.99	18.79	0.7965
13000	0.6983	61.28	22.10	0.8621

TABLE IV. The entanglement entropy in the volume-regularization.

$\sqrt{s}$ [GeV]	$\tilde{S}_{\mathrm{EE}}$
1800	0.9176
7000	1.031
8000	1.063
13 000	1.114

$\frac{d\sigma_{\mathrm{el}}}{dt}$ given by Eq. (3.25) with Eq. (3.27) in the diffraction peak model. Then the entanglement entropy (2.28) becomes

$$\tilde{S}_{\mathrm{EE}} = 1 + \ln \frac{4\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}}. \quad (3.36)$$

Evaluating this in terms of the data in Table I, we show the results in Table IV. The entanglement entropy monotonically increases according as the center-of-mass energy becomes higher.

The truncated cross sections in Table II by the step-function cutoff give a closer approximation to the experimental data in Table I better than those in Table III by the Gaussian cutoff. As shown in Fig. 1, actually the entanglement entropy obtained from the volume-regularization appears to be framed by the step-function one (above) and the Gaussian one (below).

IV. DISCUSSION, CONCLUSION, AND OUTLOOK

In our study, we have evaluated the entanglement entropy S_{EE} for the two particles elastically produced in a high-energy collision. For this sake, we have used a regularization procedure, in order to get rid of the divergences appearing in the formal derivation of S_{EE}. These divergences happen to be related to the infinite "volume" of the full two-particle Hilbert space, be there coming from the interaction or not. It can be regularized by considering the finite two-particle Hilbert space actually spanned by elastic collisions at a given energy. For the discussion we have first introduced a formulation of a finite entanglement entropy $\tilde{S}_{\mathrm{EE}}$ using the formal definition supplemented with a regularized Hilbert space volume, which is defined by

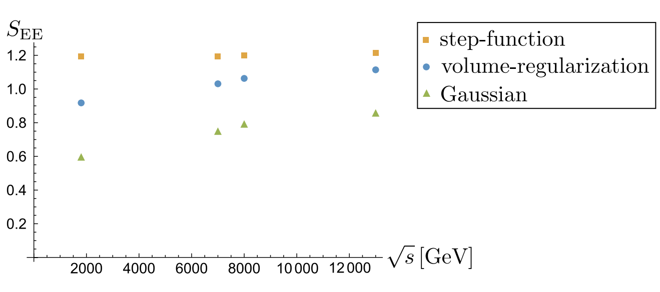

FIG. 1. The entanglement entropy in three different regularizations with respect to the center-of-mass energy.

projecting out the volume of phase space spanned by the noninteracting final states responsible of the divergence. We then considered two explicit cutoff definitions, one using a step-function and the other with a Gaussian.

Summarizing our results, we found the following:

(i) The volume-regularized formulation provides an expression of the entanglement entropy in terms of physical observables (2.28);

$$\tilde{S}_{\mathrm{EE}} = -\int_{-\infty}^{0} dt \frac{1}{\sigma_{\mathrm{el}}} \frac{d\sigma_{\mathrm{el}}}{dt} \ln\left(\frac{4\pi}{\sigma_{\mathrm{tot}}\sigma_{\mathrm{el}}} \frac{d\sigma_{\mathrm{el}}}{dt}\right).$$

(ii) In search of an adequate quantitative cutoff procedure defining the finite physical Hilbert space, we considered the case of proton-proton elastic scattering at the Tevatron and LHC energies. In a diffraction peak approximation as a simple example, we have compared the numerical results for the regularized entanglement entropy $\hat{S}_{\mathrm{EE}}$ in two different cutoffs, and we also compared them with the result for the entanglement entropy $\tilde{S}_{\mathrm{EE}}$ [see Eq. (3.36)] from the volume-regularization.

(iii) Since a cutoff dependence appears for the observables in the formula (2.28) and modifies their contribution to the entanglement entropy, the effect of the cutoff is to replace the observables in Eq. (2.28) with their expressions with the cutoff as $\hat{S}_{\mathrm{EE}}$ in Eq. (3.8). The step-function cutoff appears to give a better approximation of the real observables than the Gaussian one. However, the result for the entanglement entropy $\tilde{S}_{\mathrm{EE}}$ boils down to a framing of the volume-regularized entropy by the step-function one (above) and the Gaussian one (below).

(iv) The trend of the overall results for $\tilde{S}_{\mathrm{EE}}$ clearly demonstrates a nonzero entanglement entropy showing that a non-negligible entanglement is generated in a high-energy elastic collision, even in the presence of a large sector of inelastic reactions. Indeed, the entanglement entropy is different from zero and stays around unity, while increasing slightly with the center-of-mass energy. For instance, in the diffraction peak approximation, the volume-regularization gives Eq. (3.36);

$$\tilde{S}_{\mathrm{EE}} = 1 + 2\ln 2 + \ln\left(\frac{\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}}\right),$$

which allows one to relate the entanglement entropy simply to the ratio $\frac{\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}}$. Higher is the ratio, larger is the entanglement entropy, which seems physically sound. Moreover, it is known that this ratio stays experimentally around $1/4$, and thus $\tilde{S}_{\mathrm{EE}} \sim 1$.

As an outlook, it would be useful to find a better cutoff procedure, which would leave the observables unchanged or only slightly changed by the regularization procedure. For example, an optimization calculation could be introduced to define the cutoff less arbitrarily as those we chose in our present study. Then the full set of experimental observables could be safely introduced in the calculation of $\tilde{S}_{\mathrm{EE}}$, without cutoff dependence and diffraction peak approximation.

We have considered the entanglement entropy in the momentum Hilbert space. However the relativistic state of a particle also have a quantum number of spin (or helicity). Therefore one can consider the entanglement entropy in the momentum and spin Hilbert space, i.e., $\{|\vec{p},s\rangle_A\}\otimes\{|\vec{q},s'\rangle_B\}$. In a similar way as what we studied in this paper, such entropy will be formulated by the use of the S-matrix with respect to the helicity, which was studied in some literature [17,18]. Especially in high energy scattering of hadrons, where Pomeron exchange is dominant, the s-channel helicities of the particles are mostly conserved, and thus adding the spin boils down to extend our analysis to elastic scattering of quantum states with given momentum and given s-channel helicity.

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