Study of center and chiral symmetry realization in thermal $\mathcal{N}=1$ super Yang-Mills theory using the gradient flow

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The realization of center and chiral symmetries in $\mathcal{N}=1$ super Yang-Mills theory (SYM) is investigated on a four-dimensional Euclidean lattice by means of Monte Carlo methods. At zero temperature this theory is expected to confine external fundamental charges and to have a nonvanishing gaugino condensate, which breaks the nonanomalous Z_{2N_c} chiral symmetry. In this work we find, for the first time, a nonvanishing condensate at zero temperature and zero gaugino mass with Wilson fermions. This is achieved by means of the renormalization properties of the Yang-Mills and fermion gradient flows, which are independent of the lattice regularization. At finite temperatures, we find that the phase transitions corresponding to deconfinement and chiral restoration occur at roughly the same critical temperature for SU(2) gauge group, implying the saturation of the bound $T^c_{\mathrm{chiral}} \leq T^c_{\mathrm{dec}}$, recently obtained through anomaly matching. We furthermore discuss the agreement of our findings with conjectures from superstring theory.

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I. INTRODUCTION

Supersymmetric Yang-Mills theories (SYM) have been a useful laboratory to extend our understanding of nonperturbative phenomena. For instance, the study of supersymmetry combined with gauge symmetry has led to the discovery of duality as a feature of supersymmetric gauge theories. Besides the well-known gauge-gravity duality between the conformal $\mathcal{N}=4$ SYM and string theory on the curved space $\mathrm{AdS}_5 \times S^5$, electromagnetic duality has been conjectured for $\mathcal{N}=2$ SYM and $\mathcal{N}=1$ supersymmetric QCD [1–3]. The most interesting open challenge is the search for theories that are more similar to QCD and still allow for an analytical understanding of confinement and strong interactions due to supersymmetry. $\mathcal{N}=1$ SYM is the supersymmetric extension of the pure gauge sector of QCD describing the strong interactions of gluons and gauginos. It shares many interesting aspects with QCD while retaining minimal supersymmetry. It is expected that this theory should exhibit confinement, chiral symmetry breaking and formation of a mass-gap [4–7]. $\mathcal{N}=1$ SYM is therefore a perfect candidate for the analytical understanding of nonpertubative phenomena of non-Abelian gauge theories, like confinement and chiral symmetry breaking.

Numerical lattice simulations are the natural tool to explore nonperturbatively the phase diagram of strongly interacting gauge theories, such as $\mathcal{N}=1$ SYM, and complement the analytical investigations. The aim of this work is to apply numerical methods in a search for interesting patterns in the phase transitions of $\mathcal{N}=1$ SYM and to relate them to analytical arguments. In a previous study we have found an interesting indication for a coincidence of the chiral and deconfinement transition of this theory without an analytical explanation. In this work we find that this signal persists at finer lattice spacings and using refined numerical methods. Moreover we show that it is confirmed by analytical arguments from a string theory perspective. Another interesting open aspect of our previous study is the relation of the numerical data to the analytic prediction for the gluino condensate at zero temperature. The gradient flow methods that we apply in the current work allow the first step toward this relation: we are able to extrapolate the finite value of the gluino condensate in the chiral limit without additive renormalization. The remaining issue that is left open in our investigations is the multiplicative renormalization required to match the scheme of the analytical predictions.

The investigations of chiral symmetry breaking of $\mathcal{N}=1$ SYM on the lattice requires additional considerations.

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We use the Wilson fermion discretization that breaks chiral symmetry explicitly. In addition, the lattice discretization also unavoidably breaks supersymmetry. We have seen in our previous studies that it is possible to recover both symmetries in the continuum limit if the gaugino mass is tuned to the chiral limit. However, the breaking still leads to a divergent additive renormalization of the chiral condensate.

Recently the gradient flow has been proposed as a regularization-scheme independent smoothing technique that is able to simplify drastically the renormalization of lattice bare composite operators [8–10]. In particular, the flowed chiral condensate is free from the additive renormalization of Wilson fermions [10]. In recent years the gradient flow has found a growing spectrum of applications. In particular the investigations of finite temperature QCD including the chiral transition have shown the benefits of this method [11,12]. Furthermore, a novel approach to compute operator dimensions in conformal field theories was recently proposed in [13], which exploits the relation between the gradient flow and renormalization group transformations. The application for supersymmetric theories has been suggested in [14,15], where it can help to renormalize the supercurrent. In several works also a supersymmetric version of the method has been developed [16,17], which might even avoid the necessity of a multiplicative renormalization of the fermions.

In this work we present an extended study of the phase diagram of $\mathcal{N}=1$ SYM with the gauge group SU(2) at zero and nonzero temperature using the gradient flow. This method is summarized in Sec. IV. The details of our lattice discretization as well as the expected phase transitions are explained in Secs. II and III. We measure the chiral condensate expectation value at positive flow time. We show strong evidences that chiral symmetry is spontaneously broken at zero temperature by a nonvanishing expectation value of the gaugino condensate in Sec. V. At high temperature chiral symmetry is restored and that a phase transition occurs in the massless limit. In Sec. VI, we provide evidence that there are only two phases in the massless limit, characterized by both chiral symmetry breaking and confinement at low temperature and chiral symmetry restoration and deconfinement at high temperature. We have not found any indications of mixed phases where deconfinement occurs while chiral symmetry is broken in our numerical data. This section contains also an interpretation of our results from the point of view of superstring/M-theory.

II. $\mathcal{N}=1$ SUPER YANG-MILLS THEORY ON THE LATTICE

$\mathcal{N}=1$ super Yang-Mills theory is the minimal four-dimensional gauge theory consistent with supersymmetry. It is therefore the supersymmetric gauge theory most similar to Yang-Mills theory and QCD. The four-dimensional Euclidean on-shell action is

$$S=\int d^4x\left(\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}+\frac{1}{2}\bar{\lambda}^a \not{D}\lambda^a+m\bar{\lambda}^a\lambda^a+\frac{\theta}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{a\mu\nu}F^{a\rho\sigma}\right), \tag{1}$$

where F is the usual Yang-Mills (YM) field strength, λ the gaugino, a Majorana spinor in the adjoint representation, and D the covariant derivative. The mass term m breaks supersymmetry softly. The last term is a topological term, whose space-time integration yields the winding number Q_{top} of the gauge field, which we neglect in our investigations.

On the lattice, spinors are site variables and Yang-Mills fields A_μ are represented as links through the parallel transporter $U_\mu=\mathrm{e}^{igaA^b_\mu\tau^F_b}$, being a the lattice spacing and τ^F the Lie generators in the fundamental representation. We employ the following SYM lattice action in our simulations [18]

$$S_{\mathrm{lat}}=\sum_x\mathrm{Retr}\left\{\frac{\beta}{\mathrm{N}_c}\sum_{\mu\neq\nu}\left(\frac{5}{3}P_{\mu\nu}(x)-\frac{1}{12}R_{\mu\nu}(x)\right)\right\}+\frac{1}{2}\sum_{x,y}\bar{\lambda}(y)D_w[V_\mu](y,x)\lambda(y),$$

where the Dirac-Wilson operator

$$D_w(x,y)\lambda(y)=\lambda(x)-\kappa\sum_{\mu=1}^{4}[(1-\gamma_\mu)V_\mu(x)\lambda(x+\mu)+(1+\gamma_\mu)V^\dagger_\mu(x-\mu)\lambda(x-\mu)],$$

has parallel transporters V in the adjoint representation, defined as $V_\mu(x)_{ab}=2\mathrm{tr}(U_\mu(x)^\dagger\tau^F_aU_\mu(x)\tau^F_b)$. Here β is equal to $\frac{2N_c}{g^2}$, $P_{\mu\nu}$ and $R_{\mu\nu}$ are the plaquette and the 2×1 rectangular Wilson loop, respectively. One-level stout smearing with parameter $\rho=0.15$ has been employed for the links in the fermion action. Finally, $\kappa=\frac{1}{2m+8}$ is the hopping parameter. Our simulations also include runs with a clover improved fermion action in order to test the lattice artefacts comparing two different discretizations.

A nonzero gaugino mass breaks supersymmetry softly and the bare parameter m must be tuned to the point of a vanishing physical mass. A numerical accurate and inexpensive way to achieve that is by means of the adjoint-pion a-π. Although this is not a physical degree of freedom of the theory, its mass is related to the gaugino mass as $m^2_{\mathrm{a}-\pi}\sim m$, as was shown in [19] within the framework of partially quenched perturbation theory. The lattice discretization breaks supersymmetry, but it is recovered in the massless and continuum limit, as confirmed by supersymmetric Ward identities [20].

III. THE PHASE DIAGRAM OF $\mathcal{N}=1$ SUPER YANG-MILLS THEORY

$\mathcal{N}=1$ SYM shares several features with QCD, as it is a confining theory without scalars and it includes spinor fields interacting with a gauge field. As explained in this section, these properties transform $\mathcal{N}=1$ SYM to an important laboratory to gain a deeper understanding of nonperturbative phenomena of QCD such as chiral symmetry breaking and confinement. These phenomena are usually more accessible in supersymmetric theories due to the constraints that supersymmetry imposes.

A. Confinement of static fundamental charges

At zero temperature, the YM vacuum is expected a confining medium for external static color-electric charges. It can be probed through the Polyakov loop, which is the path ordered product of the links in the fundamental representation along a line which wraps in the compact direction

$$P_L = \frac{1}{N_t V_3} \sum_{\vec{x}} \mathrm{Tr}\left\{ \prod_{t=1}^{N_t} U_4(\vec{x}, t) \right\}, \tag{2}$$

where V_3 denotes the three-dimensional lattice volume of the noncompact directions. The Polyakov loop can be related to the exponential of the free energy of an isolated static fundamental quark. Hence confinement is detected by a vanishing vacuum expectation value of P_L, indicating that isolated fundamental quarks are states with infinite free energy. As the temperature is increased, the vacuum expectation value of the Polyakov loop becomes nonvanishing at some critical temperature, where quark deconfinement occurs.

In the case of QCD, there is no real confinement-deconfinement phase transition but a crossover. In the confined phase, the quark-antiquark potential grows linearly until it is screened by another quark-antiquark pair popping up from the vacuum. In contrast, the identification of the deconfinement phase transition is clear for $\mathcal{N}=1$ SYM, as adjoint fermion fields are unable to provide string breaking to fundamental quark-antiquark pairs. The Polyakov loop represents a good order parameter for the deconfinement phase transition for $\mathcal{N}=1$ SYM and a singularity of the partition function is expected at a critical nonzero temperature.

B. Chiral symmetry

$\mathcal{N}=1$ supersymmetric Yang-Mills theory is invariant under chiral transformations when no soft SUSY-breaking mass term is included in the action. Chiral symmetry coincides with the U(1) R-symmetry of the theory[1]

$$\lambda^a \to \mathrm{e}^{i\alpha}\lambda^a.$$

This symmetry is broken at the quantum level by instanton contributions, as the U(1) chiral rotation is equivalent to

$$\theta \to \theta - 2\mathrm{N}_c\alpha,$$

where N_c is the number of colors. The path integral is therefore invariant only for $\alpha = k\pi/\mathrm{N}_c$, i.e., the chiral symmetry of the quantum theory is actually the discrete subgroup $\mathrm{Z}_{2\mathrm{N}_c} \subset \mathrm{U}(1)$. As the gaugino condensate is not invariant with respect to the full $\mathrm{Z}_{2\mathrm{N}_c}$ group, i.e., $\langle\lambda^a\lambda^a\rangle \to \mathrm{e}^{2i\alpha}\langle\lambda^a\lambda^a\rangle$, an interesting question is whether the quantum chiral symmetry is further spontaneously broken by a nonvanishing $\langle\lambda^a\lambda^a\rangle$. Indeed, analytical calculations show that the answer is affirmative [4–7,18,21–23]. Thanks to remarkable properties of supersymmetric theories like nonrenormalization and holomorphicity of the effective superpotential with respect to fields and couplings, it was proven [24,25] that if the effective theory at long distances is massive with color-singlets as degrees of freedom, the gaugino condensate takes on the form

$$\langle\lambda^a\lambda^a\rangle = c\Lambda^3. \tag{3}$$

The global $\mathrm{Z}_{2\mathrm{N}_c}$ symmetry is therefore spontaneously broken down to its discrete subgroup Z_2, i.e., to the sign flip $\lambda^a \to -\lambda^a$. As a consequence, there are N_c degenerate vacua connected by $\mathrm{Z}_{\mathrm{N}_c}$ chiral transformations and domain walls interpolating between different vacua are expected [26].

The exact value of the proportionality constant c in (3) has been computed based on strong coupling and weak coupling instanton analysis. The differences of these approaches have been addressed in compactified SYM, where the semi-classical analysis leads to a controlled approximation [27].

There have been many attempts to generalize the calculations of the condensate towards other gauge theories. The computed value has been compared to the results of one-flavour QCD based on the orientifold planar equivalence. This comparison is based on several assumptions that are discussed in Sec. V.

C. The phase diagram at nonzero temperatures

In thermal quantum field theory, temperature corresponds to the inverse radius of a compactified direction of the path integral with thermal boundary conditions, i.e., periodic for bosons and antiperiodic for fermions. The phases of $\mathcal{N}=1$ SYM are related to the response of the mentioned properties of the zero temperature vacuum to the finite temperature. At some critical temperature the condensate is expected vanish and the spontaneously broken chiral symmetry is restored. In addition, center symmetry breaks spontaneously at the deconfinement temperature and the Polyakov loop should acquire a nonvanishing expectation value. An interesting question is whether both critical temperatures coincide, as it would be

[1]In this section λ^a is a Weyl spinor, as this notation is more natural for the massless theory.

expected if confinement and chiral symmetry breaking share the same nonperturbative origin. This question is far from trivial. Although QCD, for example, is also characterized by confinement and broken chiral symmetry at low energies, some theories have been found to exhibit only one of those [28]. In many numerical studies of QCD it has been found that chiral restoration and deconfinement phase transition occur at nearly the same critical temperature. However, taking QCD as a starting point implies the difficulty that these phase transitions are actually crossovers due to the finite-massive fundamental color charges, and no exact order parameter can be found which unambiguously signals the pseudocritical temperature.

Fortunately, both the Polyakov loop and the gaugino condensate are exact order parameters for the respective phase transitions in $\mathcal{N}=1$ SYM. Consequently, the question of the relation between the chiral and deconfinement transition can be answered by studying these quantities in a systematic way, the only remaining complications being related to the regularization and the renormalization of the chiral condensate. The lattice realization with Wilson fermions on the lattice breaks chiral symmetry, which implies besides the multiplicative renormalization factor ($Z_{\bar{\lambda}\lambda}$) an additional additive renormalization ($\mathbf{b_0}$) of the gaugino condensate

$$\langle\bar{\lambda}\lambda\rangle_{\mathrm{R}} = Z_{\bar{\lambda}\lambda}(\beta)(\langle\bar{\lambda}\lambda\rangle_{\mathrm{B}} - \mathbf{b_0}(\beta)).$$

In previous investigations [29], the additive constant $\mathbf{b_0}$ was removed by subtracting the bare condensate at zero temperature,

$$\langle\bar{\lambda}\lambda\rangle_{\mathrm{S}} = \langle\bar{\lambda}\lambda\rangle_{\mathrm{B}}^{T=0} - \langle\bar{\lambda}\lambda\rangle_{\mathrm{B}}^{T}$$

The downside of this approach is the fact that the subtracted condensate is forced to vanish at $T=0$. Although this subtraction should preserve the behavior of the order parameter near the critical temperature, it is not possible to determine whether the renormalized condensate at zero temperature is nonzero. Hence, the picture of the realization of chiral symmetry in $\mathcal{N}=1$ SYM is, following this method, incomplete. As it will be explained in the next section, the additive renormalization is not necessary when the gradient flow is used, allowing the computation of the condensate at zero temperature and, moreover, allowing a comparison to fermion discretizations which satisfy the Ginsparg-Wilson relation.

IV. THE GRADIENT FLOW

A. Flow equations

Motivated in the context of trivializing maps [30], Lüscher studied correlation functions of fields *flowed* through the equations [9][2]

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + D_\mu \partial_\nu B_\nu, \qquad B_\mu|_{t=0} = A_\mu, \tag{4}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \tag{5}$$

$$\partial_t\chi = \Delta\chi \quad \partial_t\bar{\chi} = \bar{\chi}\overleftarrow{\Delta}, \qquad \chi|_{t=0} = \psi \quad \bar{\chi}|_{t=0} = \bar{\psi}, \\ \Delta = D_\mu D^\mu, \tag{6}$$

where B (resp. A) and G are the Yang-Mills gauge field and field strength, respectively. χ and ψ are spinor fields and D_μ is the gauge-covariant derivative in the adjoint representation. The parameter $t \in \mathbb{R}$ describes a flow on the vector space of gauge fields.

The flow equations have a smoothening effect on the fields, which are Gaussian-like smeared over an effective radius $r_t = \sqrt{8t}$, as it can be easily seen by integrating equation (4) in the noninteracting limit $[B,B] \sim 0$,

$$B_{\mu,1}(t,x) = \int d^D y K_t(x-y) A_\mu(y),$$
$$K_t(z) = \int \frac{d^D p}{(2\pi)^D} \mathrm{e}^{ipz}\mathrm{e}^{-tp^2} = \frac{\mathrm{e}^{-z^2/4t}}{(4\pi t)^{D/2}},$$

where D is the number of space-time dimensions and K the heat-kernel. The term e^{-tp^2} regularizes the integral in momentum space when $t>0$, removing the UV divergences at large momenta. Some years ago it was shown that through the gradient flow the smearing property remains at all orders in perturbation theory [9]. Correlation functions of monomials of flowed gauge-fields are renormalized without extra counterterms, while spinor operators renormalize multiplicatively according to the field content [10]. An advantage of the gradient flow is its intrinsic regularization-scheme independence, and the renormalization properties of the flow hold also once discretized on the lattice. Therefore, currents and densities which are explicitly broken by the lattice discretization should be easily accessible at positive flow-time. In particular, the complications of explicit chiral symmetry breaking of the Wilson discretization of the fermion action are solved in the sense that the flowed chiral condensate renormalizes only multiplicatively.

Written explicitly, the flowed bare gaugino condensate is [10]

$$\langle\bar{\lambda}(x,t)\lambda(x,t)\rangle \\ = -\int d^D v d^D w K(t,x;0,v) S(v,w) K(t,x;0,w)^\dagger. \tag{7}$$

The flowed condensate is equivalent to the action of the heat-kernel on the fermion propagator S. The condensate at a fixed flow time that is computed in this way is proportional to any other renormalized condensate. It defines a correct order parameter for the chiral symmetry breaking

[2]The term $D_\mu\partial_\nu B_\nu$ is a gauge parameter, which is included for mere technical reasons.

even for Wilson fermions. The multiplicative factor in comparison to the $\overline{\mathrm{MS}}$ scheme can be determined from a small flow-time expansion. Consequently, it is possible to study if $\langle\bar{\lambda}\lambda\rangle \neq 0$ at $T = 0$ even with Wilson fermions. The value of $Z_{\bar{\lambda}\lambda}$ is irrelevant if the bare lattice gauge coupling is kept fixed.

The numerical integration of the flow is relatively straightforward, and the methods we employ to compute the flowed gluino condensate are described in Appendix A.

V. GAUGINO CONDENSATE AT ZERO TEMPERATURE

The gradient flow method allows for the computation of the gaugino condensate at zero temperature, without an additive renormalization. We consider four ensembles from our previous investigations of SU(2) SYM at $\beta = 1.75$ on a $32^3 \times 64$ lattice with hopping-parameter values

$$\kappa \in \{0.1490, 0.1492, 0.1494, 0.1495\}.$$

These four points are used to extrapolate to the chiral point, i.e., to vanishing renormalized gaugino mass. The scale in physical units is set by the gradient flow observables t_0, which is defined as

$$t^2\langle\varepsilon(t)\rangle|_{t=t_0} = 0.3, \qquad \varepsilon(t) = \frac{1}{4}G^a_{\mu\nu}(t)G^a_{\mu\nu}(t),$$

with ε the field energy density and $G^a_{\mu\nu}$ the flowed Yang-Mills field strength. The t_0 values for each κ value are summarized in Table I. For the computation of the flowed condensate, a mass-independent scheme was used, i.e., a common flow time was chosen corresponding to the chiral extrapolated $t_0^{\mathrm{chiral}} = 12.81(35)$. The flow equations were integrated up to t_0 and the condensate obtained from Eq. (A1). In the following we use the condensate in units of t_0, defined as:

$$\langle\bar{\lambda}\lambda\rangle \equiv t_0^{3/2}\langle\bar{\lambda}\lambda(t)\rangle|_{t=t_0}.$$

As shown in Fig. 1, we obtain a nonzero value of the gluino condensate in the chiral limit. For the numerical simulations with Wilson fermions, this is the first clear evidence that the discrete chiral symmetry is spontaneously broken at zero temperature and zero gaugino mass. In previous studies without the method of the gradient flow, only an indirect observation of the nonzero condensate has been possible from a double peak of the histogram that appears in rather unstable simulations on small volumes close to the chiral point [31,32].

TABLE I. The gradient flow scale t_0 and the values of the gluino condensate for each of the ensembles.

κ	am_π	t_0	$\langle\bar{\lambda}\lambda\rangle$
0.1490	0.23847(41)	9.851(32)	0.00025455(89)
0.1492	0.20346(54)	10.545(69)	0.00025177(80)
0.1494	0.1604(15)	11.262(72)	0.0002448(14)
0.1495	0.1294(24)	12.49(18)	0.0002386(12)

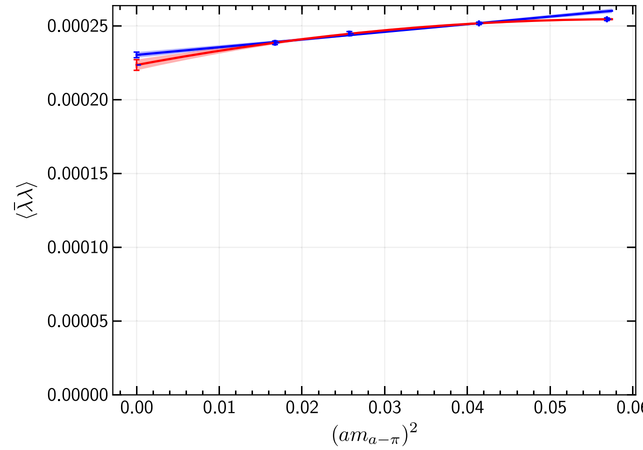

FIG. 1. Extrapolation of the chiral condensate to vanishing renormalized gaugino mass. The gaugino condensate scales almost linearly with the lattice spacing a, while some mild quadratic dependence is noticeable as the gaugino mass grows. Most importantly, the condensate does not vanish at the chiral point, which implies the spontaneously symmetry breaking of the Z_{2N_c} symmetry. The extrapolated values are $\langle\bar{\lambda}\lambda\rangle_{\mathrm{chiral}} = 0.0002304(19)$ (linear) and $\langle\bar{\lambda}\lambda\rangle_{\mathrm{chiral}} = 0.0002235(36)$ (quadratic).

Our results are of the same significance as the ones obtained with domain-wall and overlap fermions in [33–37]. In these simulations the renormalization factor of the condensate has not been determined and therefore the results are, as in our case, proportional to the renormalized condensate in $\overline{\mathrm{MS}}$ scheme. The gradient flow method opens up the possibility to study the different phases at zero temperature without the computational more expensive Ginsparg-Wilson fermions. A direct comparison of the results with different lattice actions would have been possible from the small flow-time expansion combined with finite-volume methods.

The comparison of our numerical result with strong and weak coupling instanton calculations (3) requires two further important steps, as already outlined in the comparison of one-flavor QCD with numerical results [38]. The first step is the matching of scale Λ, which, according to [38], can be set to the value in the $\overline{\mathrm{MS}}$ scheme. We have already done this analysis and found a value of $\Lambda^{\overline{\mathrm{MS}}}$ in SYM that is quite compatible to QCD [39]. The second step requires the determination of the renormalization factor $Z_{\bar{\lambda}\lambda}$ of the condensate for the exact result. The analysis of [38] shows that the renormalization scheme dependence can be rephrased in terms of a second proportionality factor $\tilde{c}$ multiplied to c in (3). This factor is assumed to have a subleading correction in the large N_c expansion, but there is so far no clear determination at small N_c. Without an analytic prediction for the additional factor, it is not possible to match the semiclassical predictions to any numerical value.

VI. DISCRETE CHIRAL SYMMETRY RESTORATION AND QUARK DECONFINEMENT

The second main purpose of this study is to investigate the realization of chiral and center symmetries in SU(2) SYM at finite temperature. Some first results in this direction were obtained in [29], and now we revise our early investigation using the gradient flow on four different new lattice ensembles at $\beta = 1.75$ and hopping parameter $\kappa \in \{0.1480, 0.1490, 0.14925\}$. To further cross-check the validity of our results, we have also analyzed a set of ensembles generated at $\beta = 1.65$ and $\kappa = 0.175$ using the tree-level clover improved fermion action with unsmeared links. Nonzero temperatures were achieved by fixing the lattice parameters and compactifying one dimension on a circle, imposing thermal boundary conditions on the fields, i.e., anti(periodic) for fermions (bosons). Thus, the lattice size was set to $24^3 \times N_t$ for $N_t \in \{4, 5, 6, \ldots, 48\}$, with the upper bound parametrizing the zero-temperature limit. For this finite-temperature analysis, a mass-dependent scale setting was used. This means that the flow time of the gaugino condensate is different for each κ value, in contrast to the zero-temperature measurements. One reason for this choice is that the integration of the adjoint flow equation gets very expensive as the flow time grows. A second reason is that the possible range of t is limited at smaller N_t. A reliable value can not be obtained once the smearing radius becomes compatible with N_t. We have checked that around the mass dependent value of t_0 the condensate shows only a very weak t dependence. Therefore we expect no relevant change of the results for a mass independent choice of t_0.

The results are summarized in Figs. 2–4 and Table II. The condensate is in general considerably reduced as the temperature increases, but a clear phase transition is difficult to identify for the larger gaugino masses corresponding to smaller values of κ. This smooth behavior of the chiral condensate at larger masses is expected since the jump of the order parameter around the pseudocritical temperature is less pronounced for a larger explicit symmetry breaking. At the smallest gaugino mass ($\kappa = 0.14925$) the signal is considerably better and a jump at the critical temperature can be identified. This suggests, as expected, that the chiral restoration becomes indeed a true phase transition and the gaugino condensate an adequate order parameter in the chiral (and supersymmetric) limit. This can be more clearly seen from the disconnected chiral susceptibility in Fig. 4.[3] It is remarkable that the phase transition appears at $T = \sqrt{t_0}/N_t \sim 0.25$ also with the improved lattice action, see Fig. 4.

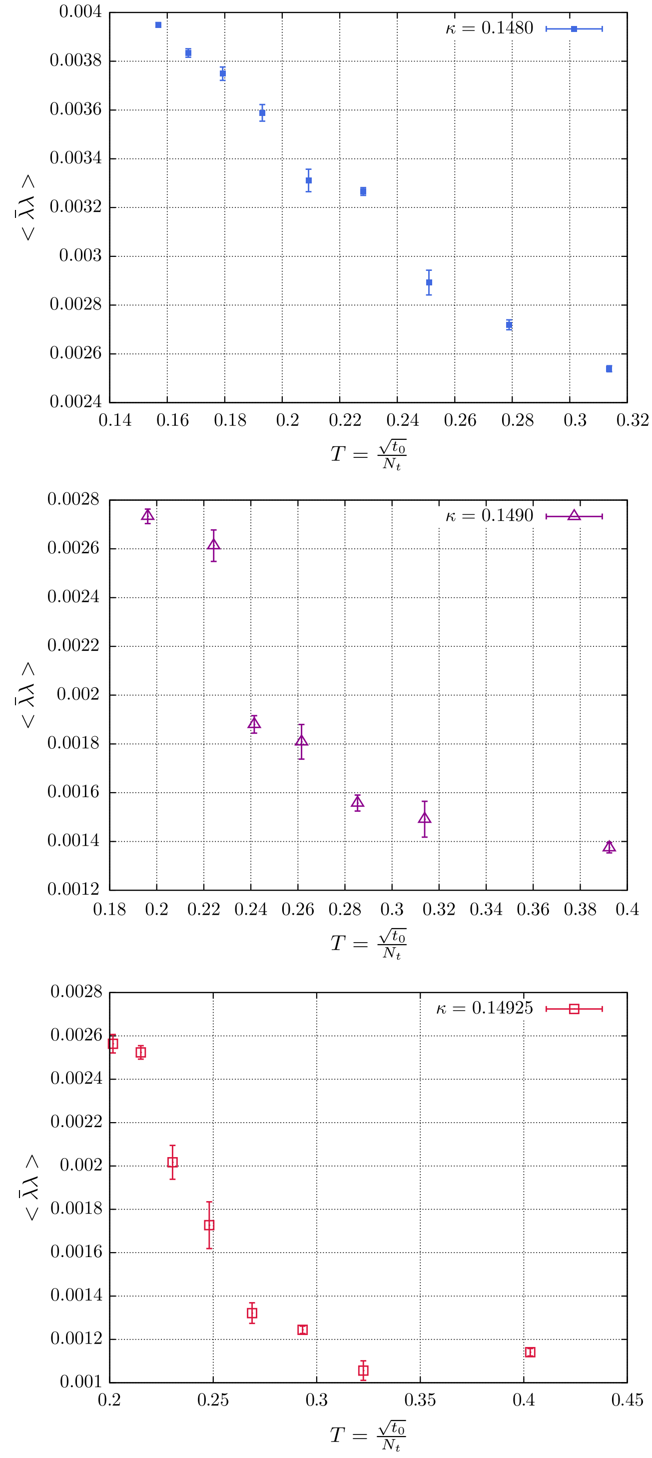

FIG. 2. Temperature dependence of the bare gaugino condensate is shown for $\kappa = 0.1480, 0.1490$ and 0.14925, with the smallest κ in the uppermost plot. On the undermost graphic, the point corresponding to the highest temperature appears to show a growth in the condensate. This is however a nonphysical oversmoothing artefact due to the fact that $\sqrt{8t_0} > N_t$ in that region.

[3] The disconnected part of the susceptibility is expected to represent the largest contribution to the phase transition's peak [29].

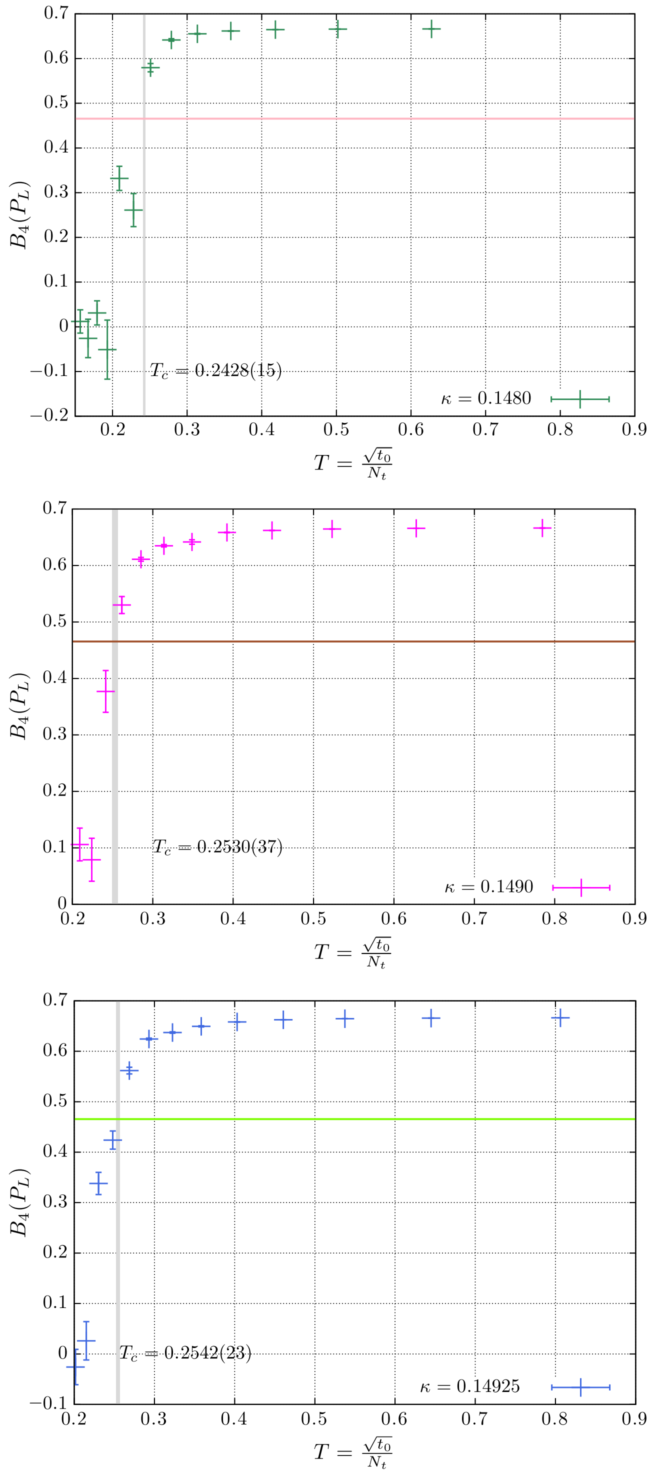

FIG. 3. Binder cumulant of the Polyakov loop for the same κ values of Fig. 2.

This temperature approximately coincides with the deconfinement phase transition, which was found in [29] to be second order and to have the critical behavior of the Z_2 Ising model. The deconfinement transition occurs thus at the point where the Binder cumulant of the Polyakov loop

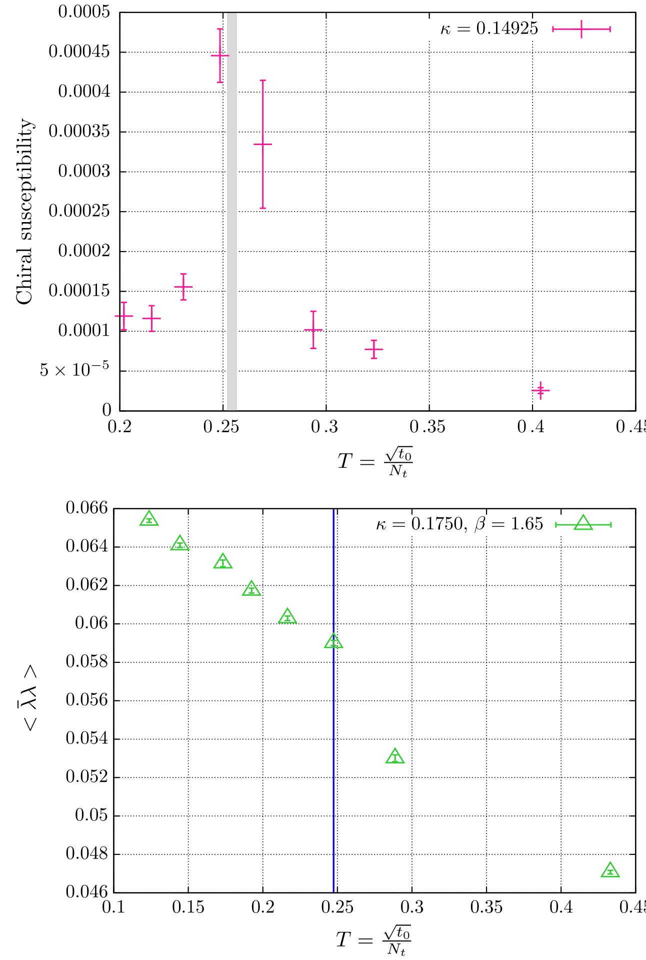

FIG. 4. At the top: Chiral susceptibility for $\kappa = 0.14925$. The gray band denotes the critical temperature of the Polyakov loop. At the bottom: gaugino condensate at $\beta = 1.65$ and $\kappa = 0.175$. The phase transition occurs at $N_t^c = 7$, which roughly agrees in dimensionless units with the critical temperature for $\beta = 1.75$.

$$B_4(P_L) = 1 - \frac{1}{3}\frac{\langle P_L^4 \rangle}{\langle P_L^2 \rangle^2}$$

reaches the critical value $B_4^c = 0.46548(5)$ [40]. This point is shown in the plots of Fig. 3. In the fixed scale approach, the temperature can be changed only by discrete steps. In order to precisely estimate the critical point of the deconfinement phase transition, we perform a linear interpolation of the Binder cumulant at two temperatures where B_4 approaches B_4^c both from the left and from the right. After finding the critical temperature for each κ value, the results were extrapolated to the chiral limit by means of a linear fit (see Fig. 5 and Table III).

Our results show that, up to numerical uncertainties, in $\mathcal{N} = 1$ SU(2) supersymmetric Yang-Mills theory, the deconfinement phase transition and restoration of the discrete Z_{2N_c} chiral symmetry occur simultaneously. This fact is far from trivial. Indeed, some nonsupersymmetric

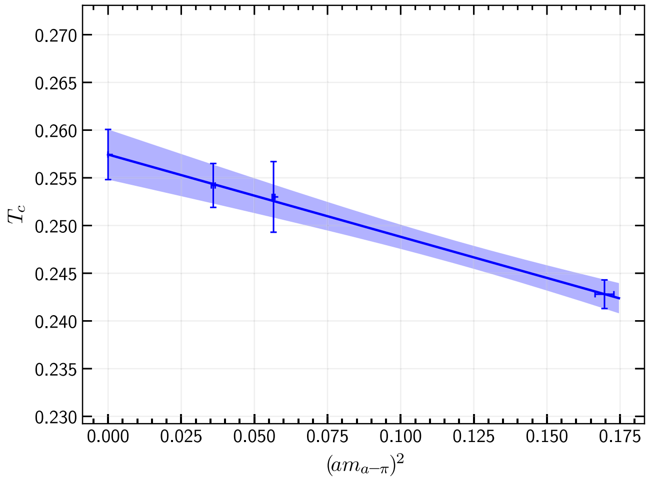

FIG. 5. Chiral extrapolation of the deconfinement phase transition. The deconfinement temperature extrapolated to the chiral limit is $T_c|_{m_{a\text{-}\pi}=0} = 0.2574(26)$.

QCD-like theories, like YM with several adjoint spinors, have been observed to exhibit phases with mixed deconfining and broken chiral symmetry phases [41]. In our previous investigations, we have not provided analytical confirmation for our finding. Here we provide arguments that predict the coincidence of the transitions in Sec. VI A.

The agreement of the two transitions is obtained up to the systematic uncertainties of our numerical determination which includes possible supersymmetry breaking lattice artefacts. Based on our previous investigations of particle spectrum and Ward identities [20], the role of these lattice artefacts is expected to be small. Furthermore our current results obtained with an improved lattice action and at a smaller lattice spacing are consistent with our earlier findings [29]. Consequently, we expect any systematic effect to be much smaller than the large difference of the two transition observed in [41].

An interesting final quantity determined by our study is the deconfinement temperature in the chiral/supersymmetric limit. We compare our result to the deconfinement critical temperature of SU(2) YM found in [42] through the ratio

$$\frac{T_c(\mathrm{SYM})}{T_c(\mathrm{YM})} = \frac{0.2574(26)}{0.3082(2)} = 0.8352(90), \qquad (8)$$

where the two temperatures are compared in dimensionless units as $\sqrt{t_0}/N_t^c$, and the scale t_0 for pure gauge has been computed in [43]. This value is compatible with our previous investigations in [29] at a smaller β. The ratio furthermore roughly agrees with an analytical prediction found in [44], where it is claimed that $\frac{T_c(\mathrm{SYM})}{T_c(\mathrm{YM})} = \sqrt{\frac{2}{3}} \sim 0.82$. It is important to stress that the ratio of Eq. (8) depends on the observable used to set the scale of the two different theories, in our case $\sqrt{t_0}$, however a general agreement is expected among scales defined from purely gluonic operators.

A. Prediction for the relation of chiral and deconfinement transition from string theory and t'Hooft anomaly matching

The result that deconfinement and restoration of chiral symmetry occur at the same critical point is far from trivial and can only be confirmed by nonperturbative methods. Lattice simulations can compute an estimation of the order parameters, but they do not provide immediately a qualitative physical interpretation of the mechanisms responsible for such results. A deep understanding of the dynamics of non-Abelian gauge theories in general and QCD in particular is, indeed, still missing and concepts like supersymmetry and string theory arose in the attempt to achieve it. An interpretation developed in the context of string theory might indeed provide such a deeper understanding of the mechanism behind our results. We provide here some rough arguments based on t'Hooft anomaly matching and a more detailed picture derived from the string theory perspective.

The coincidence of the transitions is in agreement with the predictions of [45,46]. As shown in these references based on anomaly matching arguments, the unbroken center symmetry must imply broken $\mathrm{Z}_{2\mathrm{N}_c}$ chiral symmetry in $\mathcal{N}=1$ supersymmetric Yang-Mills, meaning that $T^c_{\mathrm{chiral}} \geq T^c_{\mathrm{deconfinement}}$. The same constraint was also found from anomaly matching for $\mathrm{SU}(\mathrm{N}_c)$ with adjoint and fundamental matter in [47]. Our results indicate that this inequality is saturated, as predicted remarkably for adjoint QCD and SYM on $\mathbb{R}^3 \times S^1$ in [48–51].

In [52] Witten considered certain brane configuration consisting of two differently oriented NS five-branes and N_c D-four-branes stretching between them in weakly coupled IIA superstring theory.[4] The effective theory on the world-volume of the four-brane is a $3+1$ dimensional $\mathrm{SU}(\mathrm{N}_c)$ gauge theory with $\mathcal{N}=1$ supersymmetry. This theory has N_c vacua, which can be identified with the pattern of chiral symmetry breaking of the supersymmetric QFT. Witten succeeded to show that the confining string emanating from an external quark is topologically equivalent to the IIA fundamental string. Furthermore he found that the BPS-saturated domain wall can actually be identified with a D-brane, on which the confining strings can end. This fact directly relates confinement and chiral symmetry breaking. Indeed, the domain wall is expected to have restored chiral symmetry in its core, while a color-electric source sufficiently near the wall behaves as a free quark, and the Polyakov loop expectation value does not vanish. This was investigated further in [53] by means of an

[4] The framework is actually M-theory. The brane model is not equivalent to SYM but is in the same universality class. SYM is obtained when taking the IIA, i.e., ten dimensional limit.

effective field theory for the gaugino condensate and the Polyakov loop. The authors found that, for $\mathrm{N}_c = 3, \mathrm{Z}_3^\chi$ restoration implies Z_3^c breaking and that the results of Witten, i.e., that the confining strings end on the domain wall, can only hold if both phase transitions occur simultaneously.

VII. CONCLUSIONS AND OUTLOOK

The gradient flow has enabled us to explore the confining and the chiral properties of the $\mathcal{N} = 1$ SYM both at zero and finite temperature. We have been able to extrapolate the chiral condensate at zero temperature to the massless limit and to prove that chiral symmetry is broken. Our findings with the Wilson fermion action are in agreement with previous studies with Ginsparg-Wilson fermions, avoiding, however, the huge numerical cost of preserving chiral symmetry on the lattice. A precise quantitative comparison of the results obtained with the different fermion actions is possible once a common renormalization scheme is chosen to fix the multiplicative renormalization constant. The matching to the semiclassical predictions of the condensate would be interesting from the theoretical point of view, however a deeper understanding of the exact calculations of the gluino condensate is required. Any prediction or numerical value of $\langle\bar{\lambda}\lambda\rangle$ holds only in a given scheme, i.e., up to a multiplicative factor, and provides the same information as the condensate at a finite flow time. A comparison to the semiclassical calculations of the gluino condensate would need to find first the missing perturbative connection of the exact formulas of [6] to the $\overline{\mathrm{MS}}$ scheme.

We have also explored the phase diagram of the theory at nonzero temperature. The chiral condensate provides a signal compatible with a second order phase transition at smaller gaugino masses. In comparison to our previous investigations without gradient flow in [29], the signal of the order parameter's expectation value is clearer and thus more significant, which reflects the advantages of the method. The Binder cumulant of the Polyakov loop has been important to locate precisely the deconfinement phase transition. Through the identification of both phase transitions, we found that the critical temperatures are in fact coincident. This leads us to the remarkable conclusion that chiral symmetry restoration and deconfinement are not independent and uncorrelated nonperturbative phenomena in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, but they may obey a common underlying dynamics. As pointed out in the end of the present work, this observation is in agreement with predictions found through certain brane configuration in IIA superstring/M-theory.

We are currently working towards new applications of the gradient flow. One immediate further step is to investigate thermal SYM for SU(3) gauge group. Another very interesting direction is to take profit of the method to study the vacuum's structure of the theory at zero temperature. In addition we plan to investigate possible applications for the renormalization of the supercurrent in theories with extended supersymmetry [15].

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APPENDIX A: NUMERICAL INTEGRATION OF THE GRADIENT FLOW AND COMPUTATION OF THE GLUINO CONDENSATE

In this Appendix we describe the path for the numerical computation of the gradient flow [10,12]. First, one initial gauge configuration $V_{t=0}$ is picked and flowed by integrating the discrete version of the gauge part of Eq. (4)

$$\dot{V}_t(x,\mu) = -\{\partial_{x,\mu} S_{\mathrm{w}}(V_t)\} V_t(x,\mu),$$
$$U(x,\mu) = V_t(x,\mu)|_{t=0},$$

up to some t by means of the following Runge-Kutta integrator with step-size ϵ

$$W_0 = V_t,$$
$$W_1 = \exp{(1/4)} Z_0 W_0,$$
$$W_2 = \exp{(8/9)} Z_1 - (17/36) Z_0 W_1,$$
$$V_{t+\epsilon} = \exp{(3/4)} Z_2 - (8/9) Z_1 + (17/36) Z_0 W_2,$$

where $Z_i = -\epsilon \partial_{x,\mu} S_{\mathrm{w}}(W_i)$ and S_{w} is the Wilson plaquette action. The intermediate fields are then kept on the computer memory in order to access them during the integration of the adjoint fermion equation. As a second step one random source vector $\xi_{s=t} = \eta$ is generated on the lattice and then integrated by means of the Runge-Kutta integrator down to $s = 0$

$$\lambda_3 = \xi^{\epsilon}_{s+\epsilon},$$
$$\lambda_2 = \frac{3}{4}\Delta_2\lambda_3,$$
$$\lambda_1 = \lambda_3 + \frac{8}{9}\Delta_1\lambda_2,$$
$$\lambda_0 = \lambda_1 + \lambda_2 + \frac{1}{4}\Delta_0\left(\lambda_1 - \frac{8}{9}\lambda_2\right), \quad \text{with} \quad \xi^{\epsilon}_s = \lambda_0.$$

Further, the vector $W(v) = \sum_w D(v,w)^{-1}\xi(t;0,w)$ is computed, e.g., through conjugate gradient. Subsequently it is contracted with $\xi(t;0,v)^\dagger$ and the result averaged over the lattice sites, i.e., one calculates $-\frac{1}{N_{\mathrm{lat}}}\sum_v \xi_k(t;0,v)^\dagger W(v)$.

The discrete version of the gluino condensate $\langle\bar{\lambda}(x,t)\lambda(x,t)\rangle$ in Eq. (7) can be straightforwardly written:

$$-\sum_{v,w}\langle \mathrm{tr}\{K(t,x;0,v)(D_{\mathrm{W}}(v,w))^{-1}K(t,x;0,w)^\dagger\}\rangle,$$

where D_{W} is the Wilson-Dirac operator and the trace runs over space-time, spinor and color indices. Following [10], the trace is estimated stochastically by inserting a complete set of random complex vectors $\eta(x)$ with $\langle\eta(x)\rangle = 0$ and $\langle\eta(x)\eta(y)^\dagger\rangle = \delta_{x,y}$:

$$\langle\bar{\lambda}\lambda(t)\rangle = \frac{1}{N_\Gamma}\sum_{x\in\Gamma}\langle\bar{\lambda}\lambda(x,t)\rangle$$
$$= -\frac{1}{N_\Gamma}\sum_{v,w}\langle\xi(t;0,v)^\dagger D_{\mathrm{W}}^{-1}\xi(t;0,w)\rangle,$$
$$\xi(t;s,w) = \sum_x K(t,x;s,w)^\dagger\eta(x). \tag{A1}$$

Here $\langle\ldots\rangle$ denotes the average with respect to both the Monte-Carlo time and any internal group-symmetry representations. Finally, to compute the new vectors ξ, the so-called *adjoint* flow equation

$$\partial_s\xi(t;s,w) = -\Delta(V_s)\xi(t;s,w), \xi(t;t,w) = \eta(w), \tag{A2}$$

must be integrated from $s = t$ to $s = 0$, i.e., backwards in comparison with the flow equations presented above. Here, the gauge connection V_s in the covariant four-dimensional Laplacian Δ is flowed up to the same t as η.

APPENDIX B: FINITE TEMPERATURE DATA

TABLE II. Condensate, chiral susceptibility and Binder cumulant of the Polyakov loop at $\beta = 1.75, 24^3 \times N_t$.

κ	t_0	N_t	$\langle\bar{\lambda}\lambda\rangle$	Susceptibility	$B_4(P_L)$
0.1480	6.332(48)	8	0.002557(12)	0.000076(14)	0.65531(86)
		9	0.002738(20)	0.000082(6)	0.6417(27)
		10	0.002913(51)	0.000203(41)	0.5796(94)
		11	0.003289(16)	0.000073(13)	0.261(37)
		12	0.003334(46)	0.000214(48)	0.332(27)
		13	0.003614(34)	0.000165(34)	−0.051(66)
		14	0.003775(27)	0.000157(16)	0.031(27)
		15	0.003861(17)	0.000154(22)	−0.026(43)
		16	0.0039778(93)	0.000109(6)	0.012(26)
0.1490	9.851(32)	8	0.001374(21)	0.000031(5)	0.65851(49)
		10	0.001491(73)	0.000068(19)	0.6349(21)
		11	0.001557(32)	0.000077(11)	0.6110(33)
		12	0.001808(70)	0.000209(31)	0.530(15)
		13	0.001880(35)	0.000101(23)	0.377(37)
		14	0.002612(64)	0.000112(35)	0.079(38)
		16	0.002733(29)	0.000061(23)	0.031(43)
0.14925	10.545(69)	8	0.001144(12)	0.000025(2)	0.6581(3)
		10	0.001059(29)	0.000077(14)	0.6372(14)
		11	0.001254(23)	0.000102(24)	0.6244(18)
		12	0.001326(50)	0.000338(90)	0.5618(66)
		13	0.001742(121)	0.000450(78)	0.424(18)
		14	0.002013(92)	0.000155(21)	0.338(22)
		15	0.002533(28)	0.000108(11)	0.026(38)
		16	0.002564(57)	0.000116(17)	−0.026(35)

TABLE III. Adjoint pion masses for each κ value.

κ	$am_{a-\pi}$
0.1480	0.4119(39)
0.1490	0.23780(97)
0.14925	0.1896(17)

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