Status of isospin splittings in mesons and baryons

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Current measurements of isospin splittings in mesons and baryons are sufficiently precise that they allow estimates of the mass difference between constituent up and down quarks. Some previous results are updated in the light of these new measurements, and the importance of better measurements of some observables such as $M(K^{*\pm})$, $M(B^{*0}) - M(B^0)$, and isospin splittings in bottom baryons is noted.

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I. INTRODUCTION

Isospin-violating mass differences among hadrons are treated in the quark model as a combination of effects. The u and d quarks have an intrinsic mass difference, expressed as a direct contribution to hadron masses and via differing kinetic energies in bound states. Coulomb interactions between quarks depend on the product of their charges times the expectation value of the inverse of their separation. Strong hyperfine interactions between quarks depend on the inverse product of their masses, and electromagnetic hyperfine interactions depend both on that inverse product and on the product of quark charges. One can then write meson and baryon isospin-violating mass differences in terms of a few parameters, yielding sum rules for masses in the limit of small values of these parameters. These were exploited, for example, for mesons with heavy quarks in Ref. [1] and for baryons in Ref. [2]. Isospin splittings in baryons with two heavy quarks were examined in Refs. [3] and [4].

The experimental status of isospin splittings continues to improve. There has been a relatively new measurement of $M(D^{*+}) - M(D^+)$ [5]. Information on masses of individual charge states of charmed and bottom hadrons continues to grow, with exceptional progress in the past year for Ξ^c, Σ_b, and Ξ_b states [6] (compare PDGLive with the 2018 print version). An update of Ref. [1] was performed about 10 years ago [7]. Even the light-quark sector has seen improvements since the analysis of Ref. [2], driven by the improved precision in the Ξ^0 mass measured by the NA48 Collaboration at CERN [8]. An analysis of the present status of isospin splittings in hadrons thus seems appropriate.

We set forth our assumptions, including the interpretation of quarks as constituents with masses of several hundred MeV, in Sec. II. In Sec. III we update analyses of light-quark mesons and baryons. We treat charmed hadrons in Sec. IV, beauty hadrons in Sec. V, and the relation between the two heavy sectors in Sec. VI. We compare our results with those of several other approaches in Sec. VII and conclude in Sec. VIII.

II. ASSUMPTIONS

In a constituent-quark framework, hadron masses are governed by the sum of their quark masses, the hyperfine interactions among those quarks, and—for hadrons with more than one heavy quark (c or b)—an additional binding term between heavy quarks. This approach [9,10] successfully describes the masses of light-quark hadrons [11], those with a single charm or bottom quark [12], and the mass of the recently observed baryon with two charmed quarks [13].

When the masses of mesons and baryons are fitted with constituent-quark masses and hyperfine interactions, the quark masses in baryons are about 55 MeV heavier than those in mesons [10]. This scheme was used in Ref. [12] to predict $M(\Xi_{cc}) = (3627 \pm 12)$ MeV, in satisfactory agreement with the observed value [13] $M(\Xi_{cc}) = (3621.40 \pm 0.78)$ MeV. An alternative scheme explains the mass difference by adding a “string-junction” term of 165 MeV, allowing one to fit mesons and baryons with a universal set of quark masses [14]. However, this scheme predicts $M(\Xi_{cc})$ about 40 MeV higher, so for definiteness we shall stay with the picture of separate quark masses for mesons and baryons.

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TABLE I. Results (in MeV) of a fit to mesons and baryons with additive quark masses (different for mesons and baryons), hyperfine terms, and a binding term $B(ss)$. The label $\bar{m}$ denotes an average between m_u and m_d. The masses of ϕ, Ξ, and Ω are corrected by terms $-2B(ss)$, $-B(ss)$, and $-3B(ss)$, respectively. We find $\bar{m}^m = 307.5 \pm 0.33$ MeV, $m_s^m = 487.6 \pm 0.51$ MeV, $\bar{m}^b = 362.1 \pm 0.21$ MeV, $m_s^b = 543.9 \pm 0.45$ MeV, $b^m/(\bar{m}^m)^2 = 79.4 \pm 0.14$ MeV, $b^b/(\bar{m}^b)^2 = 50.0 \pm 0.16$ MeV, $B(ss) = 9.23 \pm 0.50$ MeV. Errors on each parameter are computed by fixing its value, minimizing χ^2 with respect to the other six parameters, and determining what values of the given parameter lead to an increase of χ^2 by one unit. The root-mean-square error of the fit is $\sqrt{\sum(\Delta M^2)/13} = 3.85$ MeV.

Meson	π	ρ	K	K^*	ϕ
Predicted	138.5	773.9	494.6	895.3	1019.9
Experiment	138.0	775.2	495.6	894.1	1019.5
ΔM^2	0.2	1.8	1.0	1.4	0.2

Baryon	N	Δ	Λ	Σ	Σ^*	Ξ	Ξ^*	Ω
Pred.	936.5	1236.4	1118.2	1185.0	1385.0	1329.7	1529.4	1670.4
Expt.	938.9	1232.	1115.7	1193.2	1384.2	1321.0	1532.5	1672.5
ΔM^2	6.0	18.9	6.2	67.3	0.2	75.3	9.9	4.5

Quark masses in [12], from Ref. [11], did not include a small binding term for a pair of s quarks, which we now take into account. The results are shown in Table I. Here the strong hyperfine term is parametrized as

$$\Delta E_{ij,\mathrm{HFs}} = b\langle \sigma_i \cdot \sigma_j\rangle/(m_i m_j). \tag{1}$$

Superscripts m and b will refer to values in mesons and baryons, respectively. The quark masses differ only slightly from those in Ref. [12].

The model we employ takes into account the intrinsic difference $\Delta = (1 - \frac{K}{m})(m_u - m_d)$ between u and d quarks, where K is a one-body kinetic energy term [15]; Coulomb interactions

$$\Delta E_{ij\,\mathrm{em}} = \alpha Q_i Q_j \langle 1/r_{ij}\rangle \tag{2}$$

between quarks; strong hyperfine (HF) interactions $\Delta E_{ij\,\mathrm{HFs}}$ as mentioned above; and electromagnetic HF interactions

$$\Delta E_{ij\,\mathrm{HFe}} = -\frac{2\pi\alpha Q_i Q_j |\Psi_{ij}(0)|^2 \langle \sigma_i \cdot \sigma_j\rangle}{3 m_i m_j}. \tag{3}$$

Symbols are defined in Ref. [2]. We may thus write the total isospin splitting as

$$\sum_{i<j} \Delta E_{ij} = \langle \Delta \rangle + a \sum_{i<j} \langle Q_i Q_j \rangle + b \sum_{i<j} \langle \sigma_i \cdot \sigma_j/(m_i m_j)\rangle + c \sum_{i<j} \langle Q_i Q_j \sigma_i \cdot \sigma_j/(m_i m_j)\rangle. \tag{4}$$

Separate parameters, labeled by superscripts m, will be used for mesons. Henceforth parameters without superscripts will refer to quantities for baryons.

III. LIGHT-QUARK HADRONS

A. Mesons

The isospin splittings of light-quark mesons, based on masses quoted in Ref. [6], are summarized in Table II. Labels denote the change in isospin associated with each mass splitting. The conflict between the quoted K^* mass splitting (which we use) and the individual K^* masses needs to be resolved before we can take our analysis as definitive.

Each splitting can be written as the sum of terms depending on the parameters Δ^m, a^m, b^m, and c^m:

$$\pi_2 = \frac{1}{2} a^m + \frac{3}{2} \frac{b^m}{(\bar{m}^m)^2} \left(\frac{\Delta^m}{\bar{m}^m}\right)^2 - \frac{3}{2} \frac{c^m}{(\bar{m}^m)^2}, \tag{5}$$

$$\rho_2 = \frac{1}{2} a^m - \frac{1}{2} \frac{b^m}{(\bar{m}^m)^2} \left(\frac{\Delta^m}{\bar{m}^m}\right)^2 + \frac{1}{2} \frac{c^m}{(\bar{m}^m)^2}, \tag{6}$$

$$K_1 = \Delta^m + \frac{1}{3} a^m + \frac{3b^m}{(\bar{m}^m)^2} \frac{\Delta^m}{m_s^m} - \frac{c^m}{(\bar{m}^m)^2} \frac{\bar{m}^m}{m_s^m}, \tag{7}$$

$$K_1^* = \Delta^m + \frac{1}{3} a^m - \frac{b^m}{(\bar{m}^m)^2} \frac{\Delta^m}{m_s^m} + \frac{c^m}{3(\bar{m}^m)^2} \frac{\bar{m}^m}{m_s^m}. \tag{8}$$

TABLE II. Isospin splittings of light-quark mesons [6].

ΔM	Label	Value (MeV)
$\pi^\pm - \pi^0$	π_2	4.5936 ± 0.0005
$\rho^\pm - \rho^0$	ρ_2	0.15 ± 0.42
$K^+ - K^0$	K_1	-3.934 ± 0.020
$K^{*+} - K^{*0}$	K_1^*	-6.7 ± 1.2[a]

[a] As quoted in pdgLive. Individual masses quoted are 891.76 ± 0.25 MeV (charged, hadroproduced); 895.5 ± 0.8 MeV (charged, τ decay); and 895.55 ± 0.20 MeV (neutral).

Here we have substituted Δ^m for $m_u^m - m_d^m$ in the terms for hyperfine splittings. We then see that four observables are expressed in terms of the three unknowns Δ^m, a^m, and c^m. [We use the value of $b^m/(\bar{m}^m)^2 = 79.4$ MeV from the fit in Table I.] A fit to the observables yields the values $\pi_2 = 4.59$ MeV, $\rho_2 = -0.07$ MeV, $K_1 = -3.93$ MeV, and $K_1^* = -3.20$ MeV, with $\Delta^m = -4.12 \pm 0.05$ MeV, $a^m = 2.20 \pm 0.62$ MeV, and $c^m/(\bar{m}^m)^2 = -2.32 \pm 0.21$ MeV. Here and subsequently the errors on each parameter are obtained by fixing its value, minimizing χ^2 with respect to the other two, and determining what values of the given parameter lead to an increase in χ^2 by one unit. The χ^2 for this fit is 8.77, nearly all (8.51) contributed by K_1^*. In view of the spread in the Particle Data Group's values for $M(K^{*\pm})$ [6], we urge further study of this state.

B. Baryons

We express the observed mass splittings among the octet baryons and the Σ^* and Ξ^* resonances [6], labeled with subscripts denoting their ΔI values, summarized in Table III, as functions of Δ (u–d mass difference with effect on kinetic energies), a (Coulomb interaction), b (strong HF interaction), and c (electromagnetic HF interaction). We have neglected effects of two-body kinetic energy operators and additional small corrections [15]. This decomposition is summarized below, where we have linearized expressions from Ref. [4] in Δ and where $\bar{m}$ is the average of m_u and m_d:

$$N_1 = \Delta + \frac{a}{3} - \frac{2b\Delta}{\bar{m}^3} + \frac{c}{3\bar{m}^2}, \tag{9}$$

$$\begin{aligned}\Sigma_1 &= 2\Delta - \frac{a}{3} + \frac{2b\Delta}{\bar{m}^3}\left[-1 + 2\frac{\bar{m}}{m_s}\right] + \frac{c}{3\bar{m}^2}\left[1 + 4\frac{\bar{m}}{m_s}\right] \\ &= N_1 + \Xi_1,\end{aligned} \tag{10}$$

$$\Sigma_1^* = 2\Delta - \frac{a}{3} - \frac{2b\Delta}{\bar{m}^3}\left[1 + \frac{\bar{m}}{m_s}\right] + \frac{c}{3\bar{m}^2}\left[1 - 2\frac{\bar{m}}{m_s}\right], \tag{11}$$

$$\Sigma_2 = a + \frac{c}{\bar{m}^2} = \Sigma_2^*, \tag{12}$$

TABLE III. Experimental mass splittings between light-quark baryons [6].

Splitting	Symbol	Value (MeV)
$M(p) - M(n)$	N_1	−1.2933
$M(\Sigma^+) - M(\Sigma^-)$	Σ_1	-8.08 ± 0.08
$M(\Sigma^+) - 2M(\Sigma^0) + M(\Sigma^-)$	Σ_2	1.535 ± 0.090
$M(\Sigma^{*+}) - M(\Sigma^{*-})$	Σ_1^*	-4.40 ± 0.61
$M(\Sigma^{*+}) - 2M(\Sigma^{*0}) + M(\Sigma^{*-})$	Σ_2^*	2.6 ± 2.1
$M(\Xi^0) - M(\Xi^-)$	Ξ_1	-6.85 ± 0.21
$M(\Xi^{*0}) - M(\Xi^{*-})$	Ξ_1^*	-3.20 ± 0.68

TABLE IV. Predicted and observed isospin splittings in light-quark baryons.

	N_1	Σ_1	Σ_1^*	Σ_2	Σ_2^*	Ξ_1	Ξ_1^*
Pred.	−1.293	−8.087	−4.685	1.529	1.529	−6.794	−3.392
Expt.	−1.293	−8.080	−4.400	1.535	2.600	−6.850	−3.200
Error	∼0	0.080	0.610	0.090	2.100	0.210	0.680
χ^2	0.000	0.007	0.218	0.004	0.260	0.072	0.079

$$\Xi_1 = \Delta - \frac{2a}{3} + \frac{4b\Delta}{\bar{m}^2 m_s} + \frac{4c}{3\bar{m}m_s}, \tag{13}$$

$$\Xi_1^* = \Delta - \frac{2a}{3} - \frac{2b\Delta}{\bar{m}^2 m_s} - \frac{2c}{3\bar{m}m_s}. \tag{14}$$

The predicted isospin splittings are very close to the observed ones, since the Coleman-Glashow relation [16] $\Sigma_1 = N_1 + \Xi_1$ is very close to being obeyed by those quantities with the smallest experimental errors. The derived parameters are $\Delta = -2.49 \pm 0.04$ MeV, $a = 3.05 \pm 0.05$ MeV, and $c/\bar{m}^2 = -1.52 \pm 0.09$ MeV. The predicted isospin splittings are compared with the observed ones in Table IV. The χ^2 for the fit is 0.64, driven mainly by the Σ^* splittings.

In comparison with Ref. [2], the following relation is satisfied to greater accuracy:

$$\begin{aligned}&\Sigma_1 - \Xi_1(= -1.23 \pm 0.22\ \text{MeV}) \\ &\quad = \Sigma_1^* - \Xi_1^*(= -1.20 \pm 0.91\ \text{MeV}).\end{aligned} \tag{15}$$

On the other hand, the relation

$$\Sigma_2(= 1.535 \pm 0.090\ \text{MeV}) = \Sigma_2^*(= 2.6 \pm 2.1\ \text{MeV}) \tag{16}$$

is still plagued with a large experimental error on the right-hand side.

C. Meson-baryon comparison

The parameters Δ, a, and $c/\bar{m}^2$ derived from fits to isospin splittings in mesons and baryons are compared in Table V. The signs are consistent, but central values are rather different. Slightly different parameters are obtained if one adopts a model in which quark masses are universal for mesons and baryons [4]. The difference between parameters obtained from mesons and baryons is not surprising in

TABLE V. Parameters describing isospin splittings in light-quark mesons and baryons.

	Δ	a	$c/\bar{m}^2$
Meson	-4.12 ± 0.05	2.20 ± 0.62	-2.32 ± 0.21
Baryon	-2.49 ± 0.04	3.05 ± 0.05	-1.52 ± 0.09

view of the large spread of values for $m_u - m_d$ obtained in various models (see Sec. IV in [2]).

IV. CHARMED HADRONS

A. Mesons

The states at our disposal are summarized in Table VI. There is not enough information to derive a set of parameters describing these mass differences. However, the total spin-dependent terms contribute in a manner proportional to $\langle \sigma_i \cdot \sigma_j \rangle$, so one may write

$$D_1 = -\Delta_c^m + \frac{2a_c^m}{3} - 3h_c^m, \tag{17}$$

$$D_1^* = -\Delta_c^m + \frac{2a_c^m}{3} + h_c^m, \tag{18}$$

where the superscript denotes charmed mesons. Eliminating the hyperfine contribution h_c^m, one finds

$$-\Delta_c^m + \frac{2a_c^m}{3} = 3.76 \pm 0.05\ \text{MeV}. \tag{19}$$

This is to be compared with the corresponding value for light-quark mesons,

$$\begin{aligned} -\Delta^m + \frac{2a^m}{3} &= [(4.12 \pm 0.05) + (1.47 \pm 0.41)]\ \text{MeV} \\ &= (5.58 \pm 0.42)\ \text{MeV}. \end{aligned} \tag{20}$$

As for the hyperfine term $h_c^m = (-0.35 \pm 0.02)$ MeV, it contains both strong b_c^m and electromagnetic c_c^m contributions, which cannot be separated from one another without further assumptions.

B. Baryons

In analogy for the light-quark baryons, we write expressions for isospin splittings of charmed baryons:

$$\Sigma_{c1} = 2\Delta_c + \frac{5}{3}a_c + \frac{2b_c}{\bar{m}^2}\frac{\Delta_c}{\bar{m}}\left[-1 + 2\frac{\bar{m}}{m_c}\right] + \frac{c_c}{3\bar{m}^2}\left[1 - \frac{8\bar{m}}{m_c}\right] \tag{21}$$

$$\Sigma_{c1}^* = 2\Delta_c + \frac{5}{3}b_c - \frac{2b_c}{\bar{m}^2}\frac{\Delta_c}{\bar{m}}\left[1 + \frac{\bar{m}}{m_c}\right] + \frac{c_c}{3\bar{m}^2}\left[1 + \frac{4\bar{m}}{m_c}\right] \tag{22}$$

$$\Sigma_{c2} = a_c + \frac{c_c}{\bar{m}^2} = \Sigma_{c2}^* \tag{23}$$

$$\Xi_{c1} = \Delta_c + \frac{1}{3}a_c + \frac{3b_c}{\bar{m}^2}\frac{\Delta_c}{m_s} + \frac{c_c}{\bar{m}m_s} \tag{24}$$

$$\Xi_{c1}' = \Delta_c + \frac{1}{3}a_c + \frac{b_c\Delta_c}{\bar{m}^2}\left(\frac{2}{m_c} - \frac{1}{m_s}\right) - \frac{c_c}{3\bar{m}}\left(\frac{1}{m_s} + \frac{4}{m_c}\right) \tag{25}$$

$$\Xi_{c1}^* = \Delta_c + \frac{1}{3}a_c - \frac{b_c\Delta_c}{\bar{m}^2}\left(\frac{1}{m_s} + \frac{1}{m_c}\right) + \frac{c_c}{3\bar{m}}\left(\frac{2}{m_c} - \frac{1}{m_s}\right). \tag{26}$$

We update a couple of relations, noted in Ref. [17], which follow from our assumptions. In 1998 the relation

$$\Sigma_{c2} \equiv M(\Sigma_c^{++}) - 2M(\Sigma_c^+) + M(\Sigma_c^0) = \Sigma_2, \tag{27}$$

appeared to be violated [2], with the left-hand side giving -2.0 ± 1.3 MeV while the right-hand side gave 1.71 ± 0.18 MeV. The present status of charmed baryon masses and isospin splittings is summarized in Table VII. The

TABLE VI. Masses and isospin splittings of charmed mesons, in MeV.

D^+	1869.65 ± 0.05
D^0	1864.83 ± 0.05
$D_1 \equiv M(D^+) - M(D^0)$	4.822 ± 0.015[a]
D^{*+}	2010.26 ± 0.05
D^{*0}	2006.85 ± 0.05
$D_1^* \equiv M(D^{*+}) - M(D^{*0})$	3.41 ± 0.07

[a] Value given separately in the D^0 section of [6].

TABLE VII. Masses and isospin splittings of charmed baryons, in MeV.

$M(\Sigma_c^{++})$	2453.97 ± 0.14
$M(\Sigma_c^+)$	2452.9 ± 0.4
$M(\Sigma_c^0)$	2453.75 ± 0.14
$\Sigma_{c1} \equiv M(\Sigma_c^{++}) - M(\Sigma_c^0)$	0.220 ± 0.013[a]
$\Sigma_{c2} \equiv M(\Sigma_c^{++}) - 2M(\Sigma_c^+) + M(\Sigma_c^0)$	1.92 ± 0.82
$M(\Sigma_c^{*++})$	$2518.41^{+0.21}_{-0.19}$
$M(\Sigma_c^{*+})$	2517.5 ± 2.3
$M(\Sigma_c^{*0})$	2518.48 ± 0.20
$\Sigma_{c1}^* \equiv M(\Sigma_c^{*++}) - M(\Sigma_c^{*0})$	0.01 ± 0.15[b]
$\Sigma_{c2}^* \equiv M(\Sigma_c^{++}) - 2M(\Sigma_c^{*+}) + M(\Sigma_c^{*0})$	1.89 ± 4.61
$M(\Xi_c^+)$	2467.93 ± 0.18
$M(\Xi_c^0)$	2470.91 ± 0.25
$\Xi_{c1} \equiv M(\Xi_c^+) - M(\Xi_c^0)$	-2.98 ± 0.22[c]
$M(\Xi_c'^+)$	2578.4 ± 0.5
$M(\Xi_c'^0)$	2579.2 ± 0.5
$\Xi_{c1}' \equiv M(\Xi_c'^+) - M(\Xi_c'^0)$	-0.8 ± 0.6[d]
$M(\Xi_c^{*+})$	2645.57 ± 0.26
$M(\Xi_c^{*0})$	2646.38 ± 0.21
$\Xi_{c1}^* \equiv M(\Xi^{*+}) - M(\Xi^{*0})$	-0.80 ± 0.26[e]

[a] Listed in [6], Σ_c section.
[b] Listed in [6], Σ_c^* section.
[c] Listed in [6], Ξ_c section.
[d] Listed in [6], $\Xi_c'^+(2578)$ section.
[e] Listed in [6], $\Xi_c(2645)$ section.

TABLE VIII. Predicted and observed isospin splittings in charmed baryons. Errors are experimental values, used in calculating χ^2 contributions.

	Σ_{c1}	Σ_{c2}	Σ^*_{c1}	Σ^*_{c2}	Ξ_{c1}	Ξ'_{c1}	Ξ^*_{c1}
Fit	0.221	1.916	−0.064	1.916	−2.827	−1.058	−1.200
Expt.	0.220	1.920	0.010	1.890	−2.980	−0.800	−0.800
Error	0.013	0.820	0.150	4.610	0.220	0.600	0.260
χ^2	0.003	0.000	0.243	0.000	0.484	0.185	2.369

sum rule is now satisfied, with the left-hand side giving 1.92 ± 0.82 MeV while the right-hand side gives 1.535 ± 0.090 MeV.

Another sum rule [17],

$$\Sigma_{c1} - 2\Xi'_{c1} = \Sigma^*_1 - 2\Xi^*_1, \tag{28}$$

is beginning to be tested, with the left-hand side yielding 1.8 ± 1.2 MeV while the right-hand side is 2.0 ± 1.5 MeV. The large errors are associated both with Ξ'_{c1} and Ξ^*_1. A further relation is

$$\Sigma^*_{c1} - 2\Xi^*_{c1} = \Sigma^*_1 - 2\Xi^*_1, \tag{29}$$

where the left-hand side is 1.61 ± 0.54 MeV. The sum rule is satisfied, with the main uncertainty coming from the right-hand side.

The information about charmed baryons is complete enough that one can perform a fit to their isospin splittings, determining parameters Δ_c, a_c, and $c_c/\bar{m}^2$ which may be compared with their light-quark counterparts. Fixed parameters in this fit (see the caption to Table I, with m_c taken from [12]) are

$$\bar{m} = 362.1\ \text{MeV}, \qquad m_s = 543.9\ \text{MeV},$$
$$b_c/(\bar{m}^2) = b/(\bar{m})^2 = 50.0\ \text{MeV}, \qquad m_c = 1710.5\ \text{MeV}. \tag{30}$$

The results of this fit are summarized in Table VIII. The derived parameters are $\Delta_c = -2.49 \pm 0.20$ MeV, $a_c = 2.77 \pm 0.23$ MeV, and $c_c/\bar{m}^2 = -0.85 \pm 0.15$ MeV. The first two are rather close to those obtained for light-quark baryons, while the last is of the same sign but only about half as large as $c/\bar{m}^2$. The χ^2 for the fit is 3.28, driven mainly by Ξ^*_{c1}.

V. BEAUTY HADRONS

A. Mesons

TABLE IX. Masses and isospin splittings of beauty mesons, in MeV.

B^+	5279.33 ± 0.13
B^0	5279.64 ± 0.14
$B_1 \equiv M(B^+) - M(B^0)$	-0.31 ± 0.07
$M(B^{*+}) - M(B^+)$	45.37 ± 0.21
$M(B^{*+})$	5324.70 ± 0.27
$M(B^{*0})$	…
$B^*_1 \equiv M(B^{*+}) - M(B^{*0})$	…

The information on beauty mesons relevant for analysis of isospin splittings is summarized in Table IX. An analysis parallel to that for charmed mesons is not possible in the absence of a value of $M(B^{*0})$. Thus in analogy to Eq. (17) all we can write is

$$B_1 = \Delta^m_b + \frac{a^m_b}{3} - 3h^m_b, \tag{31}$$

$$B^*_1 = \Delta^m_b + \frac{a^m_b}{3} + h^m_b. \tag{32}$$

Eliminating the spin-dependent term h^m_b, one finds

$$\Delta^m_b + \frac{a^m_b}{3} = \frac{1}{4}(B_1 + 3B^*_1). \tag{33}$$

Now, h^m_b contains quark charges different from those in h^m_c, but is smaller in magnitude by about a factor of $m^m_b/m^m_c \simeq 3$. Thus we probably make an error of only about 0.1 MeV in neglecting it. In that case we would predict $B^*_1 \simeq B_1 \simeq -0.31 \pm 0.07$ MeV. This is consistent with the Particle Data Group's charge-averaged value $M(B^*) - M(B) = 45.22 \pm 0.21$ MeV, to be compared with $M(B^{*+}) - M(B^+) = 45.37 \pm 0.21$ MeV [6], implying that B^* and B isospin splittings are not too different from one another. Definitive conclusions await the measurement of $M(B^{*0})$. For light-quark mesons, the combination $\Delta^m + \frac{a^m}{3}$ is equal to $(-4.12 + 0.71)$ MeV $= -3.41$ MeV.

B. Baryons

The relevant masses of beauty baryons are summarized in Table X. Here we only have information on $\Delta I = 1$

TABLE X. Masses and isospin splittings of beauty baryons, in MeV.

$M(\Sigma^+_b)$	5810.56 ± 0.25
$M(\Sigma^-_b)$	5815.64 ± 0.27
$\Sigma_{b1} \equiv M(\Sigma^+_b) - M(\Sigma^-_b)$	-5.06 ± 0.18[a]
$M(\Sigma^{*+}_b)$	5830.32 ± 0.27
$M(\Sigma^{*-}_b)$	5834.74 ± 0.30
$\Sigma^*_{b1} \equiv M(\Sigma^{*+}_b) - M(\Sigma^{*-}_b)$	-4.37 ± 0.33[a]
$M(\Xi^0_b)$	5791.8 ± 0.5[b]
$M(\Xi^-_b)$	5797.0 ± 0.9[c]
$\Xi_{b1} \equiv M(\Xi^0_b) - M(\Xi^-_b)$	-5.9 ± 0.6
$M(\Xi^{*0}_b)$	5952.3 ± 0.9
$M(\Xi^{*-}_b)$	5955.33 ± 0.13
$\Xi^*_{b1} \equiv M(\Xi^{*0}_b) - M(\Xi^{*-}_b)$	-3.03 ± 0.91

[a] From PDGLive [6].
[b] LHCb value [18].
[c] LHCb value [19].

splittings, as the neutral Σ_b and Σ_b^* masses are still unmeasured. The decomposition of isospin splittings in terms of Δ_b, a_b, b_b, and c_b is

$$\Sigma_{b1} = 2\Delta_b - \frac{1}{3}a_b - \frac{2b_b\Delta_b}{\bar{m}^3}\left[1 - \frac{2\bar{m}}{m_b}\right] + \frac{c_b}{3\bar{m}^2}\left[1 + \frac{4\bar{m}}{m_b}\right], \tag{34}$$

$$\Sigma^*_{b1} = 2\Delta_b - \frac{1}{3}a_b - \frac{2b_b\Delta_b}{\bar{m}^3}\left[1 + \frac{\bar{m}}{m_b}\right] + \frac{c_b}{3\bar{m}^2}\left[1 - 2\frac{\bar{m}}{m_b}\right], \tag{35}$$

$$\Sigma_{b2} = a_b + \frac{c_b}{\bar{m}^2} = \Sigma^*_{b2}, \tag{36}$$

$$\Xi_{b1} = \Delta_b - \frac{2}{3}a_b + \frac{3b_b\Delta_b}{\bar{m}^2 m_s} + \frac{c_b}{\bar{m}m_s}, \tag{37}$$

$$\Xi'_{b1} = \Delta_b - \frac{2}{3}a_b + \frac{\Delta_b b_b}{\bar{m}^2}\left[\frac{2}{m_b} - \frac{1}{m_s}\right] + \frac{c_b}{3\bar{m}}\left[\frac{2}{m_b} - \frac{1}{m_s}\right], \tag{38}$$

$$\Xi^*_{b1} = \Delta_b - \frac{2}{3}a_b - \frac{\Delta_b b_b}{\bar{m}^2}\left[\frac{1}{m_s} + \frac{1}{m_b}\right] - \frac{c_b}{3\bar{m}}\left[\frac{1}{m_s} + \frac{1}{m_b}\right]. \tag{39}$$

One may perform a fit to these quantities, varying Δ_b, a_b, and $c_b/\bar{m}^2$. Fixed parameters in this fit (see the caption of Table I, with m_b taken from [12]) are

$$\bar{m} = 362.1\ \text{MeV}, \qquad m_s = 543.9\ \text{MeV},$$
$$b_b/(\bar{m}^2) = b/(\bar{m})^2 = 50.0\ \text{MeV}, \qquad m_b = 5043.5\ \text{MeV}. \tag{40}$$

The results are shown in Table XI. The associated χ^2 is 0.33, so a consistent set of parameters is obtained. However, they differ from those fitting the light-quark or charmed baryons: in MeV,

$$\Delta_b = -1.56 \pm 0.34, \qquad a_b = 3.20 \pm 1.17,$$
$$c_b/\bar{m}^2 = -2.79 \pm 1.18. \tag{41}$$

TABLE XI. Predicted and observed isospin splittings in beauty baryons. Errors are experimental values, used in calculating χ^2 contributions.

	Σ_{b1}	Σ_{b2}	Σ^*_{b1}	Σ^*_{b2}	Ξ_{b1}	Ξ'_{b1}	Ξ^*_{b1}
Fit	−5.015	0.410	−4.522	0.410	−5.979	−3.095	−2.848
Expt.	−5.060	⋯	−4.370	⋯	−5.900	⋯	3.030
Error	0.180	⋯	0.330	⋯	0.600	⋯	0.910
χ^2	0.063	⋯	0.211	⋯	0.017	⋯	0.040

Two relations analogous to those for charmed baryons are predicted:

$$\Sigma_{b1} - 2\Xi'_{b1} = \Sigma^*_{b1} - 2\Xi^*_{b1} = \Sigma^*_1 - 2\Xi^*_1, \tag{42}$$

with the second holding only for equal light-quark baryon and beauty baryon parameters. The right-hand side of this relation is

$$\text{rhs} = a - \frac{2\Delta b}{\bar{m}^2}\left[\frac{1}{\bar{m}} - \frac{1}{m_s}\right] + \frac{c}{3\bar{m}^2}\left[1 + \frac{2\bar{m}}{m_s}\right], \tag{43}$$

whether for light-quark, charmed, or beauty baryons. In Sec. IV we found $\Sigma^*_1 - 2\Xi^*_1 = 2.0 \pm 1.5$ MeV. However, the large splitting between neutral and charged Ξ_b states leads the middle term of this sum rule to the value

$$\Sigma^*_{b1} - 2\Xi^*_{b1} = [-4.37 \pm 0.33 + 2(5.9 \pm 0.6)]\ \text{MeV}$$
$$= (7.4 \pm 1.2)\ \text{MeV}. \tag{44}$$

The violation of this sum rule is further evidence that one cannot always assume equal values of Δ, a, c for bottom- and lighter-quark systems.

VI. CHARM-BEAUTY RELATIONS

A. Universal parameters?

The comparison of isospin-violating parameters among light-quark, charmed, and beauty hadrons shows that one cannot regard them as universal. Suppose, first of all, that one took $\Delta^m = \Delta^m_c = \Delta^m_b$. With this assumption one could solve Eqs. (19) and (31) to obtain $a^m_c = -0.54$ MeV and $a^m_b = 11.42$ MeV. This makes little sense because the parameter a^m_c should be positive.

One could, instead, assume that the heavy-quark parameters

$$\Delta^m_Q \equiv \Delta^m_c = \Delta^m_b, \qquad a^m_Q \equiv a^m_c = a^m_b \tag{45}$$

are equal for charmed and beauty mesons. (As we shall see, this is approximately true for baryons.) Then solving Eqs. (19) and (33), assuming $h^m_b = 0$, one finds

$$\Delta^m_Q = -1.46 \pm 0.05\ \text{MeV}, \qquad a^m_Q = 3.45 \pm 0.09\ \text{MeV}, \tag{46}$$

to be compared with the light-quark meson value (see Sec. III A)

TABLE XII. Masses (in MeV) contributing to relation (48) between charmed and beauty meson hyperfine splittings.

State	Mass
$\bar{B}_s^*$	$5415.4^{+1.8}_{-1.5}$
$\bar{B}_s$	5366.88 ± 0.17
$\bar{B}_s^* - \bar{B}_s$	$48.6^{+1.8}_{-1.5}$
$\bar{B}^{*0}$	5324.70 ± 0.22[a]
$\bar{B}^0$	5279.63 ± 0.15
$\bar{B}^{*0} - \bar{B}^0$	45.07 ± 0.21[b]
D_s^*	2112.2 ± 0.4
D_s	1968.34 ± 0.07
$D_s^* - D_s$	143.86 ± 0.41
D^{*+}	2010.26 ± 0.05
D^+	1869.65 ± 0.05
$D^{*+} - D^+$	140.603 ± 0.015

[a] The charge of the state is not specified in Ref. [6]. Instead, we quote the value for a production-weighted average.

[b] Estimate based on small isospin splitting between charged and neutral $\bar{B}^*$.

$$\Delta^m = -4.12 \pm 0.05 \text{ MeV}, \qquad a^m = 2.20 \pm 0.62 \text{ MeV}. \tag{47}$$

The larger value of a makes sense, because of the deeper binding of charmed and bottom hadrons (hence a larger expectation value of $1/r$). However, the difference between Δ_Q^m [close to the value in Eq. (41)] and Δ^m is somewhat puzzling. Note that in Table V we found $\Delta = -2.49 \pm 0.04$ MeV for light-quark baryons, considerably different from the value Δ^m.

B. Relations between hyperfine splittings

Although it is not an isospin splitting, a relation between charmed meson and beauty meson hyperfine splittings makes use of the relatively new result from the *BABAR* Collaboration [5] which enters the Particle Data Group compilation. The relation [1] (updated in Ref. [7] to account for QCD corrections) is

$$\begin{aligned} &M(\bar{B}_s^*) - M(\bar{B}_s) - [M(\bar{B}^{*0}) - M(\bar{B}^0)] \\ &\quad = (m_c/m_b)\{M(D_s^*) - M(D_s) - [M(D^{*+} - M(D^+)]\}. \end{aligned} \tag{48}$$

The left- and right-hand sides of this equation, based on heavy-quark symmetry, are related to one another by $b \leftrightarrow c$. The present status of its terms is summarized in Table XII [6]. The left-hand side of Eq. (48) is 3.5 ± 1.7 MeV, while the right-hand side is $(m_c/m_b)(3.26 \pm 0.41)$ MeV$\simeq (1.09 \pm 0.14)$ MeV. A decisive test of this relation awaits separate measurements of the masses of $\bar{B}^{*+}$ and $\bar{B}^{*0}$ and a reduced error on the mass of B^*s.

TABLE XIII. Comparison of parameters governing isospin splittings in quark models.

Reference	Δ or $m_u - m_d$ (MeV)	a (MeV)	Comments
This work	$\Delta^m = -4.117$	$a^m = 2.119$	Light-quark meson octet
	$\Delta^b = -2.491$	$a^b = 3.052$	Light-quark baryons
[2]	$\Delta^b = -2.57$[a]	$a^b = 3.06$[a]	Neglecting kinetic term K
[3]		$a^m = 3.18 \pm 0.48$	Eq. (15) and Appendix A
[4]	$\Delta^b = -2.48$[a]	$a^b = 3.05$[a]	
	$\Delta^b = -2.67$[b]	$a^b = 2.83$[b]	
[6]	$m_u - m_d = -2.55 \pm 0.25$		$\overline{\text{MS}}$, $\mu_{\text{renorm.}} = 2$ GeV
[15]	$m_u - m_d = -6$		
[24]	$m_u - m_d = -3.8$		
[25]	$m_u - m_d = -2.54 \pm 0.04$		J. Franklin, priv. commun.
[26]	$m_u - m_d = -2.66$	$a^m = 1.5 \pm 0.5$	Baryon a unclear
[27]	$m_u - m_d = -4.12$		MIT bag model
[28]	$m_u - m_d = -6.7$		MIT bag model
[29]	$m_u - m_d = -4.4$	$a^b = 2.9$	
[30]	$m_u - m_d = -2.4$	Ignored	"Photon cloud" effects
[33]	$m_u - m_d = -11$	3.39	Potential models
[35]	$m_u - m_d = -1.88$	3.52	Including three-body terms
[36]	$m_u - m_d = -1.82$		
[38]	$\Delta^b = -1.84 \pm 0.16$		
[39]	$m_u - m_d = -2.5$		
[42]	$m_u - m_d = -5.7$		$\overline{\text{MS}}$, $\mu_{\text{renorm.}} = 100$ MeV
	$m_u - m_d = -4.7$		$\overline{\text{MS}}$, $\mu_{\text{renorm.}} = 200$ MeV

[a] Different masses for quarks in mesons and baryons.

[b] Universal masses for quarks in mesons and baryons.

TABLE XIV. Observables needed to refine understanding of isospin breaking.

Observable	Value, if known
$M(B^{*0})$	$\cdots$
$M(K^{*\pm})$	891.76 ± 0.25 MeV, hadroproduction 895.5 ± 0.8 MeV, τ decay
$M(B_s^*)$	$5415.4^{+1.8}_{-1.5}$ MeV
$M(\Sigma^{*+}) - 2M(\Sigma^{*0}) + M(\Sigma^{*-})$	2.6 ± 2.1 MeV
$M(\Xi_{cc}^{++}) - M(\Xi_{cc}^{+})$	Predicted in Ref. [4]

VII. COMPARISON WITH OTHER APPROACHES

Thanks to improvements in computing power, lattice quantum chromodynamics (LQCD) is beginning to be able to take into account isospin splittings in masses and decay constants. (For some references on the latter, see [20].) For LQCD approaches to light-quark splittings see Refs. [21,22] (octet baryons) and [23] (octet mesons and baryons). We look forward to LQCD calculations of isospin splittings in mesons and baryons containing at least one heavy quark.

Within quark models there is a long history of tackling isospin splittings in hadrons [2–4,15,24–41]. (Reference [40], though using chiral perturbation theory, gives an extensive list of works based on quark models.) The parameters Δ (or $m_u - m_d$) and a, when given, are compared in Table XIII. We show there also the latest estimate of $m_u - m_d$ in the current-quark picture [6].

The relation between current-quark masses (see the mini-review No. 66 in Ref. [6], and the formalism set forth in Ref. [42]) and the constituent-quark masses we are using has been discussed in [15]. However, it has been pointed out in [6] that this relation (and hence the definition of constituent-quark masses) is model dependent. We note that many of our determinations of $m_u - m_d$ in the constituent-quark picture are not that far from the current-quark value of ~ -2.5 MeV,[1] suggesting that in those cases the QCD "dressing" of current quarks may act linearly on their masses. (An exception is presented by the light-quark mesons, for which $|m_u - m_d|$ is considerably larger, and by the strange-quark mass, which is about 90 MeV heavier than the average nonstrange mass in the current-quark picture [6,43] but 180 MeV heavier than the average nonstrange mass in our constituent-quark picture (see the caption of Table I).

VIII. CONCLUSIONS

Within a constituent-quark picture, we have updated predictions of isospin splittings in hadrons with at most one c or b quark. Effects considered included an intrinsic u–d mass difference and its effect on kinetic energies (parameter Δ); Coulomb interactions among the constituent quarks (parameter a); and quark mass dependence on strong and electromagnetic hyperfine splittings (parameters b and c, respectively). The parameter Δ is found to have a non-universal value, ranging from -4.1 MeV in light-quark mesons to -1.5 MeV in heavy-quark mesons and possibly in b-quark baryons. This latter conclusion is preliminary in the absence of a direct measurement of the masses of both B^* charge states. A value of Δ near –2.5 MeV seems consistent with isospin splittings in light-quark and charmed baryons, but more negative than in bottom baryons. Most estimates of the Coulomb interaction term a lie between 2 and 3 MeV.

Quantities whose measurement would help to test relations in the present analysis include improved masses of $K^{*\pm}$ and B_s^*; some isospin splittings in beauty baryons; and $M(\Xi_{cc}^{++}) - M(\Xi_{cc}^{+})$, predicted in Ref. [4] to be (2.17 ± 0.11) MeV under the present set of assumptions [or (1.49 ± 0.12) MeV in a model with universal quark masses for mesons and baryons.] We look forward to these developments, summarized in Table XIV, in the data. Our survey of isospin-violating effects has shown that a description in terms of universal u–d mass differences and effective Coulomb and hyperfine interactions has its limitations, with effective mass splittings dependent to some degree on the hadronic environment.

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