Standard model parameters in the tadpole-free pure $\overline{\mathrm{MS}}$ scheme

Stephen P. Martin[1] and David G. Robertson[2]

[1]*Department of Physics, Northern Illinois University, DeKalb, Illinois 60115, USA*
[2]*Department of Physics, Otterbein University, Westerville, Ohio 43081, USA*

(Received 15 July 2019; published 9 October 2019)

We present an implementation and numerical study of the Standard Model couplings, masses, and vacuum expectation value (VEV), using the pure $\overline{\mathrm{MS}}$ renormalization scheme based on dimensional regularization. Here, the $\overline{\mathrm{MS}}$ Lagrangian parameters are treated as the fundamental inputs, and the VEV is defined as the minimum of the Landau gauge effective potential, so that tadpole diagrams vanish, resulting in improved convergence of perturbation theory. State-of-the-art calculations relating the $\overline{\mathrm{MS}}$ inputs to on-shell observables are implemented in a consistent way within a public computer code library, SMDR (standard model in dimensional regularization), which can be run interactively or called by other programs. Included here for the first time are the full two-loop contributions to the Fermi constant within this scheme and studies of the minimization condition for the VEV at three-loop order with four-loop QCD effects. We also implement and study the scale dependence of all known multiloop contributions to the physical masses of the Higgs boson, the W and Z bosons, and the top quark, the fine structure constant and weak mixing angle, and the renormalization group equations and threshold matching relations for the gauge couplings, fermion masses, and Yukawa couplings.

DOI: 10.1103/PhysRevD.100.073004

I. INTRODUCTION

With the discovery of the Higgs boson, the Standard Model is technically complete. This is despite indications that it will have to be extended to accommodate dark matter and to solve issues such as the hierarchy problem, the strong *CP* problem, and the cosmological constant problems. At this writing, the LHC continues to strengthen lower bounds on the masses of new particles in hypothetical ultraviolet completions such as supersymmetry. It is therefore plausible that we should view the Standard Model as a valid, complete effective field theory up to the tera-electron-volt (TeV) scale and perhaps well beyond, with nonrenormalizable terms in the Lagrangian correspondingly highly suppressed. This paper is concerned with the ongoing program of determining, as accurately as possible, the relations between the renormalizable Lagrangian parameters that define the theory and the observables and on-shell quantities that are more directly connected to experimental results. This is part of a larger goal of improving our understanding of the Standard Model at the level of accuracy required to test it with future experiments.

A convenient method of handling the ultraviolet divergences of the Standard Model is provided by dimensional regularization [1–5] followed by renormalization by modified minimal subtraction, $\overline{\mathrm{MS}}$ [6,7]. To describe the effects of electroweak symmetry breaking induced by the Higgs vacuum expectation value (VEV), there are at least two distinct ways to proceed. Consider the Higgs potential

$$V(\phi) = \Lambda + m^2 H^\dagger H + \lambda (H^\dagger H)^2, \tag{1.1}$$

where H is the canonically normalized complex Higgs doublet field. First, one may choose to organize perturbation theory by expanding the electrically neutral component of H around a tree-level VEV $v_{\mathrm{tree}}/\sqrt{2}$, defined by

$$v_{\mathrm{tree}} \equiv \sqrt{-m^2/\lambda}. \tag{1.2}$$

This is used in many works, because it has the advantage that v_{tree} is manifestly independent of the choice of gauge fixing. However, it has the disadvantage that Higgs tadpole loop diagrams do not vanish and must be included order by order in perturbation theory. This comes with a parametrically slower convergence of perturbation theory, as the tadpole contributions to other calculated quantities will include powers of $1/\lambda$ due to their zero-momentum Higgs propagators.

We choose instead to expand the Higgs field around a loop-corrected VEV v, which is defined to be the minimum of the full effective potential [8–10] in the Landau gauge.

For the Standard Model (and indeed for a general renormalizable field theory), the effective potential has now been obtained at two-loop [11,12] and three-loop [13,14] orders, with the four-loop contributions known [15] at leading order in QCD. The choice of Landau gauge is made because other gauge-fixing choices lead to unpleasant technical problems including kinetic mixing between the longitudinal components of the vector and the Goldstone scalar degrees of freedom.[1] The disadvantage of defining the VEV in this way is that calculations that make use of it are then restricted to the Landau gauge. But the advantage of this choice is that the sum of all Higgs tadpole diagrams (including the tree-level tadpole) automatically vanishes, and there are no corresponding $1/\lambda^n$ contributions in perturbation theory.

Another issue to be dealt with is that the minimization condition for the effective potential requires resummation of Goldstone boson contributions, as explained in [17,18], in order to avoid spurious imaginary parts and infrared divergences at higher loop orders. (For further perspectives and developments on this issue, see Refs. [19–25].) The end result can be written as a relation between the tree-level and loop-corrected VEVs,

$$v_{\rm tree}^2 = v^2 + \frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{1}{(16\pi^2)^n}\Delta_n, \tag{1.3}$$

with n-loop order contributions Δ_n that are free of spurious imaginary parts and infrared divergences and do not depend at all on the Goldstone boson squared mass. (The $1/\lambda$ in this equation is the source of the tadpole effects noted above if one chooses to expand in terms of $v_{\rm tree}$ rather than v.) The full three-loop contributions were given in [14] in terms of two-loop and three-loop basis integrals that can be efficiently evaluated numerically using the computer code 3VIL [26],[2] and the four-loop contribution was obtained at leading order in QCD in [15]. However, a numerical illustration of these effects was deferred. One of the purposes of the present paper is to remedy this by providing a numerical study of the three-loop and four-loop effects.

We also have a broader purpose here: to bring together in a coherent form, implemented as a public computer code, results obtained in recent years relating pole masses and other observables to the Lagrangian parameters in the tadpole-free pure $\overline{\rm MS}$ scheme. The new code, called SMDR for standard model in dimensional regularization, is a software library written in C with functions callable from user C or C++ programs. It uses the $\overline{\rm MS}$ input parameters that define[3] the Standard Model theory at a given renormalization scale Q:

$$v, \lambda, g_3, g, g', y_t, y_b, y_c, y_s, y_d, y_u, y_\tau, y_\mu, y_e, \Delta\alpha_{\rm had}^{(5)}(M_Z). \tag{1.4}$$

All of these, except the last, are defined as running parameters in the nondecoupled (high-energy) Standard Model, with gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ with gauge couplings g_3, g, and g', respectively, and six active quarks. Note that the running $\overline{\rm MS}$ Higgs squared mass parameter m^2 need not be included among these, because it is not independent, being determined in terms of λ, v, and the other parameters by the effective potential minimization condition Eq. (1.3). Also, the hadronic light-quark contribution to the fine-structure constant is given by a parameter $\Delta\alpha_{\rm had}^{(5)}(M_Z)$. In principle this is not independent of the others in Eq. (1.4), but in practice it must (at least, at present) be treated as an independent input because it depends on nonperturbative physics. The code then provides computations of the following "on-shell" output quantities:

$$\begin{aligned}
&\text{heavy particle pole masses}: M_t, M_h, M_Z, M_W,\\
&\text{running light quark masses}: m_b(m_b), m_c(m_c),\\
&\quad m_s(2\ \text{GeV}), m_d(2\ \text{GeV}), m_u(2\ \text{GeV}),\\
&\text{lepton pole masses}: M_\tau, M_\mu, M_e,\\
&\text{five-quark QCD coupling}: \alpha_S^{(5)}(M_Z),\\
&\text{Fermi constant}: G_F = 1.1663787\cdots\times 10^{-5}\ \text{GeV}^{-2},\\
&\text{fine structure constant}: \alpha_0 = 1/137.035999139\cdots\\
&\quad \text{and}\quad \Delta\alpha_{\rm had}^{(5)}(M_Z),
\end{aligned} \tag{1.5}$$

which can be viewed as dual to the $\overline{\rm MS}$ inputs. (Even though G_F and α_0 are extremely accurately known from experiment, as indicated, they are considered as outputs from the point of view of the pure $\overline{\rm MS}$ renormalization scheme.) However, note that M_W is actually extra, in the sense that the other parameters in Eq. (1.5) are already sufficient to fix the $\overline{\rm MS}$ quantities in Eq. (1.4); therefore, the computation of M_W provides a consistency check on the Standard Model. The quantity $\Delta\alpha_{\rm had}^{(5)}(M_Z)$ appears in both lists (1.4) and (1.5), due to its nonperturbative nature; it always is obtained from experiment rather than fits to other quantities. The SMDR code also computes the weak mixing

[1]The full two-loop effective potential has recently been obtained in a large class of more general gauge-fixing schemes in Ref. [16], but it is quite unwieldy, and extending it to the three-loop order is a daunting challenge.

[2]3VIL computes three-loop vacuum basis integrals numerically using the differential equations method, except in special cases for which they can be computed analytically, including the cases found in Refs. [27–47]. See Ref. [48] for an alternative evaluation of three-loop vacuum integrals based on dispersion relations.

[3]Cabibbo-Kobayashi-Maskawa mixing and neutrino mass and mixing effects are neglected in the present version. Including them would have a negligible effect on the quantities in Eq. (1.5), compared to other sources of uncertainty.

angle as defined by the Particle Data Group's Review of Particle Properties (RPP) [49] (which, unlike the present paper, uses a scheme with the top quark decoupled but the massive W boson active, corresponding to a nonrenormalizable effective theory even when the Lagrangian couplings of negative mass dimension are neglected), but this is again extra, since it is not needed in order to fix the $\overline{\text{MS}}$ quantities.

The relationship between the Sommerfeld fine-structure constant α_0 appearing in Eq. (1.5) and the couplings g and g' in Eq. (1.4) can be expressed as (see, e.g., Refs. [50–53])

$$\alpha_0 = \frac{g^2(M_Z) g'^2(M_Z)}{4\pi[g^2(M_Z) + g'^2(M_Z)]} \times [1 - \Delta\alpha^{(5)}_{\text{had}}(M_Z) - \Delta\alpha^{\text{LO}}_{\text{pert}} - \Delta\alpha^{\text{HO}}_{\text{pert}}], \qquad (1.6)$$

where the sum of one-loop contributions from t, W, τ, μ, e (but not b, c, s, d, u) are

$$\Delta\alpha^{\text{LO}}_{\text{pert}} = \frac{\alpha_0}{4\pi}\left[\frac{202}{27} + 14\ln(M_W/M_Z) - \frac{32}{9}\ln(M_t/M_Z) - \frac{8}{3}\ln(M_\tau/M_Z) - \frac{8}{3}\ln(M_\mu/M_Z) - \frac{8}{3}\ln(M_e/M_Z)\right], \qquad (1.7)$$

and the higher-order perturbative contribution $\Delta\alpha^{\text{HO}}_{\text{pert}}$ has been given as an interpolating formula in Eqs. (19)–(21) of Ref. [53]. For the running $\alpha^{\overline{\text{MS}}}(Q)$ in the decoupled theories used for the renormalization group (RG) running below M_Z [with the numbers of active (quarks, charged leptons) equal to (5, 3) or (4, 3) or (4, 2) or (3, 2)], we use the results obtained in [54], as discussed in the next section.

The pole masses M_t, M_h, M_Z, M_W, M_τ, M_μ, and M_e are each defined in terms of the complex pole in the renormalized propagator,

$$s_{\text{pole}} = M^2 - i\Gamma M. \qquad (1.8)$$

For the top-quark pole mass, the pure QCD contributions were obtained at one-loop, two-loop, three-loop, and four-loop orders in Refs. [55–59], respectively. The non-QCD contributions to M_t at one-loop and two-loop orders had also been obtained in other schemes and approximations. At one-loop order they were found in Refs. [60–62], and mixed electroweak-QCD two-loop contributions were obtained in [63–65]. Further two-loop contributions in the gaugeless limit (in which the electroweak boson masses are taken to be small compared to the top-quark mass) were found in Refs. [66–69]. Finally, the full two-loop results for M_t were provided in the tree-level VEV scheme in Ref. [70] and in the tadpole-free scheme used in the present paper in [71].

For the Higgs boson mass, we use our calculation in Ref. [72], which contains all two-loop contributions and the leading (in the limit g^2, g'^2, $\lambda \ll g_3^2$, y_t^2) three-loop contributions in the tadpole-free pure $\overline{\text{MS}}$ scheme. Earlier works on M_h at the two-loop level in other schemes and approximations include Ref. [73] which included the mixed QCD/electroweak contributions to M_h, Ref. [74] which used the gaugeless limit approximation at two-loop order, and the full two-loop approximation given as an interpolating formula in a hybrid $\overline{\text{MS}}$ /on-shell scheme in Ref. [75].

For the W and Z boson pole masses, we use the full two-loop calculations using the tadpole-free pure $\overline{\text{MS}}$ scheme given in Refs. [76,77], respectively. Previous two-loop calculations of the vector boson pole masses in other schemes (expanding around v_{tree} rather than v) appeared in Refs. [53,62,70,78]. It is important to note that for the vector bosons $V = W$ and Z, the values usually quoted, including by the RPP, are not the pole masses but the variable-width Breit-Wigner masses. These can be related to the pole masses by [79–82]

$$M^2_{V,\,\text{Breit-Wigner}} = M_V^2 + \Gamma_V^2. \qquad (1.9)$$

Thus, the Z- and W-boson pole masses defined by Eq. (1.8) are, respectively, approximately 34.1 MeV and 27.1 MeV smaller than the Breit-Wigner masses that are usually quoted.

The charged lepton pole masses are computed at two-loop order in QED, by converting the corresponding QCD formulas given in Ref. [56] and including small effects from nonzero lighter fermion masses from Ref. [83].

The running light-quark masses in Eq. (1.5) are defined in appropriate $SU(3)_c \times U(1)_{\text{EM}}$ effective field theories in which the heavier particles have been decoupled. Although it is possible to evaluate the QCD contributions to the bottom-quark and charm-quark pole masses, this is deprecated, because there is no semblance of convergence of the perturbative series relating the pole masses to the running masses for bottom and charm (and obviously for the lighter quarks as well); see Ref. [59]. Therefore we use running $\overline{\text{MS}}$ masses for all lighter quarks. Thus $m_b(m_b)$ is defined as an $\overline{\text{MS}}$ running mass in the five-quark, three-lepton QCD + QED effective theory, while $m_c(m_c)$ is similarly defined in the four-quark, two-lepton theory, and $m_s(2\text{ GeV})$, $m_d(2\text{ GeV})$, $m_u(2\text{ GeV})$ are defined in the three-quark, two-lepton theory. We follow the RPP Ref. [49] in choosing to evaluate the last three at, somewhat arbitrarily, $Q = 2$ GeV, in order to avoid larger QCD effects at smaller Q.

To obtain the five-quark, three-lepton QCD + QED effective field theory, we simultaneously decouple the heavier Standard Model particles t, h, Z, W at a common matching scale, which can be chosen at will, but should presumably be in the range from about M_W to M_t. Because W and Z are decoupled from it, this low-energy effective theory is a renormalizable gauge theory supplemented by interactions with couplings of negative mass dimension (including the Fermi four-fermion interactions).

The decouplings of the bottom quark, tau lepton, and charm quark are then performed individually.

In one mode of operation, the SMDR code takes the $\overline{\rm MS}$ input parameters of Eq. (1.4) provided by the user and outputs the on-shell quantities in Eq. (1.5). Alternatively, in a dual mode of operation, the SMDR code instead takes user input for the on-shell quantities in Eq. (1.5) (except for M_W) and determines as outputs the $\overline{\rm MS}$ quantities in Eq. (1.4) and then M_W, by doing a fit. The SMDR code also implements all known contributions to the running and decoupling of the gauge and Yukawa couplings.

In the numerical studies below, we employ a benchmark model point, chosen to yield the central values of the quantities in Eq. (1.5) (other than M_W, as noted above), as given in the 2019 update of the 2018 edition of the Review of Particle Properties in Ref. [49]:

$$\begin{aligned}
M_t &= 173.1 \text{ GeV}, \qquad M_h = 125.1 \text{ GeV}, \qquad M_{Z,\text{Breit-Wigner}} = 91.1876 \text{ GeV}, \\
G_F &= 1.1663787 \times 10^{-5} \text{ GeV}^2, \qquad \alpha_0 = 1/137.035999139, \qquad \alpha_S^{(5)}(M_Z) = 0.1181, \\
m_b(m_b) &= 4.18 \text{ GeV}, \qquad m_c(m_c) = 1.27 \text{ GeV}, \qquad m_s(2 \text{ GeV}) = 0.093 \text{ GeV}, \\
m_d(2 \text{ GeV}) &= 0.00467 \text{ GeV}, \qquad m_u(2 \text{ GeV}) = 0.00216 \text{ GeV}, \\
M_\tau &= 1.77686 \text{ GeV}, \\
M_\mu &= 0.1056583745 \text{ GeV}, \qquad M_e = 0.000510998946 \text{ GeV}, \\
\Delta\alpha^{(5)}_{\text{had}}(M_Z) &= 0.02764.
\end{aligned} \tag{1.10}$$

The $\overline{\rm MS}$ input quantities that do this are found (with default scale choices for evaluations in SMDR) to be

$$\begin{aligned}
Q_0 &= 173.1 \text{ GeV}, \\
v(Q_0) &= 246.60109 \text{ GeV}, \qquad \lambda(Q_0) = 0.12603842, \\
g_3(Q_0) &= 1.1636241, \qquad g_2(Q_0) = 0.64765961, \qquad g'(Q_0) = 0.35853877, \\
y_t(Q_0) &= 0.93480082, \qquad y_b(Q_0) = 0.015480097, \qquad y_\tau(Q_0) = 0.0099944422, \\
y_c(Q_0) &= 0.0033820038, \qquad y_s(Q_0) = 0.00029094484, \qquad y_\mu(Q_0) = 0.00058837986, \\
y_d(Q_0) &= 1.4609792 \times 10^{-5}, \qquad y_u(Q_0) = 6.7227779 \times 10^{-6}, \\
y_e(Q_0) &= 2.7929820 \times 10^{-6}.
\end{aligned} \tag{1.11}$$

This set of values obviously includes more significant digits than justified by the experimental and theoretical uncertainties; this is for the sake of reproducibility and checking when changes are made to the code, or to the default choices of matching or evaluation scales. Equation (1.11) will be referred to below as the reference model point, and a sample input file included with the SMDR distribution provides for automatic loading of these parameters. As future versions of the RPP with new experimental results become available, corresponding new versions of the reference model file will be included in new SMDR distributions; they can also be constructed easily by using functions provided. All of the figures appearing below are made using short programs (included with the SMDR distribution) that employ the SMDR library functions, in order to illustrate how the latter should be used.

II. RENORMALIZATION GROUP RUNNING AND DECOUPLING

The $\overline{\rm MS}$ renormalization group equations for the Standard Model used in this paper, and by default in the SMDR code, are the state-of-the-art ones. These include the two-loop [84–88] and three-loop [89–97] order contributions for all parameters, including the gauge couplings, the fermion Yukawa couplings, the Higgs self-coupling λ, VEV v, and negative squared mass m^2. In addition, for the strong coupling, the contributions to the beta function at four-loop order in the limit g^2, $g'^2 \ll g_3^2$, y_t^2, λ [98–102] and pure QCD five-loop order [103,104] are included. Similarly, the higher-order QCD contributions to the beta functions of the quark Yukawa couplings are included, using results found at four-loop order in Refs. [105,106] and at five-loop order in Ref. [107]. Finally, the leading QCD four-loop contribution to the beta function of the Higgs self-coupling λ is included from Refs. [15,108].

Using the reference model of Eq. (1.11) as inputs, the renormalization group running of the couplings are illustrated in Figure 1 for the range 10^2 GeV $< Q < 10^{19}$ GeV. The left panel shows the inverse gauge couplings $1/\alpha_3 = 4\pi/g_3^2$, $1/\alpha_2 = 4\pi/g^2$, and [in a grand unified theory (GUT) normalization] $1/\alpha_1 = (3/5)4\pi/g'^2$, while

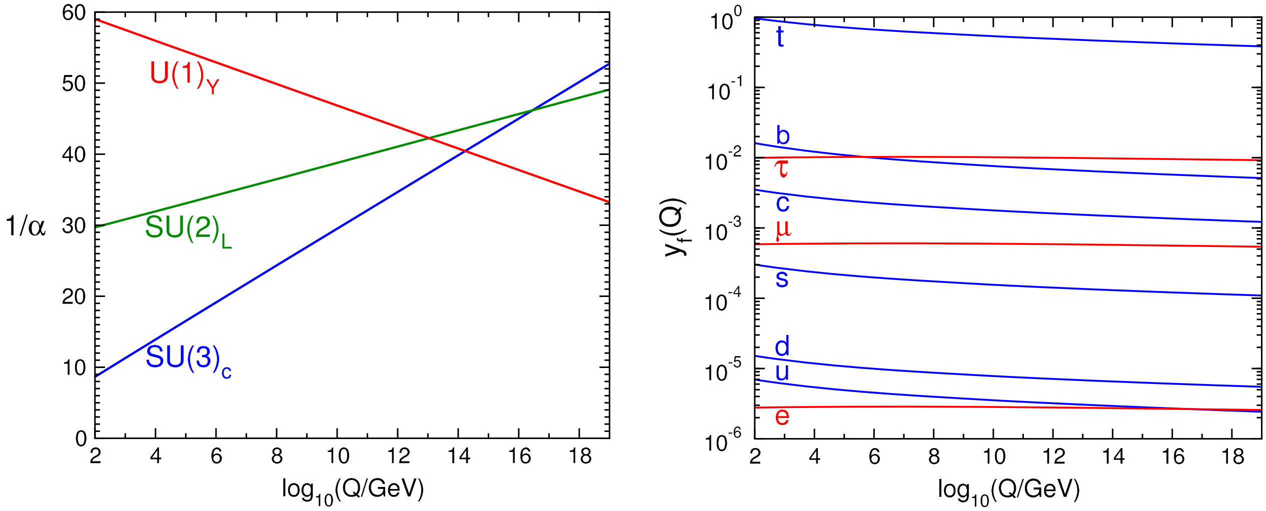

FIG. 1. Renormalization group running of the $\overline{\rm MS}$ inverse gauge couplings $1/\alpha_3$, $1/\alpha_2$, and $1/\alpha_1$ in a grand unified theory normalization (left panel) and charged fermion Yukawa couplings (right panel), as functions of the renormalization scale Q. The input parameters are given by the reference model point defined in Eq. (1.11) at $Q_0 = 173.1$ GeV.

the right panel shows the Yukawa couplings for all of the Standard Model charged fermions.

For lower scales, we use the results given in Ref. [54] to simultaneously decouple the top quark, Higgs boson, Z boson, and W boson at a common matching scale, so that the low-energy effective field theory is renormalizable and has gauge group $SU(3)_c \times U(1)_{\rm EM}$. The common matching scale is, in principle, arbitrary; by default the SMDR code uses $Q = M_Z$ for the matching but this can be modified at run-time by the user. The matching results include the two-loop matching found in [54] for the electromagnetic $\overline{\rm MS}$ coupling $\alpha(Q)$ in the theory with five quarks and three leptons, as well as the matching relation for the five-quark QCD coupling $\alpha_S(Q)$ at one-loop [109,110], two-loop [111,112], three-loop [113,114], and four-loop [115,116] orders together with the complete Yukawa and electroweak two-loop contributions obtained first in Ref. [117] (and verified and written in a different way compatible with the present paper in Ref. [54]). The pure QCD corrections to the quark mass matching relations were given at three-loop order in Refs. [113,114] and four-loop order in Ref. [118].

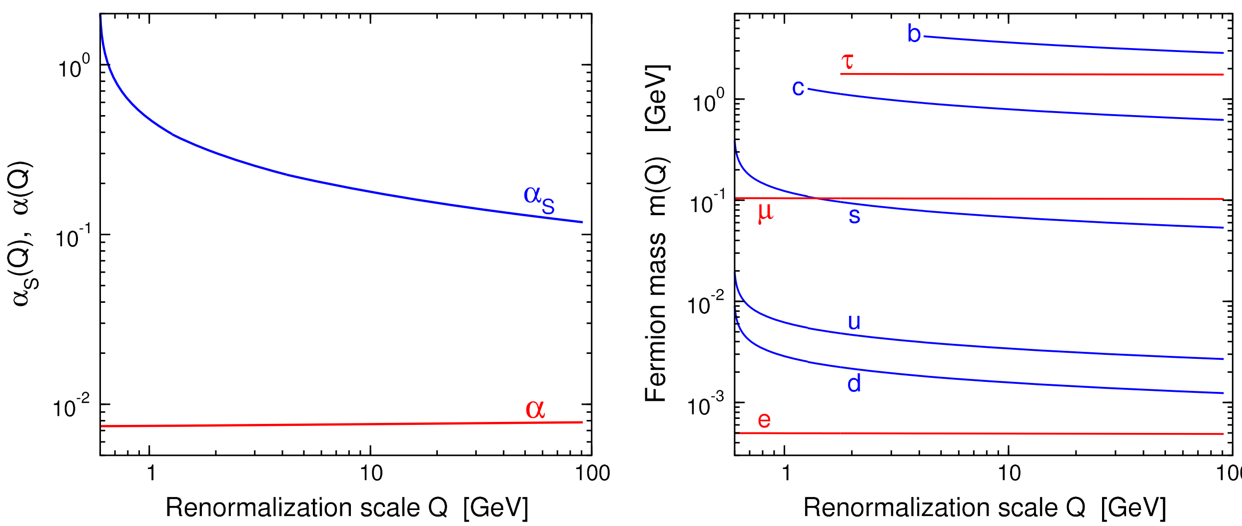

FIG. 2. Renormalization group running of the $\overline{\rm MS}$ QCD and QED gauge couplings α_S and α (left panel) and fermion masses (right panel), as functions of the renormalization scale Q. The beta functions used are five-loop order in QCD and three-loop order in QED, with active fermion contents as follows: five-quark, three-lepton for $m_b(m_b) \leq Q \leq 91.1876$ GeV; four-quark, three-lepton for $M_\tau \leq Q \leq m_b(m_b)$; four-quark, two-lepton for $m_c(m_c) \leq Q \leq M_\tau$; and three-quark, two-lepton for $Q \leq m_c(m_c)$. The matchings at $Q = m_b(m_b)$ and M_τ and $m_c(m_c)$ are done at four-loop order for the QCD coupling, two-loop order for the QED coupling, and the fermion mass matchings include effects at three-loop order in QCD and two-loop order in QED. The input parameters are defined by the reference model point given in Eq. (1.11), with t, h, Z, W simultaneously decoupled at $Q = 91.1876$ GeV.

For the QCD parts of the matching relations and beta functions, complete results had been calculated and incorporated long ago into the RunDec and CRunDec [119–121] codes. In addition, the two-loop mixed QCD/electroweak and pure electroweak contributions to the matching of the running b, c, s, d, u and τ, μ, e fermion masses were obtained in Refs. [69,70,122–124] and [54]. They are implemented in SMDR using the formulas provided in Ref. [54] consistent with the conventions of the present paper.

The running and decoupling of the QCD and QED gauge couplings and running fermion masses are shown in Fig. 2 for the sequence of effective theories with five quarks and three charged leptons [for $m_b(m_b) \leq Q \leq M_Z$], with four quarks and three charged leptons [for $M_\tau \leq Q \leq m_b(m_b)$], with four quarks and two charged leptons [for $m_c(m_c) \leq Q \leq M_\tau$], and with three quarks and two charged leptons [for $Q \leq m_c(m_c)$]. The boundaries between these effective theories are somewhat arbitrary and correspond to the default points within the SMDR code, which can be adjusted by the user. At each of the matching points $Q = m_b(m_b)$ and M_τ and $m_c(m_c)$, the parameters are actually discontinuous due to the matching mentioned above due to changing effective theories, but this cannot be discerned with the resolution of the plots.

III. MINIMIZATION OF THE EFFECTIVE POTENTIAL AND THE VACUUM EXPECTATION VALUE

We first consider a numerical illustration of the minimization condition for the effective potential, Eq. (1.3), which can be used to trade m^2 for v, when all of the other $\overline{\mathrm{MS}}$ parameters are taken to be known inputs. The quantities Δ_n have been given up to three-loop order in Ref. [14] and the four-loop order contribution at leading order in QCD is found in Ref. [15].

In Fig. 3, we start with the $\overline{\mathrm{MS}}$ quantities taken to be their benchmark reference point values defined at $Q = Q_0 = 173.1$ in Eq. (1.11). From Eq. (1.3), the value of m^2 at Q_0 for the reference model is then found to be (again including more significant digits than justified by the uncertainties)

$$m^2(Q_0) = -(92.878850\ \mathrm{GeV})^2. \tag{3.1}$$

At other renormalization group scales Q, we determine $m^2(Q)$ in two different ways. For the first way, we renormalization-group run all of the other parameters to Q, where $m^2(Q)_{\mathrm{min}}$ is then determined by again applying Eq. (1.3). The results are shown in the left panel of Figure 3, in various approximations (as labeled) for the minimization condition. The second way is to directly RG run $m^2(Q)_{\mathrm{run}}$ starting with Eq. (3.1) as its boundary condition. Note that here, the renormalization group running of $m^2(Q)_{\mathrm{run}}$ is obtained by treating it as an independent parameter in the high-energy Lagrangian. In the right panel, we show the ratio of $m^2(Q)_{\mathrm{min}}/m^2(Q)_{\mathrm{run}}$ as a function of Q. This provides a scale-invariance check yielding a lower bound on the error, because in the idealized case of calculations to all orders in perturbation theory, the ratio should be exactly 1. We find that in the case of the full three-loop plus QCD four-loop approximation,

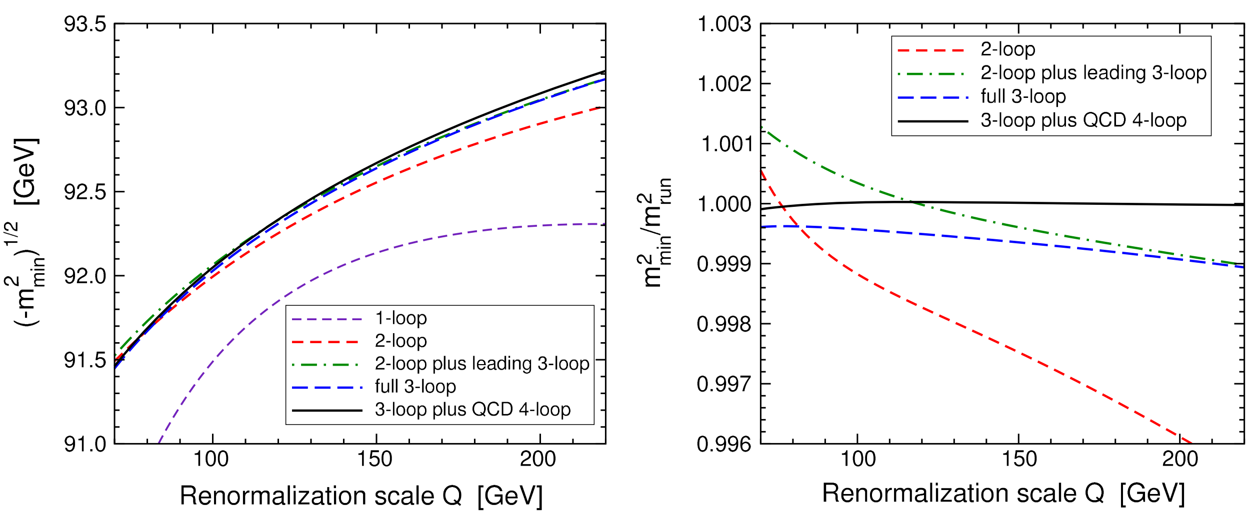

FIG. 3. The $\overline{\mathrm{MS}}$ Higgs squared mass parameter, as a function of the renormalization scale Q, for the reference model point defined at $Q_0 = 173.1$ GeV in Eq. (1.11). The other input parameters, including the VEV $v(Q)$, are obtained from the reference model by evolving them using their RG equations to the scale Q, where the Landau gauge effective potential is then required to be minimized to determine $m^2(Q)_{\mathrm{min}}$. In the left panel, results are shown for the one-loop, two-loop, two-loop plus leading three-loop, full three-loop, and three-loop plus QCD four-loop approximations to the effective potential minimization condition. The right panel shows the results for $m^2(Q)_{\mathrm{min}}/m^2(Q)_{\mathrm{run}}$, where $m^2(Q)_{\mathrm{min}}$ is determined as in the left panel, and $m^2(Q)_{\mathrm{run}}$ is obtained directly by renormalization running its input value from the reference scale $Q_0 = 173.1$ GeV.

the deviation of the ratio from unity is less than 10^{-4} for the entire range shown from 70 GeV to 220 GeV, and over most of this range the deviation is actually much smaller. Without including the four-loop QCD contribution, the scale dependence is still quite good, but is a few times 10^{-4}. In both cases, the parametric uncertainties from experimentally measured quantities would probably seem to be larger than the theoretical uncertainties, although we emphasize that the scale-dependence check can only give a lower bound on the theoretical error.

In Fig. 4, we perform the inverse of the preceding analysis. This time, we take $m^2(Q_0)$ as an input given by Eq. (3.1) and determine $v(Q)$ as an output by solving Eq. (1.3). Of course, at $Q = Q_0$, the result is exactly as given in Eq. (1.11). At other Q, we obtain $v(Q)_{\rm min}$ by first running all of the other $\overline{\rm MS}$ quantities from Q_0 to Q and then applying Eq. (1.3) again. The results are shown in the left panel of Fig. 4. We also obtain $v(Q)_{\rm run}$ by directly running it using its RG equations from Q_0. The ratio $v(Q)_{\rm min}/v(Q)_{\rm run}$ is shown in the right panel of Fig. 4. Again, in the best available approximation, the scale dependence of the ratio is much smaller than 10^{-4} over the entire range.

IV. THE FERMI DECAY CONSTANT

The Fermi weak decay constant is closely related to the vacuum expectation value, with $G_F = 1/\sqrt{2}v^2$ at tree level. Including radiative corrections, one can write

$$G_F = \frac{1+\Delta\bar{r}}{\sqrt{2}v_{\rm tree}^2} = \frac{1+\Delta\tilde{r}}{\sqrt{2}v^2}. \tag{4.1}$$

Expressions for $\Delta\bar{r}$ have been given at two-loop order in the so-called gaugeless limit (g^2, $g'^2 \ll g_3^2$, y_t^2, λ) in Refs. [69,70], using expansions in terms of $\overline{\rm MS}$ and on-shell quantities, respectively, but in both cases determined in terms of the tree-level VEV. The full two-loop version of $\Delta\bar{r}$ is quite lengthy, and to our knowledge has not appeared in print, but was obtained and presented within the public computer code mr [124]. The two-loop corrections to the closely related quantity Δr in the $\overline{\rm MS}$ scheme (but defined in terms of the W boson experimental mass) have also been discussed in Ref. [53]. We have obtained the corresponding complete two-loop result for $\Delta\tilde{r}$ in terms of v,

$$\Delta\tilde{r} = \frac{1}{16\pi^2}\Delta\tilde{r}^{(1)} + \frac{1}{(16\pi^2)^2}\Delta\tilde{r}^{(2)} + \cdots. \tag{4.2}$$

The one-loop order part is

$$\begin{aligned}\Delta\tilde{r}^{(1)} = {}& \frac{3}{4}(g^2+g'^2)[A(Z)-A(W)]/(Z-W) \\ &+ \frac{3}{4}[(4g^2-24\lambda)A(W) - g^2A(h)]/(h-W) \\ &+ 3[y_t^2A(t) - y_b^2A(b)]/(t-b) \\ &+ 2A(\tau)/v^2 - (3g^2+g'^2)/8 \\ &+ (3y_t^2+3y_b^2+y_\tau^2)/2,\end{aligned} \tag{4.3}$$

where

$$Z = (g^2+g'^2)v^2/4, \qquad W = g^2v^2/4, \qquad h = 2\lambda v^2, \tag{4.4}$$

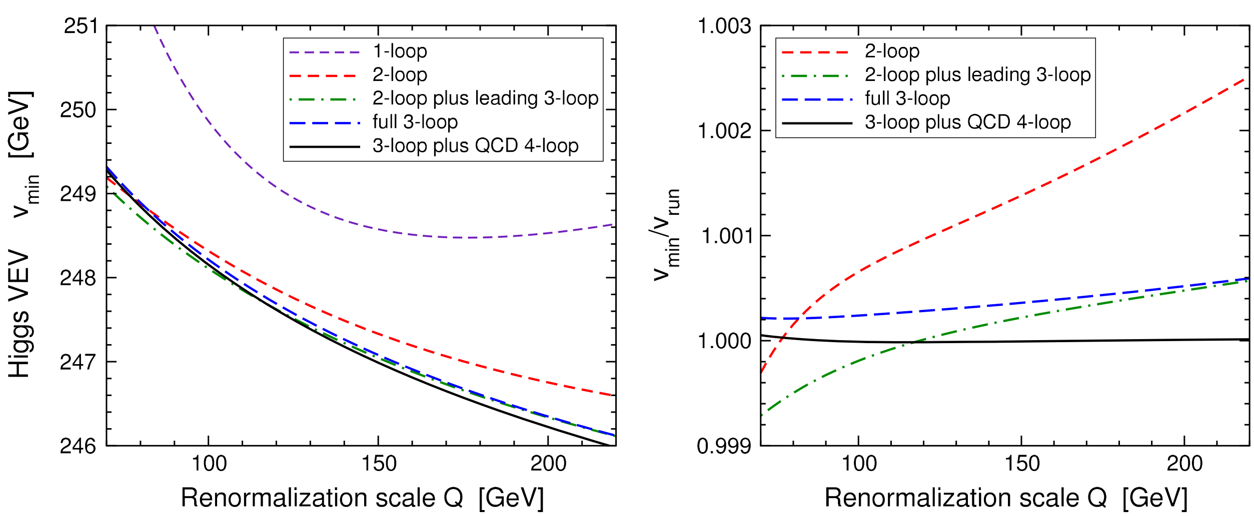

FIG. 4. The $\overline{\rm MS}$ Higgs VEV, as a function of the renormalization scale Q, for the reference model point defined at $Q_0 = 173.1$ GeV in Eq. (1.11). The other input parameters, including $m^2(Q)$, are obtained from the reference model by evolving them using their RG equations to the scale Q, where the Landau gauge effective potential is minimized to obtain $v(Q)_{\rm min}$. In the left panel, results are shown for the one-loop, two-loop, two-loop plus leading three-loop, full three-loop, and three-loop plus QCD four-loop approximations to the effective potential minimization condition. The right panel shows the results for $v(Q)_{\rm min}/v(Q)_{\rm run}$, where $v(Q)_{\rm run}$ is obtained directly by renormalization running from the reference scale $Q_0 = 173.1$ GeV.

$$t = y_t^2 v^2/2, \qquad b = y_b^2 v^2/2, \qquad \tau = y_\tau^2 v^2/2, \tag{4.5}$$

are the running $\overline{\mathrm{MS}}$ squared masses, and

$$A(x) = x\overline{\ln}(x) - x \tag{4.6}$$

with

$$\overline{\ln}(x) = \ln(x/Q^2). \tag{4.7}$$

The two-loop part is

$$\Delta\tilde{r}^{(2)} = g_3^2 y_t^2 [8\zeta_2 - 17 - 16A(t)/t - 12A(t)^2/t^2] + \Delta\tilde{r}^{(2)}_{\text{non-QCD}}, \tag{4.8}$$

where $\Delta\tilde{r}^{(2)}_{\text{non-QCD}}$ is again rather lengthy, and so is provided in its complete form as an ancillary file `Deltartilde.txt` distributed with this paper, rather than in text form here. The ancillary file `Deltartilde.txt` [125] contains the complete form of

$$\Delta\tilde{r}^{(2)}_{\text{non-QCD}} = \sum_j C_j^{(2)} I_j^{(2)} + \sum_{j\leq k} C_{j,k}^{(1,1)} I_j^{(1)} I_k^{(1)} + \sum_j C_j^{(1)} I_j^{(1)} + C^{(0)}, \tag{4.9}$$

where the lists of two-loop and one-loop basis integrals required are

$$I^{(2)} = \{\zeta_2, I(h,h,h), I(h,t,t), I(0,h,t), I(0,h,W), I(0,h,Z), I(0,t,W), I(0,t,Z), I(0,W,Z), I(h,h,W), I(h,W,W), I(h,W,Z), I(h,Z,Z), I(t,t,W), I(t,t,Z), I(W,W,W), I(W,W,Z), I(W,Z,Z)\}, \tag{4.10}$$

$$I^{(1)} = \{A(t), A(h), A(Z), A(W)\}, \tag{4.11}$$

with the two-loop vacuum integral function $I(x,y,z)$ as defined as in previous papers, e.g., [26,126,127], and the coefficients $C_j^{(2)}$, $C_{j,k}^{(1,1)}$, $C_j^{(1)}$, and $C^{(0)}$ are rational functions of t, h, Z, W, and v. (The v dependence is $1/v^4$ in each case.) The Goldstone boson contributions in $\Delta\tilde{r}$ have been resummed, so that, as explained in Refs. [14,17], the Higgs squared mass appearing here is $h \equiv 2\lambda v^2$, and not $m^2 + 3\lambda v^2$. Also, note that $\Delta\tilde{r}^{(1)}$ is well defined in the formal limits $W \to Z$, $W \to h$, and $b \to t$, despite denominators that vanish in those limits. Furthermore, although $\Delta\tilde{r}^{(2)}$ has several individual terms with λ in the denominator, one can check that the whole expression for $\Delta\tilde{r}$ is finite in the limit $\lambda \to 0$, unlike $\Delta\bar{r}$. This illustrates the absence of $1/\lambda$ effects in the tadpole-free scheme based on v; more generally, the absence of $1/\lambda$ effects provides useful checks on calculations. We have also checked that $\Delta\tilde{r}^{(2)}$ is well defined in the formal limits where $Z - 4t$, $h - W$, $W - Z$, $h - 4Z$ and $h - 4W$ vanish, despite many of the individual coefficients having denominators containing factors of these quantities. Furthermore, we have checked that $G_F = (1 + \Delta\tilde{r})/\sqrt{2}v^2$ is RG scale invariant through two-loop order, as required by its status as a physical observable. In doing this check, we have used the form of $\Delta\tilde{r}$ described above, in which G and m^2 are both completely eliminated by the Goldstone boson resummation by using Eqs. (1.2) and (1.3), and then the running with Q is computed in terms of the remaining parameters on which $\Delta\tilde{r}$ depends, namely λ, v, y_t, g_3, g, and g', using their beta functions as well as the Q dependence of the loop integral functions.

This numerical result for G_F in terms of the $\overline{\mathrm{MS}}$ quantities is shown in Fig. 5 for the benchmark reference model as a function of the scale Q at which it is computed. The scale variation is less than 1 part in 10^{-4} for Q between 100 and 220 GeV. By default, the SMDR code evaluates G_F at $Q = M_t$, and so the benchmark point there agrees exactly with the experimental value. The results can also be compared to those of formulas relating G_F to M_W given by Degrassi, Gambino, and Giardino in Ref. [53], which is larger by a fraction of about 0.0002 (or 0.0001), provided that Q in our calculation is taken to be close to M_t (or M_Z). This corresponds to a difference in the physical W-boson mass of about 8 MeV (or 4 MeV), less than the current experimental uncertainty in M_W. A further reduction in the purely theoretical sources of uncertainty in our approach

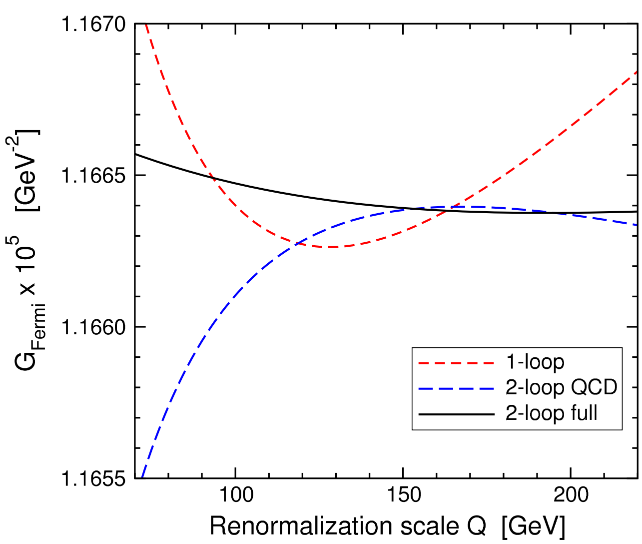

FIG. 5. The Fermi constant G_F, as a function of the renormalization scale Q at which it is computed from the $\overline{\mathrm{MS}}$ input parameters, for the reference model point defined at $Q_0 = 173.1$ GeV in Eq. (1.11). The short-dashed, long-dashed, and solid lines show the results of including the one-loop, one-loop plus two-loop QCD, and full two-loop contributions, respectively.

could come about from including the leading (in g_3 and y_t) three-loop contributions to G_F, M_Z, and M_W. There appear to be no technical obstacles to performing these calculations; when they become available, they will be included in the SMDR code.

V. PHYSICAL MASSES OF HEAVY PARTICLES

For the case of the benchmark reference model defined in Eq. (1.11), we show the pole masses of t and h and the Breit-Wigner masses of W and Z in various approximations, as a function of the renormalization scale Q used for the computation, in Fig. 6. The results shown are obtained using SMDR, which implements the formulas found in Refs. [71,72,76,77] for the tadpole-free pure $\overline{\text{MS}}$ scheme. These papers make use of the TSIL software library in order to numerically evaluate the required two-loop self-energy basis integrals, using the differential equations method as described in [127], and analytical special cases found in Refs. [56,63,127–136].

In the case of the Higgs boson pole mass, the Q dependence is seen to be of order several tens of MeV in Fig. 6, for the best available approximation, which includes the full two-loop and leading (in g_3 and y_t) three-loop contributions. However, as we argued in Ref. [72], in the specific case of M_h, a renormalization scale close to $Q = 160$ GeV should be made in order to minimize the error from other three-loop contributions, and this choice is used by default in SMDR.

In the case of the top-quark pole mass, in Fig. 6 we start with the known four-loop pure QCD approximation. Although other works often treat the top-quark pole mass using only QCD effects, the neglect of electroweak corrections is certainly not justified. Indeed, the four-loop

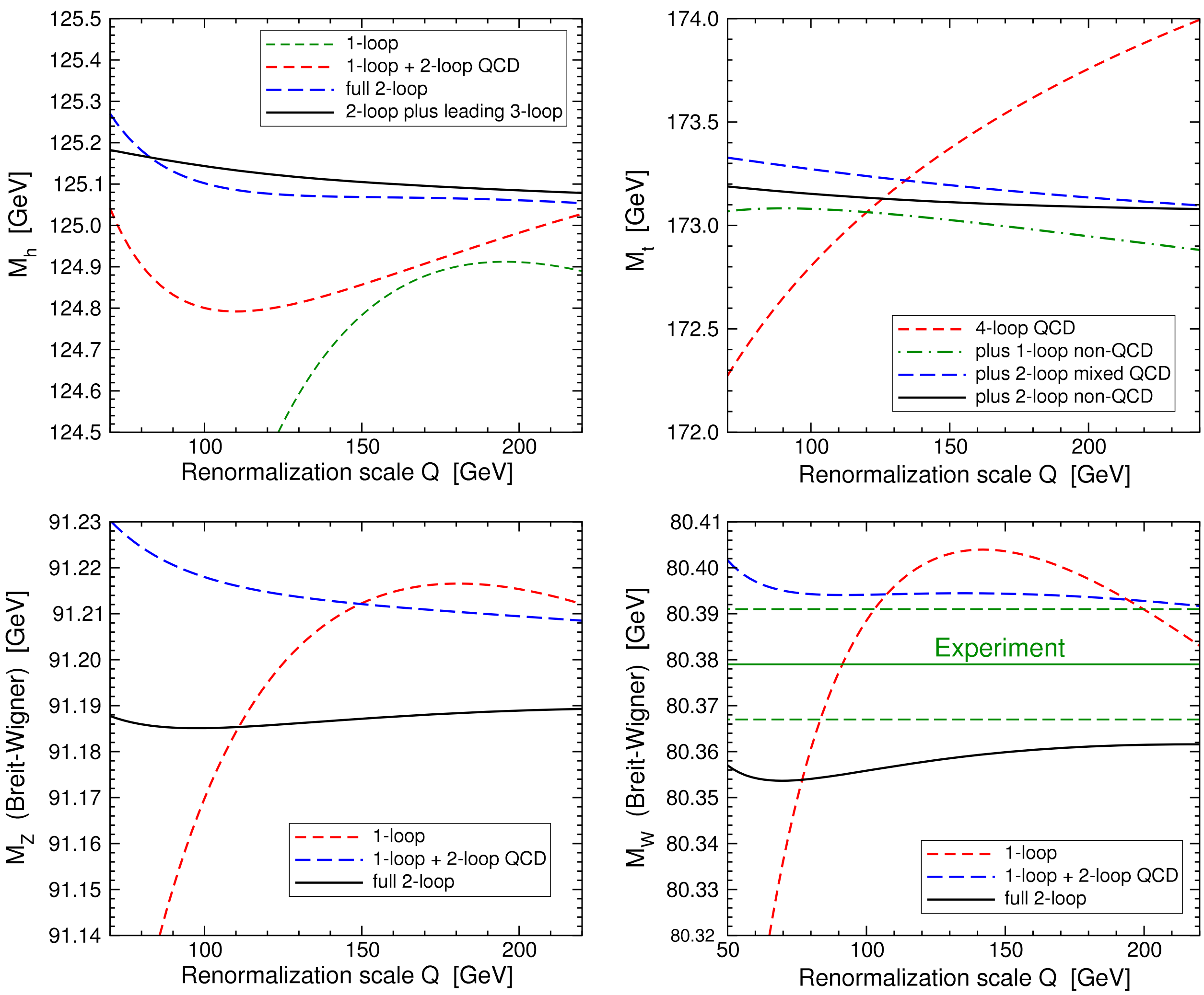

FIG. 6. Physical masses of the Higgs boson, top quark, Z boson, and W boson, as functions of the renormalization scale Q at which they are computed, in various approximations as labeled. The $\overline{\text{MS}}$ input parameters at Q are determined by RG evolution from the reference model point defined at $Q_0 = 173.1$ GeV in Eq. (1.11). In the case of M_W, we also show the present experimental central (horizontal solid line) and $\pm 1\sigma$ (horizontal dashed lines) values.

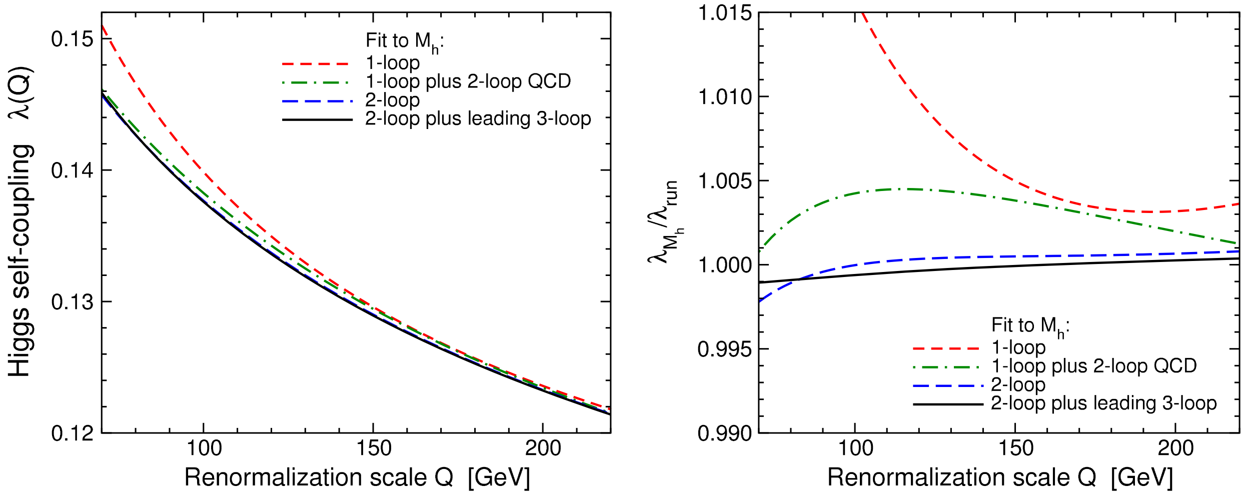

FIG. 7. The $\overline{\rm MS}$ Higgs self-coupling λ, as a function of the renormalization scale Q, for the reference model point defined at $Q_0 = 173.1$ GeV in Eq. (1.11). The other input parameters are obtained from the reference model by evolving them using their RG equations to the scale Q, where $\lambda(Q)$ is then obtained by requiring the Higgs pole mass to be 125.10 GeV. In the left panel, results are shown when the calculation of M_h is done in the one-loop, one-loop plus two-loop QCD, full two-loop, and two-loop plus leading three-loop approximations. The right panel shows the results for $\lambda(Q)_{M_h}/\lambda(Q)_{\rm run}$, where $\lambda(Q)_{M_h}$ is determined as in the left panel, and $\lambda(Q)_{\rm run}$ is obtained directly by renormalization running from the reference scale $Q_0 = 173.1$ GeV.

pure QCD approximation is seen to have a very large scale dependence of about 1.7 GeV as Q is varied from 70 GeV to 200 GeV. This shows that failing to include the electroweak contributions at one-loop order contributes a very large and scale-dependent error, although this is obscured if one also neglects the corresponding non-QCD contributions in the renormalization group running of the parameters. Even the two-loop mixed QCD/electroweak and non-QCD effects are roughly of order 200 MeV and 100 MeV, and scale dependent. By default, the SMDR code uses a scale choice $Q = M_t$ when computing M_t, but this can be changed by the user, as for example when making Fig. 6.

The lower two panels of Fig. 6 show the dependences of the Breit-Wigner M_Z and M_W on the scale Q at which they are computed, based on the full two-loop calculations in Refs. [76,77]. The Q dependences are seen to be greatly reduced by the inclusion of the two-loop contributions, as expected. The reference model shown was chosen to reproduce the experimental value of M_Z, for $Q = 160$ GeV. The result for M_W is then a prediction, since it was not used at all in the determination of the model parameters in Eq. (1.11). Note that the range of values obtained in Fig. 6 is lower than the current world average from the Review of Particle Properties in Ref. [49], which is $M_W = 80.379 \pm 0.012$ GeV. This reflects the well-known observation that the predicted central value of M_W in the Standard Model is somewhat lower than the observed range, but not by enough to draw any firm conclusions about the validity of the minimal Standard Model. (There is a long history of calculation of higher-loop contributions [32,66,129,137–156] to the ρ parameter, which gives the W boson mass in terms of the Z boson mass and other on-shell parameters.) By default, SMDR uses a choice $Q = 160$ GeV when computing both the Z and the W physical masses, but these choices can again be modified independently by the user at run-time, as of course was done when making Fig. 6.

The information from the Higgs boson mass M_h can be inverted to obtain the self-coupling λ, assuming the minimal Standard Model. This is illustrated in the left panel of Fig. 7 where we compute $\lambda(Q)$ at the renormalization scale Q by requiring it to give $M_h = 125.10$ GeV, using various approximations for the calculation of the latter. In the right panel, we then show the ratio of the value λ_{M_h} obtained in this way to the value $\lambda_{\rm run}$ obtained by RG running it from the value in the reference model at $Q_0 = 173.1$ GeV. This ratio is exactly 1 by construction at $Q = Q_0$ in the approximation used to define the reference model. In this approximation, the ratio remains less than 1 part in 10^4 over the entire range shown for Q. The parameters $\lambda(Q)$ and $m^2(Q)$ can also be run up to very high scales using the RG equations. These results are shown in Fig. 8, including the central value fit as well as the envelopes resulting from varying each of M_h, M_t, and α_S independently within their one-sigma and two-sigma experimentally allowed ranges. As is now well known (see e.g., Refs. [73–75,157] and references therein), in the best-fit case with M_h near 125 GeV, $\lambda(Q)$ runs negative at a scale intermediate between the weak scale and the Planck mass, indicating that our vacuum state may be quasistable if one makes the bold assumption that there is really no new physics all the way up to mass scales comparable to the scale Q where $\lambda(Q) < 0$.

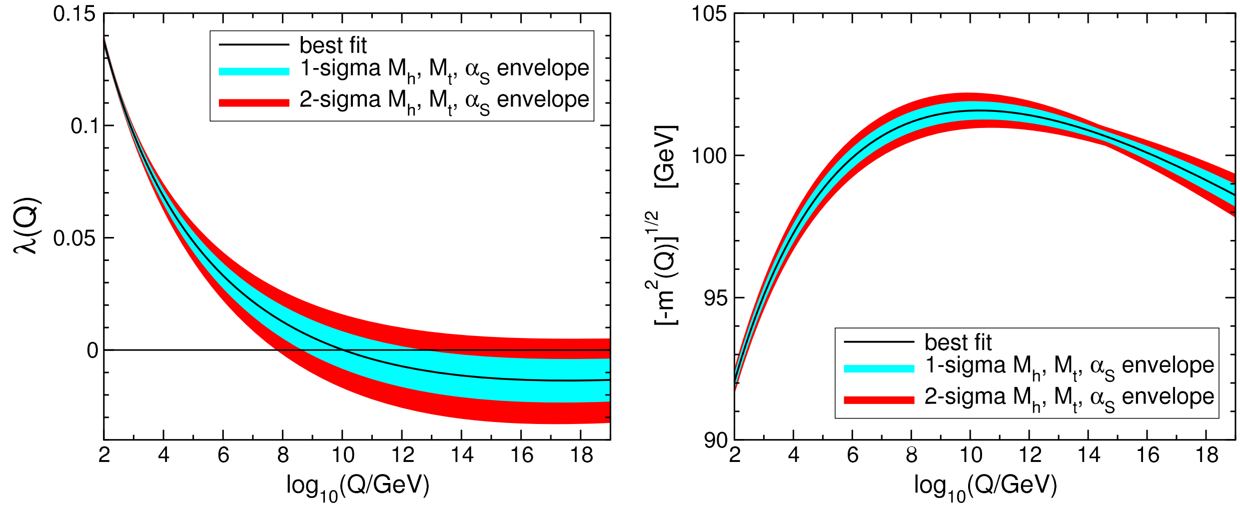

FIG. 8. Renormalization group running of the $\overline{\text{MS}}$ Higgs potential parameters λ and $\sqrt{-m^2}$, as a function of the renormalization scale Q. The black lines are the central values obtained from present experimental inputs. Also shown are the envelopes obtained by varying M_t, M_h, and $\alpha_S^{(5)}(M_Z)$ within one sigma (blue shaded region) and two sigma (red shaded region) of their central values. The slight "pinch" in the envelopes in the right panel near $Q = 10^{14}$ GeV is due to a focusing behavior of the α_S dependence of the $m^2(Q)$ renormalization group equation.

VI. THE SMDR CODE

As noted above, we have collected our results and methods in the form of a public software library written in C, which can be used interactively or incorporated into other software, and which is modular enough to be easily modified and updated.[4] A full description of how to use SMDR, and some example programs, are included with the distribution, which is available for download at [158]. For comprehensive information, we refer the reader to the file `README.txt`. In this section we give only a brief listing of some of the more common user interface variables and functions available. Note that these always begin with SMDR to avoid naming conflicts with user code.

(i) The input values of Q and the $\overline{\text{MS}}$ parameters in Eq. (1.4) are specified by global variables SMDR_Q_in, SMDR_v_in, SMDR_lambda_in, SMDR_g3_in, etc. These can be set or adjusted by the user at any time, but typically remain fixed as multiple different tasks are performed, with corresponding temporary global variables SMDR_Q, SMDR_v, SMDR_lambda, SMDR_g3, etc., used for renormalization group running to various other scales Q and subsequent individual calculations.

(ii) Renormalization group running in the full, nondecoupled theory is done with the function SMDR_RGeval_SM(). In the decoupled QCD + QED theory with five quarks and three charged leptons, the evaluation of running parameters (with simultaneous decoupling of t, h, Z, W at a scale of choice) is done by SMDR_RGeval_QCDQED_53(). Similarly, evaluation of running parameters at lower scales, including the sequential decoupling of the bottom quark, the tau lepton, and the charm quark, is done by SMDR_RGeval_QCDQED_43(), SMDR_RGeval_QCDQED_42(), and SMDR_RGeval_QCDQED_32(), respectively, where (5,3) and (4,3) and (4,2) and (3,2) refer to the numbers of active quarks and leptons.

(iii) Minimization of the effective potential to find $m^2(Q)$ from $v(Q)$, or vice versa, are accomplished with functions SMDR_Eval_m2() or SMDR_Eval_vev(), respectively. These make use of the quantity $\Delta = \sum_n \Delta_n/(16\pi^2)^n$ appearing in Eq. (1.3), which can also be computed separately with SMDR_Eval_vevDelta().

(iv) Evaluation of the complex pole masses of the four heavy particles is done with functions SMDR_Eval_Mt(), SMDR_Eval_Mh(), SMDR_Eval_MZ(), and SMDR_Eval_MW(). The last two functions also evaluate the variable-width Breit-Wigner masses of Z and W, which are the traditional ways of reporting those masses. In each case, one can specify the scale Q at which the computation is performed.

(v) Evaluation of the Fermi decay constant is done with the function SMDR_Eval_GFermi(), again with the computation performed at any specified choice of Q.

(vi) The function SMDR_Eval_Gauge() evaluates the Sommerfeld fine structure constant α_0. It also simultaneously computes the RPP "$\overline{\text{MS}}$" scheme (with only

[4]The code SMDR subsumes and replaces our earlier program SMH, which evaluated only the Higgs pole mass and was described in Ref. [72].

the top quark decoupled) values $\hat{\alpha}(Q)$ and $\hat{s}^2_W(Q)$. To the best of our knowledge, the relations of the latter two quantities to the corresponding nondecoupled $\overline{\rm MS}$ quantities are only known at two-loop order to leading order in QCD. We have implemented them in SMDR using the results of Eqs. (B.3) and (B.4) of Ref. [50].

(vii) The light quark $\overline{\rm MS}$ masses $m_b(m_b)$, $m_c(m_c)$, $m_s(2\ \mathrm{GeV})$, $m_d(2\ \mathrm{GeV})$, and $m_u(2\ \mathrm{GeV})$ are evaluated using SMDR_Eval_mbmb(), SMDR_Eval_mcmc(), and SMDR_Eval_mquarks_2 GeV().

(viii) The charged lepton physical masses can be evaluated using SMDR_Eval_Mtau_pole(), SMDR_Eval_Mmuon_pole(), and SMDR_Eval_Melectron_pole().

(ix) A function SMDR_Fit_Inputs() performs a simultaneous fit to all of the $\overline{\rm MS}$ quantities in Eq. (1.4), for specified values of the on-shell observable quantities (except for M_W) in Eq. (1.5), providing the results at a specified choice of Q.

(x) Various utility functions exist for reading parameters from and writing to electronic files.

(xi) Our programs TSIL [126] for two-loop self-energy integrals and 3VIL [26] for three-loop vacuum integrals are included within the SMDR distribution, and so need not be downloaded separately.

(xii) Interfaces for calling SMDR from external C or C++ code are included.

(xiii) A command-line program calc_all takes the $\overline{\rm MS}$ inputs of Eq. (1.4) and outputs all of the on-shell observables of Eq. (1.5).

(xiv) Another command-line program calc_fit takes the on-shell observables of Eq. (1.5) as inputs, and outputs the results of a fit to the $\overline{\rm MS}$ inputs of Eq. (1.4), by using the function SMDR_Fit_Inputs() mentioned above. This was used to obtain Eq. (1.11).

As examples, the short C programs that produced all of the data used in the figures in this paper are included within the SMDR distribution. We also include several other command line programs. These should serve to illustrate how to incorporate SMDR into new programs.

VII. OUTLOOK

In this paper, we have studied the map between the $\overline{\rm MS}$ Lagrangian parameters of the Standard Model and the observables to which they most closely correspond. In doing so, we have assumed that the minimal Standard Model is really the correct theory up to some high mass scale, so that new physics contributions effectively decouple. With the present absence of evidence at the LHC for new physics, this is at least a tenable hypothesis and plausibly will remain so for quite some time. We therefore suggest that in the future the Review of Particle Properties should provide the best-fit values of the $\overline{\rm MS}$ Lagrangian parameters of the Standard Model in the nondecoupled theory, since these fundamentally define the best model that we have to describe particle physics.

Another useful software package with rather similar aims to SMDR but a different implementation (including expansion around what we call $v_{\rm tree}$ rather than v) is mr [124]. There is also a very large number of works that test the whole space of electroweak precision observables in different ways; for an incomplete set of recent references and reviews on this approach, see Refs. [159–168]. We emphasize that our primary goal here, of obtaining the best fit to the $\overline{\rm MS}$ Lagrangian parameters, is different and complementary to that of testing the whole space of electroweak precision observables, as we are not considering possible non-negligible contributions from physics beyond the Standard Model. However, one application is to the matching to new physics models (for example, supersymmetry) characterized by some mass scale much larger than the electroweak scale. This will necessitate a matching between the high-energy theory and the Standard Model as an effective field theory, including with nonrenormalizable operators. For a very incomplete sample of recent works on this subject, see Refs. [169–184].

New theoretical refinements as well as more accurate experimental measurements will certainly come. We have therefore chosen a modular framework in which it should be straightforward to incorporate such new developments into the SMDR code. For example, we have avoided using numerical interpolating formulas from approximate fits to analytic formulas, instead opting to provide and use analytical calculations directly, up to the level of loop integrals that must then be evaluated numerically. This of course results in longer computation times, but is more transparent and easier to update. Most of the results presented in this paper are based on calculations that have appeared before, but we have provided for the first time to our knowledge a study of the impact of the three-loop contributions to the effective potential on the relation between the loop-corrected VEV and the other Lagrangian parameters. We have also provided (in Sec. IV and an ancillary file, as well as in the SMDR code) the full two-loop relation between the loop-corrected VEV and the Fermi constant, as an alternative to the relation between G_F and the tree-level VEV that was found in Refs. [69,70,124]. It is clear that significant advances will be needed in order to match the accuracy that can be obtained at proposed future e^+e^- colliders; for a recent review, see Ref. [168]. Future work in the tadpole-free pure $\overline{\rm MS}$ scheme will likely include the leading three-loop corrections to M_W, M_Z, and G_F. These and $\Delta\alpha^{(5)}_{\rm had}(M_Z)$ and M_t are the present bottlenecks to accuracy.

ACKNOWLEDGMENTS

We thank James Wells for helpful comments. This work was supported in part by the National Science Foundation Grant No. PHY-1719273. D. G. R. is supported by a grant from the Ohio Supercomputer Center.

[1] C. G. Bollini and J. J. Giambiagi, Dimensional renormalization: The number of dimensions as a regularizing parameter, Nuovo Cimento B **12**, 20 (1972); Lowest order divergent graphs in nu-dimensional space, Phys. Lett. B **40**, 566 (1972).

[2] J. F. Ashmore, A method of gauge invariant regularization, Lett. Nuovo Cimento **4**, 289 (1972).

[3] G. M. Cicuta and E. Montaldi, Analytic renormalization via continuous space dimension, Lett. Nuovo Cimento **4**, 329 (1972).

[4] G. 't Hooft and M. J. G. Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. **B44**, 189 (1972).

[5] G. 't Hooft, Dimensional regularization and the renormalization group, Nucl. Phys. **B61**, 455 (1973).

[6] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Deep inelastic scattering beyond the leading order in asymptotically free gauge theories, Phys. Rev. D **18**, 3998 (1978).

[7] E. Braaten and J. P. Leveille, Minimal subtraction and momentum subtraction in QCD at two loop order, Phys. Rev. D **24**, 1369 (1981).

[8] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, Phys. Rev. D **7**, 1888 (1973).

[9] R. Jackiw, Functional evaluation of the effective potential, Phys. Rev. D **9**, 1686 (1974).

[10] M. Sher, Electroweak Higgs potentials and vacuum stability, Phys. Rep. **179**, 273 (1989), and references therein.

[11] C. Ford, I. Jack, and D. R. T. Jones, The Standard Model effective potential at two loops, Nucl. Phys. **B387**, 373 (1992); Erratum, Nucl. Phys. **B504**, 551(E) (1997); See also C. Ford and D. R. T. Jones, The effective potential and the differential equations method for Feynman integrals, Phys. Lett. B **274**, 409 (1992); Erratum, Phys. Lett. B**285**, 399 (1992).

[12] S. P. Martin, Two loop effective potential for a general renormalizable theory and softly broken supersymmetry, Phys. Rev. D **65**, 116003 (2002).

[13] S. P. Martin, Three-loop Standard Model effective potential at leading order in strong and top Yukawa couplings, Phys. Rev. D **89**, 013003 (2014).

[14] S. P. Martin, Effective potential at three loops, Phys. Rev. D **96**, 096005 (2017).

[15] S. P. Martin, Four-loop Standard Model effective potential at leading order in QCD, Phys. Rev. D **92**, 054029 (2015).

[16] S. P. Martin and H. H. Patel, Two-loop effective potential for generalized gauge fixing, Phys. Rev. D **98**, 076008 (2018).

[17] S. P. Martin, Taming the Goldstone contributions to the effective potential, Phys. Rev. D **90**, 016013 (2014).

[18] J. Elias-Miro, J. R. Espinosa, and T. Konstandin, Taming infrared divergences in the effective potential, J. High Energy Phys. 08 (2014) 034.

[19] A. Pilaftsis and D. Teresi, Symmetry improved 2PI effective action and the infrared divergences of the Standard Model, J. Phys. Conf. Ser. **631**, 012008 (2015).

[20] A. Pilaftsis and D. Teresi, Symmetry-improved 2PI approach to the Goldstone-Boson IR problem of the SM effective potential, Nucl. Phys. **B906**, 381 (2016).

[21] N. Kumar and S. P. Martin, Resummation of Goldstone boson contributions to the MSSM effective potential, Phys. Rev. D **94**, 014013 (2016).

[22] J. R. Espinosa, M. Garny, and T. Konstandin, Interplay of infrared divergences and gauge-dependence of the effective potential, Phys. Rev. D **94**, 055026 (2016).

[23] J. Braathen and M. D. Goodsell, Avoiding the Goldstone boson catastrophe in general renormalisable field theories at two loops, J. High Energy Phys. 12 (2016) 056.

[24] A. Pilaftsis and D. Teresi, Exact RG invariance and symmetry improved 2PI effective potential, Nucl. Phys. **B920**, 298 (2017).

[25] J. Braathen, M. D. Goodsell, and F. Staub, Supersymmetric and non-supersymmetric models without catastrophic Goldstone bosons, Eur. Phys. J. C **77**, 757 (2017).

[26] S. P. Martin and D. G. Robertson, Evaluation of the general 3-loop vacuum Feynman integral, Phys. Rev. D **95**, 016008 (2017). The 3-loop Vacuum Integral Library (3VIL) is available at http://www.niu.edu/spmartin/3VIL/ or http://faculty.otterbein.edu/DRobertson/3VIL/.

[27] K. G. Chetyrkin and F. V. Tkachov, Integration by parts: The algorithm to calculate beta functions in 4 loops, Nucl. Phys. **B192**, 159 (1981); F. V. Tkachov, A theorem on analytical calculability of four loop renormalization group functions, Phys. Lett. B **100**, 65 (1981).

[28] A. I. Davydychev and J. B. Tausk, Two loop selfenergy diagrams with different masses and the momentum expansion, Nucl. Phys. **B397**, 123 (1993); A. I. Davydychev, V. A. Smirnov, and J. B. Tausk, Large momentum expansion of two loop selfenergy diagrams with arbitrary masses, Nucl. Phys. **B410**, 325 (1993); F. A. Berends and J. B. Tausk, On the numerical evaluation of scalar two loop selfenergy diagrams, Nucl. Phys. **B421**, 456 (1994).

[29] M. Caffo, H. Czyz, S. Laporta, and E. Remiddi, The master differential equations for the two loop sunrise selfmass amplitudes, Nuovo Cimento A **111**, 365 (1998).

[30] J. R. Espinosa and R. J. Zhang, Complete two loop dominant corrections to the mass of the lightest CP even Higgs boson in the minimal supersymmetric standard model, Nucl. Phys. **B586**, 3 (2000).

[31] D. J. Broadhurst, Three loop on-shell charge renormalization without integration: Lambda-MS (QED) to four loops, Z. Phys. C **54**, 599 (1992).

[32] L. Avdeev, J. Fleischer, S. Mikhailov, and O. Tarasov, $O(\alpha\alpha_s^2)$ correction to the electroweak rho parameter, Phys. Lett. B **336**, 560 (1994); Erratum, **349**, 597 (1995).

[33] J. Fleischer and O. V. Tarasov, Application of conformal mapping and Padé approximants $(\omega P's)$ to the calculation of various two-loop Feynman diagrams, Nucl. Phys. B, Proc. Suppl. **37B**, 115 (1994).

[34] L. V. Avdeev, Recurrence relations for three loop prototypes of bubble diagrams with a mass, Comput. Phys. Commun. **98**, 15 (1996).

[35] D. J. Broadhurst, Massive three-loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity, Eur. Phys. J. C **8**, 311 (1999).

[36] J. Fleischer and M. Y. Kalmykov, Single mass scale diagrams: Construction of a basis for the epsilon expansion, Phys. Lett. B **470**, 168 (1999).

[37] Y. Schröder and A. Vuorinen, High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles, J. High Energy Phys. 06 (2005) 051.

[38] A. I. Davydychev and M. Y. Kalmykov, Massive Feynman diagrams and inverse binomial sums, Nucl. Phys. **B699**, 3 (2004).

[39] M. Y. Kalmykov, About higher order epsilon-expansion of some massive two- and three-loop master-integrals, Nucl. Phys. **B718**, 276 (2005).

[40] M. Y. Kalmykov, Gauss hypergeometric function: Reduction, epsilon-expansion for integer/half-integer parameters and Feynman diagrams, J. High Energy Phys. 04 (2006) 056.

[41] V. V. Bytev, M. Kalmykov, B. A. Kniehl, B. F. L. Ward, and S. A. Yost, Differential reduction algorithms for hypergeometric functions applied to Feynman diagram calculation, arXiv:0902.1352.

[42] S. Bekavac, A. G. Grozin, D. Seidel, and V. A. Smirnov, Three-loop on-shell Feynman integrals with two masses, Nucl. Phys. **B819**, 183 (2009).

[43] V. V. Bytev, M. Y. Kalmykov, and B. A. Kniehl, Differential reduction of generalized hypergeometric functions from Feynman diagrams: One-variable case, Nucl. Phys. **B836**, 129 (2010).

[44] V. V. Bytev, M. Y. Kalmykov, and B. A. Kniehl, HYPERDIRE, HYPERgeometric functions DIfferential REduction: MATHEMATICA-based packages for differential reduction of generalized hypergeometric functions ${}_pF_{p-1}$, F_1,F_2,F_3,F_4, Comput. Phys. Commun. **184**, 2332 (2013).

[45] J. Grigo, J. Hoff, P. Marquard, and M. Steinhauser, Moments of heavy quark correlators with two masses: Exact mass dependence to three loops, Nucl. Phys. **B864**, 580 (2012).

[46] M. Steinhauser, MATAD: A program package for the computation of MAssive TADpoles, Comput. Phys. Commun. **134**, 335 (2001).

[47] P. Burda, B. Kol, and R. Shir, The vacuum seagull: Evaluating a 3-loop Feynman diagram with 3 mass scales, Phys. Rev. D **96**, 125013 (2017).

[48] A. Freitas and A. Freitas, Three-loop vacuum integrals with arbitrary masses, J. High Energy Phys. 11 (2016) 145; S. Bauberger and A. Freitas, TVID: Three-loop vacuum integrals from dispersion relations, arXiv:1702.02996.

[49] M. Tanabashi *et al.* (Particle Data Group), Review of particle physics, Phys. Rev. D **98**, 030001 (2018), with the 2019 update at http://pdg.lbl.gov/.

[50] S. Fanchiotti, B. A. Kniehl, and A. Sirlin, Incorporation of QCD effects in basic corrections of the electroweak theory, Phys. Rev. D **48**, 307 (1993).

[51] J. Erler, Calculation of the QED coupling alpha (M(Z)) in the modified minimal subtraction scheme, Phys. Rev. D **59**, 054008 (1999).

[52] G. Degrassi and A. Vicini, Two loop renormalization of the electric charge in the Standard Model, Phys. Rev. D **69**, 073007 (2004).

[53] G. Degrassi, P. Gambino, and P. P. Giardino, The $m_W - m_Z$ interdependence in the Standard Model: A new scrutiny, J. High Energy Phys. 05 (2015) 154.

[54] S. P. Martin, Matching relations for decoupling in the Standard Model at two loops and beyond, Phys. Rev. D **99**, 033007 (2019).

[55] R. Tarrach, The pole mass in perturbative QCD, Nucl. Phys. **B183**, 384 (1981).

[56] N. Gray, D. J. Broadhurst, W. Grafe, and K. Schilcher, Three loop relation of quark (Modified) Ms and pole masses, Z. Phys. C **48**, 673 (1990).

[57] K. Melnikov and T. V. Ritbergen, The Three loop relation between the MS-bar and the pole quark masses, Phys. Lett. B **482**, 99 (2000).

[58] P. Marquard, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, Quark Mass Relations to Four-Loop Order in Perturbative QCD, Phys. Rev. Lett. **114**, 142002 (2015).

[59] P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser, and D. Wellmann, $\overline{\mathrm{MS}}$-on-shell quark mass relation up to four loops in QCD and a general $\mathrm{SU}(N)$ gauge group, Phys. Rev. D **94**, 074025 (2016).

[60] M. Bohm, H. Spiesberger, and W. Hollik, On the one loop renormalization of the electroweak Standard Model and its application to leptonic processes, Fortschr. Phys. **34**, 687 (1986).

[61] R. Hempfling and B. A. Kniehl, On the relation between the fermion pole mass and MS Yukawa coupling in the Standard Model, Phys. Rev. D **51**, 1386 (1995).

[62] F. Jegerlehner, M. Y. Kalmykov, and O. Veretin, MS-bar versus pole masses of gauge bosons. 2. Two loop electroweak fermion corrections, Nucl. Phys. **B658**, 49 (2003).

[63] F. Jegerlehner and M. Y. Kalmykov, O(alpha alpha(s)) correction to the pole mass of the t quark within the Standard Model, Nucl. Phys. **B676**, 365 (2004).

[64] D. Eiras and M. Steinhauser, Two-loop O(alpha alpha(s)) corrections to the on-shell fermion propagator in the Standard Model, J. High Energy Phys. 02 (2006) 010.

[65] F. Jegerlehner, M. Y. Kalmykov, and B. A. Kniehl, On the difference between the pole and the $\overline{\mathrm{MS}}$ masses of the top quark at the electroweak scale, Phys. Lett. B **722**, 123 (2013).

[66] M. Faisst, J. H. Kuhn, T. Seidensticker, and O. Veretin, Three loop top quark contributions to the rho parameter, Nucl. Phys. **B665**, 649 (2003).

[67] F. Jegerlehner and M. Y. Kalmykov, $O(\alpha\alpha S)$ relation between pole- and MS-bar mass of the t quark, Acta Phys. Pol. B **34**, 5335 (2003).

[68] M. Faisst, J. H. Kuhn, and O. Veretin, Pole versus MS mass definitions in the electroweak theory, Phys. Lett. B **589**, 35 (2004).

[69] B. A. Kniehl and O. L. Veretin, Two-loop electroweak threshold corrections to the bottom and top Yukawa couplings, Nucl. Phys. **B885**, 459 (2014); Erratum, Nucl. Phys. **B894**, 56(E) (2015).

[70] B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, Two-loop electroweak threshold corrections in the Standard Model, Nucl. Phys. **B896**, 19 (2015).

[71] S. P. Martin, Top-quark pole mass in the tadpole-free $\overline{\mathrm{MS}}$ scheme, Phys. Rev. D **93**, 094017 (2016).

[72] S. P. Martin and D. G. Robertson, Higgs boson mass in the Standard Model at two-loop order and beyond, Phys. Rev. D **90**, 073010 (2014).

[73] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl, and M. Shaposhnikov, Higgs Boson mass and new physics, J. High Energy Phys. 10 (2012) 140.

[74] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, and A. Strumia, Higgs mass and

vacuum stability in the Standard Model at NNLO, J. High Energy Phys. 08 (2012) 098.
[75] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio, and A. Strumia, Investigating the near-criticality of the Higgs boson, J. High Energy Phys. 12 (2013) 089.
[76] S. P. Martin, Pole mass of the W boson at two-loop order in the pure $\overline{\mathrm{MS}}$ scheme, Phys. Rev. D **91**, 114003 (2015).
[77] S. P. Martin, Z-Boson pole mass at two-loop order in the pure $\overline{\mathrm{MS}}$ scheme, Phys. Rev. D **92**, 014026 (2015).
[78] F. Jegerlehner, M. Y. Kalmykov, and O. Veretin, MS versus pole masses of gauge bosons: Electroweak bosonic two loop corrections, Nucl. Phys. **B641**, 285 (2002).
[79] D. Y. Bardin, A. Leike, T. Riemann, and M. Sachwitz, Energy dependent width effects in e^+e^- annihilation near the Z boson pole, Phys. Lett. B **206**, 539 (1988).
[80] S. Willenbrock and G. Valencia, On the definition of the Z boson mass, Phys. Lett. B **259**, 373 (1991).
[81] A. Sirlin, Theoretical Considerations Concerning the Z^0 Mass, Phys. Rev. Lett. **67**, 2127 (1991).
[82] R. G. Stuart, Gauge invariance, analyticity and physical observables at the Z^0 resonance, Phys. Lett. B **262**, 113 (1991).
[83] S. Bekavac, A. Grozin, D. Seidel, and M. Steinhauser, Light quark mass effects in the on-shell renormalization constants, J. High Energy Phys. 10 (2007) 006.
[84] M. E. Machacek and M. T. Vaughn, Two loop renormalization group equations in a general quantum field theory. 1. Wave function renormalization, Nucl. Phys. **B222**, 83 (1983).
[85] M. E. Machacek and M. T. Vaughn, Two loop renormalization group equations in a general quantum field theory. 2. Yukawa couplings, Nucl. Phys. **B236**, 221 (1984).
[86] I. Jack and H. Osborn, General background field calculations with Fermion fields, Nucl. Phys. **B249**, 472 (1985).
[87] M. E. Machacek and M. T. Vaughn, Two loop renormalization group equations in a general quantum field theory. 3. Scalar quartic couplings, Nucl. Phys. **B249**, 70 (1985).
[88] M. Luo and Y. Xiao, Two Loop Renormalization Group Equations in the Standard Model, Phys. Rev. Lett. **90**, 011601 (2003).
[89] O. V. Tarasov, Anomalous dimensions of quark masses in three loop approximation, preprint JINR-P2-82-900, 1982 (in Russian) (unpublished).
[90] L. N. Mihaila, J. Salomon, and M. Steinhauser, Gauge Coupling Beta Functions in the Standard Model to Three Loops, Phys. Rev. Lett. **108**, 151602 (2012).
[91] K. G. Chetyrkin and M. F. Zoller, Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model, J. High Energy Phys. 06 (2012) 033.
[92] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops, J. High Energy Phys. 01 (2013) 017.
[93] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, Yukawa coupling beta-functions in the Standard Model at three loops, Phys. Lett. B **722**, 336 (2013).
[94] K. G. Chetyrkin and M. F. Zoller, β-function for the Higgs self-interaction in the Standard Model at three-loop level, J. High Energy Phys. 04 (2013) 091.
[95] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, Higgs self-coupling beta-function in the Standard Model at three loops, Nucl. Phys. **B875**, 552 (2013).
[96] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, Three-loop Higgs self-coupling beta-function in the Standard Model with complex Yukawa matrices, Nucl. Phys. **B879**, 256 (2014).
[97] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, Three-loop SM beta-functions for matrix Yukawa couplings, Phys. Lett. B **737**, 129 (2014).
[98] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, The four loop beta function in quantum chromodynamics, Phys. Lett. B **400**, 379 (1997).
[99] M. Czakon, The four-loop QCD beta-function and anomalous dimensions, Nucl. Phys. **B710**, 485 (2005).
[100] A. V. Bednyakov and A. F. Pikelner, Four-loop strong coupling beta-function in the Standard Model, Phys. Lett. B **762**, 151 (2016); On the four-loop strong coupling beta-function in the SM, EPJ Web Conf. **125**, 04008 (2016).
[101] M. F. Zoller, Top-Yukawa effects on the β-function of the strong coupling in the SM at four-loop level, J. High Energy Phys. 02 (2016) 095.
[102] An ambiguity due to the treatment of γ_5 in the part of the 4-loop QCD gauge coupling beta function proportional to y_t^4 has been resolved by C. Poole and A. E. Thomsen, Weyl Consistency Conditions and γ_5, Phys. Rev. Lett. **123**, 041602 (2019).
[103] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, Five-Loop Running of the QCD Coupling Constant, Phys. Rev. Lett. **118**, 082002 (2017).
[104] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, The five-loop beta function of Yang-Mills theory with fermions, J. High Energy Phys. 02 (2017) 090.
[105] K. G. Chetyrkin, Quark mass anomalous dimension to $\mathcal{O}(\alpha_S^4)$, Phys. Lett. B **404**, 161 (1997).
[106] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, The four loop quark mass anomalous dimension and the invariant quark mass, Phys. Lett. B **405**, 327 (1997).
[107] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_S^5)$, J. High Energy Phys. 10 (2014) 076.
[108] K. G. Chetyrkin and M. F. Zoller, Leading QCD-induced four-loop contributions to the β-function of the Higgs self-coupling in the SM and vacuum stability, J. High Energy Phys. 06 (2016) 175.
[109] S. Weinberg, Effective gauge theories, Phys. Lett. **91B**, 51 (1980).
[110] B. A. Ovrut and H. J. Schnitzer, The decoupling theorem and minimal subtraction, Phys. Lett. **100B**, 403 (1981).
[111] W. Bernreuther and W. Wetzel, Decoupling of heavy quarks in the minimal subtraction scheme, Nucl. Phys. **B197**, 228 (1982); Erratum, Nucl. Phys. **B513**, 758(E) (1998).
[112] S. A. Larin, T. van Ritbergen, and J. A. M. Vermaseren, The large quark mass expansion of $\Gamma(Z^0 \to \text{hadrons})$ and $\Gamma(\tau^- \to \nu_\tau + \text{hadrons})$ in the order α_s^3, Nucl. Phys. **B438**, 278 (1995).

[113] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Decoupling relations to $O(\alpha_s^3)$ and their connection to low-energy theorems, Nucl. Phys. **B510**, 61 (1998).

[114] A. G. Grozin, M. Hoeschele, J. Hoff, M. Steinhauser, M. Hoschele, J. Hoff, and M. Steinhauser, Simultaneous decoupling of bottom and charm quarks, J. High Energy Phys. 09 (2011) 066.

[115] Y. Schroder and M. Steinhauser, Four-loop decoupling relations for the strong coupling, J. High Energy Phys. 01 (2006) 051.

[116] K. G. Chetyrkin, J. H. Kuhn, and C. Sturm, QCD decoupling at four loops, Nucl. Phys. **B744**, 121 (2006).

[117] A. V. Bednyakov, On the electroweak contribution to the matching of the strong coupling constant in the SM, Phys. Lett. B **741**, 262 (2015).

[118] T. Liu and M. Steinhauser, Decoupling of heavy quarks at four loops and effective Higgs-fermion coupling, Phys. Lett. B **746**, 330 (2015).

[119] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser, RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses, Comput. Phys. Commun. **133**, 43 (2000).

[120] B. Schmidt and M. Steinhauser, CRunDec: A C++ package for running and decoupling of the strong coupling and quark masses, Comput. Phys. Commun. **183**, 1845 (2012).

[121] F. Herren and M. Steinhauser, Version 3 of RunDec and CRunDec, Comput. Phys. Commun. **224**, 333 (2018).

[122] B. A. Kniehl, J. H. Piclum, and M. Steinhauser, Relation between bottom-quark MS-bar Yukawa coupling and pole mass, Nucl. Phys. **B695**, 199 (2004).

[123] A. V. Bednyakov, B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, On the b-quark running mass in QCD and the SM, Nucl. Phys. **B916**, 463 (2017).

[124] B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, mr: a C++ library for the matching and running of the Standard Model parameters, Comput. Phys. Commun. **206**, 84 (2016).

[125] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.100.073004 for the complete form of Eq. (4.9).

[126] S. P. Martin and D. G. Robertson, TSIL: A program for the calculation of two-loop self-energy integrals, Comput. Phys. Commun. **174**, 133 (2006). The Two-loop Self-energy Integral Library (TSIL) is available at http://www.niu.edu/spmartin/TSIL/ or http://faculty.otterbein.edu/DRobertson/TSIL/.

[127] S. P. Martin, Evaluation of two loop selfenergy basis integrals using differential equations, Phys. Rev. D **68**, 075002 (2003).

[128] D. J. Broadhurst, The master two loop diagram with masses, Z. Phys. C **47**, 115 (1990).

[129] A. Djouadi, O(alpha alpha-s) vacuum polarization Functions of the Standard Model gauge bosons, Nuovo Cimento A **100**, 357 (1988).

[130] R. Scharf and J. B. Tausk, Scalar two loop integrals for gauge boson selfenergy diagrams with a massless fermion loop, Nucl. Phys. **B412**, 523 (1994).

[131] F. A. Berends and J. B. Tausk, On the numerical evaluation of scalar two loop selfenergy diagrams, Nucl. Phys. **B421**, 456 (1994).

[132] F. A. Berends, A. I. Davydychev, and N. I. Ussyukina, Threshold and pseudothreshold values of the sunset diagram, Phys. Lett. B **426**, 95 (1998).

[133] J. Fleischer, F. Jegerlehner, O. V. Tarasov, and O. L. Veretin, Two loop QCD corrections of the massive fermion propagator, Nucl. Phys. **B539**, 671 (1999); Erratum, Nucl. Phys. **B571**, 511(E) (2000).

[134] J. Fleischer, A. V. Kotikov, and O. L. Veretin, Analytic two loop results for selfenergy type and vertex type diagrams with one nonzero mass, Nucl. Phys. **B547**, 343 (1999).

[135] A. I. Davydychev and A. G. Grozin, Effect of m(c) on b quark chromomagnetic interaction and on-shell two loop integrals with two masses, Phys. Rev. D **59**, 054023 (1999).

[136] S. P. Martin, Two loop scalar self energies in a general renormalizable theory at leading order in gauge couplings, Phys. Rev. D **70**, 016005 (2004).

[137] J. J. van der Bij and F. Hoogeveen, Two loop correction to weak interaction parameters due to a heavy fermion doublet, Nucl. Phys. **B283**, 477 (1987).

[138] A. Djouadi and C. Verzegnassi, Virtual very heavy top effects in LEP/SLC precision measurements, Phys. Lett. B **195**, 265 (1987).

[139] B. A. Kniehl, Two loop corrections to the vacuum polarizations in perturbative QCD, Nucl. Phys. **B347**, 86 (1990).

[140] F. Halzen and B. A. Kniehl, Δ r beyond one loop, Nucl. Phys. **B353**, 567 (1991).

[141] R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci, and A. Vicere, Radiative correction effects of a very heavy top, Phys. Lett. B **288**, 95 (1992); Erratum, Phys. Lett. B **312**, 511 (1993).

[142] A. Djouadi and P. Gambino, Electroweak gauge bosons selfenergies: Complete QCD corrections, Phys. Rev. D **49**, 3499 (1994); QCD corrections to Higgs boson self-energies and fermionic decay widths, **51**, 218 (1995).

[143] J. Fleischer, O. V. Tarasov, and F. Jegerlehner, Two loop heavy top corrections to the rho parameter: A simple formula valid for arbitrary Higgs mass, Phys. Lett. B **319**, 249 (1993).

[144] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser, Corrections of order $\mathcal{O}(G_F M_t^2 \alpha_s^2)$ to the ρ parameter, Phys. Lett. B **351**, 331 (1995).

[145] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser, QCD Corrections from Top Quark to Relations between Electroweak Parameters to Order Alpha-s**2, Phys. Rev. Lett. **75**, 3394 (1995).

[146] G. Degrassi, P. Gambino, and A. Vicini, Two loop heavy top effects on the $m(Z)$–$m(W)$ interdependence, Phys. Lett. B **383**, 219 (1996).

[147] A. Freitas, W. Hollik, W. Walter, and G. Weiglein, Complete fermionic two loop results for the $M(W)$–$M(Z)$ interdependence, Phys. Lett. B **495**, 338 (2000); Erratum, Phys. Lett. B **570**, 265(E) (2003).

[148] J. J. van der Bij, K. G. Chetyrkin, M. Faisst, G. Jikia, and T. Seidensticker, Three loop leading top mass contributions to the rho parameter, Phys. Lett. B **498**, 156 (2001).

[149] A. Freitas, W. Hollik, W. Walter, and G. Weiglein, Electroweak two loop corrections to the $M_W - M_Z$ mass correlation in the standard model, Nucl. Phys. **B632**, 189 (2002); Erratum, Nucl. Phys. **B666**, 305(E) (2003).

[150] M. Awramik and M. Czakon, Complete Two Loop Bosonic Contributions to the Muon Lifetime in the Standard Model, Phys. Rev. Lett. **89**, 241801 (2002).
[151] A. Onishchenko and O. Veretin, Two loop bosonic electroweak corrections to the muon lifetime and $M(Z)$–$M(W)$ interdependence, Phys. Lett. B **551**, 111 (2003).
[152] M. Awramik and M. Czakon, Complete two loop electroweak contributions to the muon lifetime in the standard model, Phys. Lett. B **568**, 48 (2003).
[153] M. Awramik, M. Czakon, A. Freitas, and G. Weiglein, Precise prediction for the W boson mass in the Standard Model, Phys. Rev. D **69**, 053006 (2004).
[154] Y. Schroder and M. Steinhauser, Four-loop singlet contribution to the rho parameter, Phys. Lett. B **622**, 124 (2005).
[155] K. G. Chetyrkin, M. Faisst, J. H. Kuhn, P. Maierhofer, and C. Sturm, Four-Loop QCD Corrections to the Rho Parameter, Phys. Rev. Lett. **97**, 102003 (2006).
[156] R. Boughezal and M. Czakon, Single scale tadpoles and O(G(F m(t)**2 alpha(s)**3)) corrections to the rho parameter, Nucl. Phys. **B755**, 221 (2006).
[157] J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto, and A. Strumia, Higgs mass implications on the stability of the electroweak vacuum, Phys. Lett. B **709**, 222 (2012).
[158] The SMDR code can be downloaded from http://www.niu.edu/spmartin/SMDR/ or http://faculty.otterbein.edu/DRobertson/SMDR/.
[159] D. Y. Bardin, P. Christova, M. Jack, L. Kalinovskaya, A. Olchevski, S. Riemann, and T. Riemann, ZFITTER v.6.21: A semianalytical program for fermion pair production in e+ e- annihilation, Comput. Phys. Commun. **133**, 229 (2001); A. B. Arbuzov, M. Awramik, M. Czakon, A. Freitas, M. W. Grunewald, K. Monig, S. Riemann, and T. Riemann, ZFITTER: A semi-analytical program for fermion pair production in e+ e- annihilation, from version 6.21 to version 6.42, Comput. Phys. Commun. **174**, 728 (2006); A. Akhundov, A. Arbuzov, S. Riemann, and T. Riemann, The ZFITTER project, Phys. Part. Nucl. **45**, 529 (2014).
[160] J. Erler and S. Su, The weak neutral current, Prog. Part. Nucl. Phys. **71**, 119 (2013).
[161] M. Ciuchini, E. Franco, S. Mishima, and L. Silvestrini, Electroweak Precision Observables, New physics and the nature of a 126 GeV Higgs boson, J. High Energy Phys. 08 (2013) 106.
[162] J. D. Wells and Z. Zhang, Precision electroweak analysis after the Higgs boson discovery, Phys. Rev. D **90**, 033006 (2014).
[163] M. Baak, J. Cúth, J. Haller, A. Hoecker, R. Kogler, K. Mönig, M. Schott, and J. Stelzer (Gfitter Group), The global electroweak fit at NNLO and prospects for the LHC and ILC, Eur. Phys. J. C **74**, 3046 (2014).
[164] A. Freitas, Numerical multi-loop integrals and applications, Prog. Part. Nucl. Phys. **90**, 201 (2016).
[165] J. de Blas, M. Ciuchini, E. Franco, S. Mishima, M. Pierini, L. Reina, and L. Silvestrini, Electroweak precision observables and Higgs-boson signal strengths in the Standard Model and beyond: Present and future, J. High Energy Phys. 12 (2016) 135.
[166] J. Haller, A. Hoecker, R. Kogler, K. Mönig, T. Peiffer, and J. Stelzer, Update of the global electroweak fit and constraints on two-Higgs-doublet models, Eur. Phys. J. C **78**, 675 (2018).
[167] J. Erler and M. Schott, Electroweak precision tests of the Standard Model after the discovery of the Higgs boson, Prog. Part. Nucl. Phys. **106**, 68 (2019).
[168] A. Freitas *et al.*, Theoretical uncertainties for electroweak and Higgs-boson precision measurements at FCC-ee, arXiv:1906.05379.
[169] W. Buchmuller and D. Wyler, Effective Lagrangian analysis of new interactions and flavor conservation, Nucl. Phys. **B268**, 621 (1986).
[170] Z. Han and W. Skiba, Effective theory analysis of precision electroweak data, Phys. Rev. D **71**, 075009 (2005).
[171] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, Dimension-six terms in the Standard Model Lagrangian, J. High Energy Phys. 10 (2010) 085.
[172] J. Elias-Miro, J. R. Espinosa, E. Masso, and A. Pomarol, Higgs windows to new physics through $d = 6$ operators: constraints and one-loop anomalous dimensions, J. High Energy Phys. 11 (2013) 066.
[173] A. Pomarol and F. Riva, Towards the ultimate SM fit to close in on Higgs physics, J. High Energy Phys. 01 (2014) 151.
[174] C. Y. Chen, S. Dawson, and C. Zhang, Electroweak effective operators and Higgs physics, Phys. Rev. D **89**, 015016 (2014).
[175] J. Elias-Miro, C. Grojean, R. S. Gupta, and D. Marzocca, Scaling and tuning of EW and Higgs observables, J. High Energy Phys. 05 (2014) 019.
[176] J. Ellis, V. Sanz, and T. You, Complete Higgs sector constraints on dimension-6 operators, J. High Energy Phys. 07 (2014) 036.
[177] A. Falkowski and F. Riva, Model-independent precision constraints on dimension-6 operators, J. High Energy Phys. 02 (2015) 039.
[178] B. Henning, X. Lu, and H. Murayama, How to use the Standard Model effective field theory, J. High Energy Phys. 01 (2016) 023.
[179] J. D. Wells and Z. Zhang, Effective theories of universal theories, J. High Energy Phys. 01 (2016) 123; Renormalization group evolution of the universal theories EFT, J. High Energy Phys. 06 (2016) 122.
[180] A. Drozd, J. Ellis, J. Quevillon, and T. You, The universal one-loop effective action, J. High Energy Phys. 03 (2016) 180.
[181] S. A. R. Ellis, J. Quevillon, T. You, and Z. Zhang, Extending the universal one-loop effective action: Heavy-light coefficients, J. High Energy Phys. 08 (2017) 054.
[182] Z. Zhang, Covariant diagrams for one-loop matching, J. High Energy Phys. 05 (2017) 152.
[183] J. D. Wells and Z. Zhang, Effective field theory approach to trans-TeV supersymmetry: Covariant matching, Yukawa unification and Higgs couplings, J. High Energy Phys. 05 (2018) 182.
[184] B. Summ and A. Voigt, Extending the universal one-loop effective action by regularization scheme translating operators, J. High Energy Phys. 08 (2018) 026.