Supersymmetric Cremmer-Scherk theory

Hitoshi Nishino* and Subhash Rajpoot†
Department of Physics & Astronomy, California State University, 1250 Bellflower Boulevard, Long Beach, California 90840, USA

(Received 21 April 2019; published 20 September 2019)

We present a formulation for $N = 1$ supersymmetric Cremmer-Scherk theory in $3 + 1$ dimensions. Originally, Cremmer and Scherk presented the purely bosonic Abelian gauge-symmetry breaking with the field content of an Abelian vector A_μ and a second-rank tensor $B_{\mu\nu}$. In our present paper, we perform both the non-Abelian generalization and the supersymmetrization of Cremmer-Scherk theory. Our field content is the Yang-Mills multiplet $({A_\mu}^I, \lambda^I)$ and the non-Abelian antisymmetric tensor multiplet $({B_{\mu\nu}}^I, \chi^I, \phi^I)$, where χ^I is a Majorana spinor, and ϕ^I has spin 0^-, with the adjoint index $I = 1, 2, \ldots, \dim G$ of a non-Abelian group G. As a preliminary step, we establish supersymmetric Proca-Stückelberg formulation with the field content $({A_\mu}^I, \lambda^I)$ and $(\varphi^I, \chi^I, \phi^I)$, where φ^I (or ϕ^I) serves as the coordinates (or an adjoint representation) of the non-Abelian gauge group G. In the Abelian limit, our system is equivalent to the system with the gauged R symmetry of the chiral multiplet (φ, χ, ϕ) which has a superspace action. We next perform the duality transformation from φ^I to ${B_{\mu\nu}}^I$ to reach supersymmetric Cremmer-Scherk theory. Unlike similar formulations in the past, both of our ${B_{\mu\nu}}^I$ and ${A_\mu}^I$ fields are physical and propagating, as the most nontrivial and new ingredient in our model. Our super-Proca-Stückelberg formulation provides an important foundation of our super-Cremmer-Scherk formulation, different from conventional chiral-superfield formulations.

DOI: 10.1103/PhysRevD.100.066021

I. INTRODUCTION

There has been considerable development in building consistent interactions of the so-called "non-Abelian tensor" fields [1]. How to establish consistent theories for non-Abelian tensors had been a persistent problem, due to inconsistencies such as noninvariance of their field strengths. This problem has been recently resolved by "tensor hierarchy" formulations [1]. The key feature is that the conventional field strength of a non-Abelian tensor should be modified by a generalized Chern-Simons term. For example, the naive conventional field strength ${G^{(0)}_{\mu\nu\rho}}^I$ for a second-rank non-Abelian tensor ${B_{\mu\nu}}^I$ is ${G^{(0)}_{\mu\nu\rho}}^I \equiv 3D_{[\mu}B_{\nu\rho]}{}^I$ [2]. In tensor-hierarchy formulations [1,2], this is modified to, e.g.,[1]

$$ {G_{\mu\nu\rho}}^I \equiv 3D_{[\mu}B_{\nu\rho]}{}^I + f^{IJK}F_{[\mu\nu}{}^J C_{\rho]}{}^K \tag{1.1} $$

by the new Chern-Simons term $f^{IJK}F^J \wedge C^K$, where ${C_\mu}^I$ is a new extra vector field, and ${F_{\mu\nu}}^I \equiv 2\partial_{[\mu}A_{\nu]}{}^I + gf^{IJK}{A_\mu}^J{A_\nu}^K$ is the non-Abelian Yang-Mills (YM) field strength. Accordingly, ${C_\mu}^I$ has its proper field strength with another Chern-Simons term:

$$ {H_{\mu\nu}}^I \equiv 2D_{[\mu}C_{\nu]}{}^I + g{B_{\mu\nu}}^I. \tag{1.2} $$

One can confirm the invariances of the field strengths G and H under the tensorial transformations with the parameters β and γ, and YM-gauge transformation with the parameter α:

$$ \delta_\alpha({B_{\mu\nu}}^I, {C_\mu}^I, {A_\mu}^I) = (-f^{IJK}\alpha^J{B_{\mu\nu}}^K, -f^{IJK}\alpha^J{C_\mu}^K, +D_\mu\alpha^I), \tag{1.3a} $$

$$ \delta_\beta({B_{\mu\nu}}^I, {C_\mu}^I, {A_\mu}^I) = (+2D_{[\mu}\beta_{\nu]}{}^I, -g{\beta_\mu}^I, 0), \tag{1.3b} $$

$$ \delta_\gamma({B_{\mu\nu}}^I, {C_\mu}^I, {A_\mu}^I) = (-f^{IJK}{F_{\mu\nu}}^J\gamma^K, D_\mu\gamma^I, 0). \tag{1.3c} $$

These systematic results strongly indicate many potential applications of tensor-hierarchy formulations [1,2], such as *nonsupersymmetric* applications to higher dimensions [3]. Examples with *supersymmetrization* are such as the supersymmetrization [4,5] of Proca-Stückelberg formulations [6], or the supersymmetrization [7] of the Jackiw-Pi model [8], and even the *supersymmetric* composite models [9].

In view of these successful results, we expect many more applications. Cremmer-Scherk formulation [10] is one such example, because it contains a tensor field $B_{\mu\nu}$. The original Cremmer-Scherk theory [10] was designed for spontaneous dynamical breaking of $U(1)$ symmetry, which was different from the then-known Proca-Stückelberg [6]

*h.nishino@csulb.edu
†subhash.rajpoot@csulb.edu

[1] We use the indices $I, J, \ldots = 1, 2, \ldots, \dim G$ for a gauge group G. Accordingly, f^{IJK} is the structure constant of G.

or the Nambu-Goldstone-Higgs (NGH) mechanism [11], and is also referred to as "topologically massive gauge theory." The original motivation of [10] was to study gauge breakings in dual resonance theories [12]. As such, there are potential applications to the theories of superstring [13], supermembrane [14], or extended objects [15]. Motivated by these developments, we try in our present paper to accomplish the *"non-Abelianization"* and supersymmetrization of Cremmer-Scherk theory [10].

The field content of the original Cremmer-Scherk theory [10] was the Abelian vector A_μ and a second-rank antisymmetric tensor $B_{\mu\nu}$ with the basic Lagrangian[2]

$$\mathcal{L}_{\rm CS} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{12}(G_{\mu\nu\rho})^2 - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma} \tag{1.4a}$$

$$\overset{\nabla}{=} -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{12}(G_{\mu\nu\rho})^2 + \frac{1}{6}m\epsilon^{\mu\nu\rho\sigma}G_{\mu\nu\rho}A_\sigma, \tag{1.4b}$$

with $F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}$ and $G_{\mu\nu\rho} \equiv 3\partial_{[\mu}B_{\nu\rho]}$. In terms of the Hodge-dual field strength $N_\mu = (1/6)\epsilon_\mu{}^{\nu\rho\sigma}G_{\nu\rho\sigma}$, Eq. (1.4b) is recast into

$$\mathcal{L}'_{\rm CS} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(N_\mu)^2 + mN_\mu A^\mu + \Lambda(\partial_\mu N^\mu). \tag{1.5}$$

In (1.5), the field N_μ is a *fundamental independent* field, so that we resort to a Lagrange-multiplier Λ to *force* the Bianchi "identity" $\partial_\mu N^\mu = 0$ as a *constraint*. In modern language, this process is the so-called "duality transformation" [16]. The field equation of N_μ is *algebraic*: $N_\mu \doteq -mA_\mu + \partial_\mu \Lambda$,[3] which enables us to eliminate N_μ from (1.5) to reach the Lagrangian

$$\mathcal{L}''_{\rm CS} = -\frac{1}{4}(F'_{\mu\nu})^2 - \frac{1}{2}m^2(A'_\mu)^2, \tag{1.6}$$

with $A'_\mu \equiv A_\mu - m^{-1}\partial_\mu\Lambda$. The Lagrangian (1.6) is nothing but that for a *massive* Abelian vector A'_μ. This Cremmer-Scherk formulation [10] is an alternative way of breaking gauge symmetry, different from the conventional NGH mechanism [11]. Notice also that the massive component of A_μ is still *physical* with its *propagating* modes.

There were further works similar to the generalization of the original Cremmer-Scherk theory [10] to non-Abelian gauge groups in [17,18] with the field content $(A_\mu{}^I, B_{\mu\nu}{}^I)$ or $(A_\mu{}^I, \varphi^I)$, where $B_{\mu\nu}{}^I$ and φ^I are Hodge dual to each other. This series of formulations is sometimes referred to as "scalar-tensor theory." Also the supersymmetrizations of such systems were performed in [19–21]. Even though these works have similarity to the supersymmetrization of [10], they are not quite the same, because their YM field becomes auxiliary, lacking its propagating degrees of freedom as a vector field. For example, in the supersymmetrization in [21] based on [18], the purely bosonic field content in [18] is $(A_\mu{}^I, B_{\mu\nu}{}^I, C_\mu{}^I)$, where $A_\mu{}^I$ is the standard YM-gauge vector, while $C_\mu{}^I$ is an *extra* vector. The essential part of the Lagrangian of [18] in our notation is

$$\mathcal{L}_{\rm HK} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I F_{\rho\sigma}{}^I - \frac{1}{4}(H_{\mu\nu}{}^I)^2 - \frac{1}{2}m^2(A_\mu{}^I)^2 \tag{1.7}$$

with the conventional YM field strength: $F_{\mu\nu}{}^I \equiv 2\partial_{[\mu}A_{\nu]}{}^I + mf^{IJK}A_\mu{}^J A_\nu{}^K$, and H as the field strength of the extra vector field $C_\mu{}^I$:

$$H_{\mu\nu}{}^I \equiv 2D_{[\mu}C_{\nu]}{}^I \equiv 2\partial_{[\mu}C_{\nu]}{}^I + 2mf^{IJK}A_{[\mu}{}^J C_{\nu]}{}^K. \tag{1.8}$$

Due to the *topological* $B \wedge F$ term in (1.7) [22], the $B_\mu{}^I$-field equation yields the *pure-gauge* condition $F_{\mu\nu}{}^I \doteq 0 \Rightarrow A_\mu{}^I \doteq m^{-1}\partial_\mu\varphi^I + \mathcal{O}(\Phi^2)$,[4] resulting in the σ-model kinetic term $-(1/2)(\partial_\mu\varphi^I)^2 + \mathcal{O}(\Phi^3)$ from the $-(1/2)m^2(A_\mu{}^I)^2$ term. In other words, the YM fields in [18,21] are simply *auxiliary* lacking their kinetic terms. Because of this, we do not regard [19,21] as a genuine supersymmetrization of (1.4) or [10].

Additionally, as stressed in non-Abelian tensor-hierarchy analyses [1,2], the $B_{\mu\nu}{}^I$ kinetic term needs special care for consistent interactions, when the YM field strength is not vanishing, as in the non-Abelian generalization of [10].

In our present paper, we perform a genuine supersymmetrization of the Cremmer-Scherk theory [10], in which all YM fields have kinetic terms. In particular, we maintain the physical propagation of both $A_\mu{}^I$ and $B_{\mu\nu}{}^I$, which are the most crucial parts of the theory. Our YM field strength will not end up with the pure-gauge equation $F_{\mu\nu}{}^I \doteq 0$ as in [17]. We first consider a new $N = 1$ supersymmetric Proca-Stückelberg formulation with the field content of the YM multiplet $(A_\mu{}^I, \lambda^I)$ and a chiral multiplet (CM) $(\varphi^I, \chi^I, \phi^I)$, where φ^I are the coordinates of the group manifold G. This formulation is more economical than our previous similar formulations [4,5]. Since the scalar φ^I are the coordinates of the group manifold G, transforming under G differently from ϕ^I, conventional global R symmetry in the CM [23,24] is lost in the non-Abelian case.

We next perform a duality transformation [16] from the scalars φ^I to their Hodge-dual $B_{\mu\nu}{}^I$. Thus the resulting

[2]We use the symbol $\overset{\nabla}{=}$ for an equality up to a surface term, and our metric is $(\eta_{\mu\nu}) = \text{diag.}(-,+,+,+)$. The constant m has the dimension of mass, serving also as a minimal-coupling constant. Accordingly, the engineering dimensions for bosons (or fermions) is 0 (or 1/2).

[3]We use the symbol $\doteq$ for a field equation, distinguished from a simple equality.

[4]We use the symbol $\mathcal{O}(\Phi^n)$ for the nth order in terms of fundamental fields.

system is nothing but the $N = 1$ supersymmetrization of the Cremmer-Scherk theory [10].

As will be seen, our supersymmetric Cremmer-Scherk formulation has *nonpolynomial* structure, that has not been well covered in general non-Abelian tensor-hierarchy formulations [1,2]. The nonpolynomial feature itself, though, is not new. In fact, the aforementioned papers [17,19–21] have dealt with nonpolynomial interactions. However, Ref. [17] did not perform the supersymmetric generalization of such a system in four dimensions (4D). Also, in [19–21] the tensor multiplet is shown to be equivalent to the supersymmetric nonlinear σ model in terms of CM, in which scalars and pseudoscalars are described by a chiral superfield. In contrast, our pseudoscalar ϕ^I in a CM $(\varphi^I, \chi^I, \phi^I)$ transforms differently from the coordinate scalar φ^I. This provides the additional difference between our result and [19–21].

In the next section, we establish the non-Abelian generalization of Proca-Stückelberg theory, as the basis of our objective. In Sec. III, we give the $N = 1$ supersymmetric non-Abelian Proca-Stückelberg theory. In Sec. IV, we perform a duality transformation [16] from φ^I to its Hodge-dual $B_{\mu\nu}{}^I$. After this procedure, we reach $N = 1$ supersymmetric Cremmer-Scherk theory. Concluding remarks will be given in Sec. V. In the Appendix, we give the superspace [25] reconfirmation of supersymmetric Proca-Stückelberg theory.

II. NON-ABELIANIZATION OF PROCA-STÜCKELBERG THEORY AND CREMMER-SCHERK THEORY

Before considering supersymmetry, we perform the non-Abelian generalization of Cremmer-Scherk formulation [10], i.e., non-Abelian generalization of (1.4). Compared with [17], the difference is that our Lagrangian keeps the kinetic term of the YM field, while in [17] the YM field strength vanishes, yielding the *pure-gauge* configuration such as $A_\mu{}^I = \partial_\mu \varphi^I + \mathcal{O}(\Phi^2)$, used for the σ model.

As stated in the Introduction, we first review the purely bosonic non-Abelian Proca-Stückelberg formulation with the field content $(A_\mu{}^I, \varphi^I)$ [4,5]. The original papers on Proca-Stückelberg formulations [6] were only for the Abelian group. The basic formulation for purely bosonic non-Abelian Proca-Stückelberg theory has been already presented in our past two papers [4,5]. However, since there is subtlety with the non-Abelian case, we recapitulate the details for completeness.

The field content for non-Abelian Proca-Stückelberg theory consists of the scalar φ^I for the coordinates of the group manifold G [26], and the YM gauge field $A_\mu{}^I$. Under an infinitesimal YM-gauge-transformation δ_T with the parameter α^I, these fields transform as

$$\delta_T e^{\varphi} = -m\alpha e^{\varphi}, \qquad \delta_\alpha e^{-\varphi} = +m e^{-\varphi}\alpha, \tag{2.1a}$$

$$\delta_T A_\mu{}^I = D_\mu \alpha^I \equiv \partial_\mu \alpha^I + m f^{IJK} A_\mu{}^J \alpha^K, \tag{2.1b}$$

In (2.1a), the adjoint index I is suppressed for $e^\varphi \equiv \exp(\varphi^I T^I)$ and $\alpha \equiv \alpha^I T^I$, where T^I are the generators of G. Note that the right side of the first equation in (2.1a) is not the commutator $-[\alpha, e^\varphi]$, but $-\alpha e^\varphi$, because φ^I are the coordinates of the group manifold G. Because of the transformation (2.1a), the "field strength" for φ defined by [4,5]

$$P_\mu{}^I \equiv [(\partial_\mu e^\varphi) e^{-\varphi}]^I + m A_\mu{}^I, \tag{2.2}$$

transforms covariantly:

$$\delta_T P_\mu{}^I = -m f^{IJK} \alpha^J P_\mu{}^K. \tag{2.3}$$

The important property of the field strength P is its Bianchi identity (BId):[5]

$$D_{[\mu} P_{\nu]}{}^I \equiv \frac{1}{2} m F_{\mu\nu}{}^I + \frac{1}{2} f^{IJK} P_\mu{}^J P_\nu{}^K. \tag{2.4}$$

As a corollary, we have the gauge-transformation rule, and the covariant-derivative rule:

$$\delta_T (D_\mu e^\varphi) = -m\alpha (D_\mu e^\varphi), \tag{2.5a}$$

$$D_\mu (D_\nu e^\varphi) \equiv \partial_\mu (D_\nu e^\varphi) + m A_\mu (D_\nu e^\varphi) \tag{2.5b}$$

$$[D_\mu, D_\nu] e^\varphi = m F_{\mu\nu} e^\varphi, \tag{2.5c}$$

where the adjoint indices are suppressed. These equations are relevant to superspace formulation in the Appendix.

Because of the covariant transformation property (2.3) of $P_\mu{}^I$, we can consider the Proca-Stückelberg Lagrangian[6]

$$\mathcal{L}_{\rm PS} = -\frac{1}{4} (F_{\mu\nu}{}^I)^2 - \frac{1}{2} (P_\mu{}^I)^2 \tag{2.6}$$

invariant under non-Abelian gauge transformation: $\delta_T \mathcal{L}_{\rm PS} = 0$.

Our next step is to perform the duality transformation [16]: $\varphi^I \to B_{\mu\nu}{}^I$, namely, from the field strength $P_\mu{}^I$ to its Hodge dual $G_{\mu\nu\rho}{}^I$. To do this, we first replace $P_\mu{}^I$ by a new fundamental field $Q_\mu{}^I$, and add a constraint Lagrangian so that the B-field equation implies the BId (2.4) like $D_{[\rho} Q^I_{\sigma]} - (1/2) m F_{\rho\sigma}{}^I - (1/2) f^{IJK} Q_\rho{}^J Q_\sigma{}^K \doteq 0$ [16]. The total Lagrangian now is

[5]Equation (2.4) is equivalent to a Maurer-Cartan equation, when $F_{\mu\nu}{}^I = 0$.

[6]Except for the field strength $\mathcal{F}_{\mu\nu}{}^I$ in [4,5] replaced by $F_{\mu\nu}{}^I$, (2.6) agrees with the nonsupersymmetric $(A_\mu{}^I, \varphi^I)$ subsector of the supersymmetric Proca-Stückelberg Lagrangians in [4,5].

$$\mathcal{L}'_{\rm PS} = -\frac{1}{4}(F_{\mu\nu}{}^I)^2 - \frac{1}{2}(Q_\mu{}^I)^2 + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I\left(D_\rho Q_\sigma{}^I - \frac{1}{2}mF_{\rho\sigma}{}^I - \frac{1}{2}f^{IJK}Q_\rho{}^J Q_\sigma{}^K\right). \tag{2.7}$$

Note that the multiplier field $B_{\mu\nu}{}^I$ has its proper gauge transformation

$$\delta_\beta B_{\mu\nu}{}^I = 2D_{[\mu}\beta_{\nu]}{}^I, \tag{2.8}$$

which leaves the action $I'_{\rm PS}$ for (2.7) invariant. The confirmation $\delta_\beta I'_{\rm PS} = 0$ needs the Jacobi identity $f^{IJ[K|}f^{J|LM]} \equiv 0$.

The original Lagrange-multiplier field $B_{\mu\nu}{}^I$ is now a dynamical field as usual [16], via the algebraic Q-field equation from (2.7):

$$\Pi_\mu{}^{\nu IJ}Q_\nu{}^J \equiv \left[\delta_\mu{}^\nu\delta^{IJ} + \frac{1}{2}\epsilon_\mu{}^{\nu\rho\sigma}f^{IJK}B_{\rho\sigma}{}^K\right]Q_\nu{}^J \doteq \tilde{G}_\mu{}^{(0)I}, \tag{2.9}$$

where $\tilde{G}_\mu^{(0)\,I} \equiv (1/6)\epsilon_\mu{}^{\rho\sigma\tau}G_{\rho\sigma\tau}^{(0)\,I}$, and $G_{\mu\nu\rho}^{(0)\,I} \equiv 3D_{[\mu}B_{\nu\rho]}{}^I$, while $\Pi_\mu{}^{\nu IJ}$ stands for the square-bracket part in (2.9). Obviously, the Q-field equation is nonpolynomial, due to the involvement of the B term inside $\Pi_\mu{}^{\nu IJ}$. For simplicity, we solve (2.9) for Q up to $\mathcal{O}(\Phi^3)$ terms:

$$\begin{aligned} Q_\mu{}^I &\doteq (\Pi^{-1})_\mu{}^{\nu IJ}\tilde{G}_\nu^{(0)J} \\ &= \left[\delta_\mu{}^\nu\delta^{IJ} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^{IJK}B_{\rho\sigma}{}^K\right]\tilde{G}_\nu^{(0)J} + \mathcal{O}(\Phi^3) \\ &= \tilde{G}_\mu{}^I, \end{aligned} \tag{2.10a}$$

$$G_{\mu\nu\rho}{}^I \equiv G_{\mu\nu\rho}^{(0)\,I} - 3f^{IJK}\tilde{G}_{[\mu}^{(0)J}B_{\nu\rho]}{}^K + \mathcal{O}(\Phi^3), \tag{2.10b}$$

where $\tilde{G}_\mu{}^I \equiv (1/6)\epsilon_\mu{}^{\rho\sigma\tau}G_{\rho\sigma\tau}{}^I$. Following the usual duality-transformation procedure [16], we substitute (2.10a) back into (2.7) to yield our non-Abelian Cremmer-Scherk Lagrangian:

$$\mathcal{L}_{\rm NACS} = -\frac{1}{4}(F_{\mu\nu}{}^I)^2 - \frac{1}{12}(G_{\mu\nu\rho}{}^I)^2 - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I F_{\rho\sigma}{}^I + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^{IJK}B_{\mu\nu}{}^I\tilde{G}_\rho{}^J\tilde{G}_\sigma{}^K + \mathcal{O}(\Phi^4). \tag{2.11}$$

Since we keep only $\mathcal{O}(\Phi^1)$, $\mathcal{O}(\Phi^2)$ and $\mathcal{O}(\Phi^3)$ terms in the Lagrangian, the definition of $G_{\mu\nu\rho}{}^I$ in (2.10b) does not need $\mathcal{O}(\Phi^3)$. Needless to say, (2.11) is reduced to the Abelian case (1.4a), when $f^{IJK} = 0$.

There were some works in the past for non-Abelianization of Cremmer-Scherk theory [10], such as [27]. However, in those theories, the field strength for $B_{\mu\nu}{}^I$ is like

$$G'_{\mu\nu\rho}{}^I \equiv 3D_{[\mu}B_{\nu\rho]}{}^I + 3f^{IJK}F_{[\mu\nu}{}^J C_{\rho]}{}^K, \tag{2.12}$$

introducing the new extra vector field $C_\mu{}^I$. This is nothing but one of the general prescriptions in tensor-hierarchy formulations [1,2]. The drawback of such formulations [27] is that the field equation of the new vector $C_\mu{}^I$ yields an undesirable condition $f^{IJK}F_{\nu\rho}{}^J G'_\mu{}^{\nu\rho K} \doteq 0$, due to the lack of the $C_\mu{}^I$-kinetic term. Therefore, such formulations do not really serve as the genuine non-Abelianization of Cremmer-Scherk theory [10].

Note that the two important terms of the $B\wedge F$ and $B\wedge\tilde{G}\wedge\tilde{G}$ types coincide with the result in [17]. Notice also that (2.10) has not been covered by the general non-Abelian tensor-hierarchy formulations [1], because of the nonpolynomial structure of the field strength (2.10b). Even though similar nonpolynomial structure was already presented in [17], the main difference in our system is that the YM field has its kinetic term intact, i.e., our YM field is physical and not auxiliary.

III. $N = 1$ SUPERSYMMETRIC PROCA-STÜCKELBERG FORMULATION IN FOUR DIMENSIONS

Our next task is to *supersymmetrize* the last section. To this end, we first need to supersymmetrize the Proca-Stückelberg Lagrangian (2.6). We can in principle use our previous results in [4,5]. However, we present a new simpler supersymmetric Proca-Stückelberg formulation in this paper. Our new field content is more economical than [4,5] with only two multiplets: the non-Abelian YM multiplet $(A_\mu{}^I, \lambda^I)$ and the Proca-Stückelberg multiplet $(\varphi^I, \chi^I, \phi^I)$ with no other multiplet. The scalar φ^I parametrizes the coordinates of the gauge-group G, while a pseudoscalar ϕ^I is in the adjoint representation. Thus the two spin-zero fields 0^+ and 0^- within a CM play different roles under the same group G.

Our new action $I_{\rm SPS} \equiv \int d^4x\mathcal{L}_{\rm SPS}$ for a supersymmetric Proca-Stückelberg theory has the Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm SPS} = &-\frac{1}{4}(\mathcal{F}_{\mu\nu}{}^I)^2 + \frac{1}{2}(\bar{\lambda}^I \not{D}\lambda^I) - \frac{1}{2}(P_\mu{}^I)^2 + \frac{1}{2}(\bar{\chi}^I\not{D}\chi^I) \\ &-\frac{1}{2}(D_\mu\phi^I)^2 + m(\bar{\lambda}^I\chi^I) - \frac{1}{2}m^2(\phi^I)^2 \\ &- imf^{IJK}(\bar{\lambda}^I\gamma_5\chi^J)\phi^K - \frac{1}{2}f^{IJK}(\bar{\lambda}^I\gamma^\mu\lambda^J)P_\mu{}^K, \end{aligned} \tag{3.1}$$

where

$$P_\mu{}^I \equiv [(D_\mu e^\varphi)e^{-\varphi}]^I \equiv [(\partial_\mu e^\varphi)e^{-\varphi}]^I + mA_\mu{}^I, \tag{3.2a}$$

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^I &\equiv 2\partial_{[\mu}A_{\nu]}{}^I + mf^{IJK}A_\mu{}^J A_\nu{}^K + m^{-1}f^{IJK}P_\mu{}^J P_\nu{}^K \\ &\equiv F_{\mu\nu}{}^I + m^{-1}f^{IJK}P_\mu{}^J P_\nu{}^K, \end{aligned} \tag{3.2b}$$

$$\begin{aligned} D_\mu\lambda^I &\equiv \partial_\mu\lambda^I + mf^{IJK}A_\mu{}^J\lambda^K, \\ D_\mu\chi^I &\equiv \partial_\mu\chi^I + mf^{IJK}A_\mu{}^J\chi^K. \end{aligned} \tag{3.2c}$$

Equation (3.2a) implies that φ^I serve as the coordinates of the group manifold G. In other words, supersymmetric Proca-Stückelberg Lagrangian (3.1) is nothing but the supersymmetric gauged σ-model version of the group manifold G. The reason why we need the *modified* field strength $\mathcal{F}_{\mu\nu}{}^I$ for the $A_\mu{}^I$-kinetic term is similar to the formulation in [4,5]. This is justified by the superinvariance $\delta_Q I_{\rm SPS} = 0$, as will be described around (3.7) below. This will be further justified in our superspace formulation in the Appendix. This is also understood as the origin of the intrinsic nonpolynomial feature of our supersymmetric Cremmer-Scherk formulation in the next section.

The P and $\mathcal{F}$ field strengths satisfy their BIds which are also similar to the corresponding equations in [4,5]:

$$D_{[\mu} P_{\nu]}{}^I \equiv +\frac{1}{2} m \mathcal{F}_{\mu\nu}{}^I, \tag{3.3a}$$

$$D_{[\mu} \mathcal{F}_{\nu\rho]}{}^I \equiv +f^{IJK} \mathcal{F}_{[\mu\nu}{}^J P_{\rho]}{}^K. \tag{3.3b}$$

Our action $I_{\rm SPS}$ is invariant under $N = 1$ supersymmetry,

$$\begin{aligned} \delta_Q A_\mu{}^I &= +(\bar\epsilon\gamma_\mu\lambda^I) - m^{-1} f^{IJK} (\bar\epsilon\chi^J) P_\mu{}^K \\ &\equiv \tilde\delta_Q A_\mu{}^I - m^{-1} f^{IJK} (\bar\epsilon\chi^J) P_\mu{}^K, \end{aligned} \tag{3.4a}$$

$$\delta_Q \lambda^I = +\frac{1}{2}(\gamma^{\mu\nu}\epsilon)\mathcal{F}_{\mu\nu}{}^I + im(\gamma_5\epsilon)\phi^I + f^{IJK}\lambda^J(\bar\epsilon\chi^K), \tag{3.4b}$$

$$[(\delta_Q e^\varphi)e^{-\varphi}]^I = +(\bar\epsilon\chi^I), \tag{3.4c}$$

$$\begin{aligned} \delta_Q \chi^I &= -(\gamma^\mu\epsilon)P_\mu{}^I + i(\gamma_5\gamma^\mu\epsilon)D_\mu\phi^I \\ &\quad - ih^{IJ,KL}(\gamma_5\chi^J)(\bar\epsilon\chi^K)\phi^L, \end{aligned} \tag{3.4d}$$

$$\delta_Q \phi^I = +i(\bar\epsilon\gamma_5\chi^I), \tag{3.4e}$$

where $h^{IJ,KL} \equiv f^{IJM} f^{MKL}$.

As a corollary, we mention the transformations of $P_\mu{}^I$ and $\mathcal{F}_{\mu\nu}{}^I$. First, the general transformation rules are

$$\begin{aligned} \delta P_\mu{}^I &= D_\mu[(\delta e^\varphi)e^{-\varphi}]^I + [m(\delta A_\mu{}^I) + f^{IJK}\{(\delta e^\varphi)e^{-\varphi}\}^J P_\mu{}^K] \\ &= D_\mu[(\delta e^\varphi)e^{-\varphi}]^I + m(\tilde\delta A_\mu{}^I), \end{aligned} \tag{3.5a}$$

$$\begin{aligned} \delta \mathcal{F}_{\mu\nu}{}^I &= 2D_{[\mu}(\tilde\delta A_{\nu]}{}^I) + 2f^{IJK}(\tilde\delta A_{[\mu}{}^J)P_{\nu]}{}^K \\ &\quad - f^{IJK}[(\delta e^\varphi)e^{-\varphi}]^J \mathcal{F}_{\mu\nu}{}^K, \end{aligned} \tag{3.5b}$$

$$\tilde\delta A_\mu{}^I \equiv \delta A_\mu{}^I + m^{-1} f^{IJK}[(\delta e^\varphi)e^{-\varphi}]^J P_\mu{}^K. \tag{3.5c}$$

Note that the definition of $\tilde\delta_Q A_\mu{}^I$ in (3.4a) is consistent with (3.5c). Accordingly, we also have the convenient lemmas:

$$\delta_Q P_\mu{}^I = +(\bar\epsilon D_\mu\chi^I) + m(\bar\epsilon\gamma_\mu\lambda^I), \tag{3.6a}$$

$$\begin{aligned} \delta_Q \mathcal{F}_{\mu\nu}{}^I &= -2(\bar\epsilon\gamma_{[\mu}D_{\nu]}\lambda^I) + 2f^{IJK}(\bar\epsilon\gamma_{[\mu}\lambda^J)P_{\nu]}{}^K \\ &\quad - f^{IJK}(\bar\epsilon\chi^J)\mathcal{F}_{\mu\nu}{}^K. \end{aligned} \tag{3.6b}$$

Note that the last $m^{-1}f(\bar\epsilon\chi)P$ term in (3.4a) does not remain in $\delta_Q P_\mu{}^I$, because it is canceled by the difference term between $\tilde\delta_Q A_\mu{}^I$ and $\delta_Q A_\mu{}^I$. In the first two terms in (3.5b), the variations are $\tilde\delta A_\mu{}^I$ instead of $\delta A_\mu{}^I$, which are reflected in (3.6b).

The supersymmetric invariance $\delta_Q I_{\rm SPS} = 0$ is confirmed as follows: There are in total six sectors arising in the variation $\delta_Q I_{\rm SPS}$ up to $\mathcal{O}(\Phi^4)$: (i) $m^0\Phi^2$, (ii) $m^1\Phi^2$, (iii) $m^2\Phi^2$, (iv) $m^0\Phi^3$, (v) $m^1\Phi^3$, and (vi) $m^2\Phi^3$ up to $\mathcal{O}(\Phi^4)$.

The sector (i) has three subsectors: (a) $\lambda D\mathcal{F}$, (b) $\chi D^2\phi$, and (c) χDP. These subsectors are routine confirmations whose details we skip. The sector (ii) has three subsectors: (a) $m\chi\mathcal{F}$, (b) $m\lambda P$, and (c) $m\lambda D\phi$, which are also routine sectors, and details are skipped. The sector (iii) has only one sort of term: $m^2\chi\phi$, which is straightforward.

Sector (iv) is rather nontrivial with three subsectors: (a) $\chi\bar\lambda D\lambda$, (b) $\lambda P\mathcal{F}$, and (c) $\chi\mathcal{F}^2$. The subsector (a) needs Fierz rearrangements. There arise five different structures of γ matrices $f^{IJK}(\bar\epsilon\gamma_{[n]}\chi^I)(\bar\lambda^J\gamma^{[n]}\not{D}\lambda^K)$, where $[n] (n = 0, \ldots, 4)$ represents the number of antisymmetric γ matrices. Especially, the $\lambda\chi$ term in $\delta_Q\lambda$ plays a sophisticated role for the cancellations of these terms. The subsector (c) of (iv) for χF^2 terms is straightforward and details are skipped here.

However, the subsector (b) for $\lambda P\mathcal{F}$ terms is the most crucial sector, because this sector shows why $(\mathcal{F}_{\mu\nu}{}^I)^2$ is needed instead of the conventional $(F_{\mu\nu}{}^I)^2$ for the $A_\mu{}^I$-kinetic term.[7] There are three terms contributing to this sector, and their cancellations work like[8]

$$\begin{aligned} 0 &\stackrel{?}{=} \delta_Q\left[-\frac{1}{2}f^{IJK}(\bar\lambda^I\gamma^\mu\lambda^J)P_\mu{}^K - \frac{1}{4}(\mathcal{F}_{\mu\nu}{}^I)^2 + \frac{1}{2}(\bar\lambda^I\not{D}\lambda^I)\right]\Bigg|_{\lambda P\mathcal{F}} \\ &= -f^{IJK}\left[\left(-\frac{1}{2}\bar\epsilon\gamma^{\rho\sigma}\mathcal{F}_{\rho\sigma}{}^I\right)\gamma^\mu\lambda^J\right]P_\mu{}^K \\ &\quad - \frac{1}{2}[+2f^{IJK}(\bar\epsilon\gamma_\mu\lambda^J)P_\nu{}^K]\mathcal{F}^{\mu\nu I} \\ &\quad + \left[-\frac{1}{2}(\bar\epsilon\gamma^{\rho\sigma})\mathcal{F}_{\rho\sigma}{}^I\right]\gamma^\mu D_\mu\lambda^I|_{\lambda P\mathcal{F}} \end{aligned} \tag{3.7a}$$

$$\begin{aligned} &\stackrel{\nabla}{=} +\frac{1}{2}f^{IJK}(\bar\epsilon\gamma^{\rho\sigma}\gamma^\mu\lambda^J)P_\mu{}^K\mathcal{F}_{\rho\sigma}{}^I - f^{IJK}(\bar\epsilon\gamma_\mu\lambda^J)P_\nu{}^K\mathcal{F}^{\mu\nu I} \\ &\quad + \frac{1}{2}(\bar\epsilon\gamma^{\rho\sigma\mu}\lambda^I)D_{[\mu}\mathcal{F}_{\rho\sigma]}{}^I|_{\lambda P\mathcal{F}} \end{aligned} \tag{3.7b}$$

[7] This necessity is just the same pattern as in our previous papers [4], but just for readers who doubt the validity of our previous paper [4], we give the fresh details here.

[8] We use the symbol $\stackrel{?}{=}$ for equalities that are to be confirmed.

$$= +\frac{1}{2} f^{IJK}(\bar{\epsilon}\gamma^{\mu\rho\sigma}\lambda^I)P_\mu{}^J\mathcal{F}_{\rho\sigma}{}^K + f^{IJK}(\bar{\epsilon}\gamma^\rho\lambda^I)P^{\sigma J}\mathcal{F}_{\rho\sigma}{}^K - f^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^I)P_\nu{}^J\mathcal{F}^{\mu\nu K} + \frac{1}{2} f^{IJK}(\bar{\epsilon}\gamma^{\rho\sigma\mu}\lambda^I)P_\sigma{}^K\mathcal{F}_{\mu\rho}{}^J \tag{3.7c}$$

$$= 0 \quad \text{(Q.E.D.)}. \tag{3.7d}$$

The cancellations occurred between the first and fourth as well as second and third terms in (3.7c). Note that in (3.7b), we have used the $\mathcal{F}$-BId (3.3b) after a partial integration. In particular, if the field strength $(\mathcal{F}_{\mu\nu}{}^I)^2$ in the A_μ-kinetic term were replaced by the conventional one $(F_{\mu\nu}{}^I)^2$, there would arise no $\lambda\mathcal{F}P$ term from $\delta_Q(\mathcal{F}_{\mu\nu}{}^I)^2$ via $(\delta_Q P)\wedge P$. Therefore there would be no cancellation of $\lambda P\mathcal{F}$ terms against the contribution from the Pauli term $f^{IJK}(\bar{\lambda}^I\gamma^\mu\lambda^J)P_\mu{}^K$, which is needed independent of the choice between $\mathcal{F}$ and F in the A_μ-kinetic term. Because of these highly nontrivial mechanisms, the modified field strength $\mathcal{F}_{\mu\nu}{}^I$ is indispensable in the $A_\mu{}^I$-kinetic term. This explains why superinvariance of our action necessitates $(\mathcal{F}_{\mu\nu}{}^I)^2$ instead of $(F_{\mu\nu}{}^I)^2$ for the A_μ-kinetic term. Note also that there was no such necessity for the nonsupersymmetric case in the last section. It is not due to our "convenient choice" or "subjective taste" to use the modified field strength $\mathcal{F}_{\mu\nu}{}^I$. It is the superinvariance $\delta_Q I_{\rm SPS} = 0$ that determines the right choice between $\mathcal{F}$ and F.

The sector (v) has six subsectors: (a) $m\lambda\phi P$, (b) $m\lambda\chi^2$, (c) $m\lambda^3$, (d) $m\lambda\phi D\phi$, (e) $m\chi\phi^2$, and (f) $m\chi\phi\mathcal{F}$. Among these, the subsector (b) needs Fierz rearrangements. To be more specific, we have

$$0 \overset{?}{=} \delta_Q\left[+\frac{1}{2}(\bar{\chi}^I \not{D}\chi^I) - imf^{IJK}(\bar{\lambda}^I\gamma_5\chi^J)\phi^K + m(\bar{\lambda}^I\chi^I)\right]\Big|_{m\lambda\chi^2}$$

$$= +\frac{1}{2}mf^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^J)(\bar{\chi}^I\gamma^\mu\chi^K) - imf^{IJK}(\bar{\lambda}^I\gamma_5\chi^J)i(\bar{\epsilon}\gamma_5\chi^K) + m[f^{IJK}(\bar{\epsilon}\chi^K)\bar{\lambda}^J]\chi^I \tag{3.8a}$$

$$= -\frac{1}{2}mf^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^I)(\bar{\chi}^J\gamma^\mu\chi^K) + \left[+\frac{1}{4}mf^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^I)(\bar{\chi}^J\gamma^\mu\chi^K) + \frac{1}{8}mf^{IJK}(\bar{\epsilon}\gamma_{\mu\nu}\lambda^I)(\bar{\chi}^J\gamma^{\mu\nu}\chi^K)\right] + \left[+\frac{1}{4}mf^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^I)(\bar{\chi}^J\gamma^\mu\chi^K) - \frac{1}{8}mf^{IJK}(\bar{\epsilon}\gamma_{\mu\nu}\lambda^I)(\bar{\chi}^J\gamma^{\mu\nu}\chi^K)\right] \tag{3.8b}$$

$$= 0 \quad \text{(Q.E.D.)}. \tag{3.8c}$$

Equation (3.8b) is the result of Fierz rearrangements for the second and third terms in (3.8a). The cancellations in (3.8b) resemble that for the stereotypical YM coupling to a conventional CM. However, this justifies the contribution of our peculiar term $(\bar{\epsilon}\chi)\lambda$ in $\delta_Q\lambda^I$, and also the nontrivial mixed mass term $m(\bar{\lambda}\chi)$ in our Lagrangian $\mathcal{L}_{\rm SPS}$, as well.

The remaining subsectors (a), (c), (d), (e) and (f) are straightforward to handle. The sector (vi) has only one kind of terms: $m^2\chi\phi^2$, whose cancellation is straightforward, and its details are skipped here. The sectors (v) and (vi) with positive powers of m do not involve BIds, due to their engineering dimensions. These terms are rather straightforward except for Fierz arrangements. However, such Fierz arrangements are not special to our model, so that they are skipped here.

The total consistency of our new multiplet $(\varphi^I, \chi^I, \phi^I)$ with φ^I and ϕ^I transforming differently under gauge transformation δ_T, we give additional nontrivial supporting evidence. Namely, we can confirm the closures of supersymmetry on all fields.

To this end, we prepare the field equations of *all* fields $A_\mu{}^I, \varphi^I, \phi^I, \lambda^I$ and χ^I:[9]

$$\frac{\delta\mathcal{L}_{\rm SPS}}{\delta A_\mu{}^I} = -D_\nu\mathcal{F}^{\mu\nu I} - mP_\mu{}^I - f^{IJK}\mathcal{F}_{\mu\nu}{}^J P^{\nu K} - mf^{IJK}(\bar{\lambda}^J\gamma^\mu\lambda^K) - \frac{1}{2}mf^{IJK}(\bar{\chi}^J\gamma^\mu\chi^K) + mf^{IJK}\phi^J D^\mu\phi^K \doteq 0, \tag{3.9a}$$

$$\frac{\delta\mathcal{L}_{\rm SPS}}{[(\delta e^\varphi)e^{-\varphi})]^I} = +D_\mu P^{\mu I} - mf^{IJK}(\bar{\lambda}^J\chi^K) + f^{IJK}\left[\bar{\lambda}^J\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^K}\right)\right] \doteq 0, \tag{3.9b}$$

$$\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\phi^I} = +D_\mu^2\phi^I - m^2\phi^I - imf^{IJK}(\bar{\lambda}^J\gamma_5\chi^K) \doteq 0, \tag{3.9c}$$

$$\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^I} = +\not{D}\lambda^I + m\chi^I - imf^{IJK}(\gamma_5\chi^J)\phi^K - f^{IJK}(\gamma^\mu\lambda^J)P_\mu{}^K \doteq 0, \tag{3.9d}$$

$$\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\chi}^I} = +\not{D}\chi^I + m\lambda^I + imf^{IJK}(\gamma_5\lambda^J)\phi^K \doteq 0. \tag{3.9e}$$

Note that the coefficient of the $(\bar{\lambda}\gamma\lambda)$ term in (3.9a) is (-1), while that of the $(\bar{\chi}\gamma\chi)$ term is $(-1/2)$. This is caused by the additional contribution to the former from the $(\bar{\lambda}\gamma\lambda)P$ term in the Lagrangian. In (3.9b), the last term vanishes upon the λ-field equation.

[9] These field equations are valid up to $\mathcal{O}(\Phi^3)$, because of the Lagrangian valid up to $\mathcal{O}(\Phi^4)$.

We now start confirming the closure of supersymmetry, starting with ${A_\mu}^I$:

$$
\begin{aligned}
[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]{A_\mu}^I &= [(\bar\epsilon_2\gamma_\mu\delta_1\lambda^I) - m^{-1}f^{IJK}(\bar\epsilon_2\delta_1\chi^J){P_\mu}^K - m^{-1}f^{IJK}(\bar\epsilon_2\chi^J)\delta_1{P_\mu}^K] - (1\leftrightarrow 2)\\
&= +\bar\epsilon_2\gamma_\mu\left[\frac{1}{2}(\gamma^{\rho\sigma}\epsilon_1){\mathcal F_{\rho\sigma}}^I + im(\gamma_5\epsilon_1)\phi^I + f^{IJK}\lambda^J(\bar\epsilon_1\chi^K)\right] - (1\leftrightarrow 2)\\
&\quad - [m^{-1}f^{IJK}\bar\epsilon_2\{-(\gamma^\nu\epsilon_1){P_\nu}^J + i(\gamma_5\gamma^\mu\epsilon_1)D_\mu\phi^K\}{P_\mu}^K - (1\leftrightarrow 2)]\\
&\quad - [m^{-1}f^{IJK}(\epsilon_2\chi^J)\{(\epsilon_1 D_\mu\chi^K) + m(\epsilon_1\gamma_\mu\lambda^K)\} - (1\leftrightarrow 2)] \qquad (3.10a)\\
&= \xi^\nu{\mathcal F_{\nu\mu}}^I + D_\mu\alpha^{(1)I} + m^{-1}f^{IJK}\xi^\nu{P_\nu}^J{P_\mu}^K \qquad (3.10b)\\
&= \xi^\nu\partial_\nu{A_\mu}^I + D_\mu\alpha^{(0)I} + D_\mu\alpha^{(1)I}\\
&= \delta_P{A_\mu}^I + \delta_T^{(0)}{A_\mu}^I + \delta_T^{(1)}{A_\mu}^I = \delta_P{A_\mu}^I + \delta_T{A_\mu}^I \quad \text{(Q.E.D.).} \qquad (3.10c)
\end{aligned}
$$

Here we have used

$$\delta_1 \equiv \delta_Q(\epsilon_1), \qquad \delta_2 \equiv \delta_Q(\epsilon_2),$$
$$\xi^\mu \equiv +2(\epsilon_1\gamma^\mu\epsilon_2), \qquad \delta_P{A_\mu}^I = \xi^\nu\partial_\nu{A_\mu}^I, \qquad (3.11a)$$

$$\delta_T^{(0)} \equiv \delta_T(\alpha^{(0)}), \qquad \delta_T^{(1)} \equiv \delta_T(\alpha^{(1)}), \qquad \delta_T \equiv \delta_T^{(0)} + \delta_T^{(1)}, \qquad (3.11b)$$

$$\alpha^I \equiv \alpha^{(0)I} + \alpha^{(1)I}, \qquad \alpha^{(0)I} \equiv -\xi^\nu{A_\nu}^I,$$
$$\alpha^{(1)I} \equiv m^{-1}f^{IJK}(\bar\epsilon_1\chi^J)(\bar\epsilon_2\chi^K), \qquad (3.11c)$$

where δ_P stands for a translation operation with the parameter ξ^μ. In (3.10a), there is a pair of like terms for $(\bar\epsilon_1\gamma^\mu\lambda)(\bar\epsilon_2\chi)$ that canceled each other. Notice the nontrivial cancellation between the last $\xi P\wedge P$ term in (3.10b) and the like term out of the first $\xi\mathcal F$ term involving the like term $\xi P\wedge P$. The final form (3.10c) in terms of the translation δ_P and the gauge transformation δ_T implies the closure of supersymmetry on ${A_\mu}^I$, as desired. In the closure-confirmation computations, we keep only terms at $\mathcal O(\Phi^1), \mathcal O(\Phi^2)$ and $\mathcal O(\Phi^3)$, because our Lagrangian is valid up to $\mathcal O(\Phi^4)$.

Similarly the closure on e^φ is

$$
\begin{aligned}
&([\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]e^\varphi)e^{-\varphi}\\
&\quad = \{\delta_1[(\delta_2 e^\varphi)e^{-\varphi}] - (\delta_2 e^\varphi)(\delta_1 e^{-\varphi})\} - (1\leftrightarrow 2) \qquad (3.12a)\\
&\quad = [\delta_1(\bar\epsilon_2\chi^I) + (\delta_2 e^\varphi)e^{-\varphi}(\delta_1 e^\varphi)e^{-\varphi}] - (1\leftrightarrow 2)\\
&\quad = \bar\epsilon_2(-\gamma^\mu\epsilon{P_\mu}^I + i\gamma_5\gamma^\mu\epsilon_1 D_\mu\phi^I) - (1\leftrightarrow 2) + [(\bar\epsilon_2\chi),(\bar\epsilon_1\chi)] \qquad (3.12b)\\
&\quad = +\xi^\mu P_\mu + f^{IJK}(\bar\epsilon_2\chi^J)(\bar\epsilon_1\chi^K)T^I \qquad (3.12c)\\
&\quad = +(\delta_P e^\varphi)e^{-\varphi} + (\delta_T e^\varphi)e^{-\varphi} \quad \text{(Q.E.D.).} \qquad (3.12d)
\end{aligned}
$$

Here we regard each term as generator valued, e.g., $[(\bar\epsilon_2\chi),(\bar\epsilon_1\chi)] \equiv f^{IJK}(\bar\epsilon_2\chi^J)(\bar\epsilon_1\chi^K)T^I$, while $\delta_P e^\varphi \equiv \xi^\mu\partial_\mu e^\varphi$, and $\delta_T e^\varphi \equiv -m\alpha e^\varphi$ with $\alpha^I \equiv \alpha^{(0)I} + \alpha^{(1)I}$, consistent with (2.1a) and (3.11). In (3.12a), the second term inside the braces is to subtract the δ_1 acting on $e^{-\varphi}$. In (3.12b), the $D\phi$ term does not contribute, due to $(\bar\epsilon_2\gamma_5\gamma^\mu\epsilon_1) - (1\leftrightarrow 2) = 0$. The χ^2 term in (3.12c) is interpreted as the $\alpha^{(1)}$ term in $(\delta_T e^\varphi)e^{-\varphi}$. This highly nontrivial sophisticated and subtle rearrangement at the quadratic order for the closure provides supporting evidence for the consistency of our system. This component closure computation is reconfirmed in superspace as (A9) and (A10) in the Appendix.

As for the closure on ϕ^I, it goes as

$$
\begin{aligned}
[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]\phi^I &= \delta_1[+i(\bar\epsilon\gamma_5\chi^I)] - (1\leftrightarrow 2)\\
&= +i\bar\epsilon_2\gamma_5[-(\gamma^\mu\epsilon_1){P_\mu}^I + i(\gamma_5\gamma^\mu\epsilon_1)D_\mu\phi^I\\
&\quad - ih^{IJ,KL}(\gamma_5\chi^J)(\bar\epsilon_1\chi^K)\phi^L] - (1\leftrightarrow 2) \qquad (3.13a)\\
&= \xi^\mu\partial_\mu\phi^I - mf^{IJK}\alpha^{(0)J}\phi^K\\
&\quad - mf^{IJK}\alpha^{(1)J}\phi^K\\
&= \delta_P\phi^I + \delta_T\phi^I \quad \text{(Q.E.D.).} \qquad (3.13b)
\end{aligned}
$$

The ${P_\mu}^I$-linear term in (3.13a) does not contribute due to $(\bar\epsilon_1\gamma_5\gamma^\mu\epsilon_2) - (1\leftrightarrow 2) = 0$. The $\chi^2\phi$ term in (3.13a) produces the gauge transformation with the parameter $\alpha^{(1)I}$, consistent with (3.11c). Even though this looks simple, it is crucial that the closures on both of the spin-zero fields φ^I and ϕ^I work without trouble, despite the different δ_T transformations of ϕ^I and φ^I. The superspace reconfirmation of this closure is given as (A8) in the Appendix.

The closure on λ^I is also one of the most nontrivial, because of the λ-field equation involved:

$$[\delta_1,\delta_2]\lambda^I = \delta_1\left[+\frac{1}{2}(\gamma^{\mu\nu}\epsilon_2)\mathcal{F}_{\mu\nu}{}^I + im(\gamma_5\epsilon_2)\phi^I + f^{IJK}\lambda^J(\bar{\epsilon}_2\chi^K)\right] - (1\leftrightarrow 2) \tag{3.14a}$$

$$\begin{aligned}&= +\frac{1}{2}(\gamma^{\mu\nu}\epsilon_2)[-2(\bar{\epsilon}_1\gamma_\mu D_\nu\lambda^I) - 2f^{IJK}(\bar{\epsilon}_1\gamma_\mu\lambda^J)P_\nu{}^K - f^{IJK}(\bar{\epsilon}_1\chi^J)\mathcal{F}_{\mu\nu}{}^K] - (1\leftrightarrow 2)\\ &\quad + [im(\gamma_5\epsilon_2)i(\bar{\epsilon}_1\gamma_5\chi^I)] - (1\leftrightarrow 2)\\ &\quad + f^{IJK}\left[+\frac{1}{2}(\gamma^{\mu\nu}\epsilon_1)\mathcal{F}_{\mu\nu}{}^J + im(\gamma_4\epsilon_1)\phi^J\right](\bar{\epsilon}_2\chi^K) - (1\leftrightarrow 2)\\ &\quad + f^{IJK}\lambda^J\bar{\epsilon}_2[-(\gamma^\mu\epsilon_1)P_\mu{}^K + i(\gamma_5\gamma^\mu\epsilon_1)D_\mu\phi^K] - (1\leftrightarrow 2)\end{aligned} \tag{3.14b}$$

$$= +\xi^\mu D_\mu\lambda^I + \delta_T\lambda^I - \frac{1}{4}\xi^\mu\gamma_\mu\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^I}\right) - \frac{1}{4}\zeta^{\mu\nu}\gamma_{\mu\nu}\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^I}\right) \tag{3.14c}$$

$$\doteq \delta_P\lambda^I + \delta_T\lambda^I \quad \text{(Q.E.D.)}. \tag{3.14d}$$

Here $\zeta^{\mu\nu} \equiv (\bar{\epsilon}_2\gamma^{\mu\nu}\epsilon_1)$. In (3.14b), we have used (3.6b). Proper Fierz arrangements for the quadratic-fermionic terms and considerable cancellations among like terms in (3.14b) lead to (3.14c). By the use of the λ-field equation for the first time in (3.14c), we reach (3.14d). Since the λ-field equation is valid up to $\mathcal{O}(\Phi^3)$, we ignored a $\chi^2\lambda$ term in (3.14c).

The closure on χ^I is equally nontrivial:

$$\begin{aligned}[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]\chi^I &= \delta_1[-(\gamma^\mu\epsilon)P_\mu{}^I + i(\gamma_5\gamma^\mu\epsilon)D_\mu\phi^I] - (1\leftrightarrow 2)\\ &= [-(\gamma^\mu\epsilon_2)\{(\bar{\epsilon}_1 D_\mu\chi^I) - m(\bar{\epsilon}_1\gamma_\mu\lambda^I)\}\\ &\quad + i(\gamma_5\gamma^\mu\epsilon)i(\bar{\epsilon}_1\gamma_5 D_\mu\chi^I) + imf^{IJK}(\bar{\epsilon}_1\gamma_\mu\lambda^J)(\gamma_5\gamma^\mu\epsilon_2)\phi^K] - (1\leftrightarrow 2)\end{aligned} \tag{3.15a}$$

$$= +\xi^\mu D_\mu\chi^I - \frac{1}{2}\xi^\mu\gamma_\mu\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\chi}^I}\right) \tag{3.15b}$$

$$\doteq +\delta_P\chi^I + \delta_T\chi^I \quad \text{(Q.E.D.)}. \tag{3.15c}$$

As in the case of λ, after Fierz arrangements, cancellations of like terms in (3.15a), and upon the use of the χ-field equation in (3.15b), we reach (3.15c).[10]

We give yet another confirmation among field equations. This is performed by varying the λ and χ-field equations under supersymmetry:

$$\begin{aligned}0 \overset{?}{\doteq} \delta_Q\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^I}\right) &= \delta_Q[\not{D}\lambda^I + m\chi^I - imf^{IJK}(\gamma_5\chi^J)\phi^K - f^{IJK}(\gamma^\mu\lambda^J)P_\mu{}^K]\\ &= +(\gamma^\mu\epsilon)\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta A_\mu{}^I}\right) + f^{IJK}\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^J}\right)(\bar{\epsilon}\chi^K)\end{aligned} \tag{3.16a}$$

$$\doteq 0. \tag{3.16b}$$

We have not used any field equation except for the last equality (3.16b). Similarly for the χ-field equation, we get

$$\begin{aligned}0 \overset{?}{\doteq} \delta_Q\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\chi}^I}\right) &= \delta_Q[+\not{D}\chi^I + m\lambda^I + imf^{IJK}(\gamma_5\lambda^J)\phi^K]\\ &= -\epsilon\left[\frac{\delta\mathcal{L}_{\rm SPS}}{\{(\delta_Q e^\varphi)e^{-\varphi}\}^I}\right] - f^{IJK}\epsilon\left[\bar{\lambda}^J\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar{\lambda}^K}\right)\right] - i(\gamma_5\epsilon)\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\phi^I}\right) \doteq 0.\end{aligned} \tag{3.17}$$

As before, no field equation has been used until the last equality in (3.17).

As the final consistency confirmation, we show the divergence of the A_μ-field equation (3.8a) which is supposed to vanish by the use of all of our field equations:

[10] Just as the closure on λ, since the χ-field equation is valid up to $\mathcal{O}(\Phi^3)$, we ignored a χ^3 term.

$$
\begin{aligned}
0 &\overset{?}{\doteq} D_\mu\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta A_\mu{}^I}\right)\\
&= D_\mu\left[-D_\nu\mathcal{F}^{\mu\nu I} - mf^{IJK}(\bar\lambda^J\gamma^\mu\lambda^K) - mP^{\mu I} - \frac{1}{2}mf^{IJK}(\bar\chi^J\gamma^\mu\chi^K) + mf^{IJK}\phi^J D^\mu\phi^K - f^{IJK}\mathcal{F}_{\mu\nu}{}^J P^{\nu K}\right]\\
&= -\frac{1}{2}mf^{IJK}F_{\mu\nu}{}^J\mathcal{F}^{\mu\nu K} - 2mf^{IJK}(\bar\lambda^J \not{D}\lambda^K) - mD_\mu P^{\mu I} - mf^{IJK}(\bar\chi^J\not{D}\chi^K) + mf^{IJK}\phi^J D_\mu^2\phi^K\\
&\quad - f^{IJK}(D_\mu\mathcal{F}^{\mu\nu J})P^{\nu K} - f^{IJK}\mathcal{F}^{\mu\nu J}(D_\mu P_\nu{}^K) \qquad (3.18a)
\end{aligned}
$$

$$
\begin{aligned}
&= -2mf^{IJK}\left[\bar\lambda^J\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar\lambda^K}\right)\right] - m\left[\frac{\delta\mathcal{L}_{\rm SPS}}{\{(\delta e^\varphi)e^{-\varphi}\}^I}\right] - mf^{IJK}\left[\bar\chi^J\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar\chi^K}\right)\right] + mf^{IJK}\phi^J\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\phi^K}\right) - f^{IJK}\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta A_\mu{}^J}\right)P_\mu{}^K\\
&\doteq 0. \qquad (3.18b)
\end{aligned}
$$

In (3.18a), the first factor $F_{\mu\nu}{}^I$ is replaced by $\mathcal{F}_{\mu\nu}{}^I - f^{IJK}P_\mu{}^J P_\nu{}^K$, while the last term vanishes, because of the P-BId (3.3a). Even though each of the remaining terms in (3.18a) are directly related to the middle sides of field equations (3.9), the last equalities in (3.9) with "$\doteq$" themselves have not been used until the last equality in (3.18b). Since this confirmation involves all five field equations in our system, it would have failed, if there were any inconsistency among field equations, or that with supersymmetry and/or with gauge covariance.

For the validity of our unconventional CM, we mention the following three points: The first reasoning is rather logical: We already know that a similar situation with a tensor multiplet was presented in [4]. The tensor multiplet (TM) in [4] has the component fields $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ in terms of the notation in [4]. The reason why the TM in [4] does not follow the conventional tensor (linear) multiplet [25,28], i.e., why it cannot be described in terms of a scalar superfield L, is as follows: On the scalar superfield L [25,28], the commutator (but not anticommutator) of two spinorial derivatives gives

$$
[\nabla_\alpha, \bar\nabla_{\dot\beta}]L = c_1(\sigma^{cde})_{\alpha\dot\beta}G_{cde} + c_2{\rm tr}(W_\alpha\bar W_{\dot\beta}), \qquad (3.19)
$$

where α (or $\dot\beta$) is for the positive (or negative) chirality. Note that L is singlet under the YM group, without an adjoint index. Obviously, this is impossible for non-Abelian TM in [4], because the G term in (3.19) should carry the adjoint index, while the ${\rm tr}(W\bar W)$ term does not, due to its trace operation. The attempt to make the $W\bar W$ term be replaced by something like $f^{IJK}(W_\alpha{}^J\bar W_{\dot\beta}{}^K)$ does not work either, because such a term vanishes for an Abelian case. Because of this lack of fundamental scalar superfield, we do not have superspace action formulation at the present time.

The second reasoning is rather intuitive. Since the spin-zero fields φ^I and ϕ^I serve different tasks under G, it is obvious that this multiplet cannot be described in terms of a common superfield, such as the scalar superfield L^I carrying the common index for φ^I and ϕ^I. The third reasoning is based on the analogy of higher-dimensional supersymmetry, e.g., 11D [29] or 10D [30] with no explicit action formulation in superspace in terms of off-shell superfields. In view of this analogy, the lack of action formulation in superspace for our on-shell system is nothing bizarre, even though our system is in 4D.

Note that our results above are highly sophisticated, so that their cancellations are neither trivial results, nor accidental coincidences. In particular, the sophisticated cancellations of quadratic-order terms in the closure on φ has not been well presented by papers in the past. These computational and intuitive considerations provide the supporting evidence for two important aspects:

(1) $N = 1$ supersymmetry necessitates the modified field strength $\mathcal{F}_{\mu\nu}{}^I$ instead of the conventional one $F_{\mu\nu}{}^I$.

(2) Our nonconventional CM $(\varphi^I, \chi^I, \phi^I)$ with φ^I and ϕ^I transforming differently under δ_T is consistent with $N = 1$ supersymmetry. This has been confirmed with couplings to the YM multiplet $(A_\mu{}^I, \lambda^I)$.

Our supersymmetric Proca-Stückelberg theory given by (3.1) through (3.4) is more economical than our previous formulations [4,5]. Notice that in our CM $(\varphi^I, \chi^I, \phi^I)$, the spin-zero fields φ^I and ϕ^I play completely different roles, because the former is for the coordinates of the group manifold G, while the latter is in the adjoint representation of G. To our knowledge, this supersymmetric Proca-Stückelberg theory has not been presented before in the past.

IV. SUPERSYMMETRIC CREMMER-SCHERK THEORY BY DUALITY TRANSFORMATION

We perform next the supersymmetric duality transformation [16] from φ^I to $B_{\mu\nu}{}^I$ applied to the Lagrangian $\mathcal{L}_{\rm SPS}$ (3.1). We follow the same pattern for the nonsupersymmetric case in Sec. II. Accordingly, we reach the TM $(B_{\mu\nu}{}^I, \chi^I, \phi^I)$ coupled to YM-multiplet $(A_\mu{}^I, \lambda^I)$. As is usual in duality transformations [16], we first rewrite ${P_\mu}^I$ as ${Q_\mu}^I$ in (3.1), regarding the latter as an independent field. We next add to (3.1) a constraint Lagrangian $\mathcal{L}_{BDQ} \approx B^I \wedge [DQ^I - (1/2)m\mathcal{F}^I]$. This constraint Lagrangian "forces" the BId (3.3a) on ${Q_\mu}^I$, after the replacement of ${P_\mu}^I$ by ${Q_\mu}^I$ [16]. The ${B_{\mu\nu}}^I$ field is initially a "multiplier" field, but its field strength ${G_{\mu\nu\rho}}^I$ becomes eventually dual to ${Q_\mu}^I$ with propagation [16].

At this stage, our original supersymmetric Proca-Stückelberg Lagrangian (3.1) becomes

$$\begin{aligned}\mathcal{L}'_{\rm SPS} = &-\frac{1}{4}({\mathcal{F}_{\mu\nu}}^I)^2 + \frac{1}{2}(\bar\lambda^I \not{D}\lambda^I) - \frac{1}{2}({Q_\mu}^I)^2 + \frac{1}{2}(\bar\chi^I\not{D}\chi^I) \\ &-\frac{1}{2}(D_\mu\phi^I)^2 + m(\bar\lambda^I\chi^I) - \frac{1}{2}m^2(\phi^I)^2 \\ &- imf^{IJK}(\bar\lambda^I\gamma_5\chi^J)\phi^K - \frac{1}{2}f^{IJK}(\bar\lambda^I\gamma^\mu\lambda^J){Q_\mu}^K \\ &+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}{B_{\mu\nu}}^I\left(D_\rho {Q_\sigma}^I - \frac{1}{2}m{\mathcal{F}_{\rho\sigma}}^I\right),\end{aligned} \tag{4.1}$$

where

$${\mathcal{F}_{\mu\nu}}^I \equiv {F_{\mu\nu}}^I + m^{-1}f^{IJK}{Q_\mu}^J{Q_\nu}^K, \tag{4.2}$$

with the original $m^{-1}P^2$ term in ${\mathcal{F}_{\rho\sigma}}^I$ now replaced by $m^{-1}Q^2$.

Our next task is to get the algebraic field equation for the independent field ${Q_\mu}^I$, and eliminate it from (4.1) [16]. The variation of (4.1) by ${Q_\mu}^I$ yields its field equation

$$\begin{aligned}{\Pi_\mu}^{\nu IJ}{Q_\nu}^J \equiv &\left[{\delta_\mu}^\nu\delta^{IJ} + m^{-1}f^{IJK}{{\mathcal{F}_\mu}^{\nu K}} + \frac{1}{2}{\epsilon_\mu}^{\nu\rho\sigma}f^{IJK}{B_{\rho\sigma}}^K\right]{Q_\nu}^J \\ \doteq &\,\tilde G_\mu^{(0)I} - \frac{1}{2}f^{IJK}(\bar\lambda^J\gamma_\mu\lambda^K) + \mathcal{O}(\Phi^3).\end{aligned} \tag{4.3}$$

The ${\Pi_\mu}^{\nu IJ}$ refers to the inside of the square brackets on the second side. Equation (4.3) is nothing but the supersymmetric generalization of the purely bosonic case (2.9). As in the previous section, we keep only the $\mathcal{O}(\Phi^1)$ and $\mathcal{O}(\Phi^2)$ terms for the expression of ${Q_\mu}^I$:

$$\begin{aligned}{Q_\mu}^I &\doteq {(\Pi^{-1})_\mu}^{\nu IJ}\left[\tilde G_\nu^{(0)J} - \frac{1}{2}f^{JKL}(\bar\lambda^K\gamma_\nu\lambda^L)\right] \\ &= \left[{\delta_\mu}^\nu\delta^{IJ} - m^{-1}f^{IJK}{\mathcal{F}_{\mu\nu}}^K - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^{IJK}{B_{\rho\sigma}}^K\right] \\ &\quad\times\left[\tilde G_\nu^{(0)J} - \frac{1}{2}f^{JKL}(\bar\lambda^K\gamma_\nu\lambda^L)\right] + \mathcal{O}(\Phi^3) \\ &= {\tilde G_\mu}{}^I - \frac{1}{2}m^{-1}f^{IJK}(\bar\lambda^J\gamma_\mu\lambda^K) \\ &\quad + m^{-1}f^{IJK}{F_{\mu\nu}}^J{\tilde G_\nu}{}^K + \mathcal{O}(\Phi^3).\end{aligned} \tag{4.4}$$

This is nothing but the supersymmetrization of (2.10a).

Following [16], our next task is to substitute (4.4) into ${Q_\mu}^I$ everywhere in the Lagrangian (4.1), and reach the action $I_{\rm SCS} \equiv \int d^4x\mathcal{L}_{\rm SCS}$ for supersymmetric Cremmer-Scherk theory:

$$\begin{aligned}\mathcal{L}_{\rm SCS} = &-\frac{1}{4}({\mathcal{F}_{\mu\nu}}^I)^2 + \frac{1}{2}(\bar\lambda^I\not{D}\lambda^I) - \frac{1}{12}({G_{\mu\nu\rho}}^I)^2 \\ &+\frac{1}{2}(\bar\chi^I\not{D}\chi^I) - \frac{1}{2}(D_\mu\phi^I)^2 - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}{B_{\mu\nu}}^I{\mathcal{F}_{\rho\sigma}}^I \\ &+ m(\bar\lambda^I\chi^I) - \frac{1}{2}m^2(\phi^I)^2 - imf^{IJK}(\bar\lambda^I\gamma_5\chi^J)\phi^K \\ &-\frac{1}{2}f^{IJK}(\bar\lambda^I\gamma^\mu\lambda^J){\tilde G_\mu}{}^K + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^{IJK}{B_{\mu\nu}}^I{\tilde G_\rho}{}^J{\tilde G_\sigma}{}^K,\end{aligned} \tag{4.5}$$

up to $\mathcal{O}(\Phi^4)$ terms. The field strength ${G_{\mu\nu\rho}}^I$ is the same as (2.10b), while ${\mathcal{F}_{\mu\nu}}^I$ is defined by (4.2) with Q replaced by $\tilde G$. Or more explicitly,

$${G_{\mu\nu\rho}}^I \equiv +G_{\mu\nu\rho}^{(0)I} - 3f^{IJK}{\tilde G_{[\mu}}{}^J{B_{\nu\rho]}}^K + \mathcal{O}(\Phi^3), \tag{4.6a}$$

$$\begin{aligned}{\mathcal{F}_{\mu\nu}}^I \equiv &+2\partial_{[\mu}{A_{\nu]}}^I + mf^{IJK}{A_\mu}^J{A_\nu}^K + m^{-1}f^{IJK}{\tilde G_\mu}{}^J{\tilde G_\nu}{}^K \\ &+\mathcal{O}(\Phi^3),\end{aligned} \tag{4.6b}$$

satisfying their BIds

$$\begin{aligned}D_{[\mu}{G_{\nu\rho\sigma]}}^I \equiv &-3f^{IJK}\left(D_{[\mu}{\tilde G_{\nu|}}{}^J - \frac{1}{2}m{\mathcal{F}_{[\mu\nu|}}^J\right){B_{|\rho\sigma]}}^K \\ &+\mathcal{O}(\Phi^3),\end{aligned} \tag{4.7a}$$

$$D_{[\mu}{\mathcal{F}_{\rho\sigma]}}^I \equiv +2f^{IJK}(D_{[\mu}{\tilde G_{\nu|}}{}^J){\tilde G_{|\rho]}}{}^K + \mathcal{O}(\Phi^3). \tag{4.7b}$$

The difference between ${F_{\mu\nu}}^I$ and ${\mathcal{F}_{\mu\nu}}^I$ in the second term in (4.7a) does not matter, because the difference is only at $\mathcal{O}(\Phi^3)$. As (4.7a) shows, the definition of our G field strength involves $\tilde G$ itself. This implies that our system is intrinsically nonpolynomial, which has not been covered before by the general non-Abelian tensor-hierarchy formulations [1,2].

Our BId (4.7a) is valid only up to $\mathcal{O}(\Phi^3)$. Even though the involvement of the bare B field in the BId (4.7a) looks unconventional, it can be understood as follows: The B-field equation from (4.5) is

$$\begin{aligned}&D_{[\mu}{\tilde G_{\nu]}}{}^I - \frac{1}{2}m{\mathcal{F}_{\mu\nu}}^I + f^{IJK}(\bar\lambda^J\gamma_{[\mu}D_{\nu]}\lambda^K) \\ &\quad+ m^{-1}f^{IJK}D_{[\mu}({\mathcal{F}_{\nu]}}^{\rho J}{\tilde G_\rho}{}^K) + \mathcal{O}(\Phi^3) \doteq 0.\end{aligned} \tag{4.8}$$

This implies that the coefficient factor of the bare B field in (4.7a) vanishes by the use of B-field equation (4.8) up to $\mathcal{O}(\Phi^2)$, as desired.

Our action $I_{\rm SCS}$ is invariant under $N = 1$ supersymmetry,

$$\delta_Q A_\mu{}^I = +(\bar\epsilon\gamma_\mu\lambda^I) - m^{-1} f^{IJK}(\bar\epsilon\chi^J)\tilde G_\mu{}^K + \mathcal{O}(\Phi^3), \quad (4.9a)$$

$$\delta_Q\lambda^I = +\frac{1}{2}(\gamma^{\mu\nu}\epsilon)\mathcal{F}_{\mu\nu}{}^I + im(\gamma_5\epsilon)\phi^I + f^{IJK}\lambda^J(\bar\epsilon\chi^K), \quad (4.9b)$$

$$\begin{aligned}\delta_Q B_{\mu\nu}{}^I = &+i(\bar\epsilon\gamma_5\gamma_{\mu\nu}\chi^I) - 2im^{-1}f^{IJK}(\bar\epsilon\gamma_5\gamma_{[\mu}\lambda^J)\tilde G_{\nu]}{}^K \\ &-\frac{1}{2}\epsilon_{\mu\nu}{}^{|r\sigma}f^{IJK}(\bar\epsilon\chi^J)F_{\rho\sigma}{}^K \\ &+ f^{IJK}(\bar\epsilon\chi^J)B_{\mu\nu}{}^K + \mathcal{O}(\Phi^3), \end{aligned}\quad (4.9c)$$

$$\begin{aligned}\delta_Q\chi^I = &-(\gamma^\mu\epsilon)\tilde G_\mu{}^I + i(\gamma_5\gamma^\mu\epsilon)D_\mu\phi^I + \frac{1}{2}f^{IJK}(\gamma^\mu\epsilon)(\bar\lambda^J\gamma_\mu\lambda^K) \\ &- m^{-1}f^{IJK}(\gamma^\mu\epsilon)\mathcal{F}_\mu{}^{\nu J}\tilde G_\nu{}^K + \mathcal{O}(\Phi^3), \end{aligned}\quad (4.9d)$$

$$\delta_Q\phi^I = +i(\bar\epsilon\gamma_5\chi^I). \quad (4.9e)$$

For example, the difference between $F_{\rho\sigma}{}^K$ and $\mathcal{F}_{\rho\sigma}{}^K$ in (4.9c) will not matter, because of $\mathcal{O}(\Phi^3)$ terms ignored. Notice the peculiar $m^{-1}\mathcal{F}\tilde G$ term at $\mathcal{O}(m^{-1})$ in (4.9d) which does not arise in general non-Abelian tensor-hierarchy formulation [1].

Relevantly, we have the useful corollary:

$$\begin{aligned}\delta_Q\tilde G_\mu{}^I = &-[\bar\epsilon\gamma_\mu(\not{D}\chi^I + m\lambda^I)] + (\bar\epsilon D_\mu\chi^I) + m(\bar\epsilon\gamma_\mu\lambda^I) \\ &+ m^{-1}f^{IJK}D_\rho[(\bar\epsilon\gamma_\mu{}^{\rho\sigma}\lambda^J\tilde G_\sigma{}^K] \\ &+ m^{-1}f^{IJK}D_\nu[(\bar\epsilon\chi^J)F_{\mu\nu}{}^K] \\ &+ f^{IJK}(\bar\epsilon\chi^J)\tilde G_\mu{}^K - f^{IJK}(\bar\epsilon\gamma_\mu{}^\nu\chi^J)\tilde G_\nu{}^K \\ &-\frac{i}{2}f^{IJK}[\bar\epsilon\gamma_5\gamma_\mu{}^{\rho\sigma}(\not{D}\chi^J + m\chi^J)]B_{\rho\sigma}{}^K + \mathcal{O}(\Phi^3), \end{aligned}\quad (4.10a)$$

$$\begin{aligned}\delta_Q\mathcal{F}_{\mu\nu}{}^I = &-2(\bar\epsilon\gamma_{[\mu}D_{\nu]}\lambda^I) - 2m^{-1}f^{IJK}(\bar\epsilon\gamma_{[\mu}\not{D}\chi^J)\tilde G_{\nu]}{}^K \\ &- 2m^{-1}f^{IJK}(\bar\epsilon\chi^J)D_{[\mu}\tilde G_{\nu]}{}^K + \mathcal{O}(\Phi^3). \end{aligned}\quad (4.10b)$$

Note that the bare B-linear term in the last line in (4.10a) vanishes upon the χ-field equation: $\not{D}\chi^J + m\lambda^J \doteq \mathcal{O}(\Phi^2)$. Because if there were such a bare B term, the commutator $[\delta_\beta, \delta_Q]\tilde G_\mu{}^I$ would be problematic with the δ_β transformation:

$$\delta_\beta B_{\mu\nu}{}^I = 2D_{[\mu}\beta_{\nu]}{}^I - 2f^{IJK}\beta_{[\mu}{}^J\tilde G_{\nu]}{}^K + \mathcal{O}(\Phi^2), \quad (4.11)$$

with the derivative $D_{[\rho}\beta_{\sigma]}{}^K$ created by the bare B-term in (4.10a).

To avoid misinterpretation here, the argument in the previous paragraph is meant for the consistency of the commutator $[\delta_\beta, \delta_Q]\tilde G_\mu{}^I$. We stress that the χ-field equation is used in (4.10a) only for the consistency of commutator $[\delta_\beta, \delta_Q]\tilde G_\mu{}^I$, but not for our superinvariance confirmation $\delta_Q I_{\rm SCS} = 0$ that we will perform shortly. Since our supersymmetric formulation is on shell, we can use field equations for commutator algebras or in closure confirmations.

Relevantly, under the δ_β transformation, the G-field strength transforms as

$$\begin{aligned}\delta_\beta G_{\mu\nu\rho}{}^I &= +6f^{IJK}\beta_{[\mu|}{}^J\left(D_{|\nu|}\tilde G_{|\rho]}{}^K - \frac{1}{2}m\mathcal{F}_{|\nu\rho]}{}^K\right) + \mathcal{O}(\Phi^2) \\ &\doteq \mathcal{O}(\Phi^2). \end{aligned}\quad (4.12)$$

This vanishes up to the required order by the use of the B-field equation given below in (4.17b). Note that this invariance is highly nontrivial. In the conventional tensor-hierarchy formulations [1,2] with (1.1) and (1.2), the variation of the extra vector field $C_\mu{}^I$ cancels the unwanted term $F \wedge \beta$, as desired. In our present formulation, even though there arises no $F \wedge \beta$, the price to be paid is the term $-6f^{IJK}(D\beta^J) \wedge \tilde G^K$ arising from $3D(\delta_\beta B^I) = 3D(2D\beta^I - 2f^{IJK}\beta^J \wedge \tilde G^K)$ in (4.6a). This term is exactly canceled by another contribution from $-3f^{IJK}\tilde G^J \wedge (\delta_\beta B^K) = -3f^{IJK}\tilde G^J \wedge (2D\beta^K)$ in (4.6a). In other words, the factor $\tilde G$ is involved in the definition of G itself in such a sophisticated way that the unwanted derivative term $f^{IJK}\tilde G^J \wedge D\beta^K$ is canceled. Even though this peculiar structure had been known since [17], it does not seem to have been covered in general non-Abelian tensor-hierarchy formulations [1,2].

The supersymmetric invariance $\delta_Q I_{\rm SCS} = 0$ up to $\mathcal{O}(\Phi^4)$ can be confirmed as follows: There are in total seven sectors in $\delta_Q I_{\rm SCS}$: (i) $m^0\Phi^2$, (ii) $m^1\Phi^2$, (iii) $m^2\Phi^2$, (iv) $m^{-1}\Phi^3$, (v) $m^0\Phi^3$, (vi) $m^1\Phi^3$, and (vii) $m^2\Phi^3$.

The sector (i) has three subsectors: (a) $\lambda D\mathcal{F}$, (b) $\chi D^2\phi$, and (c) χDG. These are rather parallel to the super Proca-Stückelberg formulation in Sec. III, e.g., the previous Q is now replaced by $\tilde G$. However, we need to avoid possible misinterpretation about (4.10), i.e., we should not use the χ-field equation in there. To be more specific, subsector (c) works like

$$\begin{aligned} 0 &\stackrel{?}{\doteq} \delta_Q\left[+\frac{1}{2}(\tilde G_\mu{}^I)^2 + \frac{1}{2}(\bar\chi^I\not{D}\chi^I)]\right]\Bigg|_{\chi DG} \\ &= [(\bar\epsilon D_\mu\chi^I) - (\bar\epsilon\gamma_\mu\not{D}\chi^I)]\tilde G^{\mu I}|_{\chi DG} \\ &\quad + [+(\bar\epsilon\gamma^\mu)\tilde G_\mu{}^I]\gamma^\nu D_\nu\chi^I|_{\chi DG} \\ &\stackrel{\nabla}{=} -(\bar\epsilon\chi^I)D_\mu\tilde G^{\mu I} - (\bar\epsilon\gamma_\mu\not{D}\chi^I)\tilde G^{\mu I} + (\bar\epsilon\gamma^\mu\not{D}\chi^I)\tilde G_\mu{}^I \\ &\equiv \mathcal{O}(\Phi^3) \quad \text{(Q.E.D.)}. \end{aligned}\quad (4.13)$$

Here the last two terms cancel each other, while the G-BId (4.7a) has been used for the first term. There are $\mathcal{O}(\Phi^2)$ terms in $D_\mu\tilde G^{\mu I}$, but they contribute to subsector (d) χBDG

of (v) $m^0\Phi^3$ sector, and subsector (f) $m\chi B\mathcal{F}$ of (vi) $m\Phi^3$ sector, as will be seen shortly.

Sector (ii) has three subsectors: (a) $m\chi\mathcal{F}$, (b) $m\lambda G$ and (c) $m\lambda D\phi$. These are also parallel to the Proca-Stückelberg formulation. Similarly, sector (iii) has only one straightforward subsector: $m^2\chi\phi$.

Sector (iv) has two subsectors: (a) $m^{-1}\lambda\tilde{G}D\tilde{G}$ and (b) $m^{-1}\chi\mathcal{F}DG$. Compared with the last section, these subsectors have the contributions from $(1/2)(\tilde{G}_\mu{}^I)^2$ instead of $B\wedge DQ$-term, which is now absent. For these terms, the peculiar $\mathcal{O}(m^{-1})$ terms in $\delta_Q\tilde{G}$ and $\delta_Q\mathcal{F}$ in (4.10) play crucial roles.

Sector (v) has five subsectors: (a) $\lambda^2D\chi$, (b) $\lambda G\mathcal{F}$, (c) $\chi\mathcal{F}^2$, (d) χBDG, and (e) χG^2. Subsectors (b), (c) and (e) are parallel to Sec. III. Subsector (a) needs caveat, because it is related to (4.10a), where we should not use the χ-field equation. To be more specific, we get

$$0 \stackrel{?}{=} \delta_Q\left[+\frac{1}{2}(\bar{\lambda}^I \not{D}\lambda^I)+\frac{1}{2}(\bar{\chi}^I\not{D}\chi^I)-\frac{1}{2}f^{IJK}(\bar{\lambda}^I\gamma^\mu\lambda^J)\tilde{G}_\mu{}^K\right]\bigg|_{\lambda^2D\chi}$$
$$= +[+f^{IJK}(\bar{\epsilon}\chi^K)\bar{\lambda}^J]\not{D}\lambda^I + \left[-\frac{1}{2}f^{IJK}(\bar{\epsilon}\gamma^\mu)(\bar{\lambda}^I\gamma_\mu\lambda^J)\right]\not{D}\chi^K$$
$$-\frac{1}{2}f^{IJK}(\bar{\lambda}^I\gamma^\mu\lambda^J)(-\bar{\epsilon}\gamma_\mu{}^\nu D_\nu\chi^K) \qquad (4.14a)$$

$$\stackrel{\nabla}{=} 0 \quad \text{(Q.E.D.)}. \qquad (4.14b)$$

The third term in (4.14a) is due to the identity $-\gamma_\mu\not{D}\chi^I + D_\mu\chi^I \equiv -\gamma_\mu{}^\nu D_\nu\chi^I$ for the $D\chi$ terms in (4.10a). As in (4.13), we do not use the χ-field equation in (4.14). While we need no Fierz arrangement, we need a partial integration to arrange all terms with $(\bar{\lambda}^J\gamma^\mu\lambda^K)$ with no derivative.

Subsector (d) χBDG is also nontrivial. It goes like

$$0 \stackrel{?}{=} \delta_Q\left[+\frac{1}{2}(\tilde{G}_\mu{}^I)^2 + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^{IJK}B_{\mu\nu}{}^I\tilde{G}_\rho{}^J\tilde{G}_\sigma{}^K + \frac{1}{2}(\bar{\chi}^I\not{D}\chi^I) - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I\mathcal{F}_{\rho\sigma}{}^I\right]\bigg|_{\chi BDG} \qquad (4.15a)$$

$$= \left[(\bar{\epsilon}D_\mu\chi^I) - (\bar{\epsilon}\gamma_\mu\not{D}\chi^I) - \frac{i}{2}f^{IJK}(\bar{\epsilon}\gamma_5\gamma_\mu{}^{\rho\sigma}\not{D}\chi^J)B_{\rho\sigma}{}^K\right]\tilde{G}^{\mu I}\bigg|_{\chi BDG} + f^{IJK}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I[(\bar{\epsilon}D_\rho\chi^J) - (\bar{\epsilon}\gamma_\rho\not{D}\chi^J)]\tilde{G}_\sigma{}^K$$
$$+ \left[(\bar{\epsilon}\gamma^\mu)\frac{1}{2}\epsilon_\mu{}^{\nu\rho\sigma}f^{IJK}B_{\nu\rho}{}^J\tilde{G}_\sigma{}^K\right]\not{D}\chi^I - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I m^{-1}f^{IJK}2[(\bar{\epsilon}D_\rho\chi^J) - (\bar{\epsilon}\gamma_\rho\not{D}\chi^J)]\tilde{G}_\sigma{}^K \qquad (4.15b)$$

$$= 0 \quad \text{(Q.E.D.)}. \qquad (4.15c)$$

Out of four terms in (4.15a) the $\delta_Q[(\tilde{G}_\mu{}^I)^2]|_{\chi BDG}$ term is what we promised after (4.13) above. Relevantly, we should not use the χ-field equation in this confirmation. Even though we wrote χBDG, there are two categories of terms in (4.15b): $\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\chi^I)B_{\mu\nu}{}^JD_\rho\tilde{G}_\sigma{}^K$ and $\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\gamma_\mu\not{D}\chi^I)B_{\nu\rho}{}^J\tilde{G}_\sigma{}^K$, which are equivalent to χBDG after partial integration(s). However, there arises no term like $\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\chi^I)(D_\rho B_{\mu\nu}{}^J)\tilde{G}_\sigma{}^K$, because it is proportional to $\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\chi^I)\tilde{G}^{\sigma J}\tilde{G}_\sigma{}^K + \mathcal{O}(\Phi^4) = \mathcal{O}(\Phi^4)$. Even though the first two terms in (4.15b) seem only at $\mathcal{O}(\Phi^2)$, they contribute the aforementioned two categories $\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\chi^I)B_{\mu\nu}{}^JD_\rho\tilde{G}_\sigma{}^K$ and $\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\gamma_\mu\not{D}\chi^I)B_{\nu\rho}{}^J\tilde{G}_\sigma{}^K$, because of the BId (4.7a), and the implicit $B\tilde{G}$ term in $\tilde{G}$ as in (4.6a). In view of these nontrivial manipulations, these cancellations are by no means accidental coincidences.

Sector (vi) has seven subsectors: (a) $m\lambda\phi G$, (b) $m\lambda\chi^2$, (c) $m\lambda^3$, (d) $m\lambda\phi d\phi$, (e) $m\chi\phi\mathcal{F}$, (f) $m\chi B\mathcal{F}$, and (g) $m\lambda BG$. Subsectors (a) through (e) are similar to Sec. III, while (f) and (g) are new. Subsector (g) is rather straightforward to be skipped, while the contributions by $\tilde{G}_\mu{}^I$ to (f) $m\chi B\mathcal{F}$ is subtle, because this subsector is what we promised after (4.13):

$$0 \stackrel{?}{=} \delta_Q\left[+\frac{1}{2}(\tilde{G}_\mu{}^I)^2 - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}{}^I\mathcal{F}_{\rho\sigma}{}^I\right]\bigg|_{m\chi B\mathcal{F}}$$
$$= +(\bar{\epsilon}D_\mu\chi^I)\tilde{G}^{\mu I}|_{m\chi B\mathcal{F}} - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}[f^{IJK}(\bar{\epsilon}\chi^J)B_{\mu\nu}{}^K]\mathcal{F}_{\rho\sigma}{}^I$$
$$\stackrel{\nabla}{=} -(\bar{\epsilon}\chi^I)\left[+\frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}f^{IJK}\mathcal{F}_{\mu\nu}{}^JB_{\rho\sigma}{}^K\right]$$
$$-\frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar{\epsilon}\chi^I)B_{\mu\nu}{}^J\mathcal{F}_{\rho\sigma}{}^K \qquad (4.16a)$$

$$= 0 \quad \text{(Q.E.D.)}. \qquad (4.16b)$$

The first term in (4.15a) is from the G-BId (4.7a). Sector (vii) has only one sector $m^2\chi\phi^2$ parallel to Sec. III.

We can also confirm the mutual consistencies of all of our field equations:

$$\frac{\delta\mathcal{L}_{\rm SCS}}{\delta A_\mu{}^I} = -D_\nu\mathcal{F}^{\mu\nu I} + \frac{1}{6}m\epsilon^{\mu\nu\rho\sigma}G_{\nu\rho\sigma}{}^I + mf^{IJK}\phi^J D^\mu\phi^K + \frac{1}{2}f^{IJK}[(\bar\lambda^J\gamma^\mu\lambda^K) + (\bar\chi^J\gamma^\mu\chi^K)] + \mathcal{O}(\Phi^3) \doteq 0, \tag{4.17a}$$

$$\frac{\delta\mathcal{L}_{\rm SCS}}{\delta B_{\mu\nu}{}^I} = +\frac{1}{2}D_\rho G^{\mu\nu\rho I} - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}{}^I + \frac{1}{2}m^{-1}\epsilon^{\mu\nu\rho\sigma}f^{IJK}D_\rho(\mathcal{F}_\sigma{}^{\tau J}\tilde G_\tau{}^K) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^{IJK}(\bar\lambda^J\gamma_\rho D_\sigma\lambda^K) + \mathcal{O}(\Phi^3) \doteq 0, \tag{4.17b}$$

$$\frac{\delta\mathcal{L}_{\rm SCS}}{\delta\phi^I} = +D_\mu^2\phi^I - m^2\phi^I - imf^{IJK}(\bar\lambda^J\gamma_5\lambda^K) + \mathcal{O}(\Phi^3) \doteq 0, \tag{4.17c}$$

$$\frac{\delta\mathcal{L}_{\rm SCS}}{\delta\bar\lambda^I} = +D\!\!\!\!/\,\lambda^I + m\chi^I - imf^{IJK}(\gamma_5\chi^J)\phi^K - f^{IJK}(\gamma^\mu\lambda^J)\tilde G_\mu{}^L + \mathcal{O}(\Phi^3) \doteq 0, \tag{4.17d}$$

$$\frac{\delta\mathcal{L}_{\rm SCS}}{\delta\bar\chi^I} = +D\!\!\!\!/\,\chi^I + m\lambda^I + imf^{IJK}(\gamma_5\lambda^J)\phi^K + \mathcal{O}(\Phi^3) \doteq 0. \tag{4.17e}$$

For example, the divergence of (4.17a) is confirmed to vanish, upon the use of other field equations, as in (3.18). The most crucial consistency is the divergence of (4.17b):

$$\begin{aligned}0 \overset{?}{\doteq} D_\nu\left(\frac{\delta\mathcal{L}_{\rm SCS}}{\delta B_{\mu\nu}{}^I}\right) &= +\frac{1}{4}mf^{IJK}F_{\nu\rho}{}^J G^{\mu\nu\rho K} \\ &\quad + \frac{1}{4}m^{-1}\epsilon^{\mu\nu\rho\sigma}f^{IJK}[D_\nu, D_\rho](\mathcal{F}_\sigma{}^{\tau J}\tilde G_\tau{}^K) \\ &\quad - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}D_{[\nu}\mathcal{F}_{\rho\sigma]}{}^I + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}f^{IJK}[D_\nu, D_\sigma](\bar\lambda^J\gamma_\rho\lambda^K) \\ &\quad + \mathcal{O}(\Phi^3)\end{aligned} \tag{4.18a}$$

$$\begin{aligned}&\doteq +\frac{1}{4}mf^{IJK}\mathcal{F}_{\nu\rho}{}^J G^{\mu\nu\rho K} - \frac{1}{4}m\epsilon^{\mu\nu\rho\sigma}f^{IJK}\mathcal{F}_{\nu\rho}{}^J\tilde G_\sigma{}^K + \mathcal{O}(\Phi^3) \\ &= \mathcal{O}(\Phi^3).\end{aligned} \tag{4.18b}$$

The terms $[D,D]\mathcal{F}\tilde G$ and $[D,D](\bar\lambda\gamma\lambda)$ in the second and third lines in (4.18a) are already at $\mathcal{O}(\Phi^3)$. We also combined (4.7b) with (4.8) for the $D\mathcal{F}$ term in (4.18a) to reach (4.18b). Finally, the two terms in (4.18b) canceled each other. It is easy to see that the two terms in (4.18b) cancel each other.

For the fermionic field equations (4.17d) and (4.17e), we can vary them under supersymmetry (4.9), and get the vanishing results as desired. In particular, the peculiar term $m^{-1}\mathcal{F}G$ at $\mathcal{O}(m^{-1})$ in $\delta_Q\chi$ of (4.9d) cancels other like terms arising in these variations.

Our system looks so involved with nonpolynomial interactions with unnecessary complication. This raises the question why we cannot follow more straightforward tensor-hierarchy formulations [1,2]. In order to answer such questions, we mention the following points. First, the nonpolynomial feature of our system is traced back to the corresponding feature in the supersymmetric Proca-Stückelberg formulation in Sec. III. It is mainly caused by the nontrivial $P \wedge P$ term in $\mathcal{F}$. Second, we are formulating the non-Abelian tensor without any more extra fields. If we could use an extra vector field such as $C_\mu{}^I$ in (1.1) and (1.2), it would be much simpler. However, our objective in this paper is to supersymmetrize Cremmer-Scherk theory [10], as economically as possible, with the bosonic fields $A_\mu{}^I$ and $B_{\mu\nu}{}^I$ alone with no more vector fields. In other words, minimizing the number of supermultiplets, we have to pay the price of nonpolynomial interactions. Third, this should not be regarded as a drawback, because we are reaching a new formulation for consistent non-Abelian tensor interactions that was not covered by previous tensor-hierarchy formulations [1,2].

We can investigate now the tensorial transformation invariance associated with $B_{\mu\nu}{}^I$. To be more specific, our action $I_{\rm SCS}$ is invariant also under the tensorial transformation for the $B_{\mu\nu}{}^I$ field (4.11) with the important lemma (4.12). Since the B field itself goes down from $\mathcal{O}(\Phi^1)$ to $\mathcal{O}(\Phi^0)$ under δ_β, and our Lagrangian is fixed up to $\mathcal{O}(\Phi^4)$, the action invariance is required only up to $\mathcal{O}(\Phi^3)$. Accordingly, the transformation for $\delta_\beta B$ is required only within $\mathcal{O}(\Phi^1)$, while $\mathcal{O}(\Phi^2)$ terms are ignored. The transformation rule $\delta_\beta G$ (4.12) contains also the field strength G itself on its rhs, reflecting again the intrinsic nonpolynomial structure of our system.

Notice the absence of any derivative term $D\beta$ on the parameter β in (4.12), as desired for a field strength. This nontrivial fact is confirmed, by using $\delta_\beta B_\mu{}^I$ of (4.11) in (4.6a):

$$\begin{aligned}[\text{LHS of (4.12)}] &= +3D_{[\mu|}[D_{|\nu|}\beta_{|\rho]}{}^I - 2f^{IJK}\beta_{|\nu|}{}^J\tilde G_{|\rho]}{}^K] \\ &\quad - 3f^{IJK}\tilde G_{[\mu|}{}^J(2D_{|\nu|}\beta_{|\rho]}{}^K) + \mathcal{O}(\Phi^2)\end{aligned} \tag{4.19a}$$

$$= +6f^{IJK}\beta_{[\mu|}{}^J\left(D_{|\nu|}\tilde G_{|\rho]}{}^K - \frac{1}{2}m\mathcal{F}_{|\nu\rho]}{}^K\right) + \mathcal{O}(\Phi^2) \tag{4.19b}$$

$$= [\text{RHS of (4.12)}] \quad (\text{Q.E.D.}). \tag{4.19c}$$

Here we have kept only $\mathcal{O}(\Phi^1)$ terms, e.g., $f^{IJK}(\delta_\beta\tilde G_{[\mu}{}^J)B_{\rho]}{}^K$ is already at $\mathcal{O}(\Phi^2)$, due to $\delta_\beta\tilde G_\mu{}^J = \mathcal{O}(\Phi^1)$. Note that the

$D\beta$ terms are canceled within (4.19a), while only βDG and $m\beta F$ terms are left in (4.19b), as desired. In other words, even though the definition of G contains the bare B field in the $G \wedge B$ term, its variation with $D\beta$ is canceled by the like term arising from $D(\delta_\beta B)$ yielding $(D\beta)G$. This also justifies the nontrivial Chern-Simons term $G \wedge B$ in the G-field strength (4.6a).

After all, the invariance $\delta_\beta I_{\rm SCS} = \mathcal{O}(\Phi^3)$ is also confirmed in a nontrivial way.[11] This follows from $\mathcal{L}_{\rm SCS}$ containing the bare B term: $B \wedge \tilde{G} \wedge \tilde{G}$. The variation of the B field here has the derivative $(D\beta) \wedge \tilde{G} \wedge \tilde{G}$. After a partial integration, this term cancels exactly the variation $\beta \wedge (D\tilde{G}) \wedge \tilde{G}$ term coming out of the $(\tilde{G})^2$ term. Eventually, the existence of the bare B term like $B \wedge \tilde{G} \wedge \tilde{G}$ in our Lagrangian will not hurt.

There are differences as well as similarities compared with our previous supersymmetric non-Abelian tensor formulation [4]. The latter had two multiplets: YM multiplet $(A_\mu{}^I, \lambda^I, C_{\mu\nu\rho}{}^I)$ and a TM $(B_{\mu\nu}{}^I, \chi^I, \phi^I)$. Similarities are such as two multiplets YM and TM in both formulations. The YM symmetry is also broken in both formulations.[12] However, the most important difference is that in our present formulation, the YM field becomes massive by the dualized Proca-Stückelberg mechanism. Another difference is that in our previous formulation [4], the third-rank auxiliary field $C_{\mu\nu\rho}{}^I$ becomes massive after the gauge-symmetry breaking by absorbing $B_{\mu\nu}{}^I$. In our present formulation, the YM multiplet $(A_\mu{}^I, \lambda^I)$ has no auxiliary field.

Needless to say, $\mathcal{L}_{\rm SCS}$ in (4.5) is the non-Abelian supersymmetric completion of purely bosonic Abelian terms: $(F_{\mu\nu})^2, (G_{\mu\nu\rho})^2$ and $\epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} F_{\rho\sigma}$ in (1.4a), as a nonconventional gauge-symmetry breaking mechanism in [10].

We mention one important aspect of the duality transformation [16]: $\varphi^I \to B_{\mu\nu}{}^I$. The realization of this duality transformation is nontrivial, because this was possible thanks to the very peculiar coupling of the Nambu-Goldstone (NG) field φ^I. For any duality transformation [16] (old potential field) → (new potential field) to be possible, the old potential field should appear only as a field strength, but not as a bare field.

For example, a duality transformation from the pseudoscalar ϕ^I to its hypothetical dual tensor field $K_{\mu\nu}{}^I$ does not work. This is because there is a minimal YM coupling: $D_\mu \phi^I \equiv \partial_\mu \phi^I + m f^{IJK} A_\mu \phi^K$, so its BId corresponding to the case of φ^I in (2.4) is

$$D_{[\mu} D_{\nu]} \phi^I = \frac{1}{2} m f^{IJK} F_{\mu\nu}{}^J \phi^K, \qquad (4.20)$$

with the bare field ϕ^K at the end. The routine constraint Lagrangian [16] plus the kinetic term of ϕ^I is[13]

$$\mathcal{L}_{\rm H} = -\frac{1}{2}(H_\mu{}^I)^2 + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} K_{\mu\nu}{}^I \left(D_{[\rho} H_{\sigma]}{}^I - \frac{1}{2} m f^{IJK} F_{\rho\sigma}{}^J \phi^K \right), \qquad (4.21)$$

where $K_{\mu\nu}{}^I$ is the multiplier field, $D_\mu \phi^I$ is replaced by an independent field $H_\mu{}^I$, and the field strength of $K_{\mu\nu}{}^I$ namely $L_{\mu\nu\rho}{}^I \equiv 3 D_{[\mu} K_{\nu\rho]}{}^I$ is dual to $H_\mu{}^I$. For the duality transformation $\phi^I \to K_{\mu\nu}{}^I$ to be possible, the old field ϕ^I should be only in terms of the field strength $D_\mu \phi^I$ that is replaced by $H_\mu{}^I$. The trouble is that the bare field ϕ^I still remains in (4.21). Because of this, the field equation of $H_\mu{}^I$ does not totally eliminate old field ϕ^I. This is why a duality transformation [16] from $\phi^I \to K_{\mu\nu}{}^I$ fails. The important and novel feature of BId (2.4) compared with (4.20) is that no bare field φ^I is involved in the BId (2.4), so a duality transformation $\varphi^I \to B_{\mu\nu}{}^I$ is possible. The duality transformation $\varphi^I \to B_{\mu\nu}{}^I$ is possible thanks to the BId (2.4) without the involvement of the bare φ field. The lesson here is that we have to distinguish which field strengths can be dualized to their Hodge duals by duality transformations [16], in particular, when dealing with many modified field strengths accompanied by generalized Chern-Simons terms such as tensor hierarchies [1–2].

V. SUMMARY AND CONCLUDING REMARKS

In this paper, we have presented the non-Abelian and the genuine supersymmetric generalization of Cremmer-Scherk theory [10] in $D = 3 + 1$, by including both kinetic terms of $A_\mu{}^I$ and $B_{\mu\nu}{}^I$. This formulation is based on supersymmetric Proca-Stückelberg theory in Sec. III, which is more economical than our previous formulations with extra vector multiplets [4,5].

The salient features of our formulations are summarized as follows: In our present paper, we have succeeded in the new supersymmetric formulations of gauge-symmetry breaking originally by Cremmer-Scherk [10] for non-Abelian tensors, combining many different novel formulations in the past. The field content is the YM multiplet $(A_\mu{}^I, \lambda^I)$ and the tensor-multiplet $(B_{\mu\nu}{}^I, \chi^I, \phi^I)$.

As a very important foundation of our super-Cremmer-Scherk formulation, we have presented a new supersymmetric Proca-Stückelberg formulation with only two multiplets: YM multiplet $(A_\mu{}^I, \lambda^I)$ and CM $(\varphi^I, \chi^I, \phi^I)$.

[11] Actually, this had been known since [17] associated with nonpolynomiality, but we reconfirm it due to its importance.

[12] We use here the word "broken" in the sense of the original Proca-Stückelberg formulation [6]. This is neither conventional spontaneous breaking nor "by-hand" explicit breaking.

[13] There are other ϕ^I-dependent terms in $\mathcal{L}_{\rm SCS}$, but just for simplicity of the argument here, we look into only these terms.

This formulation is more economical than our previous formulations [4,5]. In particular, φ^I and ϕ^I playing different roles under G.

We have provided enough supporting evidence for the consistency of our CM $(\varphi^I, \chi^I, \phi^I)$ coupled to non-Abelian SYM $(A_\mu{}^I, \lambda^I)$, where φ^I and ϕ^I transform differently under G. For example, we have given detailed confirmation of the closure on all fields in (3.10) through (3.15). We have also reconfirmed the closures on ϕ^I and φ^I in the corresponding superspace formulation (A8) through (A10). The total consistency among field equations is also reconfirmed in (3.16) and (3.17), by varying fermionic field equations under supersymmetry, yielding all bosonic field equations. Additionally, the consistency of the divergence of the YM-field equation is reconfirmed in (3.18) with enough details with intermediate steps.

In order to elucidate the crucial necessity of the field strength $\mathcal{F}_{\mu\nu}{}^I$ instead $F_{\mu\nu}{}^I$ in our Lagrangian $\mathcal{L}_{\rm SPS}$ in (3.1), we have shown in (3.7) the decisive terms in the variation $\delta_Q I_{\rm SPS}$. The sector $\lambda P \mathcal{F}$ in this variation clearly showed the necessity of $\mathcal{F}_{\mu\nu}{}^I$ instead $F_{\mu\nu}{}^I$ in the A_μ-kinetic term.

Our supersymmetric Cremmer-Scherk formulation is intrinsically nonpolynomial, which has not been covered by the general non-Abelian tensor-hierarchy formulations [1,2]. In particular, the presence of the $B_{\mu\nu}{}^I$ kinetic term with nonpolynomial structure is crucial. The nonpolynomial feature itself was already known in [17], but our supersymmetrization of both $A_\mu{}^I$ and $B_{\mu\nu}{}^I$ with their physical propagations was not accomplished in $D=3+1$ in the past [17,19–21].

We have seen that despite the nonpolynomial involvement of φ^I in the field strength $P_\mu{}^I \equiv [(D_\mu e^\varphi) e^{-\varphi}]^I$, the duality transformation [16] of φ^I to its Hodge dual $B_{\mu\nu}{}^I$ is possible in a nontrivial and nonpolynomial way. In other words, our formulation provides a new link between our new supersymmetric Proca-Stückelberg theory and supersymmetric Cremmer-Scherk theory [17].

Our supersymmetric Cremmer-Scherk formulation has the term $\epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}{}^I \mathcal{F}_{\rho\sigma}{}^I$ analogous to BF theories [22]. However, we have the YM vector $A_\mu{}^I$ physically propagating with its kinetic term, and therefore it is not pure gauge: $F_{\mu\nu}{}^I \neq 0$, as opposed to [17,19–21]. In this sense, the link with the σ model is entirely different from [17,19–21].

Our superspace reformulation in the Appendix is not based on the conventional chiral multiplets [19–21]. Instead, it is based on the superspace BIds. This method may cast doubt on the validity of such a formulation, because in 4D any component field in a supermultiplet is supposed to be expressed as the $\theta = 0$ sector of the super covariant derivative of a certain (pre)potential superfield [25,31].

To such a reasonable viewpoint, we respond as follows: Our system is an on-shell system, so that the conventional superfield methods for off-shell formulations [25,31] do not apply here. A typical on-shell system is $N = 1$ supergravity in 10D [30], and no off-shell formulation has been established in 10D. In such on-shell formulations, the only known method is to satisfy all BIds of all super field strengths, as we have done in our Appendix.

Despite the lack of off-shell superspace action of our system, there is an exceptional aspect. Namely, in the Abelian limit $f^{IJK} \to 0$, our super-Proca-Stückelberg Lagrangian $\mathcal{L}_{\rm SPS}$ (3.1) coincides with the Lagrangian respecting the gauged real-shift R symmetry (3.4.35) in superspace [31] and (3.4.39) in component [31]. This special case does have the off-shell superspace-action formulation. As such, our super-Proca-Stückelberg system is interpreted as non-Abelian generalization of that with the real-shift Abelian R symmetry. The details of this statement are given in the paragraphs with (A11) through (A14) in Appendix.

As a whole, our investigation based on duality transformations [16] for supersymmetric σ models, and nonconventional gauge-symmetry breaking in supersymmetric gauge theories provides a new avenue for supersymmetric formulation of non-Abelian tensors [1,2]. Our final super-Cremmer-Scherk theory is based on super-Proca-Stückelberg theory, as its very important foundation. Our formulation has a very economical set of super multiplets, which was neither explicitly presented as an application of tensor-hierarchy formulations [1,2] nor in the context of chiral-superfield formulations [19–21]. We have opened a new direction of supersymmetry in 4D, which has not been exploited in the past.

ACKNOWLEDGMENTS

We are grateful to the referee of this paper, who made every effort to improve its manuscript.

APPENDIX: SUPERSPACE REFORMULATION OF SUPERSYMMETRIC PROCA-STÜCKELBERG THEORY

As a reconfirmation of our $N = 1$ supersymmetric Proca-Stückelberg theory, we reformulate it in superspace [25]. The importance of this superspace reformulation is summarized as five points. First, it is the good reconfirmation of Sec. III for its consistency. Second, our scalar φ^I and pseudoscalar ϕ^I transform differently under gauge group, so we cannot use the conventional superfield formulation in terms of chiral superfields [19–21]. Third, our result here will be of importance for the future applications of similar superspace formulations. Fourth, our intrinsic nonpolynomial structure of our super-Cremmer-Scherk formulation originates from the $P \wedge P$ term in the modified field strength $\mathcal{F}$ in (3.2b), which will be also reconfirmed by our superspace formulation. Fifth, our superspace BIds do not automatically provide correct field contents. Instead, the superspace reformulation provides nontrivial reconfirmation of the consistency of our system, where any inconsistency will show up explicitly.

Our fundamental superfields are ${A_A}^I$, ${\lambda_\alpha}^I$, ${B_{AB}}^I$, ${\chi_\alpha}^I$ and φ^I.[14] Our superfield strengths ${P_A}^I$ and ${\mathcal{F}_{AB}}^I$ are defined by

$$ {P_A}^I \equiv [(E_A e^{\varphi})e^{-\varphi})]^I + m{A_A}^I \equiv [(\nabla_A e^{\varphi})e^{-\varphi})]^I, \quad \text{(A1a)} $$

$$ \begin{aligned} {\mathcal{F}_{AB}}^I &\equiv E_{[A}A_{B)}{}^I - C_{AB}{}^C {A_C}^I \\ &\quad + mf^{IJK}{A_A}^J{A_B}^K + m^{-1}f^{IJK}{P_A}^J{P_B}^K \\ &= {F_{AB}}^I + m^{-1}f^{IJK}{P_A}^J{P_B}^K \\ &= {F_{AB}}^I + m^{-1}[P_A, P_B\}^I, \end{aligned} \quad \text{(A1b)} $$

where $E_A \equiv {E_A}^M\partial_M$, and ${C_{AB}}^C \equiv E_{[A}E_{B)}{}^M$, as usual [25].

Our superspace BIds for ${P_A}^I$ and ${\mathcal{F}_{AB}}^I$ are

$$ \nabla_{[A}P_{B)}{}^I - {T_{AB}}^C{P_C}^I - m{\mathcal{F}_{AB}}^I \equiv 0, \quad \text{(A2a)} $$

$$ \nabla_{[A}\mathcal{F}_{BC)}{}^I - T_{[AB|}{}^D\mathcal{F}_{D|C)}{}^I - \frac{1}{2}f^{IJK}\mathcal{F}_{[AB}{}^J P_{C)}{}^K \equiv 0, \quad \text{(A2b)} $$

similar to [4,5]. These are also superspace generalizations of the component BIds (3.3). The following superspace constraints satisfy the BIds at engineering dimensions $0 \le d \le 1$:

$$ {T_{\alpha\beta}}^c = +2(\gamma^c)_{\alpha\beta}, \qquad {\mathcal{F}_{\alpha b}}^I = -(\gamma_b\lambda^I)_\alpha, \quad \text{(A3a)} $$

$$ {P_\alpha}^I = -{\chi_\alpha}^I, \qquad \nabla_\alpha\phi^I = -i(\gamma_5\chi^I)_\alpha, \quad \text{(A3b)} $$

$$ \begin{aligned} \nabla_\alpha{\chi_\beta}^I &= -(\gamma^c)_{\alpha\beta}{P_c}^I - i(\gamma_5\gamma^c)_{\alpha\beta}\nabla_c\phi^I \\ &\quad + ih^{IJ,KL}{\chi_\alpha}^K(\gamma_5\chi^J)_\beta\phi^L, \end{aligned} \quad \text{(A3c)} $$

$$ \nabla_\alpha{\lambda_\beta}^I = +\frac{1}{2}(\gamma^{cd})_{\alpha\beta}{\mathcal{F}_{cd}}^I - im(\gamma_5)_{\alpha\beta}\phi^I + f^{IJK}{\chi_\alpha}^J{\lambda_\beta}^K. \quad \text{(A3d)} $$

All other independent components, such as ${\mathcal{F}_{\alpha\beta}}^I$ are zero.

Similarly, the following constraints are obtained from BIds at $d = 3/2$:

$$ \nabla_\alpha{P_b}^I = -\nabla_b{\chi_\alpha}^I - m(\gamma_b\lambda^I)_\alpha, \quad \text{(A4a)} $$

$$ \begin{aligned} \nabla_\alpha{\mathcal{F}_{bc}}^I &= +(\gamma_{[b}\nabla_{c]}\lambda^I)_\alpha + f^{IJK}{\chi_\alpha}^J{\mathcal{F}_{bc}}^K \\ &\quad - f^{IJK}(\gamma_{[b|}\lambda^J)_\alpha P_{|c]}{}^K. \end{aligned} \quad \text{(A4b)} $$

[14]We use the superspace coordinate indices $A = (a, \alpha)$, $B = (b, \beta), \ldots$, where $a, b, \ldots = (0), (1), (2), (3)$ (or $\alpha, \beta, \ldots =$ 1, 2, 3, 4) are for bosonic (or fermionic) coordinates. For curved coordinates we use $M = (m, \mu), N = (n, \nu), \ldots$, where $m, n, \ldots = 0, 1, 2, 3$ (or $\mu, \nu, \ldots = 1, 2, 3, 4$). Our antisymmetrization in superspace is normalized as $M_{[AB)} \equiv M_{AB} - (-1)^{AB}M_{BA}$, without the factor of 1/2. We use these superspace notations only in this Appendix.

Finally, our superfield equations are obtained from (A2)–(A4) at $d = 3/2$ and 2:[15]

$$ (\nabla\!\!\!/\lambda^I)_\alpha + m{\chi_\alpha}^I - f^{IJK}(\gamma^b\lambda^J)_\alpha{P_b}^K - imf^{IJK}(\gamma_5\chi^J)_\alpha\phi^K \doteq 0, \quad \text{(A5a)} $$

$$ (\nabla\!\!\!/\chi^I)_\alpha + m{\lambda_\alpha}^I + imf^{IJK}(\gamma_5\lambda^J)_\alpha\phi^K \doteq 0, \quad \text{(A5b)} $$

$$ \nabla_a P^{aI} - mf^{IJK}(\bar{\lambda}^J\chi^K) \doteq 0, \quad \text{(A5c)} $$

$$ \begin{aligned} &\nabla_b\mathcal{F}^{abI} + m{P_a}^I + mf^{IJK}[(\bar{\lambda}^J\gamma_a\lambda^K) + \frac{1}{2}(\bar{\chi}^J\gamma_a\chi^K)] \\ &\quad - mf^{IJK}\phi^J\nabla_a\phi^K - f^{IJK}{\mathcal{F}_a}^{bJ}{P_b}^K \doteq 0, \end{aligned} \quad \text{(A5d)} $$

$$ \nabla_a^2\phi^I - m^2\phi^I - imf^{IJK}(\bar{\lambda}^J\gamma_5\chi^K) \doteq 0. \quad \text{(A5e)} $$

For example, the λ field Eq. (A5a) is obtained by the operation

$$ \begin{aligned} (\nabla\!\!\!/\lambda)_\alpha &= +\frac{1}{2}\{\nabla_\alpha, \nabla^\beta\}{\lambda_\beta}^I - \frac{1}{2}f^{IJK}{F_\alpha}^{\beta J}{\lambda_\beta}^K \\ &= \frac{1}{2}\nabla_\alpha(\nabla^\beta{\lambda_\beta}^I) + \frac{1}{2}\nabla^\beta(\nabla_\alpha{\lambda_\beta}^I) \end{aligned} \quad \text{(A6a)} $$

$$ \begin{aligned} &= +\frac{1}{2}f^{IJK}\nabla_\alpha(\bar{\lambda}^J\chi^K) \\ &\quad + \frac{1}{2}\nabla^\beta\left[+\frac{1}{2}(\gamma^{cd})_{\alpha\beta}{\mathcal{F}_{cd}}^I - im(\gamma_5)_{\alpha\beta}\phi^I + f^{IJK}{\chi_\alpha}^J{\lambda_\beta}^K\right] \\ &= +\frac{3}{2}(\nabla\!\!\!/\lambda^I)_\alpha + \frac{1}{2}m{\chi_\alpha}^I - \frac{1}{2}f^{IJK}(\gamma^b\lambda^J)_\alpha{P_b}^K \\ &\quad - \frac{i}{2}mf^{IJK}(\gamma_5\chi^J)_\alpha\phi^K. \end{aligned} \quad \text{(A6b)} $$

Needless to say, (A3) and (A4) have been used. In (A6a), the $\chi^2\lambda$ term out of the $F\lambda$ term has been ignored because it is at $\mathcal{O}(\Phi^3)$. The same is also true for the χ-field equation by

$$ \begin{aligned} (\nabla\!\!\!/\chi)_\alpha &= +\frac{1}{2}\{\nabla_\alpha, \nabla^\beta\}{\chi_\beta}^I = \frac{1}{2}\nabla_\alpha(\nabla^\beta{\chi_\beta}^I) + \frac{1}{2}\nabla^\beta(\nabla_\alpha{\chi_\beta}^I) \\ &= -(\nabla\!\!\!/\chi^I)_\alpha - 2m{\lambda_\alpha}^I - 2imf^{IJK}(\gamma_5\chi^J)_\alpha\phi^K, \end{aligned} \quad \text{(A7)} $$

yielding (A5b).

The bosonic superfield equations (A5c) through (A5e) are obtained by applying spinorial derivatives on the fermionic superfield equations (A5a) and (A5b). Since the computations will be just parallel to the component case (3.16) and (3.17), their details are skipped here.

These superfield equations are consistent with our component Lagrangian (3.1) and field equations in (3.9).

[15]These superfield equations are valid up to $\mathcal{O}(\Phi^3)$, because our Lagrangian $\mathcal{L}_{\text{SPS}}$ is valid up to $\mathcal{O}(\Phi^4)$.

Since our superspace-formulation system is an on-shell system without any auxiliary fields, these field equations are valid modulo field equations. For example, the λ-field equation term in (3.9b) does not show up explicitly in superspace in (A6), because they are parts of superspace "constraints" satisfying BIds at dimensions $d \ge 3/2$.

As has been also mentioned in Sec. III, because of the lack of fundamental scalar superfield such as L, which was valid for an Abelian TMs [28], we do not have a superspace action formulation at the present time.

In order to dispel skepticism about the coexistence of two fields φ^I and ϕ^I with the different transforming properties under gauge group G, we first reconfirm the closure (3.13) on ϕ^I in superspace:

$$\begin{aligned} 2(\gamma^c)_{\alpha\beta}\nabla_c\phi^I &= \{\nabla_\alpha,\nabla_\beta\}\phi^I - mf^{IJK}F_{\alpha\beta}{}^J\phi^K \\ &= \nabla_{(\alpha}\nabla_{\beta)}\phi^I - mf^{IJK}F_{\alpha\beta}{}^J\phi^K \\ &= \nabla_{(\alpha|}[-i(\gamma_5)_{|\beta)}{}^\gamma\chi_\gamma] \\ &\quad - mf^{IJK}(-m^{-1}f^{JLM}\chi_\alpha{}^L\chi_\beta{}^M)\phi^K \qquad (A8a) \\ &\overset{?}{=} -i(\gamma_5)_{(\beta|}{}^\gamma[-(\gamma^d)_{|\alpha)\gamma}P_d{}^I - i(\gamma_5\gamma^d)_{|\alpha)\gamma}\nabla_d\phi^I \\ &\quad + ih^{IJ,KL}\chi_{|\alpha)}{}^K(\gamma_5\chi^J)_\gamma\phi^L] \\ &\quad - h^{IK,LM}\chi_\alpha{}^L\chi_\beta{}^M\phi^K \qquad (A8b) \\ &= +2(\gamma^d)_{\alpha\beta}\nabla_d\phi^I \quad \text{(Q.E.D.)} \qquad (A8c) \end{aligned}$$

up to $\mathcal{O}(\Phi^3)$. In (A8a), the P-linear term vanishes, due to the antisymmetry $(\gamma_5\gamma^d)_{(\alpha\beta)} = 0$. Also in (A8a), we used $\mathcal{F}_{\alpha\beta}{}^I = F_{\alpha\beta}{}^I + f^{IJK}\chi_\alpha{}^J\chi_\beta{}^K$ and $\mathcal{F}_{\alpha\beta}{}^I = 0$. In (A8b), the $\chi^2\phi$ terms cancel each other as desired, thanks to the Jacobi identity $h^{I[J,K]L} = -h^{IL,JK}$.

To see the consistency of the coexistence of φ^I and ϕ^I, we finally reconfirm the closure (3.12): $[\nabla_{(\alpha}(\nabla_{\beta)}e^\varphi)]e^{-\varphi}$ by two different methods. The first one is using the definition of $P_\alpha{}^I$:

$$\begin{aligned} [\nabla_{(\alpha}(\nabla_{\beta)}e^\varphi)]e^{-\varphi} &= \nabla_{(\alpha|}[(\nabla_{\beta)}e^\varphi)e^{-\varphi}] + (\nabla_{(\alpha}e^\varphi)(\nabla_{\beta)}e^{-\varphi}) \\ &= \nabla_{(\alpha}P_{\beta)} - (\nabla_{(\alpha}e^\varphi)e^{-\varphi}(\nabla_{\beta)}e^\varphi)e^{-\varphi} \qquad (A9a) \\ &= -[-2(\gamma^c)_{(\alpha\beta)}P_c - i(\gamma_5\gamma^c)_{(\alpha\beta)}\nabla_c\phi] \\ &\quad - m^{-1}P_{(\alpha}P_{\beta)}, \qquad (A9b) \\ &= +2(\gamma^c)_{\alpha\beta}(\nabla_c e^\varphi)e^{-\varphi} - m^{-1}\chi_{(\alpha}\chi_{\beta)}, \qquad (A9c) \end{aligned}$$

where all terms are generator valued, carrying T^I, e.g., $\nabla_{(\alpha}P_{\beta)} \equiv (\nabla_{(\alpha}P_{\beta)}{}^I)T^I$, etc. In (A9a), we used $P_\alpha = -\chi_\alpha$, while in (A9b), we also used that $(\gamma_5\gamma^c)_{(\alpha\beta)} = 0$.

The second method is to use the anticommutator $\{\nabla_\alpha,\nabla_\beta\}$:

$$\begin{aligned} [\nabla_{(\alpha}(\nabla_{\beta)}e^\varphi)]e^{-\varphi} &= [\{\nabla_\alpha,\nabla_\beta\}e^\varphi]e^{-\varphi} \\ &= (+T_{\alpha\beta}{}^c\nabla_c e^\varphi + mF_{\alpha\beta}e^\varphi)e^{-\varphi} \qquad (A10a) \\ &= [+2(\gamma^c)_{\alpha\beta}\nabla_c e^\varphi \\ &\quad + m(\mathcal{F}_{\alpha\beta} - m^{-1}P_\alpha P_\beta)e^\varphi]e^{-\varphi} \qquad (A10b) \\ &= +2(\gamma^c)_{\alpha\beta}(\nabla_c e^\varphi)e^{-\varphi} - m^{-1}\chi_{(\alpha}\chi_{\beta)}. \qquad (A10c) \end{aligned}$$

In Eq. (A10a), we used $\{\nabla_\alpha,\nabla_\beta\}e^\varphi = T_{\alpha\beta}{}^c\nabla_c e^\varphi + mF_{\alpha\beta}e^\varphi$ which is the superspace version of (2.5c). Note that the gauge field strength $F_{\alpha\beta}$ in (A10a) differs from $\mathcal{F}_{\alpha\beta}$, and the difference is corrected in (A10b). Needless to say, we also used the constraint $\mathcal{F}_{\alpha\beta} = 0$ in (A10b).

As desired, the two results (A9c) and (A10c) based on two different methods agree with each other, showing the consistency of our system. It is crucial that the χ^2 terms at the quadratic order are consistent, which are by no means a simple accidental coincidence. It also involves the subtle difference between $\mathcal{F}_{\alpha\beta}$ and $F_{\alpha\beta}$, which should not be screwed up. The closure (3.12) in component language was tricky, but we also saw this subtlety is reflected also in superspace language in an equally sophisticated way.

The confirmations (A9) and (A10) are equivalent to the satisfactions of superspace BId (A2a) at the engineering dimension $d = 1$. In particular, the closures (A9) and (A10) justify the correct structure of $\nabla_\alpha\chi_\beta{}^I$ in (A3c), and equivalently $\delta_Q\chi^I$ in (3.4d) in component. This is hardly trivial results, because it further verifies the total consistency of the coexistence of φ^I and ϕ^I transforming differently under the gauge group G. The satisfaction of our BIds is not the result of accidental coincidences.

At the present time, we have neither a superspace action formulation nor an off-shell formulation with auxiliary fields. This is similar to higher-dimensional supersymmetry, where only on-shell formulations are known in superspace. Typical examples are such as 10D supergravity [30] or 11D supergravity [29].

Even though we have no superspace action formulation, at least in the Abelian limit ($f^{IJK} \to 0$), our system can be shown to be equivalent to that with gauged [31] R symmetry [24], as follows. First, we generalize the original R symmetry in [24] to the Abelian automorphism symmetry of 4D, $N = 1$ Poincaré superalgebra. It is realized in terms of the superspace Grassmann coordinates together with relevant fermionic component fields. Certain freedoms exist in representations due to extra transformations on superfields. To be more specific, a chiral superfield Φ is transformed under R symmetry, either by a phase: $\delta_\beta\Phi = i\beta\Phi$, real shift: $\delta_\alpha\Phi = \alpha$, or pure imaginary shift:

$\delta_\gamma \Phi = i\gamma$. The chiral superfields with different realizations of R symmetry are formally related as $\Phi = e^{\Phi'}$ or $\Phi = e^{i\Phi''}$. For example, the free chiral superfield Lagrangian $\Phi\bar{\Phi}$ is invariant (up to a total divergence) under all these three types of the R-symmetry representations. In the Abelian limit $f^{IJK} \to 0$, the real-shift R symmetry is

$$\delta_\alpha \varphi = -m\alpha, \qquad \delta_\alpha A_a = \partial_a \alpha, \qquad \delta_\alpha \phi = 0, \qquad \text{(A11)}$$

where φ (or ϕ) is identified with the real (or imaginary) part of the spin-0 fields of Φ. Our action I_{SPS} (3.1) in the Abelian limit $f^{IJK} \to 0$ is actually invariant under δ_α because of $\delta_\alpha P_a = \delta_\alpha(\partial_a \varphi + mA_a) = 0$.

Second, we can regard Eq. (3.4.35) in superfield formulation in [31] as the gauging of the shift realization of R symmetry. In other words, Eq. (3.4.35) in [31],

$$I_{\text{Abelian}} = \frac{1}{2}\int d^6 z W^\alpha W_\alpha + m^2 \int d^8 z V^2 + im \int d^8 z V(\Phi - \bar{\Phi}) + \frac{1}{2}\int d^8 z \bar{\Phi}\Phi, \qquad \text{(A12)}$$

can be interpreted as supersymmetric Abelian Proca-Stückelberg theory. Here we are following the conventional superspace notation [31], such as $W_\alpha \equiv -(1/4)\bar{D}^2 D_\alpha V$. The real-shift R symmetry is gauged in (A12), as I_{Abelian} is invariant under [31]

$$\delta_\alpha \Phi = m\Lambda, \qquad \delta_\alpha \bar{\Phi} = m\bar{\Lambda}, \qquad \delta_\alpha V = \frac{i}{2}(\bar{\Lambda} - \Lambda). \qquad \text{(A13)}$$

The component Lagrangian of (A12) is equivalent to (3.4.39) in [31]:

$$\mathcal{L}_{\text{Abelian}} = -\frac{1}{4}(F_{ab})^2 + \frac{1}{2}(\bar{\lambda}\partial\!\!\!/\lambda) - \frac{1}{2}m^2(A_a)^2 + \frac{1}{2}(\bar{\chi}\partial\!\!\!/\chi) - \frac{1}{2}(\partial_a\phi)^2 + m(\bar{\lambda}\chi) - \frac{1}{2}m^2\phi^2, \qquad \text{(A14)}$$

up to appropriate normalizations. The only minor difference is that (3.4.39) in [31] keeps auxiliary fields F and D, while using the gauge $\varphi = 0$. The gauge $\varphi = 0$ is equivalent to the absorption of our φ field into the longitudinal component of A_a: $P_a \equiv \partial_a \varphi + mA_a \to +mA_a$. Modulo these points, (A14) is nothing but our Lagrangian $\mathcal{L}_{\text{SPS}}$ (3.1) in theit Abelian limit $f^{IJK} \to 0$. In other words, in the Abelian limit of (3.1), we have the superspace action formulation (A12) in terms of the chiral Φ and real V superfields. Hence in the Abelian limit, the two superfields V and Φ are fully disentangled off-shell and prior to any gauge fixing.

From this viewpoint, our supersymmetric Proca-Stückelberg Lagrangian $\mathcal{L}_{\text{SPS}}$ (3.1) can be interpreted as the non-Abelian generalization of the Abelian case (A12), (A14) or (3.4.39) in [31].

[1] B. de Wit and H. Samtleben, Fortschr. Phys. **53**, 442 (2005); B. de Wit, H. Nicolai, and H. Samtleben, J. High Energy Phys. 02 (2008) 044; C.-S. Chu, arXiv:1108.5131; H. Samtleben, E. Sezgin, and R. Wimmer, J. High Energy Phys. 12 (2011) 062.

[2] H. Nishino and S. Rajpoot, Phys. Rev. D **85**, 105017 (2012).

[3] For higher-dimensional applications, see Appendix of Ref. [2].

[4] H. Nishino and S. Rajpoot, Nucl. Phys. **B872**, 213 (2013).

[5] H. Nishino and S. Rajpoot, Nucl. Phys. **B887**, 265 (2014).

[6] A. Proca, J. Phys. Radium **7**, 347 (1936); E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 225 (1938); see, e.g., D. Feldman, Z. Liu, and P. Nath, Phys. Rev. Lett. **97**, 021801 (2006); for reviews, see, e.g., H. Ruegg and M. Ruiz-Altaba, Int. J. Mod. Phys. A **19**, 3265 (2004).

[7] H. Nishino and S. Rajpoot, Phys. Lett. B **747**, 93 (2015).

[8] R. Jackiw and S.-Y. Pi, Phys. Lett. B **403**, 297 (1997); R. Jackiw, arXiv:hep-th/9705028.

[9] H. Nishino and S. Rajpoot, Phys. Lett. B **757**, 237 (2016).

[10] E. Cremmer and J. Scherk, Nucl. Phys. **B72**, 117 (1974).

[11] Y. Nambu, Phys. Rev. D **117**, 648 (1960); J. Goldstone, Nuovo Cimento **19**, 154 (1961); P. W. Higgs, Phys. Rev. Lett. **13**, 508 (1964).

[12] See, e.g., J. H. Schwarz, Phys. Rep. **8**, 269 (1973), and references therein.

[13] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Vols. I and II (Cambridge University Press, Cambridge, England, 1986).

[14] E. Bergshoeff, E. Sezgin, and P. K. Townsend, Phys. Lett. B **189**, 75 (1987); Ann. Phys. (N.Y.) **185**, 330 (1988).

[15] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, England, 2007).

[16] H. Nicolai and P. K. Townsend, Phys. Lett. **98B**, 257 (1981).

[17] D. Z. Freedman and P. K. Townsend, Nucl. Phys. **B177**, 282 (1981).

[18] M. Henneaux and B. Knaepen, Phys. Rev. D **56**, R6076 (1997); U. Theis, Phys. Lett. B **609**, 402 (2005).

[19] T. E. Clark, C. H. Lee, and S. T. Love, Mod. Phys. Lett. A **04**, 1343 (1989).

[20] K. Furuta, T. Inami, H. Nakajima, and M. Nitta, Prog. Theor. Phys. **106**, 851 (2001).

[21] F. Brandt and U. Theis, Fortschr. Phys. **48**, 41 (2000).

[22] G. T. Horowitz, Commun. Math. Phys. **125**, 417 (1989); for more references, see, e.g., D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, Phys. Rep. **209**, 129 (1991).

[23] A. Salam and J. Strathdee, Nucl. Phys. **B87**, 85 (1975).

[24] P. Fayet, Nucl. Phys. **B90**, 104 (1975).
[25] J. Wess and J. Bagger, *Superspace and Supergravity* (Princeton University Press, Princeton, NJ, 1992); S. J. Gates, Jr., M. T. Grisaru, M. Roček, and W. Siegel, Front. Phys. **58**, 1 (1983).
[26] See, e.g., A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982).
[27] A. Lahiri, Phys. Rev. D **55**, 5045 (1997); E. Harikumar, A. Lahiri, and M. Sivakumar, Phys. Rev. D **63**, 105020 (2001).
[28] S. Ferrara, B. Zumino, and J. Wess, Phys. Lett. **51B**, 239 (1974); W. Siegel, Phys. Lett. **85B**, 333 (1979); U. Lindstrom and M. Roček, Nucl. Phys. **B222**, 285 (1983); for reviews of linear multiplet coupled to SG, see, e.g., P. Bintruy, G. Girardi, and R. Grimm, Phys. Rev. D **343**, 255 (2001), and references therein; P. Binetruy, G. Girardi, and R. Grimm, Phys. Rep. **343**, 255 (2001).
[29] E. Cremmer and S. Ferrara, Phys. Lett. **91B**, 61 (1980).
[30] B. E. W. Nilsson, Nucl. Phys. **B188**, 176 (1981).
[31] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity, of A Walk through Superspace* (IOP, Bristol, UK, 1998).