Quantum propagation in Smolin's weak coupling limit of 4D Euclidean gravity

Madhavan Varadarajan
Raman Research Institute Bangalore-560 080, India

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Two desirable properties of a quantum dynamics for loop quantum gravity (LQG) are that its generators provide an anomalyfree representation of the classical constraint algebra and that physical states which lie in the kernel of these generators encode propagation. A physical state in LQG is expected to be a sum over graphical $SU(2)$ spin network states. By propagation we mean that a quantum perturbation at one vertex of a spin network state propagates to vertices which are "many links away" thus yielding a new spin network state which is related to the old one by this propagation. A physical state encodes propagation if its spin network summands are related by propagation. Here we study propagation in an LQG quantization of Smolin's weak coupling limit of Euclidean gravity based on graphical $U(1)^3$ "charge" network states. Building on our earlier work on anomalyfree quantum constraint actions for this system, we analyse the extent to which physical states encode propagation. In particular, we show that a slight modification of the constraint actions constructed in our previous work leads to physical states which encode robust propagation. Under appropriate conditions, this propagation merges, separates and entangles vertices of charge network states. The "electric" diffeomorphism constraints introduced in previous work play a key role in our considerations. The main import of our work is that there are choices of quantum constraint constructions through LQG methods which are consistent with vigorous propagation thus providing a counterpoint to Smolin's early observations on the difficulties of propagation in the context of LQG type operator constructions. Whether the choices considered in this work are physically appropriate is an open question worthy of further study.

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I. INTRODUCTION

Loop quantum gravity [1–3] is an attempt at nonperturbative canonical quantization of general relativity based on a classical Hamiltonian description in terms of triads (or, equivalently $SU(2)$ electric fields) and conjugate connections. While the $SU(2)$ rotation constraint and the 3d diffeomorphism constraints can be satisfactorily represented and solved in quantum theory, the construction of the Hamiltonian constraint operator involves an infinity of choices. In order to identify the correct choice of the Hamiltonian constraint, these choices should be confronted with physical criteria. Two such criteria are that the constraint action be compatible with an anomalyfree representation of the classical constraint algebra and that the constraint action should be consistent with the propagation of quantum perturbations. Whereas the "anomalyfree" criterion ensures spacetime covariance, the "propagation" criterion is motivated by the existence of classical solutions to general relativity which describe propagating degrees of freedom.

A clear statement of the propagation criterion in the context of LQG states which live on graphs was first provided by Smolin [4] as follows. Given a spin network state, the action of a constraint deforms the state in the vicinity of one of its vertices to yield a "perturbed" spin net state. If the quantum dynamics is such that this perturbation moves to distant (in terms of graph connectivity) vertices of the original spin net state, the quantum dynamics will be said to encode propagation and the initially perturbed state and the final state can be said to be related by propagation. Further, Smolin envisaged putative propagation as being generated by successive actions of the Hamiltonian constraint and concluded that LQG techniques were unable to generate propagation when viewed in this way. The underlying reason for the inability of successive actions of the Hamiltonian constraint to generate propagation is the *ultralocality* of such actions. More in detail, LQG techniques yield constraint actions on spin network states which are nontrivial only at the vertices of the spin network state being acted upon. Further, the action at one vertex is independent of that at any other vertex and results in a deformation of the vertex structure of the spin net in the immediate vicinity of the vertex being acted upon. Further actions at this vertex yield further deformations in its immediate "ultralocal" vicinity. Since such ultralocal actions do not involve more than one vertex, such actions cannot propagate perturbations to vertices far away.

One way out of this, as noted by Smolin, is to define the constraint action to involve more than one vertex. However, as also noted by Smolin, it seems impossible to construct operators with such actions as direct quantizations of the classical expression for the Hamiltonian constraint.[1]

However, the study of parametrized field theory (PFT) [6] implies that rather than visualizing propagation as being generated by successive actions of constraint operators, propagation should be seen as a property of physical states as follows. Physical states lie in the joint kernel of the quantum constraints and are constructed as sums of infinitely many spin net states. Given one such spin net summand, if there exists another summand which is related to the first by propagation then we shall say that the physical state encodes propagation. Note that this notion of propagation as encoded by physical states is logically distinct from that of spin net deformations generated by successive actions of constraint operators. The former involves the structure of *solutions* to the constraint whereas the latter involves the action of the constraint *operator*. Hence it is possible in principle (and seen in practice in [6]) that physical states can encode propagation despite the inability of successive operator actions to generate such propagation. Note also that while the notion of propagation in terms of physical states is not *generated* by the repeated ultralocal action of the constraints, it is nevertheless crucially dependent on the properties of this action. The reason is as follows. The properties of the action of the constraint operators determine the structural properties of physical states because physical states are constructed precisely so as to be annihilated by this action. More in detail since every physical state is a sum over spin network summands, how these summands relate to each other is then determined by properties of the constraint action. As mentioned above, propagation is encoded in these relations and it is in this indirect way that the properties of the constraint operator action dictate if propagation ensues or not. It is in this sense that the propagation criterion restricts the choice of the Hamiltonian constraint operator.[2]

Next, consider the anomalyfree criterion. While LQG does provide an anomalyfree representation of a very nontrivial subalgebra of the constraint algebra, namely that of the spatial diffeomorphism constraints [7–9], the implementation of a *nontrivial* anomalyfree commutator corresponding to the Poisson bracket between a pair of Hamiltonian constraints is still an open problem [3,8,10]. A key identity discovered in [11], implies that this Poisson bracket is the same as that of a sum of Poisson brackets between pairs of *diffeomorphism* constraints smeared with *electric field dependent* shifts which we shall call electric shifts. Such diffeomorphism constraints are called *electric diffeomorphism constraints*. Hence an implementation of nontrivial anomalyfree commutators between Hamiltonian constraints is equivalent to the imposition of this identity in quantum theory. This requires the construction not only of the Hamiltonian constraint operator but also these electric diffeomorphism operators in such a way that the commutator between a pair of Hamiltonian constraints equals the appropriate sum of commutators of electric diffeomorphism constraints. This requirement is extremely nontrivial. However, precisely because of this fact, the anomalyfree criterion is expected to prove extremely restrictive for the choice of Hamiltonian constraint operator.

Given the nontriviality of the two criteria and the involved nature of full blown LQG, it is useful to first develop intuition for structural properties of the Hamiltonian constraint which are compatible with these criteria in simpler toy models. In this regard Smolin's weak coupling limit of Euclidean gravity [12] offers an ideal testing ground. Since this system may be obtained simply be replacing the $SU(2)$ electric and connection fields of Euclidean gravity by their $U(1)^3$ counterparts, we shall refer to this model as the $U(1)^3$ model. Its constraint algebra is isomorphic to that of Euclidean gravity and hence displays structure functions. An LQG quantization of this system [11,13–15] leads to a representation space for holonomy- flux operators spanned by $U(1)^3$ spin network states which we call "charge" network states. These states are labeled by graphs whose edges are colored by representations of $U(1)^3$. Our recent work [15] concerns the imposition of the *anomalyfree criterion*, as articulated above, in the context of this model. Here we confront the ideas of [15] with the *propagation criterion* as applied to physical states. In doing so we shall not be concerned with the enormous amount of technical detail entailed in the constructions of [15]. Rather, we shall abstract what we regard as the key features of those constructions and base our analysis of propagation on these features and minimal generalizations and modifications thereof. Whether this broad treatment can then be endowed with the level of technical detail in [15] so as to demonstrate compliance with anomaly freedom is an open question.

We now turn to an account of the key features of the work in [15]. The Hamiltonian constraint operator

[1]While this brief account represents the broad lesson drawn by Smolin, it is a drastic oversimplification of Smolin's analysis. The reader is urged to consult Ref. [4] for a detailed account of this analysis. It is also pertinent to note here that there is a hitherto unnoticed obstruction to his beautiful arguments. We shall report on this separately in a note with Thiemann [5].

[2]As we shall see in later in this paper as well as in [5] the key property of the constraint operator action responsible for propagation in physical states is that this action should be consistent with *nonunique parentage*. A spin net is said to have nonunique parentage if it can be obtained (up to diffeomorphisms) by the action of the Hamiltonian constraint on two (diffeomorphically) distinct spin nets; if we refer to the spin net in question as a "child," since this child is obtained by the action of the constraints on two distinct "parents," we may describe the child spin net as having "nonunique parentage").

constructed in that work acts nontrivially on a charge network state only at those of its vertices which have valence greater than 3 provided these vertices are non-degenerate in a precise sense.[3] The Hamiltonian constraint action on any such N ($N > 3$) valent vertex of a parent charge net results in a sum over deformed child charge nets. The graph underlying a child charge net is obtained by deforming the parental graph in the vicinity of its vertex along some parental edge at that vertex. Roughly speaking, this deformation along a parental edge corresponds to a "singular diffeomorphism" wherein the remaining $N-1$ edges are pulled "almost" along this edge so as to form a cone with axis along this edge. The resulting "child graph" now has the original N valent parent vertex and an N valent child vertex, the child vertex structure being conical in the manner described. In addition the charges in the vicinity of these two vertices are altered, these alterations arisng from flipping the signs of certain parental charges due to which the original parent vertex is now no longer non-degenerate in the child. Thus, the constraint acts through a combination of "singular diffeomorphisms" and "charge flips." For obvious reasons, and for future reference, we shall refer to the Hamiltonian and electric diffeomorphism constraint actions as $N \to N$ actions.

Reference [15] constructs a space of anomalyfree states which support the action of this Hamiltonian constraint operator so as to yield an anomalyfree constraint algebra in the detailed sense articulated in that work and sketched briefly above. Each such state is a specific linear combination of certain charge net states. Thus each such state is specified by a "ket set" of charge net states and a set of coefficients, one for each element of the ket set associated with the anomalyfree state. The sum over all the elements of the ket set with these coefficients then yields the anomalyfree state so specified, on which constraint commutators are anomaly free in the sense described above. In particular the Hamiltonian constraint commutators are shown to equal the appropriate sum of electric diffeomorphism constraint commutators in accordance with the identity discovered in [11]. The electric diffeomorphism constraint operators can be constructed very naturally in a manner similar to the Hamiltonian constraint. The operators so constructed move the original parent vertex by exactly the same "singular diffeomorphisms" as employed by the Hamiltonian constraint, but with *no* charge flips, with the singular nature of the singular diffeomorphisms arising from the distributional nature of the quantum electric shift.

[3]The Hamiltonian and electric diffeomorphism constraint actions of [15] are specified on charge net states with only a single vertex whereas the discussion of propagation necessarily involves multivertex chargenet states. However, since these actions at different vertices are independent of each other, the actions of [15] automatically define actions on multivertex charge nets through a sum over actions on individual vertices.

The constructions of [15] also imply that in the special case that the coefficients are chosen to be unity for all elements of the ket set, it turns out that the anomalyfree state is killed by the Hamiltonian and diffeomorphism constraints as well as the electric diffeomorphism constraints. Such a state is then a physical state which supports a trivial anomalyfree realization of the commutators. Nontrivial anomalyfree commutators arise only if the coefficients are chosen in a specific nontrivial way and the resulting state is then an *off shell* state. From [15], such an anomalyfree state can be thought of as an off-shell deformation of the physical state obtained with unit coefficients. The ket set associated with a physical state and its off shell deformations in [15] satisfy the following properties. The first property is that the ket set is closed with respect to deformations generated not only by the action of the Hamiltonian constraint but also by the electric diffeomorphism constraints. Thus, in the parent-child language used above, this property says that if a certain charge net is in the ket set then so are all its deformed children produced by the action of the Hamiltonian and electric diffeomorphism constraints. The second property is more subtle and can be phrased succinctly in the parent-child language as follows. If a chargenet is in the ket set then so are all its possible parents. Here a "possible parent" p of a charge net c refers to any charge net which when acted upon by the Hamiltonian or electric diffeomorphism constraints gives rise to deformed children, one of which is the charge net c. In addition to these properties, the ket set is also closed with respect to semianalytic diffeomorphisms; amongst other things, this is necessary for an anomalyfree representation of the commutators involving the (usual) diffeomorphism constraints smeared with c-number shifts. We summarize these properties in the form of the statement (a) below:

(a) If the ket set has a certain charge net then

(a.1): All possible children generated by the action of the Hamiltonian and electric diffeomorphism constraints on this charge net are also in the ket set.

(a.2): All possible *parents* of this charge net (i.e., all charge nets which when acted upon by these constraints generate children one of which is the charge net in question) are also in the ket set.

(a.3): The ket set is closed with respect to the action of semianalytic diffeomorphisms.

In terms of ket sets subject to property (a),[4] our discussion of propagation may be restated as follows. Smolin's visualisation of propagation is based on the repeated action of constraint operators. Such actions concern property (a1) but not (a2). The key new element to be

[4]Reference [15] only constructs ket sets each of whose elements have single non-degenerate vertex. Here we implictly assume that the considerations of [15] can be generalized to multivertex ket sets. We shall discuss this further in Sec. V A.

confronted when we analyze physical states are the summands which owe their existence due to property (a2).

The fact that the sum, with unit coefficients, over elements of the ket set subject to Property (a) defines an *anomalyfree, physical* state, is a direct consequence of a particular structural property of the Hamiltonian and electric diffeomorphism constraint approximants employed in [15]. We shall describe this property in Sec. II G. We note here, that if we drop the anomalyfree requirement on physical states, there is no reason to consider the electric diffeomorphism constraints. Then, as will be apparent in Sec. II G, for Hamiltonian constraints with this particular structural property, physical states may be constructed as sums over elements of ket sets with a weaker version of property (a) wherein any mention of the electric diffeomorphism constraint is removed so that all children and "possible" parents are only with reference to the Hamiltonian constraint. However such states will not, in general, support anomalyfree commutators since we have no control on the "right-hand side" of the key identity of [11]. If we now construct electric diffeomorphism constraint operators with the structural property described in Sec. II G, then physical states which support anomalyfree commutators may be naturally constructed as sums over elements of ket sets with unit coefficients, these ket sets being subject to property (a) in which both the Hamiltonian and electric diffeomorphism constraints play a role. Such physical states are then killed by both the Hamiltonian and electric diffeomorphism constraints and thereby provide a consistently trivial implementation of the anomalyfree requirement. To summarize: for Hamiltonian and electric diffeomorphism constraint actions with the structural property described in Sec. II G, anomalyfree physical states can be constructed as sums, with unit coefficients, over elements of ket sets subject to property (a) above.

In this work, it will prove necessary to slightly modify the constraint approximants of [15] so as to engender propagation. The modified actions also have the special structural property of Sec. II G. Hence the ket sets we consider in this work will all be subject to Property (a) and will define anomalyfree physical states. Whether propagation ensues or not for a particular choice of such actions is then dependent entirely on the properties of the possible parents in property (a2).[5] Hence our strategy in this work is to analyse whether the ket sets relevant to choices of constraint actions with the structure described in Sec. II G have possible parents which facilitate propagation. If they do, it follows that the physical states obtained as sums over elements of such ket sets encode propagation. In what follows we shall often use a more direct language and simply say that such ket sets encode propagation. As stated above, while physical states constructed as sums over elements of ket sets are guaranteed to be anomaly free with respect to the associated constraint actions of the type discussed in Sec. II G, whether off shell deformations of these physical states can be constructed which support *nontrivial* anomalyfree commutator brackets is an open question which we leave for future work.

We are now in a position to discuss the work done in this paper. As a warm up exercise, we start with an exploration of propagation of specific perturbations between the vertices of a simple 2 vertex state. This exercise serves to illustrate the notion of propagation (or lack thereof) in the language of ket sets and possible parents articulated above. The simple 2 vertex state $|s_{A,B,N}\rangle$ that we study consists of a pair of vertices A, B joined by N edges with charges subject to a genericity condition. We create a "perturbation" of this state in the vicinity of the vertex A by the action of the Hamiltonian constraint. We show that this perturbation cannot "propagate" to vertex B and be "absorbed" there in the context of the constraint actions constructed in [15]. In the language of ket sets, we show that the minimal ket set which satisfies property (a) with respect to the constraint actions of [15] and which contains $|s_{A,B,N}\rangle$ *does not* encode propagation of this specific type of perturbation. Equivalently, the physical state annihilated by the constraint actions of [15] and obtained by summing over the elements of this ket set *does not* encode propagation of this specific perturbation. Next, we study physical states subject to a further physically reasonable condition. This condition implies that physical states satisfy additional operator equations which are also of the form discussed in Sec. II G. A natural class of anomalyfree physical states which are annihilated by the constraints of [15] and satisfy the new condition may then be constructed as sums over elements of ket sets which satisfy, in addition to property (a), the property of closure with respect to children and possible parents appropriate to the new condition. Once again, we study the minimal ket set which contains the 2 vertex state $|s_{A,B,N}\rangle$. In this case, we study the perturbation created at the vertex A by the action of operators involved in the specification of this additional condition. We show that this perturbation can propagate to vertex B and be absorbed there. In the language of ket sets, this minimal ket set *does* encode propagation of this specific perturbation from A to B by virtue of the richer class of children and possible parents whose existence is traced to the new condition. Thus, there is a natural class of physical states which satisfy an additional physical condition and which do encode propagation between vertices of a generic 2 vertex states of the type $|s_{A,B,N}\rangle$. We note here that the new condition is closely related to the combination of constraints appearing in Ref. [16] (see vi), pg 85, Chap. 6 of

[5]The structural property of constraint approximants alluded to above was first discovered in the context of PFT [6]. While we did not phrase that analysis explicitly in terms of ket sets, it is straightforward to check that such a rephrasing is immediate and that the key lesson of that analysis is the role played by the properties of possible parents of (a2) in enabling propagation.

this reference). This concludes our study of propagation of specific types of perturbations of this simple 2 vertex state.

Next, we consider generic multivertex states with more than 2 vertices. We show that the "$N \to N$" constraint actions of [15] cannot generate propagation of any perturbations between pairs of vertices of different valence. We argue that at best, even for states subject to the additional condition mentioned above, only a certain "1d" propagation may be possible for special multivertex states. Therefore, in order to engender a more vigorous, 3d and long range propagation, we slightly modify the $N \to N$ constraint actions of [15]. The key features of the modified constraint actions can be seen to arise from valid quantization choices and differ from those of [11,13,15] in that they *change* the valence of the vertex on which they act. It then turns out that ket sets satisfying condition (a) with respect to this modified action have a significantly richer structure which encodes vigorous propagation. The key property of the modified action is that it changes the valence of both the original parent vertex as well as the child vertex in the deformed children charge nets relative to [15]. This is achieved by visualizing the "singular diffeomorphism" deformation involved in the action of the Hamiltonian and electric diffeomorphism constraints slightly differently from that described above as follows.

We imagine the generalized diffeomorphisms to act by pulling all but 3 of the remaining edges *exactly* along the Ith edge with the remaining 3 edges being pulled "almost" along the Ith edge in a conical manner. This results in a child graph in which the original parent vertex valence drops to $N - 3$ and the child vertex has a valence of 4. With the incorporation of this modified action into that of the Hamiltonian and electric diffeomorphism constraints, it turns out that the ket sets subject to condition (a) do encode propagation. We shall refer to this modification of the constraint action as an $N \to 4$ modification. As we shall see, the deformations generated by the electric diffeomorphism constraints play a crucial role in this encoding of propagation. Since the only reason the electric diffeomorphism constraints appear in our considerations is to ensure compliance with the anomalyfree criterion, this suggests that the two criteria of anomaly freedom and propagation work in unision.

The layout of the paper is as follows. Section II starts with a brief review of earlier material in [6,11,13,15] which is of direct relevance to our work here. Specifically, in Secs. II A–II E we review the constraint actions of [15]. In the interests of pedagogy we suppress certain important details in our treatment; these details are collected and described in Sec. II F and may be skipped by readers unfamiliar with [15]. In Sec. II G we review the structural property of constraint approximants alluded to above which is connected with property (a). Section III studies propagation in the context of the $N \to N$ actions of [15]. In Sec. III A we study propagation of specific perturbations in a simple 2 vertex state as discussed above. In Sec. III B we show that the constraint actions of [15] cannot engender propagation between pairs of vertices of different valence in multivertex states and argue that, at best, a "1d propagation" may be possible for a very restrictive class of multivertex states. In Sec. IV we describe the $N \to 4$ modification of the constraint action and show that the ket set compatible with this modified action encodes vigorous propagation. As mentioned above, a comprehensive proof that the constraint action considered in Sec. IV has a nontrivial anomalyfree implementation would be at least as involved as the considerations of [15] and is out of the scope of this work. In Sec. V we discuss the new challenges to be confronted relative to Ref. [15] in the construction of such a putative proof as well as certain technicalities related to our treatment of propagation hitherto. In Sec. VI we discuss an important consequence of the $N \to 4$ action, namely the phenomenon of vertex mergers. Section VII contains a discussion of our results and of open issues.

Our work here may be considered as a continuation of that in the series of papers [11,13,15]. While a detailed understanding of the considerations of this work, especially that of Secs. II F and V, requires familiarity with these works, an understanding of the broad features of this work requires familiarity only with the reasonably self contained expositon of Secs. II A–II E and II G of this work. Readers not familiar with [11,13,15] may skip Secs. II F and V on a first reading. Further, the reader interested mainly in the long range 3d propagation results may skip Sec. III A entirely.

II. BRIEF REVIEW OF RELEVANT MATERIAL FROM REFS. [6,15]

A. Elements of the classical theory

The phase space variables $(A_a^i, E_i^a, i = 1, 2, 3)$ are a triplet of $U(1)$ connections and conjugate density weight one electric fields on the Cauchy slice Σ with canonical Poisson brackets $\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^3(x, y)$. The Gauss law, diffeomorphism, and Hamiltonian constraints of the theory are

$$G[\Lambda] = \int d^3x \Lambda^i \partial_a E_i^a \tag{2.1}$$

$$D[\vec{N}] = \int d^3x N^a (E_i^b F_{ab}^i - A_a^i \partial_b E_i^b) \tag{2.2}$$

$$H[N] = \frac{1}{2} \int d^3x N q^{-1/3} \epsilon^{ijk} E_i^a E_j^b F_{ab}^k, \tag{2.3}$$

with $F_{ab}^i \coloneqq \partial_a A_b^i - \partial_b A_a^i$, $qq^{ab} \coloneqq \sum_i E_i^a E_i^b$, $q = \det q_{ab}$. A key identity [11] holds on the Gauss law constraint surface:

$$\{H[N], H[M]\} = (-3)\sum_{i=1}^{3}\{D[\vec{N}_i], D[\vec{M}_i]\} \quad (2.4)$$

where the electric shifts N_i^a are defined as:

$$N_i^a = N E_i^a q^{-1/3} \quad (2.5)$$

and the electric diffeomorphism constraints $D(\vec{N}_i)$ by

$$D[\vec{N}_i] = \int d^3x N_i^a E_j^b F_{ab}^j \quad (2.6)$$

B. Quantum kinematics

A charge network label c is the collection $(\alpha, \vec{q}_I, I = 1, \ldots, N)$ where α is an oriented graph with N edges, the Ith edge e_I colored with a triplet of $U(1)$ charges $(q_I^1, q_I^2, q_I^3) \equiv \vec{q}_I$ such that the net outgoing charge at every vertex vanishes. The gauge invariant holonomy associated with c is h_c,

$$h_c := \prod_{I=1}^{N} e^{i\kappa\gamma q_I^j \int_{e_I} A_a^j dx^a}, \quad (2.7)$$

where κ is a constant with dimensions ML^{-1}, γ is a dimensionless Immirzi parameter. Henceforth we use units such that $\kappa\gamma = 1$. The Hilbert space is spanned by charge network states $|c\rangle$ which are eigen states of the electric field operator. The eigen value of the electric shift operator $\hat{N}_i^a(x)$ (see (2.5)) is nonzero only at vertices of the charge net state and requires a regulating coordinate patch at each of these vertices for its evaluation:

$$\hat{N}_i^a(v)|c\rangle = N_i^a(v)|c\rangle := \sum_{I_v} N_{I_v i}^a |c\rangle,$$
$$N_{I_v i}^a := \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} q_{I_v}^i \hat{e}_{I_v}^a. \quad (2.8)$$

Here v is a vertex of c, I_v refers to the I_vth edge at v, and $\hat{e}_{I_v}^a$ to the unit I_vth edge tangent vector, unit with respect to the coordinates $\{x\}$ at v and $N(x(v))$ denotes the evaluation of the density weighted lapse N at v in this coordinate system. $\nu_v^{-2/3}$ is proportional to the eigenvalue of the $\hat{q}^{-1/3}$ operator, this eigenvalue being (possibly) nontrivial only for vertices of valence greater than 3. We refer to the eigenvalue $N_i^a(v) = \sum_{I_v} N_{I_v i}^a$ as the *quantum shift*. We emphasise that for each vertex of valence $N > 3$ we need a choice of regulating coordinates to evaluate this quantum shift.

C. Discrete Hamiltonian constraint from P1

The action of the discrete approximant to the Hamiltonian constraint operator of [15] is motivated as follows. A charge net state can be thought of heuristically as a wave function of the connection which is itself a holonomy. Accordingly we use the following notation for the this wave function:

$$c(A) = h_c(A) = \prod_{I=1}^{N} e^{i\kappa\gamma q_I^j \int_{e_I} A_a^j dx^a} = \exp\left(\int d^3x c_i^a A_a^i\right) \quad (2.9)$$

where we have defined:

$$c^{ai}(x) := c^{ai}(x; \{e_I\}, \{q_I\})$$
$$= \sum_{I=1}^{M} i q_I^i \int dt_I \delta^{(3)}(e_I(t_I), x) \dot{e}_I^a(t_I). \quad (2.10)$$

Holonomy operators act by multiplication and the electric field operator by functional differentiation on charge net wave functions. Using the identity $N_i^a F_{ab}^k = £_{\vec{N}_i} A_b^k - \partial_b(N_i^c A_c^i)$, the classical Hamiltonian constraint can be written on the Gauss Law constraint surface as:

$$H[N] = \frac{1}{2}\int_\Sigma d^3x \epsilon^{ijk} N_i^a F_{ab}^k E_j^b + \frac{1}{2}\int_\Sigma d^3x N_i^a F_{ab}^i E_i^b$$
$$= \frac{1}{2}\int_\Sigma d^3x \left(-\epsilon^{ijk}(£_{\vec{N}_j} A_b^k) E_i^b + \sum_i (£_{\vec{N}_i} A_b^i) E_i^b\right) \quad (2.11)$$

where we have added the classically vanishing second term on the right-hand side of the first line. The action of the corresponding operator on the state $c(A)$ is obtained by replacing the electric shift by the action of its operator correspondent (2.8) which is, in turn, replaced by its eigenvalue $N_i^a(v) = \sum_{I_v} N_{I_v i}^a$ to yield:

$$\hat{H}[N]c(A) = \sum_{I_v} -\frac{\hbar}{2i} c(A) \int_{\triangle_{\delta(v)}} d^3x A_a^i (\epsilon^{ijk} £_{\vec{N}_j^{I_v}} c_k^a + £_{\vec{N}_i^{I_v}} c_i^a) \quad (2.12)$$

where for the purposes of our heuristics we have replaced the quantum shift N_i^a, which is strictly speaking non zero only at the point $x = v$ on the Cauchy slice Σ, by some regulated version thereof which is of small compact support $\Delta_\delta(v)$ of coordinate size δ^3 about v (in the coordinates we used to define the quantum shift). Next, we approximate the Lie derivative with respect to the quantum shift in $\Delta_\delta(v)$ by the difference of the *pushforward* action of a small diffeomorphism and the identity as follows:

$$(£_{\vec{N}_i^I} c_j^a) A_a^k = -\frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \times \frac{\varphi(q_{I_v}^i \vec{\hat{e}}_I, \delta)^* c_j^a A_a^k - c_j^a A_a^k}{\delta} + O(\delta). \quad (2.13)$$

where we imagine extending the unit coordinate edge tangents $\vec{\hat{e}}_I$ to $\Delta_\delta(v)$ in some smooth compactly supported way and define $\varphi(q^i_{I_v}\vec{\hat{e}}_I,\delta)$ to be the finite diffeomorphism corresponding to translation by an affine amount $q^i_{I_v}\delta$ along this edge tangent vector field. Using (2.13), we obtain:

$$\hat{H}[N]c(A) = \frac{1}{\delta}\frac{\hbar}{2\mathrm{i}}c(A)\frac{3}{4\pi}\sum_v N(x(v))\nu_v^{-2/3} \times \sum_{I_v,i}\int_\Sigma \mathrm{d}^3x[\cdots]^{I_v,i}_\delta + O(\delta), \tag{2.14}$$

$$\begin{aligned}
[\cdots]^{I_v,1}_\delta &= [(\varphi c^a_2)A^3_a - c^a_2A^3_a] + [(\varphi\bar{c}^a_3)A^2_a - \bar{c}^a_3A^2_a] \\
&\quad + [(\varphi c^a_1)A^1_a - c^a_1A^1_a] \\
[\cdots]^{I_v,2}_\delta &= [(\varphi c^a_3)A^1_a - c^a_3A^1_a] + [(\varphi\bar{c}^a_1)A^3_a - \bar{c}^a_1A^3_a] \\
&\quad + [(\varphi c^a_2)A^2_a - c^a_2A^2_a] \\
[\cdots]^{I_v,3}_\delta &= [(\varphi c^a_1)A^2_a - c^a_1A^2_a] + [(\varphi\bar{c}^a_2)A^1_a - \bar{c}^a_2A^1_a] \\
&\quad + [(\varphi c^a_3)A^3_a - c^a_3A^3_a],
\end{aligned} \tag{2.15}$$

where we have written $\bar{c}^a_i \equiv -c^a_i$ and where we have suppressed the labels I_v, i to set $\varphi c^a_j \equiv \varphi(q^i_{I_v}\vec{\hat{e}}_{I_v},\delta)^*c^a_j$. The integral in (2.14) is of order δ and we approximate it by its exponential minus the identity to get our final expression:

$$\hat{H}[N]c(A) = \frac{\hbar}{2\mathrm{i}}c(A)\frac{3}{4\pi}\sum_v N(x(v))\nu_v^{-2/3} \times \sum_{I_v}\sum_i \frac{\mathrm{e}^{\int_\Sigma[\cdots]^{I_v,i}_\delta} - 1}{\delta} + O(\delta). \tag{2.16}$$

For each fixed (I_v, i) the exponential term is a product of edge holonomies corresponding to the chargenet labels specified through (2.15). This product may be written as

$$h^{-1}_{c_{(i,\mathrm{flip})}}h_{c_{(i,\mathrm{flip},I_v,\delta)}}, \tag{2.17}$$

where $c_{(i,\mathrm{flip},I_v,\delta)}$ is the deformation of $c_{(i,\mathrm{flip})}$ by $\varphi(q^i_{I_v}\vec{\hat{e}}_I,\delta)$ and $c_{i,\mathrm{flip}}$ has the same graph as c but "flipped" charges. To see what these charges are, fix $i = 1$ and some edge I_v corresponding to the first line of (2.15). In $c_{(1,\mathrm{flip})}$, the connection A^3_a corresponding to the 3rd copy of $U(1)$ is multiplied by the charge net c^a_2 corresponding to the second copy of $U(1)$. This implies that in the holonomy $h_{c_{(1,\mathrm{flip})}}$ the charge label in the 3rd copy of $U(1)$ for any edge is exactly the charge label in the second copy of $U(1)^3$ of the same edge in c i.e., in obvious notation $q^3|_{c_{(1,\mathrm{flip})}} = q^2|_c$ where we have suppressed the edge label. A similar analysis for all the remaining terms in (2.15) indicates that the charges ${}^{(i)}q^j$, $j = 1, 2, 3$ on any edge of $c_{(i,\mathrm{flip})}$ are given by the following "i- flipping" of the charges on the same edge of c.

$${}^{(i)}q^j = \delta^{ij}q^j - \sum_k \epsilon^{ijk}q^k \tag{2.18}$$

The exact nature of the deformed chargenet $c_{(i,\mathrm{flip},I_v,\delta)}$ depends on the definition of the deformation. Since the deformation is of compact support around v, the combination $h^{-1}_{c_{(i,\mathrm{flip})}}h_{c_{(i,\mathrm{flip},I_v,\delta)}}$ is the identity except for a small region around v. From (2.16), this term multiplies $c(A)$. We call the resulting chargenet as $c_{(i,I_v,1,\delta)}$[6] so that in terms of holonomies we have that:

$$h_{c_{(i,I_v,1,\delta)}}(A) = h^{-1}_{c_{(i,\mathrm{flip})}}(A)h_{c_{(i,\mathrm{flip},I_v,\delta)}}(A)h_c(A). \tag{2.19}$$

Our final expression for the discrete approximant to the Hamiltonian constraint then reads:

$$\hat{H}[N]_\delta c(A) = \frac{\hbar}{2\mathrm{i}}\frac{3}{4\pi}\sum_v N(x(v))\nu_v^{-2/3}\sum_{I_v}\sum_i \frac{c_{(i,I_v,1,\delta)} - c}{\delta}. \tag{2.20}$$

A similar analysis for the action of the electric diffeomorphism constraint yields the following counterpart of (2.16):

$$\hat{D}[\vec{N}_i]c(A) = \frac{\hbar}{\mathrm{i}}c(A)\frac{3}{4\pi}\sum_v N(x(v))\nu_v^{-2/3} \times \sum_{I_v}\frac{\mathrm{e}^{\int_\Sigma \varphi(q^i_{I_v}\vec{\hat{e}}_I,\delta)^*c^a_jA^j_a - c^a_jA^j_a} - 1}{\delta} + O(\delta). \tag{2.21}$$

which then yields the final result

$$\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}\sum_v N(x(v))\nu_v^{-2/3}\sum_{I_v}\frac{1}{\delta}(c_{(I_v,i,0,\delta)} - c) \tag{2.22}$$

where $c_{(I_v,i,0,\delta)}$ is obtained from c only by a singular deformation without any charge flipping so that

$$(c_{(I_v,i,0,\delta)})^a_i(x) \coloneqq \varphi(q^i_{I_v}\vec{\hat{e}}_{I_v},\delta)^*c^a_i(x). \tag{2.23}$$

It remains to specify the deformation $\varphi(q^i_{I_v}\vec{\hat{e}}_{I_v},\delta)$. We do so in the next section. As we shall see, this deformation is visualized as an abrupt pulling of the vertex structure along

[6]The 1 in the subscript refers to the "positive" i-flip (2.18) as distinct from a "negative" i-flip which we shall encounter in (2.25) below.

the I_vth edge. Due to its "abruptness" we refer to this deformation as a *singular diffeomorphism*. In this language, Eqs. (2.20) and (2.22) imply that whereas the action of the Hamiltonian constraint is a combination of charge flips and singular diffeomorphisms, the action of the electric diffeomorphism constraints is exactly that of singular diffeomorphisms without any charge flips.

D. Linear vertices, upward and downward cones and negative charge flips

The following discussion implies that the detailed specification of the deformation $\varphi(q^i_{I_v}\vec{\hat{e}}_{I_v},\delta)$ is only needed for a special class of chargenet vertices which are called *linear* vertices. In this regard, recall from Sec. I that any state of interest is associated with a corresponding ket set and is built out of linear combinations of charge net states in this ket set. The action of the Hamiltonian and electric diffeomorphism constraints on any such state is then determined by their action on elements of the ket set. Charge net elements of these ket sets are characterised by a certain *linearity* property [15]. In order to define this linearity property recall from Sec. II C that the action of these constraints on an element c of the ket set requires the evaluation of the quantum shift (2.8) at vertices of c which have valence greater than 3 which, in turn, requires the choice of a coordinate patch around each such vertex. Let v be any N valent ($N > 3$) vertex of an element c of a ket set. Then the following linearity property holds: there exists a small enough neighborhood of v such that the edges of c at v in this neighborhood are *straight lines* with respect to the coordinate patch at v. Such vertices are called *linear* with respect to the coordinate patches associated with them and these coordinate patches are referred to as linear coordinate patches. Thus, for our purposes, it suffices to specify the deformations $\varphi(q^i_{I_v}\vec{\hat{e}}_{I_v},\delta)$ for linear vertices.

Accordingly consider any such vertex of c.[7] From the discussion above this deformation must distort the graph underling c in the vicinity of its vertex v in such a way that its vertex is displaced by a coordinate distance δ along the I_vth edge direction to leading order in δ. Due to the vanishing of the quantum shift everywhere except at v, this regulated deformation is visualized to abruptly pull the vertex structure at v along the I_vth edge. Due to the abrupt pulling the original edges develop kinks signaling the point from which they are suddenly pulled. Since these kinks are points at which the edge tangents differ we call them C^0 kinks. The final picture of the distortion is one in which the displaced vertex lies along the I_vth edge and is connected to the kinks on the remaining edges by edges which point almost exactly opposite to the I_vth one. The structure in the vicinity of the displaced vertex is exactly that of a "downward" cone formed by these edges with axis along the I_vth one. For small enough δ the linear nature of the vertex provides the necessary linear structure to define this conical deformation, with the cone getting stiffer as the regulating parameter δ decreases.

The downward conical structure is appropriate for vertex displacement by $\varphi(q^i_{I_v}\vec{\hat{e}}_{I_v},\delta)$ along the *outgoing* "upward" direction $\vec{\hat{e}}_{I_v}$ along the Ivth edge which, in turn, is appropriate for positive $q^i_{I_v}$. For negative $q^i_{I_v}$, the displacement is downward along an *extension* of the I_vth edge past v, with the remaining $N-1$ edges forming an *upward* cone around the cone axis along the I_vth edge.

Note also that we can equally well replace Eq. (2.13) through the judicious placement of negative signs by:

$$(\pounds_{\vec{N}^I_i} c^a_j)A^k_a = \frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\frac{\varphi(q^i_{I_v}(-\vec{\hat{e}}_I),\delta)^* c^a_j A^k_a - c^a_j A^k_a}{\delta} + O(\delta). \tag{2.24}$$

This would then result in *upward* conical deformations for $q^i_{I_v} > 0$ and downward ones for $q^i_{I_v} < 0$.

A similar use of negative signs in equation (2.11) offers a different starting point for our heuristics and leads to negative charge flips for the deformed charge nets generated by the Hamiltonian constraint approximant:

$$^{(-i)}q^j = \delta^{ij}q^j + \sum_k \epsilon^{ijk}q^k \tag{2.25}$$

We denote the negative i-flipped child of the parent c by

$$c_{(i,I_v,-1,\delta)}, \tag{2.26}$$

the -1 denoting the negative flip (2.25).

To summarize: Using the parent-child language of Sec. I, we have that (a) legitimate approximants to the Hamiltonian and electric diffeomorphism constraints generate both upward and downward conically deformed children irrespective of the sign of the edge charge labels and (b) legitimate approximants to the Hamiltonian constraint generate both positive and negative flipped charge net children. While our notation for deformed children will not reflect the choice of upward and downward deformation (which we shall specify explicitly as and when required), our notation will reflect the choice of positive or negative charge flip as follows. We have already denoted a positive flipped child by $c_{(i,I_v,1,\delta)}$ with "1" signifying a positive i-flip. We shall denote a negative flipped child by $c_{(i,I_v,-1,\delta)}$ with the "−1" signifying a negative flip. As in Sec. II A we shall denote the holonomies associated with these states by $h_{c_{(i,I_v,1,\delta)}}$, $h_{c_{(i,I_v,-1,\delta)}}$.

[7]The deformations described in this section are appropriate for linear vertices subject to a further restriction, namely that at such vertices no triple of edge tangents is linearly dependent. In the language of [11,13,15] and of the next section, such vertices are called "GR" vertices.

Next, we note that it turns out [15] that, by virtue of the linearity of the conical deformation in the vicinity of the displaced vertex, the displaced vertices are also linear. In addition, it also turns out that the constraint approximants (2.20), (2.22) preserve a second set of properties called GR and CGR properties of the vertex structure. We describe these properties in the next section.

E. GR and CGR vertices

A linear GR vertex is defined as a linear vertex which has valence greater than 3 and at which no triple of edge tangents is linearly dependent. A linear vertex v of a charge net c will be said to be linear CGR if:

(i) The union of 2 of the edges at v form a single straight line so that v splits this straight line into 2 parts

(ii) The set of remaining edges together with any one of the two edges in (i) constitute a GR vertex in the following sense. Consider, at v, the set of out going edge tangents to each of the remaining edges together with the outgoing edge tangent to one of the two edges in (i). Then any triple of elements of this set is linearly dependent.

In (i) and (ii) above, the notion of straight line is with respect to the linear regulating coordinate system associated with the linear vertex v. The edges in (i) are called conducting edges, the line in (i) is called the conducting line and the remaining edges are called nonconducting edges.

It is straightforward to see that the conical deformations generated by the Hamiltonian and electric diffeomorphism constraint approximants on parental vertices which are GR result in displaced vertices in the children which are either GR or CGR. While the displaced vertices in the children generated by the electric diffeomorphism constraints preserve the GR nature of the parental vertex, those generated by the Hamiltonian constraint, depending on the charge labelings and the edge along which the deformation is generated, could be GR or CGR.

If the parental vertex is CGR, the conical deformations are, in general, constructed slightly differently from those encountered in Sec. II D. However in the specific case that the deformation at a CGR vertex is along a conducting edge, this deformation is identical to that described in Sec. II D, in that the nonconducting edges lie on a cone with axis along the conducting line. In the main body of this work[8] we shall only encounter parental vertices which are GR or CGR, and in the latter case, will only encounter deformations along conducting edges. We depict these deformations in Figs. 1 and 2. More in detail, Fig. 1 depicts downward conical deformations of a GR vertex. For simplicity of depiction we have chosen the valence to be $N = 4$. Figure 2 shows a deformation along the set of collinear "conducting" edges at a CGR vertex.

[8]A minor exception is the state depicted in Fig. 6; however we do not discuss its deformations in any detail.

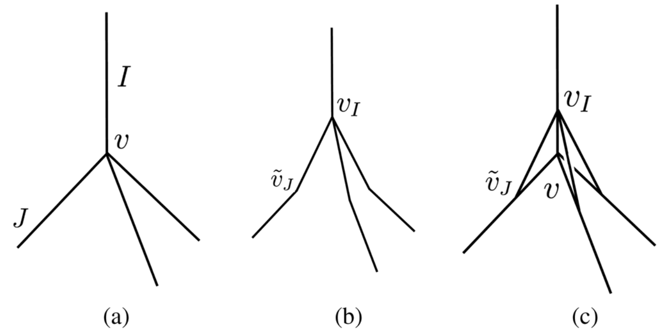

FIG. 1. Fig. 1(a) shows an undeformed GR vertex v of a chargenet c with its Ith and Jth edges as labeled. The vertex is deformed along its Ith edge in Fig. 1(b) wherein the displaced vertex v_I and the C^0 kink, $\tilde{v}_J$ on the Jth edge are labeled. Figure 1(c) shows the result of a Hamiltonian type deformation obtained by multiplying the chargenet holonomies obtained by coloring the edges of Fig. 1(b) by flipped images of charges on their counterparts in c, Fig. 1(a) by negative of these flipped charges and Fig. 1(a) by the charges on c. If the edges of Fig. 1(b) are colored by the charges on their counterparts in c then one obtains an electric diffemorphism deformation.

In our review hitherto we have skipped certain technicalities and, more importantly, *extrapolated* some of the results and structures of [15] in a manner plausible to us. We comment on these matters in the next section.

F. Technical caveats to our hitherto broad exposition

(1) *Interventions, upward directions and C^1, C^2 kinks*: The singular diffeomorphisms which are responsible for the deformations underlying Eqs. (2.20) and (2.22) are of the

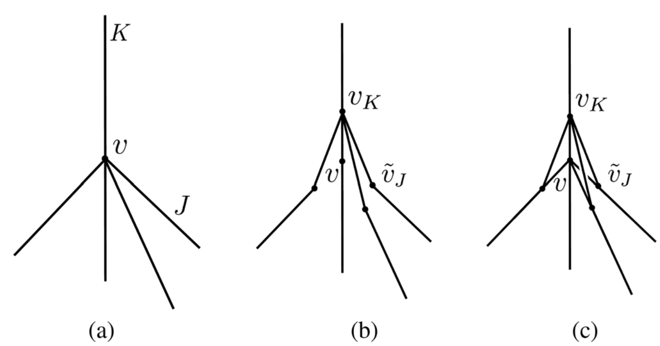

FIG. 2. In Fig. 2(b) the vertex structure of Fig. 2(a) is deformed along its Kth edge and the displaced vertex v_K and the C^0 kink $\tilde{v}_J$ on the Jth edge are as labeled. Figure 2(c) shows the result of a Hamiltonian type deformation obtained by multiplying the 3 chargenet holonomies obtained by coloring the edges of Fig. 2(b) by the flipped images of the charges on their counterparts in c, the edges of Fig. 2(a) by the negative of these flipped charges and the edges of Fig. 2(a) by the charges on c. If the edges of Fig. 2(b) are colored by the charges on their counterparts in c then one obtains an electric diffemorphism deformation.

form $\varphi(q^i_{I_v}\vec{e}_{I_v},\delta)$. Here $\varphi(q^i_{I_v}\vec{e}_{I_v},\delta)$ approximates the Lie derivative with respect to the quantum shift through equation (2.13). As sketched above one may choose to approximate the Lie derivative of the quantum shift through equation (2.24) in which case (2.20), (2.22) would be modified by appropriate negative signs. In both cases the upward direction for the cone was defined to be along the outward point edge tangent $\vec{e}_{I_v}$. We note here that this is *not* how we proceeded in [15].

There we assigned an upward direction to each edge at the vertex v of c based on the graph topology of c. This direction was either outward or inward pointing. In the latter case, we first multiplied the parental charge net by judiciously chosen small loop holonomies which we called *interventions*. The loops were chosen so that (a) these interventions were classically equal to unity to $O(\delta^2)$ and (b) the action of the intervening holonomies modified the parental vertex structure so as to replace the edges which were assigned inward pointing directions by edges whose outward pointing directions at v coincided with the assigned upward directions (c) the interventions resulted in the conversion of CGR vertices to GR ones. The resulting modified parent state was then acted upon by (2.20), (2.22) and then multiplied by the inverse of the intervening holonomies.[9] As a result the deformations of the parental state were dictated by the assigned upward directions rather than the outward pointing parental tangents and no extra negative signs were introduced. In addition certain "C^1 and C^2" kinks were placed on the parental edge (and/or its extension) along which the child vertex was displaced so as to serve as markers for the choice of upward direction [15]. The demonstration of anomalyfree action in [15] was based on this complicated choice of approximant and off-shell and physical states were constructed from ket sets satisfying property (a) with respect to these choices of approximants.

As we shall discuss further in Sec. V A, we believe that it is possible to repeat the demonstration of anomalyfree action by interpreting the choice of upward direction as a regulating choice rather than as being fixed once and for all by the graph topology as in [15]. In other words, we may specify the choice of upward directions at the parental vertex being acted upon by a product of constraint operators as inward or outward for each edge freely. We shall *assume* that with this freedom of choice, we will still be able to provide a demonstration of anomalyfree constraint action along lines similar to that in [15] albeit without the introduction of the C^1 and C^2 kinks referred to above. The ket set satisfying property (a) appropriate to the incorporation of this freedom of choice then contains conically deformed children for cone axes which may be chosen along or opposite to the outward pointing parental edges at any vertex v of the parent *independent* of the sign of the parental edge charges $q^i_{I_v}$.[10]

(2) *Multivertex states:* The detailed demonstration of anomaly freedom in [15] is in the context of ket sets with elements which have *only a single vertex* where the Hamiltonian and electric diffeomorphism constraints act nontrivially. However the notion of propagation between vertices can only be formulated for *multivertex* charge nets. Note that the action of the constraints as derived in Sec. II A at one vertex is *independent* of the action at a distinct vertex. Hence the action of the constraints derived in [15] can be easily generalized to multivertex charge nets and the sum over v in (2.20), (2.22) constitutes exactly this generalization. It is then necessary to also generalise the detailed demonstration of anomaly freedom in [15] to the case of ket sets satisfying property (a) whose elements have multiple vertices on which constraint approximants act nontrivially in accordance with this generalization. While such a demonstration is outside the scope of this work, its existence does seem plausible to us and we shall *assume* this existence for the considerations in this paper. We comment on this matter further in Sec. V A.

G. A key structural property of constraint actions of interest

The structural property of constraint approximants $\hat{C}_\delta$ connected with property (a) of Sec. I and alluded to in that section is that any such approximant takes the following form:

$$\hat{C}_\delta|s\rangle = \sum_v \sum_{\text{deformation},v} a_{\text{deformation},v}\frac{\hat{O}_{\text{deformation},v} - \mathbf{1}}{\delta}|s\rangle. \tag{2.27}$$

Here $|s\rangle$ is (the appropriate counterpart of) a spin net state. The operator $\hat{O}_{\text{deformation},v}$ is a kinematically well defined operator which deforms the vertex structure of the "parent" $|s\rangle$ in a δ coordinate sized vicinity of its vertex v in a specific way and yields a deformed "child" spin net $\hat{O}_{\text{deformation},v}|s\rangle$, and $a_{\text{deformation},v}$ is a nonzero complex coefficient. The sums are over different deformations at each vertex and then over all vertices.

We now show that this form implies that the state obtained as the sum, with unit coefficients, over all elements of any ket set which satisfies property (a) is an

[9] As discussed in [15] the displaced vertex in the resulting child is either CGR or GR.

[10] Thus, the ket sets considered here differ from those of [15] in that (a) there is no placement of C^1, C^2 kinks in children (b) children which arise from *both* directions of conical deformations of parental vertices irrespective of the signs of parental edge charges are in the ket set rather than children which arise only from uniquely prescribed choices of these directions as in [15]; in this sense the ket sets here are slightly larger than those of [15].

anomalyfree physical state. More precisely, since the sum is kinematically non-normalizable, it is more appropriate to define the state as a sum over *bra* correspondents of elements of the ket set. Such a state Ψ lies in the algebraic dual space of complex linear mappings on the finite span of (the appropriate analog of) spin network states. The constraints operators act through dual action on such a state. We show below that such a state is an anomalyfree physical state with respect to the dual action of the constraint operators of the form (2.27).

The constraint approximants act by dual action on such a state as follows:

$$\Psi(\hat{C}_\delta|s\rangle) = \Psi\left(\sum_v \sum_{\text{deformation},v} a_{\text{deformation},v} \frac{\hat{O}_{\text{deformation},v} - \mathbf{1}}{\delta}|s\rangle\right) \tag{2.28}$$

and their continuum limit action is defined as

$$\lim_{\delta\to 0}\Psi(\hat{C}_\delta|s\rangle) = \lim_{\delta\to 0}\Psi\left(\sum_v \sum_{\text{deformation},v} a_{\text{deformation},v} \frac{\hat{O}_{\text{deformation},v} - \mathbf{1}}{\delta}|s\rangle\right). \tag{2.29}$$

We show that the contribution of each term in the sum vanishes i.e., we show that

$$\Psi(|s\rangle) = \Psi(\hat{O}_{\text{deformation},v}|s\rangle)\ \forall\ |s\rangle. \tag{2.30}$$

First let $|s\rangle$ lie in the complement of the ket set. Then the left-hand side (lhs) of (2.30) vanishes. The right-hand side (rhs) involves the action of Ψ on a child of $|s\rangle$. This vanishes by virtue of property (a2) of the ket set, for if it did not vanish, that would imply the existence of a possible parent $|s\rangle$ of the child $\hat{O}_{\text{deformation},v}|s\rangle$ such that this possible parent is *not* in the ket set even though its child is. Next let $|s\rangle$ be in the ket set. Then the lhs is equal to 1 because Ψ is a superposition of (bra correspondents of) elements of the ket set with *unit* coefficients. The rhs is then also equal to 1 by virtue of property (a1). Thus Ψ is in the kernel of the electric diffeomorphism and Hamiltonian constraint operators. Finally, Ψ is diffeomorphism invariant by virtue of property (a3). Since the diffeomorphism invariant state Ψ is killed by the Hamiltonian constraint, Ψ is a physical state. Since it is also killed by the electric diffeomorphism constraint, constraint commutators consistently trivialize and the state is also anomaly free. This completes the proof.

To summarize: The constraint approximants considered in this work will all have the structure (2.27). For any ket set which satisfies property (a) with respect to these constraint approximants, the state obtained by summing over elements of this ket set with unit coefficients is a physical state i.e., it is a diffeomorphism invariant state annihilated by these constraint approximants, and, hence, by their continuum limits. Such a state also supports trivial anomaly free constraint commutators. As discussed in Sec. I (see also [6]), whether a physical state based on a specific ket set supports propagation depends crucially on the nature of the possible parents of property (a2). In contrast to the single vertex ket sets considered in [15], the ket sets considered in this work are based on multivertex kets because the very notion of propagation as that between vertices is defined only for the multivertex case.

As mentioned in Sec. I, whether off-shell deformations of these multivertex physical states can be constructed in a manner similar to the single vertex case of [15] so as to support nontrivial anomalyfree commutators is a question which is outside the scope of the work in this paper. We return to this point in Sec. V A.

III. INSUFFICIENT PROPAGATION

In Sec. III A, in order to illustrate the various structures involved in our discussion of propagation in Sec. I, we study these structures in the context of a simple example, namely that of a 2 vertex charge network state, each vertex having the same valence. In Sec. III A 1 we consider a perturbation created by the action of a single Hamiltonian constraint on this state. We show that the minimal ket set, consistent with the constraint actions of [15], which contains this state does *not* encode propagation of this perturbation. In Sec. III A 2 we enlarge this ket set by requiring that the physical state it defines be subject to additional conditions. We consider a specific perturbation of the simple 2 vertex state created by the action of an operator associated with these additional conditions. We show that the enlarged ket set *does* encode propagation of this perturbation from one vertex of the state to the other. Besides their pedagogic value in illustrating our articulation of propagation in terms of ket sets, the considerations of Sec. III A 2 display an intriguing connection with the existence of a certain elegant combination of constraints in Ref. [16] (see footnote 11 in this regard).

In Sec. III B we investigate propagation between vertices of different valence. Specifically, we show that the $N \to N$ constraint actions of [15] are *inconsistent* with propagation between vertices of different valence of a multivertex state, and that, at best these actions may engender '1d' propagation between vertices of special multivertex states. The arguments in Sec. III B are simple and robust and the reader mainly interested in the motivation for the $N \to 4$ modification may skip the slightly more involved considerations of Sec. III A.

A. Propagation in a simple 2 vertex state

Consider the simple case of a $U(1)^3$ gauge invariant parent charge net state p with 2 N-valent vertices connected

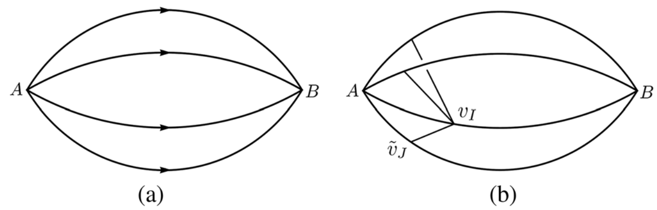

FIG. 3. Fig. 3(a) shows the simple 2 vertex chargenet of interest. Figure 3(b) shows the result of a Hamiltonian constraint deformation of the chargenet of Fig. 3(a) along its Ith edge with the displaced vertex v_I and the C^0 kink $\tilde{v}_J$ as labeled.

by N edges as depicted in Fig. 3(a). Let the set of $U(1)^3$ edge charges be $\vec{q}_J, J = 1, .., N$ with $\vec{q}_J = (q_J^1, q_J^2, q_I^3) \in U(1)^3$. Consider the "generic" case where p has no symmetries and none of its charge components vanishes so that:

$$q_J^i \neq 0 \quad i = 1, 2, 3 \quad J = 1, \ldots, N. \tag{3.1}$$

We are interested in establishing propagation or the lack thereof of specific perturbations between vertices A and B in the context of $N \to N$ constraint actions.

1. No propagation

Here we consider the smallest ket set subject to property (a) which contains p. A single $N \to N$ action of the Hamiltonian constraint on this parent state yields various children in this ket set. We focus on the child c obtained by deforming the vertex structure at A along its Ith edge as shown in Fig. 3(b). The deformation renders the original parent vertex at A degenerate so that it has vanishing volume eigenvalue and creates the new nondegenerate vertex v_I. The vertex v_I is connected in c to the C^0 kinks $\{\tilde{v}_{J\neq I}\}$ on the parental edges $\{e_{J\neq I}\}$ by the deformed counterparts of the latter as show in Fig. 3(b).

We are interested in the existence of other possible parents of c in the ket set. By a possible parent we mean a chargenet p' whose deformation by at least one $N \to N$ constraint action yields c upto diffeomorphisms. By "other" we mean that p' is not diffeomorphic to p. Thus, we are interested in the existence of p' not diffeomorphic to p such that c is generated from p' by any combination of at least one Hamiltonian or electric diffeomorphism constraint deformation, ordinary diffeomorphisms and, possibly, further Hamiltonian/electric diffeomorphism deformations.

Since c contains only one set of *trivalent* kinks, it can only be generated (up to diffeomorphisms) by a *single Hamiltonian constraint* action on a state p' with 2 N valent vertices and no such kinks so that $c = \phi' \widehat{\text{Ham}} \phi p'$ where $\widehat{\text{Ham}}$ refers to a Hamiltonian constraint deformation. Clearly by redefining p' appropriately we may set ϕ equal to the identity with no loss of generality. Let the vertices of p' be A', B' and let $\widehat{\text{Ham}}$ act at A' to yield c' so that we have that $\phi' c' = c$.

Next, denoting the flipped charges on the deformed edges in c by the subscript flip we have the following:

(a) By virtue of the genericity condition (3.1) on the charge labels of p, it follows straightforwardly that the charges $q_J^k - (q_{\text{flip}})_J^k$ on the segments between A and $\tilde{v}_J$ are nonvanishing, thus implying that these segments are present in c.

(b) the parental graph can be immediately reconstructed from that of c simply by removing the deformed edges in c from v_I to each $\tilde{v}_J$.

(c) the parental vertex whose deformation yields c can be identified uniquely as A by virtue of A being degenerate in c.

(d) the parental edge charges $q_{J\neq I}^k$ in p can be uniquely identified with the charges $q_{J\neq I}^k$ on the segments in c from $\tilde{v}_J$ to B and the parental edge charges on the Ith edge in p can be idnetified with those on the edge from v_I to B in c.

From (a)–(d), p can be uniquely reconstructed from c. Since ϕ' preserves kink structure, vertex degeneracy, and colorings, it immediately follows that $\phi' p' = p$. Hence this example illustrates the lack of propagation of this particular perturbation.

2. Propagation from an additional condition

As discussed in Secs. I and II G, anomalyfree physical states are annihilated by the diffeomorphism, Hamiltonian and electric diffeomorphism constraints. Here, we demand that these states be further annihilated by certain operator implementations of the linear combinations

$$H_{\pm}(N) = (\pm H(N)) + \frac{1}{2} \sum_{i=1}^{3} D(\vec{N}_i) \tag{3.2}$$

of the Hamiltonian and electric diffeomorphism constraints.[11] If the operators $\hat{H}_{\pm}(N)$ are regulated simply as sums of the regulated versions (2.20) and (2.22) of the individual Hamiltonian and electric diffeomorphism constraints, this condition is already satisfied by anomalyfree physical states by virtue of their being annihilated by the individual constraints. Here we do not regulate $\hat{H}_{\pm}(N)$ in this trivial way. Instead we proceed as follows.

[11]As mentioned earlier these combinations are reminiscent of the elegant combinations of the diffeomorphism and Hamiltonian constraints for Lorentzian gravity constructed in [16]. These combinations in that work obtain an elegant form when expressed in terms of spinors. The trace part of the combination yields the Hamiltonian constraint and the trace free part yields electric diffeomorphism constraints smeared with an additional electric field. For details see (see vi), pg. 85, Chapter 6 of [16]).

From Eqs. (2.16) and (2.21), it immediately follows that

$$\hat{H}_+[N]c(A) = \frac{\hbar}{2\mathrm{i}} c(A) \frac{3}{4\pi} \sum_v N(x(v)) \nu_v^{-2/3} \times \sum_{I_v} \sum_i \frac{\mathrm{e}^{\int_\Sigma [\cdots]_\delta^{I_v,i} + \varphi(q^i_{I_v} \vec{e}_I, \delta)^* c^a_j A^j_a - c^a_j A^j_a} - 1}{\delta} + O(\delta) \tag{3.3}$$

$$= \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} \sum_v N(x(v)) \nu_v^{-2/3} \times \sum_{I_v} \sum_i \frac{\mathrm{e}^{\int_\Sigma [\cdots]_\delta^{I_v,i} + \varphi(q^i_{I_v} \vec{e}_I, \delta)^* c^a_j A^j_a} - c(A)}{\delta} + O(\delta) \tag{3.4}$$

$$= \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} \sum_v N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{(i,I_v,+,\delta)}(A) - c(A)}{\delta} + O(\delta) \tag{3.5}$$

where in the second line we used that $c(A) = \exp \int_\Sigma c^a_i A^i_a$. and in third we defined the deformed state $c_{(i,I_v,+,\delta)}(A)$ as

$$c_{(i,I_v,+,\delta)}(A) \coloneqq \mathrm{e}^{\int_\Sigma [\cdots]_\delta^{I_v,i} + \varphi(q^i_{I_v} \vec{e}_I, \delta)^* c^a_j A^j_a}. \tag{3.6}$$

In the notation of Eqs. (2.20) and (2.22), we have that:

$$\hat{H}_+[N]_\delta c(A) = \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} \sum_v N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{(i,I_v,+,\delta)} - c}{\delta} \tag{3.7}$$

It is straightforward to check that in the notation developed in the beginning of Sec. II A, the holonomy underlying the deformed state $c_{(i,I_v,+,\delta)}$ is obtained as the product of the holonomy corresponding to an i-flipped child generated by the Hamiltonian constraint and the holonomy corresponding to an electric diffeomorphism child as follows:

$$\begin{aligned} h_{c_{(i,I_v,+,\delta)}}(A) &= (h^{-1}_{c_{i,\mathrm{flip}}}(A) h_{c_{i,\mathrm{flip},I_v,\delta}}(A)) h_{c_{(i,I_v,0,\delta)}}(A) \\ &= (h_{c_{(i,I_v,1,\delta)}}(A) h_c^{-1}(A)) h_{c_{(i,I_v,0,\delta)}}(A). \end{aligned} \tag{3.8}$$

Here, from (2.17) the term in brackets in the first equality corresponds to the $\mathrm{e}^{\int_\Sigma [\cdots]_\delta^{I_v,i}}$ contribution to equation (3.6), and the second equality follows from (2.19).

It is also straightforward to check that if the charge flip (2.18) underlying the term $[\cdots]_\delta^{I_v,i}$ in (2.16) is replaced by the negative charge flip (2.25), then the line of argumentation which leads to (3.3)–(3.5) yields the following regulated action of $\hat{H}_-(N)$:

$$\hat{H}_-[N]_\delta c(A) = \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} \sum_v N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{(i,I_v,-,\delta)} - c}{\delta}. \tag{3.9}$$

The holonomy underlying the deformed state $c_{(i,I_v,-,\delta)}$ is given by the product:

$$h_{c_{(i,I_v,-,\delta)}}(A) = (h_{c_{(i,I_v,-1,\delta)}}(A) h_c^{-1}(A)) h_{c_{(i,I_v,0,\delta)}}(A), \tag{3.10}$$

where we have used the notation $c_{(i,I_v,-1,\delta)}$ as in (2.26). The discussion of Sec. II D may then be repeated in the context of the deformed children generated by $\hat{H}_\pm(N)$. It follows that legitimate approximants to these operators can be constructed so as to generate both upward and downward conically deformed children irrespective of the sign of the edge charge labels. Figure 4 depicts a downward conical deformation of a parental GR vertex by these operators. The holonomy underlying the deformed child is obtained as the product of holonomies based on the graphs depicted in the figure. The graphs involved are the same irrespective of whether the child is generated by $\hat{H}_+(N)$ or $\hat{H}_-(N)$; however the colorings in the two cases differ and are as described in the figure caption.

Next, the discussion of Sec. II G can be applied to Eqs. (3.7) and (3.9) to conclude the following. The ket set appropriate to these equations contains all possible upward and downward deformed children generated by the action of $\hat{H}_\pm(N)$ on any parent in the ket set as well all possible parents of any child in the ket set. The state obtained by

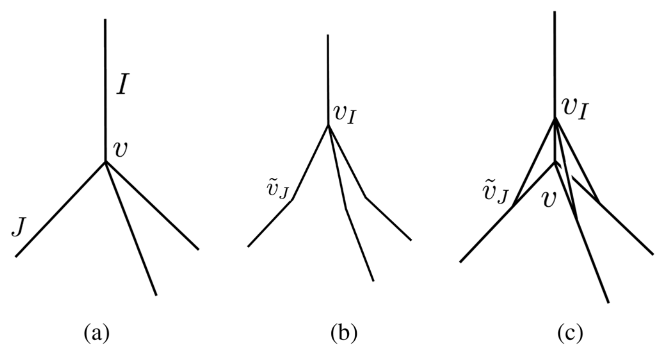

FIG. 4. Figure 4(a) shows an undeformed GR vertex v of a chargenet c with its Ith and Jth edges as labeled. The vertex is deformed along its Ith edge in Fig. 4(b) wherein the displaced vertex v_I and the C^0 kink, $\tilde{v}_J$ on the Jth edge are labeled. Figure 4(c) shows the result of an $H_\pm$ type deformation obtained by multiplying the chargenet holonomies obtained by coloring the edges of Fig. 4(b) by flipped images of charges on their counterparts in c, Fig. 4(a) by negative of these flipped charges and Fig. 4(b) by the charges on c. If the flip is positive, the deformation is generated by H_+ and if negative, by H_-. As result the charges on this deformed chargenet on the deformed edges are the *sum* of the $\pm$-flipped and unflipped charges and the segments from v to the kinks $\tilde{v}_J$ carry the negative of the $\pm$-flipped images of their charges in c.

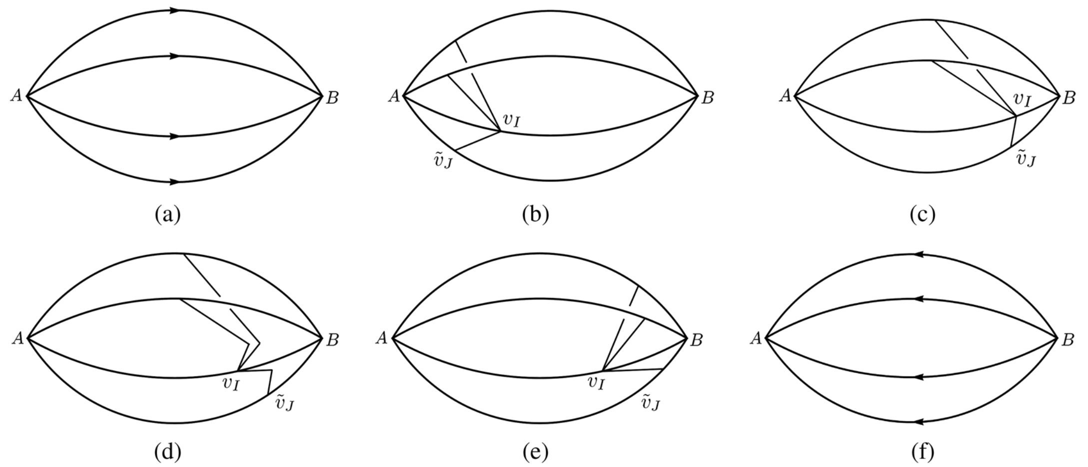

FIG. 5. The figures show the sequence of ket set elements Fig. 5(a) to Fig. 5(f) which encode propagation from vertex A to vertex B as described in the main text. The ket set is the minimal one containing Fig. 5(a). It underlies a physical state subject to the additional physical conditions of Sec. III B.

summing over all elements of this ket set is killed by the actions (3.7), (3.9).

Since the requirement that $\hat{H}_{\pm}(N)$ annihilated states of interest is imposed in addition to the demand that such states be anomalyfree physical states with respect to the Hamiltonian and diffeomorphism (and Gauss Law) constraints, the ket set of interest satisfies the closure properties described in the previous paragraph and also satisfies property (a) as articulated in Sec. I. We shall use the properties described in the previous paragraph together with property (a3) (i.e., the closure of the ket set with respect to diffeomorphisms) to show that the minimal ket set containing the simple 2 vertex state of Sec. III A 1 does encode propagation of a perturbation created by the action of $\hat{H}_{+}(N)$ at one of its vertices. In what follows, for notational convenience we rename this simple 2 vertex state (called p hitherto) as c.

Our argumentation is primarily diagrammatical and described through Fig. 5 as follows:

(1) We start at the left with the simple 2 vertex charge net c with N valent vertices A, B connected through N edges in Fig. 5(a). The outgoing charge on the Kth edge emanating from vertex A is denoted by q^k_K.

(2) This parent chargenet is deformed by the action of $\hat{H}_{+}(N)$ at the vertex A to give the child $c_{(i,I_A,+,\delta)}$ shown in Fig. 5(b). Since the vertex A is GR, the deformation is of the type depicted in Fig. 4(c). As in that figure, the index J will be used for edges which are different from the Ith one. The charges on the child may be inferred from (3.8). Denoting the kth component of the outgoing charge label from a vertex v to a vertex $\bar{v}$ by $q^k_{v\bar{v}}$ it is straightforward to infer that:

$$q^k_{A\tilde{v}_J} = -(q_{\mathrm{flip}})^k_J \quad q^k_{\tilde{v}_J B} = q^k_J \quad q^k_{v_I\tilde{v}_J} = q^k_J + (q_{\mathrm{flip}})^k_J$$
$$q^k_{Av_I} = -(q_{\mathrm{flip}})^k_I \quad q^k_{v_I B} = q^k_I \tag{3.11}$$

Here by q^k_{flip} we mean the positive i-flip (2.18).[12]

(3) The charge net of Fig. 5(b) is acted upon by a seminanalytic diffeomorphism so as to "drag" the deformation from the vicinity of vertex A to the vicinity of vertex B.[13] We slightly abuse notation and denote the images of v_I, $\tilde{v}_J$ by this diffeomorphism by the same symbols v_I, $\tilde{v}_J$.

(4) The charge net of Fig. 5(c) is deformed by the action of an appropriate electric diffeomorphism at v_I to yield the charge net of Fig. 5(d). This transforms the conical deformation in Fig. 5(c) which is downward with respect to the Ith line from A to B to one which is upward conical with respect to this line in Fig. 5 (d). As a result, the deformation is now *downward* conical with respect to the (oppositely oriented) line from B to A.[14]

(5) The charge net c' of Fig. 5(f) has the same graph as that of the charge net c but its charges from B to A are different from those of c. Denoting these charges by q'^k_I, these charges are related to those on c by

$$q'^k_I = (q_{\mathrm{flip}})^k_I. \tag{3.12}$$

Thus the outgoing charges from B in c' are just the positive i-flipped images of the *incoming* charges at

[12]The vertex v_I is CGR (see Sec. II F). Equations (3.11) imply that the net outgoing charges at this vertex are $q^k_K + (q_{\mathrm{flip}})^k_K$, $K = 1, \ldots, N$. We assume that the charges q^k_K are such that the CGR vertex is nondegenerate. For the definition of nondegeneracy of a CGR vertex, see [15].

[13]We assume that the state $c_{(i,I,+1,\delta)}$ is such that it can be transformed via an appropriate diffeomorphism to the state depicted in Fig. 5(c). We shall comment further on this in Sec. V B.

[14]For a downward deformation of a CGR vertex see Fig. 2(b). An upward deformation may be visualized by turning Figs. 2(a) and 2(b) upside down; see [15] and figures therein for details.

B in c. This state will play the role of a possible parent.

(6) The charge net c' of Fig. 5(f) is deformed by the action of $\hat{H}_{-}(N)$ at the vertex B. The deformed child $c'_{(i,I_B,-,\delta)}$ is depicted in Fig. 5(e). Once again we have abused notation and re-used the symbols v_I, $\tilde{v}_J$. The charges on this state can be inferred from (3.10) and (3.12). Using the fact that a negative i-flip is the inverse of a positive i-flip, these charges turn out to be identical to their counterparts in Fig. 5(b):

$$q^k_{A\tilde{v}_J} = -(q_{\rm flip})^k_J \quad q^k_{\tilde{v}_J B} = q^k_J \quad q^k_{v_I\tilde{v}_J} = q^k_J + (q_{\rm flip})^k_J \tag{3.13}$$

$$q^k_{Av_I} = -(q_{\rm flip})^k_I \qquad q^k_{v_I B} = q^k_I. \tag{3.14}$$

(7) The chargenet $c'_{(i,I_B,-1,\delta)}$ of Fig. 5(e) is deformed by the action of an appropriate electric diffeomorphism to give exactly the chargenet of Fig. 5(d).[15]

The minimal ket set containing the chargenet c of Fig. 5(a) must contain the charge nets depicted in Figs. 5(b)–5(f). Steps (1)–(7) imply that the chargenet of Fig. 5(d) has 2 possible ancestors, one depicted in Fig. 5(a) and one in Fig. 5(f). The sequence of elements Figs. 5(a)–5(f) is then one which encodes the 'emmission' of a conical perturbation at the vertex A of c depicted in Fig. 5(b) and its propagation and final 'absorption' by vertex B to yield the chargenet c' in Fig. 5(f). Thus the imposition of appropriate additional physical conditions on anomalyfree states can engender propagation. Unfortunately, as we now argue, this propagation seems to be, at best, only "1 dimensional" and even this "best case" requires very special states.

B. No propagation between generic vertices of multivertex states

The $N \to N$ deformations of [15] used hitherto create N valent (CGR or GR) vertices from N valent parental ones. Consider a pair of GR vertices v_1, v_2 in a multivertex graph of different valences N_1, N_2. Any child vertex created from a deformation of v_1 has a valence N_1 and any child vertex created from a deformation of v_2 has valence $N_2 \neq N_1$. Hence the set of children obtained through multiple deformations of v_1 and v_2 split into two disjoint classes, namely those with an N_1 valent child vertex and those with an N_2 valent child vertex. The former are unambiguously associated with v_1 and their creation can be visualized through a lineage associated with v_1. Similarly any lineage for the latter is associated with v_2. Thus no possible parent of any child in the latter lineage can be part of the former lineage. This implies the impossibility of propagation between two such vertices.

Next, consider a pair of GR N valent vertices v_1, v_2 in a graph which are connected through $M < N$ edges leaving $N - M$ edges free to connect with other parts of the graph. For generic graph connectivity, once again children obtained through deformations of v_1, v_2 fall into two disjoint sets by virtue of their connectivity with these 2 sets of free edges and there is no propagation.

We digress here to note that the vertex deformations defined in [15] can be naturally extended to the case of a linear "multiply CGR" vertex. We define the multiply CGR property as follows. An N valent linear M-fold multiply CGR vertex v is one with the following vertex structure:

(a) There exists an open neighborhood U of v and a coordinate patch thereon such that the edges at v are coordinate straight lines in U.
(b) There are M pairs of edges at v such that the union of each such pair forms a coordinate straight line in U with v splitting this line into this pair of edges. There are $N - M$ edges which are not of this type.
(c) Consider the set of outgoing edge tangents to the remaining $N - M$ edges, together with one edge tangent from each of the M collinear pairs. Then any triple of edge tangents from this set is linearly independent.

Deformations of such vertices can again be made, similar to the CGR case by transforming them to GR vertices through interventions [15]. Such deformations then create child vertices whose valence (in the generalized sense described above) is the same as that of the parent vertex.[16]

The arguments in the first two paragraphs of this section indicate that long range propagation can at best be possible for special graphs. Since the bottlenecks to propagation arise from free edge connectivity and varying vertex valence, one best case scenario where propagation could conceivably occur between multiple vertices is as follows:

(i) These vertices are of the same valence, say N.
(ii) These vertices are connected to each other by N edges.

This implies that these vertices must be N-fold CGR with no two edges connecting a pair of such vertices being collinear at either of the two vertices so connected. This leads to a graph connectivity depicted in Fig. 6 which is intuitively 1 dimensional.

We are unable to construct other examples of graph connectivity which could, conceivably, display propagation. It would be good to construct a proof that no such examples exist. In any case, the arguments above indicate that propagation in the context of $N \to N$ deformations can occur, at best in graphs with very special connectivity. Hence, notwithstanding the propagation in the simple 2

[15] The positions of the points $\tilde{v}_J$ in Figs. 5(e) and 5(d) should be identical. For reasons of visual clarity, these figures do not reflect this fact.

[16] As mentioned in Sec. II F, whether these deformations can be shown to be anomaly free in the sense of [15] is an open question.

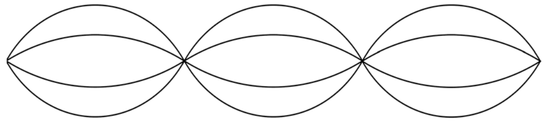

FIG. 6. Fig. 6 shows a best case graph structure for the purposes of putative propagation based on $N \to N$ deformations.

vertex graph described in Sec. III A 2 above, we seek a modification of the $N \to N$ deformation of [15] so as to engender vigorous, 3d, long range propagation for generic graphs. As we shall see in the next section, the $N \to 4$ modification described in Sec. I has this property.

IV. VIGOROUS PROPAGATION FROM $N \to 4$ DEFORMATIONS

In Sec. IV A we define a modified $N \to 4$ implementation of the singular diffeomorphism encountered in Sec. II C. In Sec. IV B we show that this modification enables communication between vertices of different valence as well between vertices which have free edges, thus overcoming these bottle necks to propagation in the $N \to N$ case. An immediate consequence is that of 3d long range propagation between vertices of a chargenet based on a graph which is dual to a *triangulation* of the Cauchy slice. In such a graph a vertex is connected to a nearest vertex by a single edge leaving 3 edges free (which in turn are connected to other nearest vertices). Whereas the $N \to N$ deformations do not engender propagation between vertices of such a graph, this sort of graph structure is not a barrier to propagation for the $N \to 4$ deformation. We discuss this explicitly in Sec. IV C. We note here that such graphs underlie spin nets which have a ready semiclassical interpretation in the $SU(2)$ case [17]. Our argumentation in Secs. IV B and IV C is largely pictorial and similar in character to that of Sec. III A 2.

A. The $N \to 4$ deformation

The argumentation of Sec. II C applies unchanged in the case of the $N \to 4$ deformations described here. Hence the action of the constraint operators of interest is still built out of singular diffeomorphisms and, in the case of the Hamiltonian constraint, charge flips; all that changes is the implementation of the singular diffeomorphisms.

We first define a downward conical $N \to 4$ deformation of an N valent GR vertex v of a charge net c. This $N \to 4$ deformation replaces the $N \to N$ deformation of Fig. 1(b). As in the $N \to N$ case, this $N \to 4$ deformation corresponds to that generated by an electric diffeomorphism action. The deformed charge nets generated by the Hamiltonian constraint can be obtained by combining this deformation with charge flips exactly as in the $N \to N$ case as described in the figure caption accompanying Fig. 1 with the $N \to N$ deformed charge net of Fig. 1(b) replaced by its $N \to 4$ counterpart which we now construct and which is displayed in Fig. 7(d).

To construct the $N \to 4$ downward conical deformation of an N valent linear GR vertex v [depicted in Fig. 7(a)] by the singular diffeomorphism $\varphi(q^i_{I_v}\vec{e}_{I_v}, \delta)$ (with $q^i_{I_v}$ assumed to be positive as is appropriate for downward conicality), we first fix 3 edges $e_{J^i_v \neq I_v}$, $i = 1, 2, 3$. We deform these 3 edges exactly as for the $N \to N$ downward conical deformation with $N = 4$. This part of the deformation is depicted in Fig. 7(b). The remaining edges are pulled *exactly* along the I_vth edge as depicted in Fig. 7(c). The $N \to 4$ deformation combines both these deformations and is depicted in Fig. 7(d). Dropping the v subscripts to the edge indices in what follows, if the outward-going edge charges at the (undeformed) vertex v are $q^i_K, K = 1, .., N$, then in the (obvious) notation used in (3.11), the charges on the deformed charge net of Fig. 7(d) can be readily inferred from Figs. 7(b) and 7(c) to be

$$q^k_{v_I \tilde{v}_{J^i}} = q^k_{J^i}, \quad i = 1, 2, 3$$
$$q^k_{v_I v} = \sum_{J \neq I, J^1, J^2, J^3} q^k_J = -q^k_{v v_I}, \tag{4.1}$$

with the charges on the remaining parts of the graph being exactly those on these parts of the graph in the undeformed parent state c of Fig. 7(a).

As mentioned above, the deformed vertex structure of Fig. 7(d) is created from the undeformed one of Fig. 7(a) by an action of the electric diffeomorphism constraint. The deformed vertex structure created by the Hamiltonian constraint can then be constructed exactly as for the $N \to N$ case by combining the $N \to 4$ deformation of Fig. 7(d) with appropriate charge flips as depicted in Fig. 7(e) and described in the accompanying figure caption. If the charge q^i_I is negative, the vertex v is displaced along the extension of the Ith edge and the conical deformation of the 3 chosen edges is then upward conical. We do not discuss upward conical deformations in detail as we do not need them here; the details are straightforward and we leave their working out to the interested reader.

Due to the choice of 3 preferred edges in this deformation, the resulting charge net is now denoted by $c_{(i, I_v, \vec{J}_v \beta_v, \delta)}$ where $\beta = +1, -1, 0$ for positive, negative and no flips and the particular choice of edge triple is indicated by $\vec{J}_v \equiv (J^1_v, J^2_v, J^3_v)$. The action of the constraint is then obtained by summing over all possible triples of such edges so that the Hamiltonian constraint action is

$$\hat{H}[N]_\delta c(A) = \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} \sum_v \beta_v N(x(v)) \nu_v^{-2/3} \sum_{I_v} \frac{1}{\binom{N-1}{3}} \times \sum_{\vec{J}_v} \sum_i \frac{c_{(i, I_v, \vec{J}_v, \beta_v, \delta)} - c}{\delta} \tag{4.2}$$

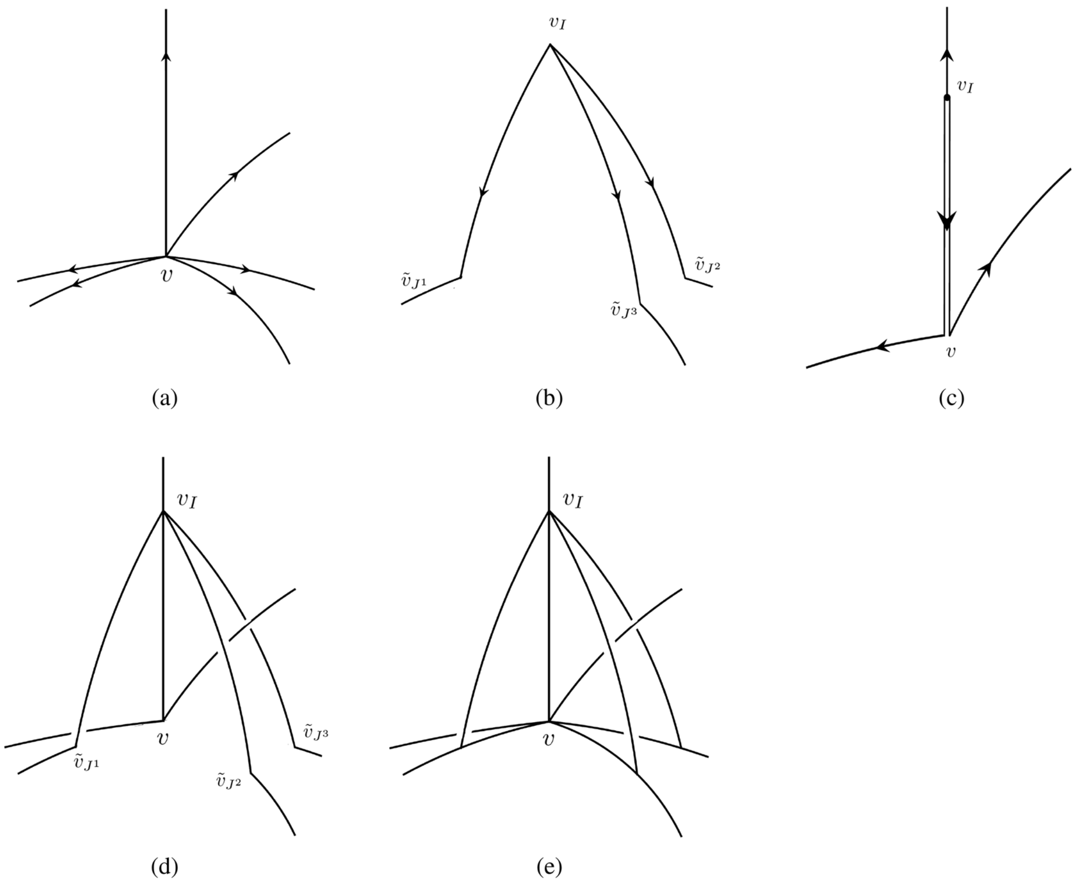

FIG. 7. Figure 7(a) shows an undeformed GR vertex v of a chargenet c. Figure 7(b) shows the conical deformation of the 3 preferred edges e_{J^i}, $i = 1, 2, 3$ along the Ith edge, the displaced vertex v_I and the 3 C^0 kinks, $\tilde{v}_{J^i}$, $i = 1, 2, 3$ are as labeled. Figure 7(c) shows the remaining edges being pulled exactly along the Ith edge. The deformations of Figs. 7(b) and 7(c) combine to yield the $N \to 4$ deformation shown in Fig. 7(d). Figure 7(e) shows the result of a Hamiltonian type deformation obtained by multiplying the chargenet holonomies obtained by coloring the edges of Figs. 7(b) and 7(c) by the flipped images of charges on their counterparts in c, of Fig. 7(a) by negative of these flipped charges and Fig. 7(a) by the charges on c. If the edges of Figs. 7(b) and 7(c) are colored by the charges on their counterparts in c then they combine to yield the holonomy of Fig. 7(d), this being the result of an electric diffemorphism deformation.

and the electric diffeomorphism constraint action is

$$\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}\sum_v N(x(v))\nu_v^{-2/3}\sum_{I_v}\frac{1}{\binom{N-1}{3}} \times \sum_{\vec{J}_v}\frac{1}{\delta}(c_{(I_v,i,0,\vec{J}_v,\delta)} - c). \tag{4.3}$$

In Eq. (4.2), $\beta_v = \pm 1$ depending on whether a positive or negative flip is chosen for the deformations at v. In both the above equations we have implicitly chosen the appropriate downward/upward conical deformation dictated by the sign of the edge charge $q^i_{I_v}$. However these deformations can be chosen to be either upward or downward provided, as discussed in Sec. II D, we insert minus signs at appropriate places in these equations. The main implication of all this is that the set of children obtained from the action of constraint deformations are generated through positive and negative charge flips as well as upward and downward conical deformations.

In what follows we shall also require the deformation generated by an electric diffeomorphism constraint on a 4 valent CGR vertex along its collinear edges. Since $N = 4$ this deformation coincides with the $N \to N$ deformation with $N = 4$ depicted in Fig. 2(b). While Fig. 2(b) depicts a downward conical deformation, an upward conical deformation can be visualized by viewing Figs. 2(a) and 2(b) upside down; for details see [15] and figures therein. The charges on the deformed edges for such deformations are exactly the same as those on their undeformed counterparts.[17]

We note here that the charges on the deformed child in the case of a parental GR vertex can be quickly inferred as

[17]As mentioned in Sec. II F, the derivation of these deformations and charge labelings as well as the deformations along other edges which contribute to the action of the constraint at this vertex proceed through the use of interventions [15] which convert the parental CGR vertex to a GR one. The interested reader may consult [15] for details with regard to the intervention procedure.

follows without going through the holonomy multiplication of Fig. 7. The charges on the deformed edges $e_{v_I\tilde{v}^i_J}, i = 1, 2, 3$, are exactly the same as for the $N \to N$ edges with $N = 4$. Thus in the case of Hamiltonian constraint deformation these charges are obtained through positive or negative i-flips of the charges on their undeformed counterparts in c whereas for an electric diffeomorphism deformation these charges are identical to those on their undeformed counterparts in c. The remaining charges maybe inferred from $U(1)^3$ gauge invariance together with the fact that the deformation is confined to a δ size vicinity of the parental vertex.

B. Propagation between vertices with different valence and with free edges

Consider two linear GR vertices A, B of a charge network c with valences N_A, N_B. connected by $M < N$ edges leaving $N_A - M$ and $N_B - M$ edges free at v_A and v_B. We now show that the $N \to 4$ constraint action engenders propagation between the vertices A, B. Our argumentation is primarily diagrammatical and described through Fig. 8 as follows:

(1) We start at the left with the "unperturbed" charge network structure described above depicted in Fig. 8(a). We shall be interested in a Hamiltonian constraint generated deformation along the Ith edge emanating from A and connecting to B. In order to keep the figure uncrowded, it explicitly depicts only this single edge between A, B and only a few more edges at these vertices. The reader may think of $M - 1$ of the edges emanating from A and $M - 1$ of those from B as being connected so as to yield $M - 1$ more edges connecting A, B. The connectivity of the remaining free edges does not affect the argumentation.

We denote the outgoing edge charges at A by $q^k_J, J = 1, .., N$. In the $N \to 4$ deformation at A along e_I, a choice of 3 edges $e_{J^i}, i = 1, 2, 3$ has to be made. As we shall see below, propagation generically ensues irrespective of which choice we make.

(2) The parent chargenet c is deformed in a downward conical manner along the Ith edge at A by the action of $\hat{H}(N)$ at the vertex A to give the child $c_{(i,I,\vec{J},\beta,\delta)}$ shown in Fig. 8(b), where we have used the notation of (4.2) and dropped the "vertex" suffix to avoid notational clutter. Here β can be $+1$ or -1. In either case, we refer to the relevant flipped image of the charge q^i (2.18), (2.25) as q^i_{flip}. The deformation of the GR vertex A is exactly that of Fig. 7(e) with charges in the vicinity of vertex A obtained exactly as described in the figure caption accompanying Fig. 7(d). These charges in obvious notation are

$$q^k_{v_I\tilde{v}_{J^i}} = (q_{\text{flip}})^k_{J^i}, \quad i = 1, 2, 3 \tag{4.4}$$

$$q^k_{A\tilde{v}_{J^i}} = q^k_{J^i} - (q_{\text{flip}})^k_{J^i} \tag{4.5}$$

$$q^k_{Av_I} = q^k_I - (q_{\text{flip}})^k_I - \sum_{J\neq I,J^1,J^2,J^3} (q_{\text{flip}})^k_J = q^k_I + \sum_{i=1}^{3} (q_{\text{flip}})^k_{J^i} = -q^k_{v_IA}, \tag{4.6}$$

with the charges on the remaining part of the graph remaining unchanged and where we have used gauge invariance of at A in the unperturbed charge net c to go from the first equality to the second in (4.6).

(3) The charge net of Fig. 8(b) is acted upon by a semianalytic diffeomorphism so as to drag the deformation from the vicinity of vertex A to the vicinity of vertex B.[18] We slightly abuse notation and denote the images of v_I, $\tilde{v}_J$ by this diffeomorphism by the same symbols v_I, $\tilde{v}_J$.

(4) The charge net of Fig. 8(c) is deformed by the action of an appropriate electric diffeomorphism at the CGR vertex v_I as described in Fig. 2(b) to yield the charge net of Fig. 8(d). This transforms the conical deformation in Fig. 8(c) which is downward with respect to the Ith line from A to B to one which is upward conical with respect to this line in Fig. 8(d). As a result, the deformation is now *downward* conical with respect to the (oppositely oriented) line from B to A.

(5) The charge net c' of Fig. 8(e) is based on a graph which is obtained by adding 3 edges to the graph underlying the unperturbed state c. These edges emanate from the vertex B and terminate at the 3 kinks $\tilde{v}_{J^i}$ of Fig. 8(c). The charges on this charge net in the vicinity of vertices A, B are as follows.

$$q^k_{B\tilde{v}_{J^i}} = (q_{\text{flip}})^k_{J^i}, \quad i = 1, 2, 3 \tag{4.7}$$

$$q^k_{A\tilde{v}_{J^i}} = q^k_{J^i} - (q_{\text{flip}})^k_{J^i} \tag{4.8}$$

$$q^k_{AB} = q^k_I + \sum_{i=1}^{3} (q_{\text{flip}})^k_{J^i} = -q^k_{BA}, \tag{4.9}$$

with the charges on the remaining parts of the graph being the same as in c.

(6) The charge net c' of Fig. 8(e) is deformed in a downward conical manner by the action of the electric diffeomorphism $\hat{D}(\vec{N}_i)$ at its $N_B + 3$ valent

[18] We assume that the state $c_{(i,I,\vec{J},\beta,\delta)}$ is such that it can be transformed via an appropriate diffeomorphism to the state depicted in Fig. 8(c). We shall comment further on this in Sec. V B.

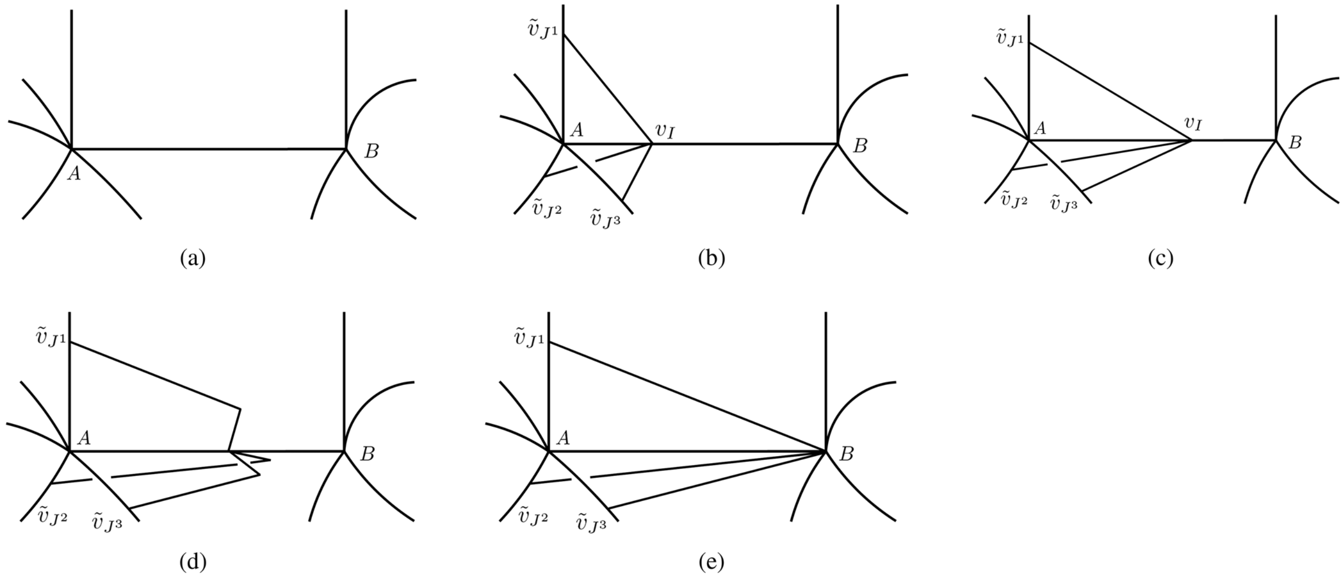

FIG. 8. The figures show the sequence of ket set elements Fig. 8(a) to Fig. 8(e) which encode propagation from vertex A to vertex B of different valence as described in the main text. The ket set is the minimal one containing Fig. 8(a) appropriate to the $N \to 4$ deformation.

vertex B along the edge from B to A with the chosen 3 edges being exactly the edges from B to each of $\tilde{v}_{J^i}$, $i = 1, 2, 3$ to give exactly the state in Fig. 8(d).

The minimal ket set containing the chargenet c of Fig. 8(a) must contain the charge nets depicted in Figs. 8(b)–8(e). Steps (1)–(6) imply that the chargenet of Fig. 8(d) has 2 possible ancestors, one depicted in Fig. 8(a) and one in Fig. 8(e). The sequence of elements Figs. 8(a)–8(e) is then one which encodes the "emission" of a conical perturbation at the vertex A of c depicted in Fig. 8(b) and its propagation and final "absorption" by vertex B to yield the chargenet c'. The result of this absorption is an additional connectivity in the graph which additionally entangles the vertices A and B. Further, the valence of vertex B as a result of this absorption has increased by 3. It is in this sense that the $N \to 4$ action generates propagation between vertices of different valence as well as in the presence of free edges.

Here we have implicitly assumed that the vertices v in Fig. 8(a), v_I in Fig. 8(b) and B in Fig. 8(e) are nondegenerate. The first two assumptions are simply assumptions on the charge labelings of vertex A in c. The third is an assumption on the labelings of the vertex B of the charge net in Fig. 8(e). As mentioned above the vertex structure at B in Fig. 8(e) is obtained by adding 3 extra edges to the original vertex B in c. These are positioned in the vicinity of B so as to render B GR in c'. It seems reasonable to us that exploiting the available freedom in positioning these 3 edges relative to the original edges at B would enable us to choose an edge configuration such that B is nondegenerate in c'.

C. 3d propagation

Let the unperturbed charge network c in the previous section be based on a graph dual to a triangulation of Σ by tetrahedra. Every vertex of this graph is then (linear) GR and 4 valent. Each vertex is connected to 4 other vertices each such connection being through a single edge. In the language of the previous section, each vertex then has 3 free edges. Figure 9(a) shows the graph structure of c in the vicinity of 3 of its vertices A, B, C.

Repeating the considerations of the previous section, we "perturb" c at its vertex A through the action of the Hamiltonian constraint to yield $c_{(i,I,\vec{J},\beta,\delta)}$ shown in Fig. 9(b) and then "evolve" this perturbation at A in c to B yielding the chargenet c' of Fig. 9(c) in which the vertex B is now 7 valent. We shall rename c' as c_{AB} in what follows so as to remind us that the perturbation has traversed the path $A - B$ in c to yield $c' = c_{AB}$. Here we show how to further evolve this perturbation beyond the vertex B through the exclusive use of electric and semianalytic diffeomorphisms, once again through a primarily diagrammatic argument (see Fig. 9).

In what follows we use the (obvious) notation wherein an edge connecting the point P to the point Q is denoted by e_{PQ} and the charge thereon by q^k_{PQ}.

(1) We act by an electric diffeomorphism at B of c_{AB} which deforms the vertex structure at B along the edge e_{BC} as depicted in Fig. 9(d). The 3 additional edges chosen for this deformation are exactly the edges $e_{B\tilde{v}_{J^i}}$. This results in a reduction of valence of B back to 4 together with creation of the displaced vertex which we call v' The vertex v' is now directly connected to the edges e_{J^i} emanating from A at the points $\tilde{v}_{J^i}$. Each of the segments $l_{v'\tilde{v}_{J^i}}$ which connect v' to $\tilde{v}_{J_i}$ are obtained by deforming the edges $e_{B\tilde{v}_{J^i}}$ and thus each of these segments has a kink at $\tilde{v}'_{J_i}$ as shown in Fig. 9(d). The charges on these segments

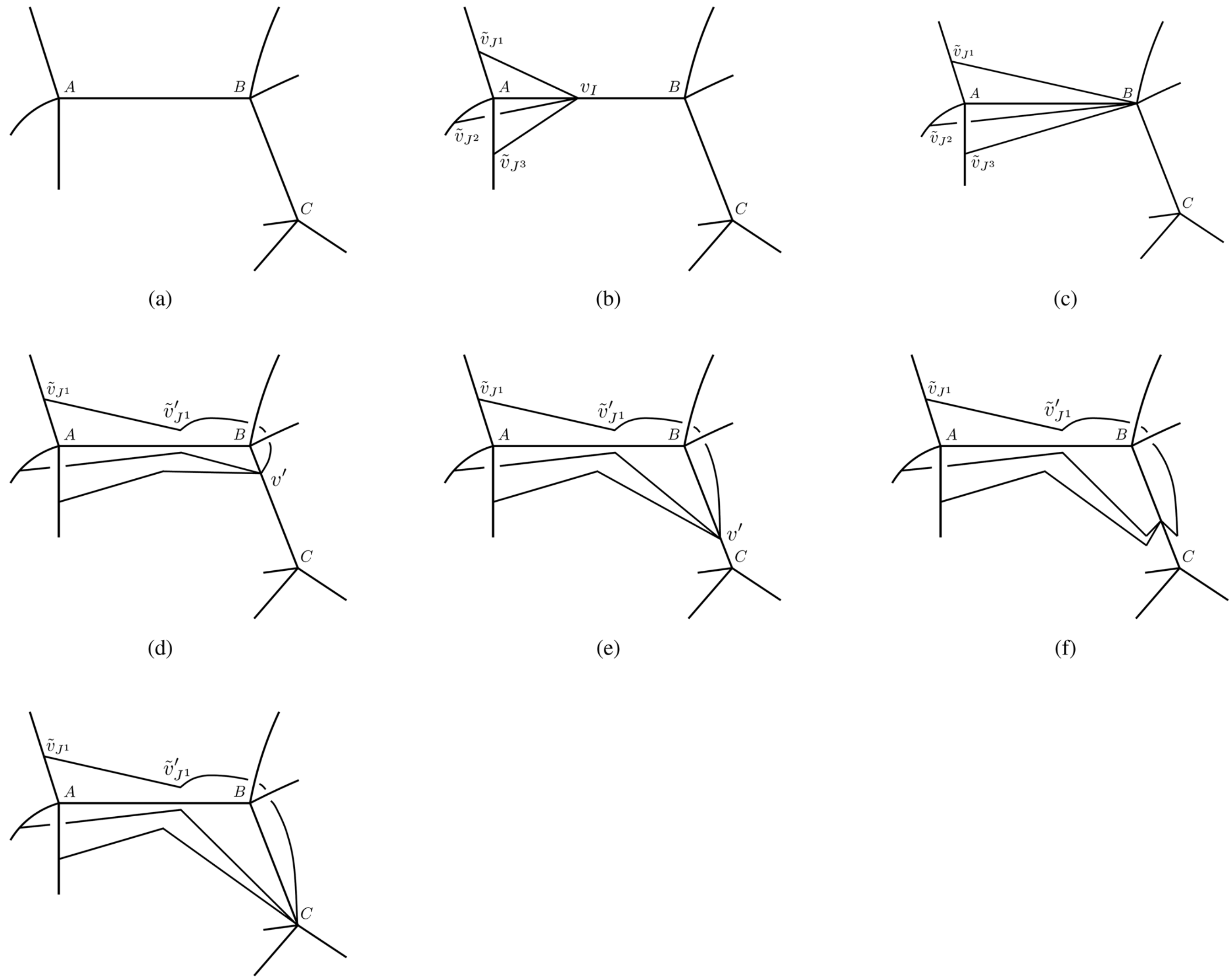

FIG. 9. The figures show the sequence of ket set elements Fig. 9(a) to Fig. 9(g) which encode propagation from vertex A to vertex C as described in the main text. The unperturbed chargenet in Fig. 9(a) is based on a graph which is dual to a triangulation of the Cauchy slice. The ket set is the minimal one which contains Fig. 9(a) and is appropriate to the $N \to 4$ deformation.

are exactly the same as those on the edges $e_{B\tilde{v}_{J^i}}$. From (4.7), we have (in obvious notation)

$$q^k_{v'\tilde{v}_{J^i}} = q^k_{B\tilde{v}_{J^i}} = (q_{\rm flip})^k_{J^i}, \quad i = 1, 2, 3 \qquad (4.10)$$

From gauge invariance and the fact that the deformation only affects the immediate vicinity of B we have from (4.10) that:

$$q^k_{v'B} = -\sum_{i=1}^{3} (q_{\rm flip})^k_{J^i} - q^k_{v'C}, \quad q^k_{v'C} = q^k_{BC} \qquad (4.11)$$

where we have denoted the charge on e_{BC} in c (and hence also in c_{AB}) by q^k_{BC}. The charges on the remaining parts of the graph are exactly those on these parts of the graph in c_{AB}

(3) We act by a semianalytic diffeomorphism ϕ so as to move the vertex v' to the vicinity of C as shown in Fig. 9(e).[19] Abusing notation slightly we continue to refer to the moved vertex as v'.
(4) Similar to (4) of Sec. IV B we act by an electric diffeomorphism at v' so as to change the sign of the conicality of the vertex structure to obtain the chargenet depicted in Fig. 9(f).
(5) Similar to (5) of Sec. IV B, the charge net depicted in Fig. 9(f) can be obtained by the action of an electric diffeomorphism at the 7- valent vertex C of the

[19] An assumption similar to that in footnote 18 applies.

chargenet c_{ABC} depicted in Fig. 9(g), with charge labels as follow:

$$^{(G)}q^k_{C\tilde{v}'_{J^i}} = (q_{\text{flip}})^k_{J^i}, \quad i = 1, 2, 3, \tag{4.12}$$

$$^{(G)}q^k_{CB} = -\sum_{i=1}^{3}(q_{\text{flip}})^k_{J^i} + q^k_{CB}. \tag{4.13}$$

Here the left superscript (G) indicates charges on the chargenet c_{ABC} depicted in Fig. 9(g), q^k_{CB} is the charge on the edge e_{CB} of the unperturbed charge net c of Fig. 9(a) and, as in (4.10), $(q_{\text{flip}})^k_{J^i}, i = 1, 2, 3$ denote the flipped images of the charges $q^k_{J^i}, i = 1, 2, 3$ in c on the three chosen edges $e_{J^i}, i = 1, 2, 3$ for the $N \to 4$ deformation. The charges on the remaining parts of the graph in Fig. 9(g) are unchanged relative to their values on these parts of the graph in Fig. 9(f).

Once again, the charge nets in Figs. 9(a)–9(g) must all be in the ket set and the sequence Figs. 9(a)–9(g) describes the propagation of the perturbation from vertex B to vertex C of c_{AB}. Here similar to the case of c' in Sec. IV B, we have assumed that the 7-valent vertex C of c_{ABC} is nondegenerate (the nondegeneracy of the vertex v' follows from that of the assumed degeneracy of v_I in Sec. IV B).

To summarize: the sequence

$$c \to c_{(i,I,\vec{J},\beta,\delta)} \to c_{AB} \to c_{ABC} \tag{4.14}$$

represents propagation of a perturbation from the vertex A to the vertex C which is two links away from A. The path of propagation is A–B–C. Clearly, with assumptions of nondegeneracy similar to (5) above for the 7-valent vertices encountered in the course of propagation, together with assumptions on the existence of appropriate semianalytic diffeomorphisms similar to that in footnote 18, we may propagate this perturbation along a path joining A to a vertex A_{final} as far away as we desire. This shows that the $N \to 4$ action generically engenders 3d long range propagation.

V. PROPAGATION AND ANOMALYFREE ACTION

In Sec. V A we discuss the new challenges to be confronted in rendering the constraint actions discussed in this paper consistent with the requirement of anomaly-free commutators as articulated in [15]. One outcome of this discussion is the suggested enlargement of the ket sets considered hitherto by replacing the role of semianalytic diffeomorphisms in property (a3), Sec. I by a larger set of transformations which we call linear vertex preserving homeomorphisms or lvh transformations. In Sec. V B we describe certain technical issues which we overlooked in our treatment of propagation hitherto and suggest that these issues may be alleviated as by the use of these transformations. In Sec. V C we discuss the issue of "fake" propagation.

A. New challenges for a proof of anomalyfree action consistent with propagation

The physical states considered hitherto are based on ket sets subject to property (a). The considerations of Sec. II G imply that these physical states provide a trivial anomaly-free representation space for the Hamiltonian (and spatial diffeomorphism) constraints. Several new challenges must be confronted relative to [15] in order to construct off shell deformations of these states which support a nontrivial anomalyfree implementation of constraint commutators. We describe the ones we are able to anticipate below.

(1) New issues for the $N \to N$ action:

(1a) Multiple vertices[20]: The work in [15] constructs the desired off shell states based on ket sets with elements each of which has only a single vertex where the Hamiltonian constraint acts non-trivially through $N \to N$ deformations. The first challenge is to generalize this construction to the case of multivertex ket sets. Due to the independence of the action of the Hamiltonian and electric diffeomorphism constraints at independent vertices, we feel that it should be possible to generalize the considerations of [15] to the multivertex case with $N \to N$ deformations. The candidate ket set subject to property (a) would consist of multivertex kets. We would then seek to construct appropriate off shell states as linear combinations of (the bra correspondents of) these kets with coefficients which were products, over vertices, of the single vertex coefficients of [15]. The new feature of the multiple action of constraints would be the appearance of contributions from different vertices. The new challenge would be developing an efficient book keeping of these contributions and defining appropriate "Q-factors" [15] to ensure the existence of anomalyfree continuum limit commutators.

(1b) Coordinate patch specification: In addition to (1a), an appropriate diffeomorphism covariant specification of coordinate patches for each vertex of each multivertex ket in the ket set is necessary in order both to define the action of the (higher density) constraints as well as to evaluate the off-shell state coefficients mentioned above. Such a specification exists for the single vertex states of [15]. Its detailed construction rests on the choice of certain fiducial coordinate patches with respect to which certain preferred classes of kets in the ket set known as primaries, have linear vertices (see [15] for details). It then turns out that the physical states constructed in [15] have an imprint of the choice of these fiducial coordinate patches; this is unsatisfactory from the point of view of the "background independent" philosophy of LQG. Work in progress suggests that this imprint can be removed by enlarging

[20] This is a slightly more detailed discussion of (2), Sec. II F.

the single vertex ket sets of [15] so that the semianalytic diffeomorphisms of property (a3) in Sec. I are replaced by a larger set of transformations, which we tentatively identify as homeomorphisms which preserve the linearity of linear vertices and which we refer to as '*linear-vertex preserving homeomorphisms*' or *"lvh" transformations*.

The coordinate patches at such vertices are then related by transformations which have a much more local character than diffeomorphisms. Since many of the considerations of [15] rest on local Jacobian transformations, it seems that the constructions of [15] can be generalized to the context of these larger ket sets. Assuming such a generalization is successful, we anticipate that a further generailization of this coordinate patch specification to the multivertex case should not face any significant obstacles.

(1c)Upward direction specification[21]: As mentioned in Sec. II F, the constructions of [15] involve a unique specification of upward or downward conical deformations at a vertex deriving from the signs of the charges at the vertex and the unique identification of an upward direction at the vertex from the graph structure in the vicinity of the vertex. The graph structure involved in the unique identification of an upward direction is that of the placement and type of kinks (i.e., C^0, C^1, C^2 [15]) in the vicinity of the vertex. We believe that the creation and placement of the C^1, C^2 kinks is an unnecessary feature of the constructions of [15] and that the specification of an upward direction at the parental vertex being acted upon can be made freely. What seems to be important in defining the action of the products of regulated constraint operators on a parent state is that the upward directions defining each operator in this product be suitably correlated with the free choice of upward direction associated with the action of the first operator in the product on a parental vertex. In this way, the choice of an upward direction is a further regulator choice in defining constraint operator products. Anomalyfree commutators then refer to the equality of 2 operator products related through a replacement of Hamiltonian constraint commutators by electric diffeomorphism commutators in accordance with the identity (2.4), wherein the 2 operator products are assigned the same choice of upward direction at the parental vertex. The end result is that the ket set would contain both upward and downward deformations of parents independent of the sign of the parental charges. While our intuition is that it should be possible to demonstrate all this, it is of course essential to explicitly construct such a demonstration.

(1d) States subject to additional physical requirements: As seen in Sec. III A the $N \to N$ constraint action does not engender propagation. This is slightly ameliorated by the imposition of further physical requirements in Sec. III B. While we feel that the considerations of [15] should generalize to multivertex ket sets *not subject to these requirements*, for ket sets subject to this requirement still new challenges arise related to the fact that the action of $\hat{H}_{\pm}(N)$ of Sec. III B do not generically preserve the number of nondegenerate vertices.[22] However since even with the additional requirements, generic ket sets of Sec. III B do not engender vigorous propagation we shall not discuss this case further. Instead we proceed to a discussion of challenges in the context of the $N \to 4$ action.

(2) The $N \to 4$ action: Coordinate patch specification remains an issue and the discussion in (1b) above applies equally well to the $N \to 4$ action. New issues arise from the fact that, unlike the $N \to N$ action, the $N \to 4$ action does not preserve the number of nondegenerate vertices. For example, if we have a single vertex charge net with a nondegenerate vertex of valence N, a Hamiltonian constraint $N \to 4$ action on this vertex yields a state with an N valent vertex and a 4 valent vertex, which for $N \geq 7$ are both generically non-degenerate. Thus unlike the $N \to N$ case, it is not meaningful to talk of "fixed vertex sectors." Related to this, the second action of a constraint acquires possible contributions not only from the 4 valent child vertex created by the first action but also from the N valent vertex at the location of the original parental vertex. Since the off shell states constructed in [15] were geared to the preservation of the number (and valence) of the nondegenerate vertices by the $N \to N$ action, accommodating this new feature is the key new challenge. Its confrontation requires a better understanding of the potential children created by the action of the second constraint on the N valent vertex at the location of the original parental vertex. A detailed analysis reveals that a second key challenge arises from the ubiquitous existence of 4 valent vertices of nonunique parentage (such vertices being necessary for propagation).

To summarize: There are at least 2 new and nontrivial challenges to be overcome relative to the $N \to N$ case. In this sense it is certainly that the $N \to N$ case is potentially much better suited to anomaly freedom by virtue of its relative simplicity. However since it seems to lack the crucial physical property of being consistent with long range propagation, it is essential to attempt to overcome the new challenges and provide a proof of anomaly freedom for the $N \to 4$ action. While we remain optimistic that this can be done (perhaps with minor propagation preserving modifications of the $N \to 4$ action), an explicit proof presents an open problem.

[21]This issue and the related exposition involves an assumed familiarity with fine technical issues in [15].

[22]Note however that the action of the Hamiltonian and diffeomorphism constraints do preserve the number of nondegenerate vertices modulo the "eternal nondegeneracy" assumption of [15]. This is in contrast to the $N \to 4$ constraint action; see the next point (2) for details.

B. Issues related to the movement and absorption of child vertices

Our demonstrations of propagation hitherto overlooked issues concerned with the specification of coordinate patches at vertices where the Hamiltonian and electric diffeomorphism constraints act. More in detail, propagation ensues due to our demonstration of nonunique ancestry of the same child. Our arguments purport to show that the same child can be generated, up to diffeomorphisms, by constraint actions in the vicinity of one ancestral vertex and equally well by constraint actions in the vicinity of a neighboring ancestral vertex. In our arguments we have made two assumptions, one explicit and one implicit. The first is an assumption that semianalytic diffeomorphisms can be used to "move" child vertices in the vicinity of one ancestral vertex to the vicinity of a neighboring ancestral vertex, and is explicitly mentioned in footnotes 13, 18 and 19. The second, implicit, assumption relates to the "absorption" of the moved vertex by the second ancestral vertex resulting in a distinct "possible" ancestor, or, equivalently, the creation of this moved vertex by constraint actions at this second ancestral vertex in this distinct ancestor. These actions create deformations which are conical with respect to the coordinate patch associated with this second ancestral vertex and this patch is in general different from the one associated with the first ancestral vertex so it is not clear if the moved vertex can be conical with respect to the patch associated with this second ancestral vertex. If it is not, then the candidate "possible ancestor" cannot be an ancestor of such a child and propagation does not ensue. Hence the implicit assumption is that the child vertex can be moved from the vicinity of one ancestral vertex to a neighboring ancestral vertex in such a way that it does represent a deformation of this neighboring vertex in the candidate possible ancestor which is conical with respect to the ancestral coordinate patch at this neighboring vertex in this possible ancestor.

We believe that both these assumptions are unnecessary if we admit the lvh transformations described in (1b) of Sec. V A. In other words, these lvh transformations can be used to move a child vertex from the vicinity of one ancestral vertex to the vicinity of a neighboring ancestral vertex in such a way that the moved child vertex can indeed be created by the candidate possible ancestor used in our arguments for propagation in Secs. III A 2, IV B and IV C. Showing this requires developments along the lines of (1b), Sec. V A and constitutes a problem for future work.

C. The issue of "fake" propagation

As discussed in Sec. I, propagation is said to be encoded by a physical state if in the set of its kinematic summands there exists a "propagation" sequence of states starting with a parent p and describing the propagation of its perturbations from one parental vertex to another followed by absorption of these perturbations to yield another possible parent p'. In order that this notion of propagation be strongly tied to the properties of the quantum dynamics (i.e., to the properties of the constraint operators), it is essential that the coefficients in the sum over kinematic summands are *uniquely* determined (up to an overall constant factor) by the requirement that the physical state in question is annhilated by the constraints. This rules out *fake* propagation wherein the physical state in question is itself an arbitrary linear combination of physical states such that each physical state in this combination contains only a subset of elements of the desired propagation sequence as summands. Such fake propagation arises only because of the "artificial" choices of linear combination of individual physical states and is not strongly tied to the dynamics. Thus it is important to establish that the examples of propagating physical states we construct in Sec. IV are ones whose summand coefficients *are* uniquely determined.

In this regard, we note that the physical states which encode propagation in Sec. IV are "minimal" physical states containing a parent p of interest. By a minimal physical state containing a charge net p we mean one which is built out of a sum (with unit coefficients) over all elements of the minimal ket set containing p. In turn, by the minimal ket set containing p we mean the smallest ket set subject to property (a) which contains p. Section IV then constructs examples of a minimal ket set containing a specific parent p of interest such that this ket set also contains propagation sequences of states which start with p. The question is then if the specific $N \rightarrow 4$ implementation of the constraints uniquely fixes the coefficients in a linear superposition of elements of a minimal ket set to be unity up to an overall factor. A proof of an affirmative answer to this question is desirable but should be straightforward to construct given that both upward and downward conical deformations are generated independent of the sign of charge labels, if need be by the use of interventions (see Sec. II F and [15]) together with judicious sign insertions [see the discussion around Eq. (2.24)]. The idea is to then construct legitimate expressions for the constraints in which the "(deformed child—parent)" contributions to (2.29) can occur with either positive or negative signs independently of each other so that the condition that a physical state Ψ is annihilated by all these expressions reduces to (2.30).

While a proof along the lines sketched above is appropriate with regard to minimality in the context of property (a) of Sec. I, there is a further fine technical point which arises due to the desired enlargement of these minimal ket sets through the replacement of semianalytic diffeomorphisms in the articulation of property (a3) by the larger set of lvh transformations advocated in (1b) and (2) of Sec. V A. We shall refer to properties (a1), (a2) together with this replacement of (a3) as the *lvh modification of property (a)*. We note that the enlargement of the ket set through the requirement of consistency with the lvh modified property (a) is *not* a direct result of the implementation of the Hamiltonian, electric diffeomorphism and semianalytic diffeomorphism constraints but is motivated by the general requirement of background independence as discussed in (1b). More in detail, that this requirement indicates that physical states should not depend on ad-hoc regulating coordinate choices. Since we strongly believe that any trace of *ad-hoc* regulating structures in our physical state space would lead to unphysical consequences, we are only interested in propagation for those physical states which do not bear any trace of such regulating structures. In the context of our discussion of fake propagation we then adopt a working definition for the engendering of propagation as follows.

The *minimal lvh extended ket set containing* p is defined to be the smallest ket set containing p which is consistent with the lvh modification of property (a). A physical state Ψ_{lvh} will be said to encode propagation from a parent chargenet state p if:

(i) Ψ_{lvh} is constructed as the sum over elements, with unit coefficients, of elements of the ket set $S_{\mathrm{ket},lvh}$.
(ii) $S_{\mathrm{ket},lvh}$ is the minimal lvh extended ket set containing p.
(iii) There is a propagation sequence in $S_{\mathrm{ket},lvh}$ starting from p with elements of this sequence being related by the action of Hamiltonian constraint deformations, electric diffeomorphism deformations and lvh transformations.

Clearly this definition requires clarity on the exact nature of the lvh transformations alluded to in (1b), Sec. V A. As indicated in Sec. V A, this clarification along with other issues discussed in Sec. V A constitute open problems worthy of further study.

VI. VERTEX MERGERS AND THE ISSUE OF COARSE GRAINING

While, in Sec. IV C, we illustrated propagation in the context of a parent charge net based on a graph which is dual to a triangulation of the Cauchy slice, it is instructive (and fun) to explore other consequences of the $N \to 4$ action for states based on graphs with this, as well as different, vertex structures. In this section we focus on the phenomenon of *vertex merger*.

For parent graphs, in which a pair of parental vertices are connected by a single line which extends beyond these vertices (rendering the vertices "CGR"), it turns out that this action can effectively *merge* parental vertices, the chargenets with these merged vertices corresponding to possible parents of appropriately deformed children of the parent chargenet. To see this imagine that in Fig. 9(a) the edge e_{AB} extends beyond A and connects to some other vertex D in the graph. Applying an downward conical electric diffeomorphism deformation with respect to e_{BC} to the (now CGR) parental vertex A, we effectively "move" A towards B. We bring the deformed vertex in the vicinity of B (see Sec. V B for pertinent discussion), reverse the conicality of the deformed vertex structure by an appropriate electric diffeomorphism and then get B to "absorb" this conical deformation through the use of an electric diffeomorphism on a possible parent in which B is now 7 valent and A has disappeared. For special graphs with appropriately connected (multiply) CGR parental vertices (see Sec. III B for the definition of a multiply CGR vertex), a repeated application of appropriate electric and semianalytic diffeomorphisms can cause these parental vertices to merge to form a single high valent vertex reminiscent of classical "collapse." Since we have not explored the *classical dynamics* of $U(1)^3$ theory adequately, we do not know if this dynamics admits such collapsing solutions. In case (as we are inclined to believe) it does not, we would view the (physical state based on the minimal ket containing the) parental state above as not of physical interest. Of course the identification of vertex merging with collapse is only intuitive because, similar to the notion of propagation, it is based on the behaviour of charge net summands in the sum which defines a physical state rather than on the behavior of physical expectation values of Dirac observables. On the other hand, one may speculate that if the $N \to 4$ action can be incorporated into the action of the constraints for gravity, such an incorporation could possibly carry the seeds of both graviton propagation and gravitational collapse.

From Figs. 8 and 9, the simple modification from the $N \to N$ to the $N \to 4$ action not only creates and merges vertices it also creates new connections between structures in the vicinity of vertices thus "entangling" them. Thus this modification has the desirable property of a simple local rule which leads to rich nonlocal structure. Due to this property, the $U(1)^3$ model offers an ideal testing ground for *coarse graining* proposals in the context of the $N \to 4$ action. Specifically we have in mind a $U(1)^3$ implementation of the proposals of Livine and Charles [18] for the $SU(2)$ case. Since coarse graining and the recovery of effectively smooth fields is a key foundational issue in LQG, the $U(1)^3$ model hereby acquires additional significance as a toy model for LQG.[23]

Reverting to the example of a charge net based on a graph which is dual to a triangulation, it is possible to construct an evolution sequence from the unperturbed state c of Fig. 9 which leads to a single vertex of high valence

[23] Recall however that the $U(1)^3$ model maybe understood as a weak coupling limit of Euclidean gravity [12]. Since a well-posed initial value formulation, and the consequent integrability of Hamiltonian vector fields of the constraints are not expected to exist for Euclidean theory, it may be that the same holds for the $U(1)^3$ model. If this is true, then it is pertinent to point out that despite the appropriateness of the $U(1)^3$ model as a toy model to probe *structural* properties of the quantum constraints as done in this work and in [11,13,15], the model may not serve as a useful testing ground for issues connected with semiclassicality.

surrounded by parental vertices which are rendered degenerate. A sketch of this construction is as follows. We start from the parental chargenet c of Fig. 9(a) and generate the chargenet c_{AB} of Fig. 9(b). In doing so we have increased the valence of vertex B to 7 with a concomittant loss of nondegeneracy of vertex A cause by the vanishing of the ith component of all the outgoing charges there (see the discussion in Sec. II). We then deform c_{AB} as follows:

(a) We perturb some nearest nondegenerate parental vertex D connected to A by the action of a Hamiltonian constraint. Since the connection must be through one of the 3 edges e_{J^i}, the perturbation will encounter the trivalent kink $\tilde{v}_{J^i}$ for some fixed i (more precisely this kink is a CGR bivalent vertex) before getting to A.

(b) We may then, through the exclusive use of semianalytic diffeomorphisms and electric diffeomorphisms together with appropriate assumptions on nondegeneracy of vertices of possible parents:

(b1) get $\tilde{v}_{J^i}$ to "absorb" the conical perturbation rendering it 5 valent,

(b2) emit it towards A rendering $\tilde{v}_{J^i}$ bivalent CGR as before,

(b3) get the vertex A to absorb this perturbation so as to render A 7 valent, and, (b4) get A to emit this perturbation and B to absorb it rendering A degenerate and 4 valent, and B, 10 valent and connected to D, in addition to its prior connectivity with A.

Thus at the end of (a)–(b4) we have again increased the valence of B by 3 at the cost of rendering a nondegenerate vertex, in this case vertex D, degenerate. We may then, with appropriate nondegeneracy assumptions, repeat this procedure for any path connecting a nondegenerate vertex to B, thereby increasing the valence of B by 3 each time and simultaneously rendering the nondegenerate vertex, degenerate. The end result is the vertex B of high valence with all vertices in a region surrounding B rendered degenerate. Note that this process of vertex merger is distinct from the case alluded to above wherein the original parental vertices merge through the exclusive application of electric and semianalytic diffeomorphisms. Indeed, since no parental vertex of a graph dual to a triangulation [see Fig. 9(a)] is CGR, that mechanism of vertex merger fails. Instead, as sketched above, we need to employ the Hamiltonian constraint deformation of step (a) as well.

It is not clear if the state obtained from repeated applications of the steps (a)–(b4), with many degenerate parental vertices and a single nondegenerate one at the "microscopic" level, represents a classically singular configuration precisely because any such interpretation is intertwined with the issue of coarse graining of such a state. This discussion points to a clear need for an unambiguous interpretation of physical states and their kinematic summands. If, following such a putative interpretation, generic physical states display properties at variance with generic classical solutions, one may need to further modify the dynamics perhaps by a suitable mixture of $N \to 4$ and $N \to N$ deformations. For example, if we want to avoid the state obtained through repeated applications of (a)–(b4), we may create an obstruction to the accumulation of valence by B beyond some fixed valence $N_{\max}$ as follows. We define a dynamics which generates $N \to 4$ deformations for all nondegenerate vertices of valence $N < N_{\max}$ and which generates $N \to N$ deformations for vertices of valence $N \geq N_{\max}$. Clearly, if through repeated applications of (a)–(b4), we increase the valence of B to N such that $N_{\max} > N \geq N_{\max} - 3$, a further such application will be obstructed at step (b4) because any possible parental vertex of valence $N \geq N_{\max}$ can only yield a child vertex of valence N rather than one of valence 4. Thus, for such a dynamics, evolution sequences for generic parental graphs will not generate vertices of valence greater than or equal to $N_{\max}$ by vertex mergers. Note however that the evolution sequence describing long range 3d propagation in Sec. IV C remains consistent with such a dynamics provided $N_{\max} > 7$, a valence of 7 being the maximum valence encountered in this evolution sequence. Also note that our discussion of vertex mergers above is subject to the same technical caveats discussed in Sec. V B in connection with propagation.

VII. DISCUSSION

Early pioneering works [19] on the quantization of the Hamiltonian constraint for gravity in the late 1980s and early 1990s together with the development of a rigorous quantum kinematics (see for e.g., [20] as well as [1,2,7] and references therein) and some ideas from other researchers [21] lead to the detailed framework for the construction of a Hamiltonian constraint operator in Thiemann's seminal work [22]. This framework organizes this construction as the continuum limit of quantum correspondents of classical approximants to the Hamiltonian constraint. The resulting operator carries an imprint of the choice of these approximants and is hence infinitely nonunique. Current work seeks to subject the resulting operator to physically and mathematically well motivated requirements so as to reduce this nonuniqueness. On the other hand, since these requirements are extremely nontrivial, the mathematical tools and techniques needed to confront them also have to be constantly upgraded. For example, the work [23] seeks to construct this operator such that it is well defined on a *Hilbert* space rather than a representation space with no natural inner product. This leads to the consideration of new Hilbert spaces which lie between the kinematic Hilbert space of LQG and the linear representation space known as the habitat [8]. The works [9,11,13,15] seek constructions which are consistent with nontrivial anomalyfree commutators and lead to new tools such as electric field dependent holonomy approximants [9], the use of electric diffeomorphism deformations arising from the discovery of a new classical identity [11], diffeomorphism covariant choices of coordinate patches [13], interventions [15] and a new mechanism for diffeomorphism covariance [15].

In this work we confront the Hamiltonian constraint of the $U(1)^3$ model by the requirement of propagation. The new mathematical elements which allow us to bypass the obstructions to propagation (modulo the caveats discussed in Sec. V B) pointed out by Smolin [4] are the structural property of the constraint discussed in Sec. II F (and uncovered in [6]) together with the $N \to 4$ action. As with all new additions to our toolkit, it is necessary to accumulate intuition as to what they do and, if necessary improve them further or discard them. It is in this general context that the developments presented in this paper should be viewed.

Thus, while on the one hand the $N \to 4$ action engenders vigorous propagation thereby showing the basic LQG framework for the construction of the Hamiltonian constraint is powerful enough to bypass the "no propagation" folklore in the field (see also, however, the discussion at the end of this section), it also leads to the phenomenon of vertex merger discussed in Sec. VI. More generally, the $N \to 4$ action separates vertices, merges vertices and increases graphical connectivity leading to rich nonlocal structure. The work in this paper studies aspects of this structure in the context of a few examples of parental graph structures. It is necessary to study a larger diversity of such examples so as to explore the full power of the $N \to 4$ action, and if necessary subject it to further improvement.

In this regard, it is an open issue as to whether the $N \to 4$ or some other choice of improved constraint actions is physically appropriate for the $U(1)^3$ model. In view of the discussion in Secs. VI and I, any resolution of this issue involves (a) an understanding of coarse graining of kinematic states and a consequent interpretation of physical states, (b) an understanding of the classical solution space of the model, and, (c) an analysis of constraint commutator actions on a suitable off shell state space so as to check if the chosen actions are anomaly free in the sense of [15]. In relation to (c), focusing exclusively on the $N \to 4$ action, we have discussed the challenges inherent in a putative demonstration of nontrivial anomalyfree action in Sec. V A. If these challenges can be overcome, we believe that a mixture of $N \to 4$ and $N \to N$ actions of the type discussed towards the end of Sec. VI should then also not present significant obstacles to such a demonstration. With regard to issues (a) and (b), as indicated in Sec. VI, we believe that the $U(1)^3$ model provides a valuable toy model to test proposals for coarse graining in LQG such as those in [18].

The next step beyond the $U(1)^3$ model is that of the construction of a satisfactory quantum dynamics for full blown LQG. By "satisfactory" we mean, at the very least, "anomalyfree" and "consistent with propagation." By "anomalyfree," we mean a constraint action which admits the construction of a space of off shell states which support nontrivial anomalyfree commutators in the sense of [15]. In this regard, we believe that it is important that the constraint action be such that a second constraint action acts nontrivially on the deformed structure created by a first such action; this property (which holds for the $N \to 4$ deformation) is crucial for the putative emergence of the desired $M\partial_a N - N\partial_a M$ dependence of the commutator on the lapses M, N which smear the two constraints. By "consistent with propagation" we mean the existence of sequences of kinematic state summands which describe propagation.

We note here that while the seminal constructions of the Hamiltonian constraint by Thiemann [22,24] do not have the property that second constraint actions act on deformations generated by the first, it turns out that *contrary to common belief* [4], there is no "in principle" obstruction to the encoding of propagation by physical states in the kernel of Thiemann's constraint [5]. More in detail, Smolin provided a beautiful characterization of propagation in Ref. [4] and applied this characterization to physical states in the kernel of Thiemann's constraint, the latter formulated as in [22,24]. As mentioned in footnotes 1 and 2 the no propagation argumentation of Ref. [4] faces a hitherto unnoticed obstruction in the form of states of non-unique parentage [5]. Thus despite the fact that Thiemann's constraint does not display the detailed structure (2.27), its action shares the property of consistency with non-unique parentage (see footnote 2) with the constraint actions considered in this work and it is this property which serves as an obstruction to the arguments of [4]. This observation represents ongoing work with Thiemann and we shall report on this elsewhere [5].

Returning to the issue of anomalyfree actions consistent with propagation for the case of full blown LQG, we briefly detail our current understanding of the situation for the case of Euclidean LQG wherein the $U(1)^3$ gauge group of this work is replaced by $SU(2)$. We may think of the challenges to be overcome in the construction of such actions for Euclidean LQG to be of 2 types: the first to do with the graph structure of spin net work states and the second to do with the non-Abelian nature of SU(2). Roughly speaking, we believe that the work here together with that in [11,13,15] will be adequate (modulo the open issues sketched above) to confront challenges of the first type. The new challenges will be of the second type. In this regard, there already certain structures available to us which are $SU(2)$ analogs of key $U(1)^3$ ones. First, the key Poisson bracket identity (2.4) also holds unchanged for the $SU(2)$ case [11]. Second, recall that we made crucial use of the identity $N_i^a F_{ab}^k = \pounds_{\vec{N}_i} A_b^k - \partial_b (N_i^c A_c^i)$, to motivate the action of the constraint in Sec. II. The key feature of this identity is that it allows us to interpret the classical evolution equations in terms of electric diffeomorphisms and charge flips. It turns out that there exists a beautiful $SU(2)$ analog of this interpretation for the classical equations of Euclidean gravity [25]. The availability of these two structures provides a starting point for an analysis of the $SU(2)$ case. To summarize: our hope is that the framework of [22,24] together with the structures developed in [11,13,15] and in this work as well as the geometrical interpretation of classical evolution [25] will prove useful for the putative construction of an anomalyfree quantum dynamics, consistent with propagation, for Euclidean, and finally, for Lorentzian gravity.

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