Linear superposition of regular black hole solutions of Einstein nonlinear electrodynamics

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In this paper we focus on certain properties of the Schwarzschild metrics that, to the best of our knowledge, have gone unnoticed. These properties include (i) the existence of a generalized Birkhoff theorem for general nonlinear electrodynamics, (ii) the existence of two double eigenvalues of the Einstein tensor $G^{\mu}{}_{\nu}$ coincident in number with the two pairs of eigenvalues of the electrodynamics energy-momentum tensors, (iii) the partial fitting of the dominant energy conditions, (iv) linear superpositions of solutions of the Einstein electrodynamics equations through the inverse integration method, and (v) the generating technique of solutions with a Reissner-Nordström limit at spatial infinity. Each of these aspects are treated in detail, and a class of regular black hole solutions is reported and studied to some extent.

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I. INTRODUCTION

Nonlinear electrodynamics (NLE), formulated by Born and Infeld (BI) in 1934 [1], provides the electron with a spatial volume containing a charge such that its self-energy is finite. Incidentally, two years later Heisenberg and Euler [2] formulated their nonlinear electrodynamics in the framework of quantum electrodynamics (QED) to determine the nonlinear interactions between photons. A generalization of the classical NLE appeared later in the work of Plebański [3]; see also Ref. [4]. All Petrov type-D solutions of Einstein-Born-Infeld electrodynamics allowing for stationary and axial symmetries (among them the static spherically symmetric solution) were published by García, Salazar, and Plebański [5]. They developed NLE allowing for duality rotations [6] and studied the birefringence properties of the theory [7]. A renewed interest in BI-NLE arose after its appearance in string theory in its low-energy limit; see Refs. [8,9]. The first regular static symmetric charged solution (published by Ayón-Beato and García [10]) in which electromagnetic fields are described in terms of NLE potentials was a major contribution to the study of regular solutions in Einstein and non-Einsteinian gravity. These authors found the nonlinear electrodynamics source (magnetic potential) [11] for the Bardeen model reported in 1968 [12]: the first metric structure that fit the tensor energy conditions exhibiting everywhere-regular Riemannian invariants. An interesting solution described by means of a magnetic field is the Hayward regular solution [13]. In the last decade, there have been numerous attempts to derive stationary rotating solutions with relative success; the number of articles is large and this topic is outside of the scope of this paper, although we can mention some recent works by Toshmatov and collaborators [14,15] and the references therein devoted to the construction and analysis of regular rotating black hole and no-horizon spacetimes based on generic regular black hole spacetimes related to electric or magnetic charge for nonlinear electrodynamics coupled to general relativity.

In 1967 Wheeler coined the term “black hole” for the first exact solution derived by Schwarzschild in 1916 of a point mass m, a few months after Einstein’s publication of *The Foundation of the Generalised Theory of Relativity.* The charged point mass solution was reported by Reissner (in 1916) and Nordström (in 1918). The static spherically symmetric representation of the Schwarzschild metric exhibits an “apparent singularity” or “Schwarzschild singularity” at the Schwarzschild radius $r = 2m$, a fact that was noticed in the 1920s by various researchers (Eddington and Lemaître, among others) who introduced new coordinates to remove the “apparent singularity.” At the end of the 1930s Oppenheimer *et al.* [16,17] studied the collapse of a massive static spherical star and came to the conclusion that in the process of approaching its critical surface the star’s curvature will increase indefinitely, the light radiated by an imploding star will be redshifted, and as the star reaches its critical radius the redshift will become infinite and the star will disappear from the observer’s sight; hence, the star becomes black. A complete understanding of the Schwarzschild metric was achieved by the introduction of null coordinates by Finkelstein, Kruskal, and Szekeres for its maximal extension and the interpretation of the Schwarzschild radius as the surface bound to the event

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horizons. Since then, the theory of black holes has established itself as an important part of theoretical physics. In 1963, in a half-page *Physical Review Letters* article, Kerr reported the rotating black hole solution, which was later charged by Newman and collaborators. To gain insight into black holes Penrose introduced spinors into the description of spacetimes [18] and developed the so-called Penrose diagram procedure. The thermodynamics of black holes was developed by Hawking [19] in the beginning of the 1970s. Meanwhile, in differential geometry various tools were developed, as well as the Newman-Penrose tetrad formalism. The algebraic classification of the Riemann and Weyl tensors by Petrov in 1966 allowed for a deeper understanding of the structure of spacetime. The search for solutions of specific Petrov types became an area of intense research in the 1960s and 1970s, together with the creation of methods and techniques to generate solutions via group theory [20,21]. Chandrasekhar's contributions to the mathematical theory of black holes can be found in Ref. [22]. An excellent and pleasant historical account of the developments of black holes was published by Thorne [23] in 1994, who took part in some of the "golden age" achievements.

In spite of all of these theoretical achievements, the experimental discovery of black holes was elusive until recently: on September 14, 2015 the Laser Interferometer Gravitational-wave Observatory (LIGO) detected the gravitational wave GW150914 [24], which was emitted by a binary system of rotating black holes many thousands of years ago. This gravitational-wave detection showed indirectly the existence of spinning black holes, and certainly new studies in the field of experimental black hole physics will be undertaken. This last paragraph shows, to some extent, how hard it is to achieve experimental (astrophysical) success in this area.

In the framework of the above-mentioned theories—NLE and Einstein gravity—we shall focus on the search for solutions for Schwarzschild-like metrics, in particular regular black holes. With this purpose in mind, in what follows we shall develop various topics:

(1) A demonstration of the Birkhoff theorem in NLE.
(2) The determination of the algebraic types of the Einstein and electrodynamics energy-momentum tensors.
(3) A definition of the inverse integration method.
(4) The generation of electromagnetic solutions via the linear superposition method.
(5) A definition and characterization of a multiparametric solution to Einstein NLE.

The well-known Birkhoff theorem states that any spherically symmetric vacuum solution (gravitational field with a cosmological constant Λ, if any) is static. Generalizations of this theorem for a Maxwell field (with a Λ term) can be found in Sec. 13.4 of Ref. [21] and references therein. Here we generalize this theorem to nonlinear electrodynamics.

Since the Einstein tensor of Schwarzschild-like-metrics (for short Schwarzschild metrics) allows for two pairs of double eigenvalues, it permits electrodynamic fields besides the vacuum solution with a Λ term. Moreover, the traceless Ricci tensor allows for equal pairs of quadruple eigenvalues that are opposite in sign.

The inverse integration method consists in expressing the Lagrangian and electromagnetic field tensor components in terms of the structural functions via the Einstein electrodynamics equations; thus, for a set of given structural functions one evaluates the Lagrangian and field tensor which, by construction, fulfill the dynamical equations and conservation field equations. For the Schwarzschild metrics, there are two independent Einstein equations, which are used to determine the electromagnetic Lagrangian and field through the metric functions; the electromagnetic field equations hold identically.

In the general theory of electrodynamics, one assumes a relation $\mathcal{L} = \mathcal{L}(\mathcal{F}, \check{Q})$ depending on the invariants $\mathcal{F} \sim (\mathbf{E})^2 - (\mathbf{B})^2$ and $\check{Q} \sim \mathbf{E} \cdot \mathbf{B}$, where $\mathbf{E}$ and $\mathbf{B}$ denote, respectively, the electric and magnetic vectors. In practice, it is quite hard to *a priori* guarantee that the proposed Lagrangian satisfies the Maxwell limit and, at the same time, the energy-momentum tensor fulfills the energy conditions. In this respect, see Ref. [6] and references therein. In the inverse procedure proposed above, one overcomes these difficulties in part: there is no need to solve for the coordinate r in terms of $\mathcal{F}$, and later substitute $r(\mathcal{F})$ into $\mathcal{L}$ to get $\mathcal{L}(\mathcal{F})$. Recall that in the standard integration method the Lagrangian is given in terms of the invariant $\mathcal{L}(\mathcal{F})$ and substituted into the dynamical equations, and from there one integrates to obtain the structural functions.

A generating solution technique is given for the Schwarzschild metrics based on a seed metric and a set of "distorting" functions. The seed metrics are the Reissner-Nordström (RN)-like solutions, and the distorting functions F_I, $I = \{\epsilon, M, q, \Lambda\}$ are such that at spatial infinity, $r \to \infty$, their leading term is 1: $F_I(r \to \infty) \to 1$. Then, the new structural function

$$Q(r) = \epsilon F_\epsilon - 2M/rF_M + q^2/r^2 F_q - \Lambda r^2/3F_\Lambda, \quad (1)$$

which has the correct limiting Reissner-Nordström-like solution at infinity, determines a class of solutions for nonlinear electrodynamics, which in turn becomes Maxwell electromagnetism at infinity; the Lagrangian $\mathcal{L}$ equals the invariant $\mathcal{F} = (F_{\mu\nu}F^{\mu\nu})/4$, $\mathcal{L} = \mathcal{F}$, i.e., the weak limit of the generated electrodynamics is the Maxwell one.

Applying the generating procedure outlined above to the RN structural function, a "multiparametric solution" is derived; it is denoted as $S(\epsilon/e, M/m, E_q, \Lambda/l|p, a, b, s)$, where ϵ, M, E_q, and Λ are the curvature parameter (related to α), mass, charge, and cosmological constant, respectively, which are equipped with the exponents p, a, b, s and numerical parameters e, m, l which denote the number of charges q.

This family of solutions branches out a chain of solutions, among them the regular black hole class, which do not exhibit a singularity at the origin of the coordinate system and possess an event horizon; the regularity of the solutions is established by determining the range of the parameters for which the absence of singularities of the Riemann algebraic invariants in the entire spacetime holds. In 1998 the first regular solution to Einstein NLE was published and analyzed in detail by Ayón and García [10]. Since that publication many works on regular solutions have been published; see the list of citations to Ref. [10] in spires. Nowadays, regular black holes occupy an outstanding place in exact solutions of Einstein gravity.

II. BIRKHOFF THEOREM FOR NONLINEAR ELECTRODYNAMICS

A spherically symmetric Schwarzschild metric that is coupled to electrodynamics of any kind (depending on the electromagnetic invariant $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}/4$) and fulfills the equations derived from the Lagrangian $\mathcal{L}$ constructed on $\mathcal{F}$, $\mathcal{L}(\mathcal{F})$, in the presence of a Λ term (if any) is static. The extension to Schwarzschild-like metrics is immediate.

No conditions on the components of the electromagnetic fields $F_{\mu\nu}$ are imposed, except for their antisymmetry, $F_{\mu\nu} \coloneqq 2A_{[\nu,\mu]}$, and thus at this stage all components are present. The electrodynamics is derived from a Lagrangian $\mathcal{L}$ depending on a single invariant $\mathcal{F}$,

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \mathcal{L} = \mathcal{L}(\mathcal{F}). \tag{2}$$

The energy-momentum electromagnetic tensor for $\mathcal{L}(\mathcal{F})$ is given by

$$T_{\mu\nu} = -\mathcal{L}g_{\mu\nu} + \frac{d\mathcal{L}}{d\mathcal{F}}F_{\mu\sigma}F_{\nu}{}^{\sigma}. \tag{3}$$

The metric is assumed to be spherically symmetric (flat or pseudospherically symmetric) and of the form

$$g = \frac{y^2dx^2}{X(\alpha,x)} + y^2X(\alpha,x)d\phi^2 + \frac{dy^2}{Q(y,t)} - P(y,t)dt^2, \tag{4}$$

where $X(\alpha,x) \coloneqq (a_0 - \alpha x^2)$. The spatial sector $\{x,\phi\}$ allows for various possible two-dimensional spaces of constant curvature depending on the function $X(\alpha,x)$:

$$X(1,\cos\theta):\ \text{spherical space};\ ds_2{}^2 = K^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$X(0,-):\ \text{flat space};\ ds_2{}^2 = K^2(dx^2 + dz^2),$$
$$X(-1,\cosh\theta):\ \text{pseudospherical space};$$
$$ds_2{}^2 = K^2(d\theta^2 + \sinh^2\theta d\phi^2),$$

where K stands for the constant Gaussian curvature; see, for instance, Eqs (21.15)–(21.17) of Ref. [25]. The Einstein equations

$$E^{\mu}{}_{\nu} \coloneqq R^{\mu}{}_{\nu} - R\delta^{\mu}{}_{\nu}/2 + \Lambda\delta^{\mu}{}_{\nu} - \kappa T^{\mu}{}_{\nu} = 0 \tag{5}$$

for this metric and an energy-momentum tensor associated to electrodynamics give rise to a number of independent equations of the form

$$E^x_y:\ P(y,t)F_{x\phi}F_{y\phi} - y^2X(\alpha,x)F_{xt}F_{yt} = 0,$$
$$E^x_t:\ y^2Q(y,t)X(\alpha,x)F_{xy}F_{yt} + F_{x\phi}F_{\phi t} = 0. \tag{6}$$

It is easy to detect which components ought to vanish to fulfill the algebraic system. The zero solutions are $F_{xt} = 0 = F_{y\phi}$ and $F_{xy} = 0 = F_{\phi t}$. Hence, the remaining components $F_{x\phi}$ and F_{yt} are free. Under these conditions, the equation

$$E^{y}{}_{t} \coloneqq y\left(\frac{\partial}{\partial t}Q(y,t)\right)X(\alpha,x) - \frac{d\mathcal{L}}{d\mathcal{F}}QF_{y\phi}F_{\phi t} + \frac{d\mathcal{L}}{d\mathcal{F}}Q(y,t)F_{xy}F_{xt}X(\alpha,x)^2 \tag{7}$$

is equal to zero, by virtue of the previous results, if $Q(y,t)$ is time independent,

$$\frac{\partial}{\partial t}Q(y,t) = 0 \to Q(y,t) \to Q(y). \tag{8}$$

From the equation $E^{t}{}_{t} - E^{y}{}_{y} = 0$, to eliminate $\mathcal{L}$ one arrives at the equations

$$yX(\alpha,x)\left(Q\frac{\partial}{\partial y}P - P\frac{\partial}{\partial y}Q\right) - \frac{d\mathcal{L}}{d\mathcal{F}}(PQF_{y\phi}{}^2 + F_{\phi t}{}^2) - \frac{d\mathcal{L}}{d\mathcal{F}}(PQF_{xy}{}^2 + F_{xt}{}^2)X(\alpha,x)^2 = 0.$$

Substituting here all the previously derived results, one writes the above equation as

$$\frac{\partial}{\partial y}\left(\frac{P(y,t)}{Q(y)}\right) = 0 \to P(y,t) = F(t)Q(y). \tag{9}$$

By a scale transformation of the time coordinate one obtains $F(t) = 1$. Hence, the spherically symmetric metric (4) becomes static, and consequently we have demonstrated an extension of the Birkhoff theorem to the case of electrodynamics of any kind.

Incidentally, also in the general theory of nonlinear electrodynamics where one assumes a relation $\mathcal{L} = \mathcal{L}(\mathcal{F},\check{Q})$ depending on both invariants $\mathcal{F} \sim (\mathbf{E})^2 - (\mathbf{B})^2$ and $\check{Q} \sim \mathbf{E}\cdot\mathbf{B}$, an extension of the Birkhoff theorem also holds: for general NLE with

$$\mathcal{L} = \mathcal{L}(\mathcal{F}, \check{\mathcal{Q}}), \qquad \mathcal{F} = F_{\mu\nu}F^{\mu\nu}/4, \qquad \check{\mathcal{Q}} = F_{\mu\nu}F^{\star\mu\nu}/4,$$

with $F^{\star}{}_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$, where $\epsilon_{\alpha\beta\mu\nu}$ is the totally antisymmetric Levi-Civita pseudotensor, the spherically symmetric metric coupled to gravity is static. Its demonstration follows a similar pattern to the one exhibited above.

In what follows, the function $Q(y)$ is replaced by $Q(y)/y^2$. The static metric structure, in coordinates $\{x, y, \phi, t\}$, then assumes the form

$$g = \frac{y^2 dx^2}{X(\alpha, x)} + y^2 X(\alpha, x) d\phi^2 + \frac{y^2 dy^2}{Q(y)} - \frac{Q(y)}{y^2} dt^2. \tag{10}$$

III. EINSTEIN EQUATIONS FOR NONLINEAR ELECTRODYNAMICS

The Einstein equations coupled to a matter field tensor $T_{\mu\nu}$ and a cosmological constant Λ are

$$E_{\mu\nu} \coloneqq G_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = 0. \tag{11}$$

The evaluation of the Einstein tensor $G^{\mu}{}_{\nu}$ and the curvature scalar R for the metric (10) yields

$$\begin{aligned} G^{\mu}{}_{\nu} &= G^{x}{}_{x}(\delta^{\mu}_{x}\delta^{x}_{\nu} + \delta^{\mu}_{\phi}\delta^{\phi}_{\nu}) + G^{t}{}_{t}(\delta^{\mu}_{y}\delta^{y}_{\nu} + \delta^{\mu}_{t}\delta^{t}_{\nu}), \\ G^{x}{}_{x} &= \frac{1}{2}\frac{\ddot{Q}}{y^2} - \frac{\dot{Q}}{y^3} + \frac{Q}{y^4}, \qquad G^{t}{}_{t} = \frac{\dot{Q}}{y^3} - \frac{Q}{y^4} - \frac{\alpha}{y^2}, \\ R &= 2\frac{\alpha}{y^2} - \frac{\ddot{Q}}{y^2}. \end{aligned} \tag{12}$$

From the point of view of the eigenvalue problem, this allows for two different double eigenvalues: $\lambda_1 = \lambda_3 = G^{x}{}_{x}$ and $\lambda_2 = \lambda_4 = G^{t}{}_{t}$. Thus, the related energy-momentum tensor may describe electrodynamics; see, for instance, Ref. [26]. Other way to arrive at this conclusion is by means of the search for the eigenvalues of the traceless Ricci tensor $S^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{R}{4}\delta^{\mu}{}_{\nu}$, which amounts to

$$\begin{aligned} S^{\mu}{}_{\nu} &= S(\delta^{\mu}_{x}\delta^{x}_{\nu} - \delta^{\mu}_{y}\delta^{y}_{\nu} + \delta^{\mu}_{\phi}\delta^{\phi}_{\nu} - \delta^{\mu}_{t}\delta^{t}_{\nu}), \\ S &= \frac{\ddot{Q}}{4y^2} - \frac{\dot{Q}}{y^3} + \frac{Q}{y^4} + \frac{\alpha}{2y^2}, \end{aligned} \tag{13}$$

where $(S^{\mu}{}_{\nu}) = S\,\mathrm{diag}(1,-1,1,-1)$. The relation $G^{x}{}_{x} - G^{t}{}_{t} = 2S$ is remarkable. Hence, according to the Plebański classification of matter tensors [27] this algebraic structure corresponds to the electrodynamics, regardless of whether it is nonlinear or of Maxwell type. In the standard nomenclature, the traceless Ricci tensor canonical structure for electrodynamics is

$$[(11)(1,1)] \sim [2\mathrm{S} - 2\mathrm{T}]_{(11)}, \qquad \lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = S.$$

A straightforward consequence of the coincidence of the algebraic types of the Einstein tensor and the energy-momentum tensor in electrodynamics for static Schwarzschild metrics is the following *Theorem:* besides the vacuum with Λ solutions, static Schwarzschild-like metrics only allow electromagnetic solutions to the Einstein (linear or nonlinear) electrodynamics equations.

In a certain sense, we are facing a theorem about the uniqueness of classes of electrodynamics solutions: besides the vacuum with Λ solution, the static spherical (pseudospherical) metric only allows electrodynamics solutions; in such a case, for each electromagnetic invariant Lagrangian $\mathcal{L}$ related to the electromagnetic invariant $\mathcal{F}$ the solution is unique. Due to the established existence of two pairs of eigenvalues there is no room for fluids; see Chap. 2 of Ref. [26] regarding the structure of fluids. Any attempt to accommodate other kind of fields different from electrodynamics or vacuum in the above Schwarzschild metric (10) is spurious, although in the literature one finds "solutions" (let us call them better spacetime models) for fluids.

For the above metric structure (10), it is natural to choose the electromagnetic field tensor to be dependent on the electric field component $F_{yt}(y) \coloneqq \mathcal{E}$ only, as a function of y, namely,

$$F_{\mu\nu} = 2F_{yt}(y)\delta_{y[\mu}\delta_{\nu]t}, \qquad \mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(F_{yt})^2, \tag{14}$$

leaving aside the magnetic field $F_{x\phi}$ which happens to be constant. Therefore, the components of the nonvanishing electromagnetic energy-momentum tensor (3) are given in tensor form as

$$\begin{aligned} T^{\mu}{}_{\nu} &= T^{x}{}_{x}(\delta^{\mu}_{x}\delta^{x}_{\nu} + \delta^{\mu}_{\phi}\delta^{\phi}_{\nu}) + T^{t}{}_{t}(\delta^{\mu}_{y}\delta^{y}_{\nu} + \delta^{\mu}_{t}\delta^{t}_{\nu}), \\ T^{t}{}_{t} &= -\mathcal{L} - \frac{d\mathcal{L}}{d\mathcal{F}}(F_{yt})^2, \qquad T^{x}{}_{x} = -\mathcal{L}, \end{aligned} \tag{15}$$

from which the electrodynamics eigenvalue property becomes apparent. The contraction of this tensor gives $T^{\mu}{}_{\mu} = -4\mathcal{L} - 2\frac{d\mathcal{L}}{d\mathcal{F}}(F_{yt})^2$. The field equation $(\frac{d\mathcal{L}}{d\mathcal{F}}F^{\mu\nu})_{;\nu} = 0$ yields

$$y^2\frac{d\mathcal{L}}{d\mathcal{F}}F_{yt} = -Q_0 \to y^2\frac{d\mathcal{L}}{dy} - Q_0\frac{dF_{yt}}{dy} = 0, \tag{16}$$

where Q_0 is an integration constant related to the charge.

One only has three equations: two Einstein equations arising from the components $E^{x}{}_{x}$ and $E^{y}{}_{y}$, namely,

$$\begin{aligned} E^{y}_{y} = E^{t}_{t} &= \frac{\dot{Q}}{y^3} - \frac{Q}{y^4} - \frac{\alpha}{y^2} + \Lambda + \mathcal{L} + \frac{d\mathcal{L}}{d\mathcal{F}}(F_{yt})^2 = 0, \\ E^{x}_{x} = E^{\phi}_{\phi} &= \frac{\ddot{Q}}{2y^2} - \frac{\dot{Q}}{y^3} + \frac{Q}{y^4} + \Lambda + \mathcal{L} = 0, \end{aligned} \tag{17}$$

and the operational electromagnetic field equation (16). The difference of E^{y}_{y} and E^{x}_{x} gives rise to

$$Q_0 F_{yt}(y) = -\frac{\ddot{Q}}{2} + 2\frac{\dot{Q}}{y} - 2\frac{Q}{y^2} - \alpha, \tag{18}$$

which can be considered as an independent equation. Using the inverse integration process, this last equation (18) is considered as a relation to determine the electromagnetic field F_{yt} for a given metric structure function $Q(y)$. In a similar manner, $\mathcal{L}(y)$ in terms of the function $Q(y)$ is given by $E^{x}{}_{x}$,

$$\mathcal{L}(y) = -\frac{\ddot{Q}}{2y^2} + \frac{\dot{Q}}{y^3} - \frac{Q}{y^4} - \Lambda. \tag{19}$$

The substitution of F_{yt} from Eq. (18) and $\mathcal{L}$ from Eq. (19) into the field equation (16) leads to an identity; there are no further differential restrictions at all except for the equations (18) and (19).

The optimal and time-saving way to characterize the gravitational field is to use the null tetrad Newman-Penrose formalism consisting in deriving the curvature quantities in the null tetrad basis, where the metric is determined as

$$g = 2\mathbf{e^1 e^2} - 2\mathbf{e^3 e^4} = g_{ab}\mathbf{e^a e^b}, \tag{20}$$

$$(g_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathbf{e^a} = e^{a}{}_{\mu}\mathbf{dx}^{\mu},$$

$$\left.\begin{matrix}\mathbf{e^1}\\ \mathbf{e^2}\end{matrix}\right\} = \frac{1}{\sqrt{2}}\left(\frac{y\mathbf{dx}}{\sqrt{a_0 - \alpha x^2}} \pm i\sqrt{a_0 - \alpha x^2}\, y\mathbf{d}\phi\right),$$

$$\left.\begin{matrix}\mathbf{e^3}\\ \mathbf{e^4}\end{matrix}\right\} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{Q(y)}\mathbf{dt}}{y} \pm \frac{y\mathbf{dy}}{\sqrt{Q(y)}}\right). \tag{21}$$

The Newman-Penrose Weyl curvature tensor scalars reduce to the single Weyl complex component

$$-12y^4\Psi_2 = y^2\ddot{Q} - 6y\dot{Q} + 12Q - 2\alpha y^2, \tag{22}$$

and thus the gravitational field is of Petrov type D or conformally flat. The nonvanishing traceless Ricci tetrad components are $S_{12} = S_{34} = S = 2\Phi_{11}$, and finally one adds to this set the scalar Riemann curvature R. All of these quantities (Ψ_2, S, Φ_{11}, R) are *per se* invariants; all other algebraically constructed Riemann invariants (quadratic and so on) can be expressed through them. Notice the invariant relations $2S = -Q_0 F_{yt}/y^2$ and the relation between the quadratic Weyl tensor invariants $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} =: \mathcal{W}^2 = 48{\Psi_2}^2$.

IV. LINEAR SUPERPOSITION OF SOLUTIONS IN ELECTRODYNAMICS

It has been established in the previous section that for any given function $Q(y)$ one determines a Lagrangian function $\mathcal{L}(y)$ and the corresponding electric field component $F_{yt}(y) =: \mathcal{E}(y)$ according to the equations (18) and (19). Since these equations depend linearly on the structural function $Q(y)$ and its derivatives, a linear superposition of structural functions $Q_i(y)$ yields a linear superposition of the Lagrangian functions $\mathcal{L}(Q_i(y))$ and the fields $\mathcal{E}(Q_i(y))$, and consequently one arrives at the following result.

Theorem.—For static Schwarzschild metrics coupled to electrodynamics (linear and nonlinear) and a Λ term (if any), any linear superposition of structural functions leads to linear superpositions of Lagrangian functions and the corresponding electromagnetic field functions. The resulting set is a solution of the Einstein electrodynamics field equations:

$$E^{\mu}_{\nu}(\{Q_i; \mathcal{L}(Q_i), \mathcal{E}(Q_i)\}) = 0,$$
$$\mathcal{Q} = \sum Q_i(y) \to \{\mathcal{L} = \sum \mathcal{L}(Q_i), \mathcal{E} = \sum \mathcal{E}(Q_i)\};$$
$$E^{\mu}_{\nu}(\{\mathcal{Q}; \mathcal{L}, \mathcal{E}\}) = 0.$$

Therefore one may, for instance, add the Born-Infeld structural function to the RN one; for each structural function one evaluates the corresponding Lagrangian and field function. Each set of functions in turn fulfills the Einstein electrodynamics equations, and hence their superposition should be a solution too.

This theorem explains why one can add a cosmological term to any solution of the Einstein electrodynamics equations with no effort. Moreover, each structural function Q of the set

$$\{\alpha y^2 - 2my, \alpha y^2 - \Lambda y^4/3, \alpha y^2 - 2my - \Lambda y^4/3\},$$

on its own, is a solution of the Einstein–vacuum–cosmological constant equations, and thus there is no need to add the term associated with the two-dimensional curvature constant, the mass term, or the term related to Λ to any chosen structural function every time; it is enough to make these additions (if needed) to the total structural function once. In a certain sense, the Kottler structural function $Q_K = \alpha y^2 - 2my - \Lambda y^4/3$, like the solution to homogeneous equations (i.e., in the absence of electromagnetic quantities), can be added to the structural function Q responding to the electromagnetic fields. Thus, in the "Einstein-Electrodynamics Zoo Park" one can accommodate large families of species.

In the gravitational theory, the fields ought to fulfill the *sine qua non* "dominant energy conditions" which demand, for an observer with time like vector u^μ, $u^\mu u_\mu \leq 0$, the projection of the energy-momentum tensor $T_{\mu\nu}$ must ensure that a) the local energy density $T_{\mu\nu}u^\mu u^\mu \geq 0$ is non-negative, and b) the local energy flow vector $V_\mu \coloneqq T_{\mu\nu}u^\nu$ is a nonspacelike vector $V_\mu V^\mu \leq 0$ (see Sec. 5.3 of Ref. [21]).

For the electrodynamics under consideration, the natural choice of the timelike vector is $u^\mu = \delta^\mu{}_t y/\sqrt{Q}$, while the electromagnetic local energy density is $T_{\mu\nu}u^\mu u^\nu = \mathcal{L} + \frac{d\mathcal{L}}{d\mathcal{F}}(F_{yt})^2$ and it has to be positive. The local energy flow vector is timelike,

$$V_\mu \coloneqq T_{\mu\nu}u^\nu = \delta^t{}_\mu \sqrt{Q}\left[\mathcal{L} + \frac{d\mathcal{L}}{d\mathcal{F}}(F_{yt})^2\right]/y,$$
$$V_\mu V^\mu = -\left[\mathcal{L} + \frac{d\mathcal{L}}{d\mathcal{F}}(F_{yt})^2\right]^2 \leq 0. \tag{23}$$

Therefore, one has to wonder about the positiveness of the local energy. Besides the energy conditions, some classes of solutions must satisfy some physical conditions, such as fitting some weak field behavior or being regular throughout the entire spacetime.

V. MULTIPARAMETRIC SOLUTION TO EINSTEIN NONLINEAR ELECTRODYNAMICS

In what follows we deal with a multiparametric solution $S(\epsilon/e, M/m, E_q, \Lambda/l; p, a, b, s)$ of Einstein nonlinear electrodynamics theory. The structure of this gravitational field is determined by the metric (10) with the structural function

$$Q(y) = \epsilon y^2 \frac{y^p}{(y^2+e^2q^2)^{p/2}} - 2M\frac{y^{a+1}}{(y^2+m^2q^2)^{a/2}} + E\frac{y^b}{(y^2+q^2)^{b/2}} - \frac{\Lambda}{3}\frac{y^4 y^s}{(y^2+l^2q^2)^{s/2}}, \tag{24}$$

where ϵ, M, E_q, and Λ stand for the curvature parameter (related to α), mass, charge, and a cosmological constant, respectively; the distorting functions are equipped with exponents denoted by p, a, b, s. and the numerical parameters e, m, l associated with the terms with ϵ, M, Λ, respectively, denote the number the corresponding charges q. The curvature parameter ϵ, related to the spherical (flat, pseudospherical) character of the metric, and the charge $E \to q^2$ are introduced for "transition solution" purposes; this point will become apparent later in the transition limit to the RN solution. To simplify notation, auxiliary functions of the form

$$Y(n,v) \coloneqq (y^2+n^2q^2)^{-2-v/2}$$

are introduced. The evaluation of the field F_{yt} yields

$$\begin{aligned}
Q_0F_{yt} &= F_\epsilon + F_\Lambda + F_M + F_{E_q} - \alpha,\\
F_\epsilon &= \epsilon p e^2q^2y^p(3y^2+e^2q^2-pe^2q^2)Y(e,p)/2\\
&\quad + \epsilon y^p(y^2+e^2q^2)^2Y(e,p),\\
F_\Lambda &= \frac{1}{6}\Lambda l^2q^2s(3l^2q^2+sl^2q^2+y^2)y^{2+s}Y(l,s),\\
F_M &= aMm^2q^2[m^2q^2(a-3)-5y^2]y^{a-1}Y(m,a),\\
F_{E_q} &= -\frac{E}{2}bq^2(q^2b-5q^2-7y^2)y^{-2+b}Y(1,b)\\
&\quad - 2E(y^2+q^2)^2y^{-2+b}Y(1,b),
\end{aligned} \tag{25}$$

where Q_0 is proportional to the charge q; it can be chosen as $Q_0 = q$ or $\sqrt{2}q$.

The Lagrangian amounts to

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_\epsilon + \mathcal{L}_\Lambda + \mathcal{L}_M + \mathcal{L}_{E_q},\\
\mathcal{L}_\epsilon &= \frac{\epsilon}{2}pe^2q^2(y^2-pe^2q^2-e^2q^2)y^{p-2}Y(e,p),\\
\mathcal{L}_\Lambda &= \frac{\Lambda}{6}sl^2q^2(3y^2+sl^2q^2+5l^2q^2)y^sY(l,s)\\
&\quad - \Lambda + \Lambda(y^2+l^2q^2)^2y^sY(l,s),\\
\mathcal{L}_M &= aMm^2q^2[m^2q^2(a-1)-3y^2]y^{a-3}Y(m,a),\\
\mathcal{L}_{E_q} &= -\frac{E}{2}bq^2(bq^2-3q^2-5y^2)y^{b-4}Y(1,b),\\
&\quad - E(y^2+q^2)^2y^{b-4}Y(1,b).
\end{aligned} \tag{26}$$

The nonvanishing Weyl component Ψ_2 is given by

$$\begin{aligned}
\Psi_2 = &-\frac{\epsilon e^2q^2p}{12}[e^2q^2(p-3)-5y^2]y^{p-2}Y(e,p) - \frac{\epsilon}{6}(y^2+e^2q^2)^2y^{p-2}Y(e,p) + \frac{1}{36}l^2q^2\Lambda s(l^2q^2+sl^2q^2-y^2)Y(l,s)y^s\\
&+ \frac{aM}{6}m^2q^2[m^2q^2(a-5)-7y^2]Y(m,a)y^{a-3} + M(y^2+m^2q^2)^2Y(m,a)y^{a-3} - \frac{E}{12}bq^2(bq^2-7q^2-9y^2)Y(1,b)y^{b-4}\\
&- E(y^2+q^2)^2Y(1,b)y^{b-4} + \frac{1}{6}\frac{\alpha}{y^2}.
\end{aligned} \tag{27}$$

The structures of the traceless Ricci tensor eigenvalue function S and the scalar curvature R follow a similar pattern; these quantities are given as

$$
\begin{aligned}
S &= S_\epsilon + S_\Lambda + S_m + S_{E;q} + S_\alpha,\\
S_\epsilon &= -\frac{1}{4}\epsilon p e^2 q^2 y^{p-2}(3y^2 + e^2q^2 - pe^2q^2)Y(e,p)\\
&\quad + \frac{\epsilon y^{-2}}{2(y^2+e^2q^2)^{p/2}}[(y^2+e^2q^2)^{p/2} - y^p],\\
S_\Lambda &= -\frac{1}{12}\Lambda s l^2 q^2 (3l^2q^2 + sl^2q^2 + y^2)y^s Y(l,s),\\
S_m &= \frac{M}{2} a m^2 q^2 [m^2q^2(3-a) + 5y^2] y^{a-3} Y(m,a),\\
S_{E;q} &= \frac{1}{4} E b q^2 (q^2 b - 5q^2 - 7y^2) y^{b-4} Y(1,b)\\
&\quad + E(y^2+q^2)^2 y^{b-4} Y(1,b),\\
S_\alpha &= \frac{1}{2y^2}(\alpha - \epsilon).
\end{aligned} \tag{28}
$$

Finally, the scalar curvature can be given as

$$
\begin{aligned}
R &= R_\epsilon + R_\Lambda + R_m + R_{E;q} + R_\alpha,\\
R_\epsilon &= -\epsilon p e^2 q^2 y^{p-2}(y^2 + 3e^2q^2 + pe^2q^2)Y(e,p)\\
&\quad + 2\frac{\epsilon y^{-2}}{(y^2+e^2q^2)^{p/2}}[(y^2+e^2q^2)^{p/2} - y^p],\\
R_\Lambda &= \frac{\Lambda}{3} s l^2 q^2 (7l^2q^2 + sl^2q^2 + 5y^2) y^s Y(l,s)\\
&\quad + 4\Lambda - 4\frac{\Lambda}{(y^2+l^2q^2)^{s/2}}[(y^2+l^2q^2)^{s/2} - y^s],\\
R_m &= 2aMm^2q^2[m^2q^2(1+a) - y^2] y^{a-3} Y(m,a),\\
R_{E;q} &= -bEq^2(bq^2 - q^2 - 3y^2) y^{b-4} Y(1,b),\\
R_\alpha &= \frac{2}{y^2}(\alpha - \epsilon).
\end{aligned} \tag{29}
$$

It should be pointed out that the first link in the chain $S(\epsilon, m, E/q, \Lambda; p, a, b, s)$ is the regular black hole solution $S(1, m, q, \Lambda; 0, 3, 4, 0)$ [10]; one can add a series of these solutions with different amounts of charges and masses, and again it would be a solution, although not necessarily regular or with the de Sitter–anti de Sitter (dS-AdS) limit at infinity.

A. Ranges of regularity

As far as the class of *regular* fields is concerned one has to investigate the regular (nonsingular) behavior of the algebraic invariants in the whole spacetime, in particular, as the spatial coordinate $y(=r)$ approaches the origin $y = r = 0$ of the coordinate system. The invariants $\mathcal{F}(=-(F_{yt})^2/2), \mathcal{L}, \Psi_2, S$, and R are well-behaved functions of y if

$$\epsilon = \alpha, \qquad s \geq 0, \qquad p = 0, \qquad a \geq 3, \qquad b \geq 4.$$

Consequently, the solutions with exponents fitting those ranges may describe regular black holes, with the event horizon radius fulfilling the condition $Q(y) = 0$, which ought to be the outermost horizon.

B. Limiting transition

In gravitation, to establish the physical character of a solution one searches for its weak partner. For instance, the weak partner of the Schwarzschild solution is the field for a central Newtonian mass in the weak limit of gravity. For the Reissner-Nordström solution its limit is the charged point particle, and so on. Thus, to judge the physical character of this solution we derive its limiting transitions by looking at the solution in the far region, i.e., at $y \to \infty$. Taking into account the behavior at $y \to \infty$ of terms like

$$\frac{y^p}{(y^2+n^2q^2)^{p/2}} \approx 1 - \frac{1}{2}\frac{pn^2q^2}{y^2} + \frac{n^4q^4}{8y^4}(2p+p^2) + \cdots,$$

one establishes, via their substitutions in $Q(y)$, the Reissner-Nordström structural function limit

$$Q_{RN} = -\frac{\Lambda}{3}y^4 + \epsilon_0 y^2 - 2My + E_0,$$

where the constant ϵ and the electric constant E have been replaced, respectively, by

$$
\begin{aligned}
\epsilon &= \epsilon_0 - \frac{\Lambda}{6} l^2 q^2 s,\\
E &= E_0 + \frac{\Lambda}{24} l^2 q^4 s[(s+2)l^2 - 2e^2p] + \frac{\epsilon_0}{2} e^2 q^2 p.
\end{aligned}
$$

Notice that the Lagrangian $\mathcal{L}$ and the electric field F_{yt} as y approaches infinity behave as

$$\mathcal{L}_{y\to\infty} \approx -\frac{1}{y^4}E_0, \qquad Q_0 F_{yt_{y\to\infty}} \approx -\frac{2}{y^2}E_0 + \epsilon_0 - \alpha. \tag{30}$$

Therefore, by replacing $Q_0 \to q$, $E_0 \to q^2$, $\epsilon_0 \to \alpha$, and $y \to r$ one arrives at the RN solution of the Maxwell theory in the presence of a cosmological constant Λ, i.e., the RN solution immersed in the dS or AdS spacetime. For $\epsilon_0 = 1$, $x = \cos\theta$, and $a_0 = 1$, one obtains the standard RN static spherically symmetric metric in Schwarzschild coordinates.

VI. FINAL REMARKS

There are various theoretical aspects that deserve further attention that are outside the scope of this work, such as the motion of test particles and light rays through the study of the geodesic equations, the determination of the geodesic deviation for null geodesics, and the study of birefringence, among others. In the theory of black holes, the thermodynamics of black holes is highly relevant to determine to what extend the nonlinearity of the fields distorts the physical quantities in comparison to those considered as the linear or weak limits of the theory, such as the modified Smarr relation and the first law of thermodynamics. As far as experimental effects are concerned, we only recently (2015) received indirect insight into the interaction of rotating black holes through the detection of gravitational waves; these experiment require high accuracy, fine-tunings, and extremely precise measurements. These experimental achievements may enhance new studies, such as the study of the deflection of light or the deviation of the perihelia of a test star in the neighborhood of a massive nonlinear electrodynamically charged astrophysical object. It is clear that the most observable effect on the trajectories (of photons, and charged and neutral particles) will be caused by the central mass followed by the effect of the charge and momentum, if any. In the last decade plasma physics (plasma waves) and powerful laser instrumentation have become areas of great interest; see, for instance, the studies on particle acceleration in nonlinear electrodynamics in Ref. [28] and references therein. In astrophysics, nonlinear classical electromagnetic fields have been used in the description of electron emission by massive stars, and also to explain the accelerated expansion of the Universe. NLE has been coupled to inflationary models to avoid the big bang.

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Note added in proof.—An alternative point of view on the existence—non existence of spherically symmetric electric nonlinear electrodynamics in Einstein gravity has been published by [29].

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