Light propagation in the field of the *N*-body system and its application in the TianQin mission

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Given current high-precision modern space missions, a precise relativistic modeling of observations is required. By solving the eikonal equation within the post-Newtonian approximation, we use an iterative method to determine light propagation in the gravitational field of an isolated, gravitationally bound *N*-body system. Different from traditional *N* bodies that are independent of each other within a system, our system includes the velocities, accelerations, gravitational interactions, and tidal deformations of the gravitational bodies. The light delays of these factors are then precisely determined by the analytical solutions. These delays are significant and are likely to reach a detectable level for *strong* gravitational fields, such as binary pulsars and some gravitational-wave sources. The result's application in the vicinity of the Earth provides a relativistic framework for modern space missions. From the relativistic analysis in the TianQin mission, we find possible tests for alternative gravitational theories, such as a possible determination of the post-Newtonian parameter γ at the level of some scalar-tensor theories of gravity.

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I. INTRODUCTION

The growing accuracy of radioscience, laser, and astrometric observations requires a detailed modeling of light propagation in curved spacetime. By solving the null geodesic equations or eikonal equations in a given metric, the problem of light propagation has been studied in several contexts, e.g., light propagation in the gravitational field of *N* arbitrarily moving bodies within the first post-Newtonian (1PN) and 1.5PN approximations [1,2], the second post-Newtonian (2PN) effects of one arbitrarily moving point-like body [3], and post-Minkowskian (PM) effects on the gravitational field of static bodies endowed with arbitrary intrinsic mass-multipole and spin-multipole moments [4–6]. Furthermore, light travel time has been studied using methods based on the Synge world function [7] and time transfer function (TTF) [8,9], such as the 3PM TTF in the field of a static monopole [10]. These solutions are useful when dealing with light propagation in the Solar System's gravitational field. When considering realistic celestial objects the nonrigid characteristic of bodies cannot be ignored, which could lead to tidal deformations in an *N*-body system. This means that gravitational interactions and tidal deformations must be taken into account in high-precision observables and experiments. Corresponding calculations of the light travel time have not been reported yet. They should be treated carefully since their contributions may be non-negligible for space missions or astrometric observations in the foreseeable future. In particular, the influences due to tidal deformations are significant in binary pulsar systems and some recent gravitational-wave (GW) sources.

The first direct observation of GW events (GW150914) made by Advanced LIGO has opened the era of observational GW astronomy [11–13] and marks the beginning of a new era in gravitational physics [14,15]. Subsequently, several more GW events have been detected by Advanced LIGO and Advanced VIRGO [16,17]. At present, ground-based GW detectors are able to detect GWs in the high-frequency regime (10 to 10^3 Hz). Aiming to provide more observations of GW events and complement ground-based detectors, space-based detectors are being developed to make GW observations in the low-frequency regime (10^{-4} to 1 Hz) where picometer-level accuracy is required.

An alternative gravitational-wave mission in space may use heliocentric or geocentric orbits for spacecraft. The best representative of the former orbit option is the LISA mission [18]. For the latter option, early versions of SAGITTARIUS [19] and OMEGA [20] were proposed to the ESA in 1993 and NASA in 1996, respectively. Recently, the TianQin mission was proposed by Luo *et al.* [21]. It relies on an equilateral triangular constellation with an inter-spacecraft distance of about 1.7×10^5 km and is planned to be launched in 2035. Compared to heliocentric options, these geocentric missions have several major advantages in terms of propulsion requirements,

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telecommunication systems, and the time required to inject the mission into its final orbit. Unfortunately, they are confronted with two technological issues: keeping sunlight from getting into the telescopes, and generating an extremely stable clock frequency. For TianQin, the first issue is cured by a relatively short science run [21]. Another issue comes from the influence of the Earth-Moon system's gravitational field, which remains to be solved.

The larger effects due to the Earth-Moon system are the main difference between heliocentric and geocentric options. The extremely stable clocks used in GW space missions are significantly affected by gravitational fields and geocentric orbits. The orbit-parametrized frequency shift between spacecraft should determined by using a full general-relativistic treatment in the Solar System barycentric coordinate reference system (BCRS). Moreover, time-delay interferometry (TDI) [22–25] requires accurate knowledge of light propagation delays, and in simulation code time delays should be generated as realistically as possible. These factors require us to rigorously determine light propagation in the gravitational field of the Solar System. In the DSX formalism [26,27], the relativistic phase in the gravitational field of an N-body system can be determined by solving the eikonal equation [28,29]. Different from classical treatments in static fields, the influences of the N-body system's velocity, acceleration, gravitational interactions, and tidal deformation are also determined in this formalism. These formulas can be applied to calculations for the Earth-Moon system.

The propagation delays and frequency shift for the LISA mission due to the gravitational field of the Sun were studied in Ref. [30]. Further models of the range and frequency measurements in geocentric space missions like GRACE Follow-On [31] and ACES [32] were developed to analyze relativistic observables. Considering the different satellite configurations and orbital options, it is necessary to perform a detailed relativistic analysis for TianQin. Meanwhile, for TianQin, some effects on the frequency shift come from coordinate effects of the BCRS that depend on the coordinate chart. These effects cancel each other and thus are not detectable. These quantities should be avoided in the simulation code and scientific missions. Finally, in addition to TianQin's primary goals, the analysis of the relativistic effects may provide a formalism to test fundamental physics, such as the post-Newtonian parameter γ (which describes the measure of spacetime curvature produced by a unit rest mass) [33] and local Lorentz invariance [34,35].

The rest of this paper is organized as follows. In Sec. II, we discuss light propagation in the gravitational field of the Solar System and derive the corresponding relativistic solution for the light phase. In Sec. III, we develop the general-relativistic phase model for TianQin. Also, the frequency shift between spacecraft and coordinate effects are discussed. We give our conclusion in Sec. IV. The method of solving the eikonal equation is given in Appendix A. In Appendix B, we present the instantaneous coordinate distance and corresponding derivatives. In Appendix C, some relationships for Keplerian orbits are presented.

II. RELATIVISTIC PHASE OF LIGHT IN THE GRAVITATIONAL FIELD OF THE SOLAR SYSTEM

A. Space-time reference system

Einstein's general relativity is a covariant theory in which coordinate charts are merely labels, and thus the physical observables should be coordinate-independent quantities. This means that one has a wide freedom to choose the coordinate system when describing the outcome of a particular experiment. In fact, any reference system covering the spacetime region of the experiment can be used to describe the results of that experiment. However, some available coordinate systems that are associated with a particular celestial body or laboratory have important advantages when describing the observations of precision experiments. By using the harmonic gauge conditions and conservation laws, the relativistic, proper reference frame can be determined. In order to conveniently formulate the coordinate picture of the measurement procedure or offer a simpler mathematical description of the experiments under consideration, one should pick a specific coordinate system to model the observables. For the TianQin mission, the choice is the Solar System BCRS.

The BCRS is a particular implementation of a barycentric reference system in the Solar System with the space-time coordinates $x^\mu \equiv (ct, \mathbf{x})$, which has its origin at the Solar System barycenter. Given IAU recommendations [36,37], the metric of the BCRS can be written in a form that only depends on two harmonic potentials,

$$\begin{aligned} g_{00} &= 1 - \frac{2w}{c^2} + \frac{2w^2}{c^4} + O(c^{-6}), \\ g_{0i} &= \frac{4\delta_{ik}w^k}{c^3} + O(c^{-5}), \\ g_{ij} &= -1 - \delta_{ij}\frac{2w}{c^2} - \frac{3\delta_{ij}}{2}\frac{w^2}{c^4} + O(c^{-6}), \end{aligned} \tag{1}$$

where w and $\mathbf{w}$ are the scalar and vector harmonic potentials, respectively, written as

$$\begin{aligned} w = \sum_b \frac{GM_b}{r_b}\bigg\{1 + \frac{1}{c^2}\bigg(2v_b^2 - \sum_{c\neq b}\frac{GM_c}{r_{cb}} - \frac{1}{2}(\mathbf{n}_b\cdot\mathbf{v}_b)^2 \\ - \frac{1}{2}(\mathbf{r}_b\cdot\mathbf{a}_b)\bigg)\bigg\} + w_l + O(c^{-3}), \end{aligned} \tag{2}$$

$$\mathbf{w} = \sum_b \frac{GM_b}{r_b}\left(\mathbf{v}_b + \frac{(\mathbf{s}_b\times\mathbf{r}_b)}{2r_b^2}\right) + O(c^{-2}), \tag{3}$$

where GM_b is the gravitational constant of the body b, $\mathbf{s}_b$ is the angular momentum per unit of mass of body b, $r_b = |\mathbf{r}_b| = |\mathbf{x} - \mathbf{x}_{b0}|$ with $\mathbf{x}_{b0}$ being the barycentric position of body b, $\mathbf{r}_{bc} = \mathbf{x}_{c0} - \mathbf{x}_{b0}$ is the distance vector pointing to body c from b, and $\mathbf{v}_b = d\mathbf{x}_{b0}/dt$ and $\mathbf{a}_b = d\mathbf{v}_b/dt$ are the barycentric velocity and acceleration of body b, respectively. Last, w_l contains the contributions from higher gravitational potential coefficients characterizing the shape of body b, and b should represent all bodies in the Solar System.

B. Relativistic phase model

In the framework of general relativity, solving the null geodesic equation is the standard method to obtain all information about light propagation between two point events [9]. Different approaches are also available for calculating light propagation delays, such as the Synge world function [7], time transfer functions [9], and the eikonal equation [28]. Here, our choice is based on solving the eikonal equation, which is closely related to the Synge world function [7]. As discussed by Ashby *et al.* in Ref. [28], the problem of light travel time can be reduced to geometrical optics by using the eikonal equation. This implies that when we consider the light time between two point events, it is enough to solve the problem by using the eikonal equation. As a scalar function, the phase φ of an electromagnetic wave is invariant under a set of general coordinate transformations, which satisfies the eikonal equation [28,31,38]

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 0. \tag{4}$$

This equation can be derived from Maxwell's equations in which the solution φ describes the wave front of an electromagnetic wave propagating in the curved spacetime. To obtain the solution $\varphi(t,\mathbf{x})$, we introduce a covector describing the electromagnetic wavefront in the curved spacetime, $K_\mu = \partial_\mu\varphi$. For light, it satisfies the equation $g_{\mu\nu}K^\mu K^\nu = 0$ with the vector $K^\mu = g^{\mu\nu}\partial_\nu\varphi$ tangent to the light ray. Assuming that the phase $\varphi(t,\mathbf{x})$ is known, one can straightforwardly study the properties of light propagation.

To find a solution of the eikonal equation, we expand the phase φ using the perturbation method,

$$\varphi(t,\mathbf{x}) = \varphi_0 + \int k_\mu dx^\mu + \varphi_{\mathrm{GF}}(t,\mathbf{x}), \tag{5}$$

where φ_0 is a constant, $k_\mu = k^0(1,\mathbf{k})$ (satisfying the relation $\eta^{\mu\nu}k_\mu k_\nu = 0$) is a constant null vector along the direction of propagation of the unperturbed electromagnetic plane wave, and φ_{GF} represents the perturbation due to the gravitational field. Since we can define the time component $k^0 = \omega/c$, where ω is th e constant angular frequency of the unperturbed electromagnetic wave, the vector $\mathbf{k}$ ($|\mathbf{k}| = 1$) is the unit vector along the light's path.

As a consequence of the perturbation method (5), the wave vector K^μ of light in curved spacetime can be expressed in the form

$$K^\mu(t,\mathbf{x}) = g^{\mu\nu}\partial_\nu\varphi = k^\mu + k^\mu_{\mathrm{GF}}(t,\mathbf{x}), \tag{6}$$

where $k^\mu_{\mathrm{GF}}(t,\mathbf{x})$ is the perturbation of the wave vector due to the gravitational field.

To determine φ_{GF} we can use asymptotic perturbation theory [28], which is described in Appendix A. In the metric given by Eq. (1), the solutions of φ_{GF} are given by Eqs. (A6) and (A7). For the convenience of calculations and presentation, we take a particular part of the perturbation from $\varphi^{(2)}$ and place it into $\varphi^{(1)}$. This part is the contribution due to the G^2 term in the scalar potential w. Therefore, φ_{GF} is rewritten as $\varphi_{\mathrm{GF}} = \varphi^1_{\mathrm{GF}} + \varphi^2_{\mathrm{GF}}$, where φ^1_{GF} is the sum of $\varphi^{(1)}$ and the above-mentioned part from $\varphi^{(2)}$. We assume that a unique light ray connects two point events $(ct_A, \mathbf{x}_A)$ and $(ct_B, \mathbf{x}_B)$ with coordinate time relationship $t_A < t_B$. From Eqs. (A6) and (A7), φ^1_{GF} is expressed as

$$\varphi^1_{\mathrm{GF}}(t,\mathbf{x}) = -\frac{R_{AB}k_0}{2}\int_0^1\left(\frac{4w}{c^2} - \frac{8\mathbf{w}\cdot\mathbf{N}_{AB}}{c^3}\right)d\lambda, \tag{7}$$

where the integral is calculated along the straight line between $\mathbf{x}_A$ and $\mathbf{x}_B$, defined by the parametric equations

$$x^i = \lambda(x^i_B - x^i_A) + x^i_A, \qquad 0 \le \lambda \le 1, \tag{8}$$

and $\mathbf{N}_{AB} = (\mathbf{x}_B - \mathbf{x}_A)/R_{AB}$ with $R_{AB} = |\mathbf{x}_B - \mathbf{x}_A|$. For calculations in the vicinity of celestial bodies, tidal deformations of sources should be taken into account, which means that we should consider potential variations caused by the tidal deformations. By using the body's tidal deformations to correct the two harmonic potentials, it is recommended to rewrite Eqs. (2) and (3) in the following forms:

$$w(t,\mathbf{x}) = w_0(t,\mathbf{x}) + w_{\mathrm{tid}}(t,\mathbf{x}), \tag{9}$$

$$\mathbf{w}(t,\mathbf{x}) = \mathbf{w}_0(t,\mathbf{x}) + \mathbf{w}_{\mathrm{tid}}(t,\mathbf{x}), \tag{10}$$

where w_0 and $\mathbf{w}_0$ are the previous scaler and vector potentials in Eqs. (2) and (3), respectively. w_{tid} and $\mathbf{w}_{\mathrm{tid}}$ represent the tidal-deformation potentials in the N-body system. Since the vector potential effects are much smaller than the scalar potential effects, we neglect the contribution of $\mathbf{w}_{\mathrm{tid}}$. Generally, w_{tid} can be expressed in the form of Love numbers [39],

$$w_{\mathrm{tid}}(t,\mathbf{x}) = \sum_b (k_2)_b \frac{R_b^5}{r_b^3}\sum_{c\neq b}\frac{GM_c}{2r^3_{cb}}[3(\mathbf{n}_{cb}\cdot\mathbf{n}_b)^2 - 1], \tag{11}$$

where $(k_2)_b$ is the Love number of body b, R_b is the equatorial radius of body b, $\mathbf{n}_{cb} = \mathbf{r}_{cb}/r_{cb}$, and $\mathbf{n}_b = \mathbf{r}_b/r_b$. Since r_b^3 appears in the denominator, its influence is significant only in the vicinity of a gravitational body. Inserting it into Eq. (7) allows us to calculate the deformation effects.

Subsequently, we calculate the mass multipoles w_l. Using the Blanchet-Damour moments, w_l can be theoretically determined from the distribution of mass and matter currents. As an example, Earth's contribution to w_l is represented here. Usually, it is useful to present these moments of the Earth as parameters evaluated by numerically fitting to various kinds of experimental data, such as satellite motion tracking, geodetic measurements, and gravimetry. An approximate expansion of the Earth's multipoles $w_{l,E}$ is the spherical harmonic expansion, which is given by [40]

$$w_{l,E}(\mathbf{x}) = \frac{GM_E}{r_E}\sum_{l=2}^{\infty}\sum_{k=0}^{+l}\left(\frac{R_E}{r_E}\right)^l P_{lk}(\cos\theta) \times (\mathcal{C}_{lk}^E\cos(k\phi) + \mathcal{S}_{lk}^E\sin(k\phi)), \tag{12}$$

where R_E is the Earth's equatorial radius, P_{lk} are associated with the Legendre polynomials, $\mathcal{C}_{lk}^E$ and $\mathcal{S}_{lk}^E$ are spherical harmonic coefficients characterizing the Earth, and $\mathcal{C}_{l0}^E = -J_l^E$ is the mass multipole moment of the Earth.

To solve Eq. (7), we use Eqs. (9)–(12) to calculate the perturbed phase. The perturbation φ_{GF}^1 is further written as

$$\varphi_{\mathrm{GF}}^1(t,\mathbf{x}) = -2R_{AB}k_0\int_0^1\left(\sum_b\frac{GM_b}{c^2r_b}\left\{1+\frac{1}{c^2}\left(2v_b^2-\sum_{c\neq b}\frac{GM_c}{r_{cb}}-\frac{1}{2}(\mathbf{n}_b\cdot\mathbf{v}_b)^2-\frac{1}{2}(\mathbf{r}_b\cdot\mathbf{a}_b)\right)\right\}+\frac{w_l}{c^2}+\frac{w_{\mathrm{tid}}}{c^2}-\frac{2\mathbf{w}\cdot\mathbf{N}_{AB}}{c^3}\right)d\lambda. \tag{13}$$

It may be noted that for the mass multipoles w_l, the J_2 moment is the main contribution, and the higher moments can be neglected since their influences are much smaller than that of the quadrupole term. After some mathematical manipulations, φ_{GF}^1 is obtained as

$$\begin{aligned}\varphi_{\mathrm{GF}}^1(t,\mathbf{x}) = \sum_b\Bigg\{&-\frac{2GM_bk_0}{c^2}\left(1-\sum_{c\neq b}\frac{GM_c}{c^2r_{cb}}\right)\ln\frac{r_{bA}+r_{bB}+R_{AB}}{r_{bA}+r_{bB}-R_{AB}}+\varphi_v^b(t,\mathbf{x})+\varphi_{\mathbf{r}\cdot\mathbf{a}}^b(t,\mathbf{x})+\varphi_{\mathrm{tid}}^b(t,\mathbf{x})\\ &-\frac{GM_bk_0R_b^2R_{AB}J_2^b}{c^2(r_{bA}r_{bB}+\mathbf{x}_{bA}\cdot\mathbf{x}_{bB})}\left[\frac{1-(\mathbf{I}_b\cdot\mathbf{n}_{bA})^2}{r_{bA}}+\frac{1-(\mathbf{I}_b\cdot\mathbf{n}_{bB})^2}{r_{bB}}-\left(\frac{1}{r_{bA}}+\frac{1}{r_{bB}}\right)\frac{[\mathbf{I}_b\cdot(\mathbf{n}_{bA}+\mathbf{n}_{bB})]^2}{1+\mathbf{n}_{bA}\cdot\mathbf{n}_{bB}}\right]\\ &+\frac{4GM_bk_0}{c^3}\frac{\mathbf{R}_{AB}\cdot[\mathbf{s}_b\times(\mathbf{n}_{bA}+\mathbf{n}_{bB})]}{(r_{bA}+r_{bB})^2-R_{AB}^2}\Bigg\},\end{aligned} \tag{14}$$

where $\mathbf{I}_b$ is the unit vector along the body b's rotation axis, $\varphi_v^b(t,\mathbf{x})$ represents the correction from the velocity, $\varphi_{\mathbf{r}\cdot\mathbf{a}}^b(t,\mathbf{x})$ is the acceleration contribution to the phase, and $\varphi_{\mathrm{tid}}^b(t,\mathbf{x})$ comes from the tidal deformations. This equation describes the gravitational contributions of an N-body system in phase including the influences of the system's mass, oblateness J_2, velocity, acceleration, gravitational interactions, rotation, and tidal deformations. After further calculation we obtain $\varphi_v^b(t,\mathbf{x})$,

$$\begin{aligned}\varphi_v^b(t,\mathbf{x}) = \frac{GM_bk_0}{c^2}R_{AB}\Bigg\{&\left[\frac{4\mathbf{N}_{AB}\cdot\mathbf{v}_b}{cR_{AB}}-\frac{4v_b^2}{c^2R_{AB}}+\frac{(\mathbf{N}_{AB}\cdot\mathbf{v}_b)^2}{c^2R_{AB}}\right]\ln\frac{r_{bA}+r_{bB}+R_{AB}}{r_{bA}+r_{bB}-R_{AB}}\\ &+\frac{(\mathbf{r}_{bA}\cdot\mathbf{v}_b)[2r_{bA}(\mathbf{r}_{bB}\cdot\mathbf{v}_b)+(r_{bB}-r_{bA})(\mathbf{r}_{bA}\cdot\mathbf{v}_b)]}{c^2r_{bA}^2r_{bB}^2((\mathbf{n}_{bA}\cdot\mathbf{n}_{bB})+1)}-\frac{r_{bA}^2(\mathbf{N}_{AB}\cdot\mathbf{v}_b)^2[(r_{bB}-r_{bA})+2r_{bB}(\mathbf{n}_{bA}\cdot\mathbf{n}_{bB})]}{c^2r_{bA}^2r_{bB}^2((\mathbf{n}_{bA}\cdot\mathbf{n}_{bB})+1)}\Bigg\},\end{aligned} \tag{15}$$

with $\mathbf{N}_{AB} = (\mathbf{x}_B - \mathbf{x}_A)/R_{AB}$. Clearly, its value is zero for a static gravitational body. The first term is a direct relativistic correction to the Shapiro term in which the largest correction is proportional to the velocity of the gravitational body. The latter two terms are indirect corrections to the Shapiro delay. Integrating the acceleration-dependent term, the acceleration perturbation in Eq. (14) is given by

$$\varphi_{\mathbf{r}\cdot\mathbf{a}}^b(t,\mathbf{x}) = \frac{GM_bk_0}{c^4}\left\{(r_{bB}-r_{bA})(\mathbf{N}_{AB}\cdot\mathbf{a}_b)+[(\mathbf{r}_{bA}\cdot\mathbf{a}_b)-(\mathbf{N}_{AB}\cdot\mathbf{a}_b)(\mathbf{N}_{AB}\cdot\mathbf{r}_{bA})]\ln\frac{r_{bA}+r_{bB}+R_{AB}}{r_{bA}+r_{bB}-R_{AB}}\right\}. \tag{16}$$

The acceleration contributions also include direct and indirect corrections to the Shapiro delay, which do not exist in the case of a static gravitational field. Equations (15) and (16) are sufficient to describe the effects of motion on light propagation delays for the gravitational field of an isolated, gravitationally bound N-body system.

By introducing the impact vector $\mathbf{d}_b = \mathbf{N}_{AB} \times (\mathbf{x}_{bA} \times \mathbf{N}_{AB})$ with $d_b = |\mathbf{d}_b|$, the term $\varphi^b_{\rm tid}$ is

$$\varphi^b_{\rm tid}(t,\mathbf{x}) = -k_0 (k_2)_b R_b^5 \sum_{c,c\neq b} \frac{GM_c}{c^2 r_{cb}^3} \left\{ \frac{r_{bA}^2}{r_{bB}^2} \frac{(r_{bB}^3 - r_{bA}^3)\mathcal{B}_{c1} + 3r_{bA}^2 R_{AB}\mathcal{B}_{c2} + R_{AB}(R_{AB}^2 + 3\mathbf{x}_{bA}\cdot\mathbf{R}_{AB})\mathcal{B}_{c3}}{d_b^4 r_{bB}} + \frac{1}{d_b^2}\left(\frac{\mathbf{N}_{AB}\cdot\mathbf{x}_{bA}}{r_{bA}} - \frac{\mathbf{N}_{AB}\cdot\mathbf{x}_{bB}}{r_{bB}}\right)\right\}, \tag{17}$$

where

$$\mathcal{B}_{c1} = s_A^2 c_{cA}^2 c_A + 2c_{cA}c_c(1+c_A^2) + 2c_A(c_{cA}^2 + c_c^2), \tag{18}$$

$$\mathcal{B}_{c2} = (1+c_A^2)(c_{cA}^2 + 2c_A c_{cA} c_c) + 2c_A^2 c_c^2, \tag{19}$$

$$\mathcal{B}_{c3} = (1+c_A^2)c_c^2 + 2c_{cA}^2 + 4c_A c_{cA} c_c, \tag{20}$$

with $c_A = \mathbf{n}_{Ab}\cdot\mathbf{N}_{AB}$, $c_{cA} = \mathbf{n}_{bA}\cdot\mathbf{n}_{bc}$, $c_c = \mathbf{N}_{AB}\cdot\mathbf{n}_{bc}$, and $s_A = \sqrt{1-c_A^2}$. The tidal effects are in direct proportion to the Love number $(k_2)_b$ that is dependent on body b, and $(k_2)_E$ is about 0.3 for the Earth. A more massive source of tidal force will lead to more a obvious deformation on the surface of the body, which produces a greater potential variation. Then, the corresponding light delay arising from this potential variation may become non-negligible. GW missions recently observed events from binary neutron star inspirals. In these systems, as well as in binary pulsar systems, tidal deformations are significant due to the strong gravitational interactions.

Since $\varphi^1_{\rm GF}$ has been determined from the above equations, we consider another term: $\varphi^2_{\rm GF}$. $\varphi^2_{\rm GF}$ can be determined from Eq. (A7). For the sake of discussion, we rewrite it in two parts: the contribution of the square of the Newtonian potential $\varphi^2_{\rm GF}$ and the contribution of the coupling terms φ^2_c. The velocity- and acceleration-term contributions are higher order than c^{-4}, and thus can be safely ignored. $\varphi^2_{\rm GF}$ consists of the squared terms $G^2M_b^2$ [7],

$$\varphi^2_{\rm GF} = \sum_b \frac{G^2 M_b^2 k_0 R_{AB}}{c^4 r_{bA} r_{bB}} \left[\frac{4}{1+\cos\theta_b} - \frac{15\theta_b}{4\sin\theta_b}\right], \tag{21}$$

with $\cos\theta_b = \mathbf{n}_{bA}\cdot\mathbf{n}_{bB}$. For φ^2_c, we consider contributions from the terms in w^2 like $(GM_b/c^2 r_b)(GM_c/c^2 r_c)$ of two sources, and other contributions are of the same magnitude. The analytical solution requires cumbersome calculations. For our calculations in the vicinity of one source, such as a body b, another distance r_c can be expressed in the form

$$\frac{1}{r_c} = \frac{1}{r_{bc}} + \frac{\mathbf{n}_{bc}\cdot\mathbf{r}_b}{r_{bc}^2} + O(r_{bc}^{-3}). \tag{22}$$

This allows us to compute the coupling term effect φ^2_c,

$$\varphi^2_c = -\frac{GM_b k_0}{4c^4}\sum_c \frac{GM_c}{r_{bc}}\left[\frac{(\mathbf{n}_{bc}\cdot\mathbf{N}_{AB})(r_{bB}-r_{bA})}{r_{bc}} + \frac{r_{bc} + \mathbf{x}_{bA}\cdot\mathbf{n}_{bc} - (\mathbf{x}_{bA}\cdot\mathbf{N}_{AB})(\mathbf{n}_{bc}\cdot\mathbf{N}_{AB})}{r_{bc}} \times \ln\frac{r_{bA}+r_{bB}+R_{AB}}{r_{bA}+r_{bB}-R_{AB}}\right]. \tag{23}$$

This equation describes the potential-coupling effect when the light signal passes near body b. Its contribution for TianQin is smaller than that of Eq. (21), and thus can be neglected.

Finally, using the above method we consider light propagation between the TianQin spacecraft (as described in Sec. III). At the instant of reception on spacecraft B, the signal's phase received from the interferometer on spacecraft A can be expressed as

$$\varphi(t_B,\mathbf{x}_B) = \varphi(t_A,\mathbf{x}_A) + \varphi_{\rm GW} + \varphi_{\rm noise} + \frac{2\pi}{c} f_A \left(\frac{d\tau_A}{dt}\right)_{t_A} \times [c(t_B - t_A) - \mathcal{R}_{AB}(\mathbf{x}_A,\mathbf{x}_B)], \tag{24}$$

where $\mathbf{x}_A = \mathbf{x}_A(t_A)$, $\mathbf{x}_B = \mathbf{x}_B(t_B)$, f_A is the proper frequency of the transmitter on spacecraft A, $\mathcal{R}_{AB}$ is the total geodesic distance without considering gravitational waves, $\varphi_{\rm GW}$ is the phase contribution of gravitational waves, and $\varphi_{\rm noise}$ is the noise term containing various kinds of noise, such as laser-frequency noise, clock noise, optical path-length noise, etc., Clearly, the perturbations of the gravitational field in phase have been merged into $\mathcal{R}_{AB}$.

C. Estimating the magnitudes of different gravitational terms

Using the procedure described in Sec. II B we can estimate the magnitudes of various terms in phase in the context of the TianQin mission. TianQin's spacecraft are placed in orbit around the Earth so that the influences of the Earth-Moon system on laser signal propagation are significant. A convenient method is to split the gravitational delays into the Earth-Moon-system contribution and an external contribution (excluding the Earth and Moon). At the instant of reception on spacecraft B, the relativistic phase is determined by Eq. (24). The term $\mathcal{R}_{AB}$ contains light propagation delays, which is required in TDI. To study the various gravitational contributions in $\mathcal{R}_{AB}$, it is convenient to express the term $\mathcal{R}_{AB}$ as

$$\mathcal{R}_{AB} = R_{AB} + \Delta^{1\mathrm{EM}} + \Delta^{1\mathrm{ext}} + \Delta_{\mathrm{tid}} + \Delta^{2}_{\mathrm{GF}}, \quad (25)$$

where the last four terms represent the gravitational contributions derived from Eqs. (14), (17), and (21). With the nominal orbital parameters of TianQin, we subsequently use their values to evaluate the order of various gravitational terms.

We start with the second term in Eq. (25). In order to calculate its magnitude, this term can be expressed as

$$\Delta^{1\mathrm{EM}} = \Delta^{1\mathrm{EM}}_{m} + \Delta^{1\mathrm{EM}}_{J_2} + \Delta^{1\mathrm{EM}}_{s} + \Delta^{1\mathrm{EM}}_{v} + \Delta^{1\mathrm{EM}}_{a}. \quad (26)$$

The first term in Eq. (26) is the Shapiro term from the contribution of the Earth and Moon's mass monopoles, which is given by

$$\Delta^{1\mathrm{EM}}_{m} \approx \frac{2GM_E}{c^2}\ln\frac{r_{EA}+r_{EB}+R_{AB}}{r_{EA}+r_{EB}-R_{AB}} + \frac{2GM_M}{c^2}\frac{R_{AB}}{d_M}, \quad (27)$$

with $d_M = (r_{MA} + r_{MB})/2$. If we consider the gravitational delays in the parametrized post-Newtonian (PPN) formalism, this Shapiro term should be revised as $(1+\gamma)\Delta^{1\mathrm{EM}}_{m}/2$, which can be used to test the PPN parameter γ. A classical experiment to test γ in the Solar System was based on the Sun's Shapiro delay [41]. For the Earth, the Shapiro term is almost a constant (about 2.34 cm), due to the stable triangular constellation of spacecraft with respect to the Earth. For the Moon, the Shapiro delay term is at the level of 5×10^{-5} m. However, the Moon's Shapiro delay is not a constant, but rather varies with d_M. The amplitude of the variation part reaches several micrometers, with a frequency that is about the same as the orbital frequency of the spacecraft. Assuming that $\theta_{MA/B} = \mathbf{n}_{ME}\cdot\mathbf{n}_{EA/B}$, the Moon's contribution can be expressed as

$$\Delta^{1M}_{m} = \frac{2GM_M}{c^2}\frac{R_{AB}}{r_{\mathrm{EM}}}\left(1+\frac{r_A\cos\theta_{MA}+r_B\cos\theta_{MB}}{2r_{\mathrm{EM}}}\right). \quad (28)$$

For an order of magnitude estimate, we can assume that the spacecraft's orbit plane coincides with that of the Moon. Equation (28) can be rewritten as

$$\Delta^{1M}_{m} = \frac{2GM_M}{c^2}\frac{R_{AB}}{r_{\mathrm{EM}}}\left(1+\frac{r_A}{r_{\mathrm{EM}}}\cos\omega_{ms}t\right), \quad (29)$$

where ω_{ms} is the summation of angular frequencies of the spacecraft's orbit and the Moon's orbit (for inverse orbital directions). The constant part is about 49 μm and the amplitude of the variation part is about 6 μm. If we compute this term without this approximation (the spacecraft orbital plane coincides with that of the Moon), we would get a slightly longer delay, which is greater than the estimate of Eq. (29) by about 0.5 μm. Equation (29) can be used to estimate the magnitude; however, in the actual calculations and applications, we use Eq. (27). To demonstrate the influence of the Earth-Moon system's Shapiro delay on the TianQin mission, we use Fourier analysis for the rough estimations. From the Fourier analysis of Eq. (27), we find that the effects of the Earth-Moon system's Shapiro delay lead to a contribution of $3\times10^{-13}\ \mathrm{m/Hz^{1/2}}$ at 6 mHz, which is smaller than TianQin's position-measurement accuracy level of $1\ \mathrm{pm/Hz^{1/2}}$ at 6 mHz. In the low-frequency regime (10^{-4}–1 Hz), all contributions from the Shapiro delay are smaller than $1\ \mathrm{pm/Hz^{1/2}}$. Therefore, the influence of the light delays due to the Earth-Moon system's gravitational field is negligible for the TianQin mission. Moreover, the Shapiro delay may have the potential to improve the accuracy of the post-Newtonian parameter γ, whose current best value of $\gamma = 1 + (2.1\pm2.3)\times10^{-5}$ was reported by the Cassini mission [41]. From the Moon's contribution to the Shapiro delay, this gives a 5.6 μm amplitude at the orbital frequency. If the position-measurement accuracy can reach $1\ \mathrm{pm/Hz^{1/2}}$ at the μHz level, the uncertainty in the PPN parameter γ can be tested with an accuracy of 1.8×10^{-7}, which approaches the level where some scalar-tensor theories of gravity predict that a deviation from general relativity might be expected [42,43]. A smaller deviation from general relativity at the level of 10^{-9} is predicted by heuristic string-theory arguments, which may be tested by space missions like BEACON [44] or LATOR [43].

The second term $\Delta^{1\mathrm{EM}}_{J_2}$ in Eq. (26) is the contribution from the mass's quadrupole moment J_2. Since the Moon's mass is much smaller than that of Earth, it is sufficient to neglect the Moon's quadrupole moment. Considering the equilateral triangular constellation, we can adopt the values $\mathbf{n}_{EA}\cdot\mathbf{n}_{EB} = -1/2$ and $r_{EA} = r_{EB}$ for this term. From the penultimate term of Eq. (14), the delay due to the Earth's oblateness can be computed as

$$\Delta^{E}_{J_2} = 1.35\times10^{-7}\ \mathrm{m}\cdot\Big[1-\frac{5}{2}(\mathbf{I}_E\cdot\mathbf{n}_{EA})^2-\frac{5}{2}(\mathbf{I}_E\cdot\mathbf{n}_{EB})^2 - 4(\mathbf{I}_E\cdot\mathbf{n}_{EA})(\mathbf{I}_E\cdot\mathbf{n}_{EB})\Big]. \quad (30)$$

This shows that the Earth's quadrupole contribution to the delay is large enough to be observed.

The third term in Eq. (26) is the delay due to the angular momentum of the Earth. The last term in Eq. (14) gives

$$\Delta_s^E = -\frac{4GM_E}{c^3}\frac{\mathbf{R}_{AB}\cdot[\mathbf{s}_E\times(\mathbf{n}_{EA}+\mathbf{n}_{EB})]}{(r_{EA}+r_{EB})^2-R_{AB}^2} = 1\times10^{-9}\ \mathrm{m}\cdot\mathbf{N}_{AB}\cdot[\mathbf{I}_E\times(\mathbf{n}_{EA}+\mathbf{n}_{EB})]. \quad (31)$$

The Earth-rotation contribution does not exceed 1 nm, and we can amplify or inhibit by choosing an optimized orbit. In an ideal situation, it may be used to test gravitomagnetic effects on light propagation.

Next, we look at the fourth term in Eq. (26), which is due to the velocity effect of gravitational sources. Since the velocity and acceleration corrections are higher-order effects, it is sufficient to use the relationship $r_{EA}=r_{EB}$. From Eqs. (14) and (15), the Earth-velocity-dependent term gives

$$\Delta_v^{1E} = \frac{2GM_E}{c^2}\left\{\left(-\frac{2\mathbf{N}_{AB}\cdot\mathbf{v}_E}{c}+\frac{2v_E^2}{c^2}-\frac{(\mathbf{N}_{AB}\cdot\mathbf{v}_E)^2}{2c^2}\right)\times\ln\frac{r_{EA}+r_{EB}+R_{AB}}{r_{EA}+r_{EB}-R_{AB}} - \frac{R_{AB}}{c^2r_{EB}}[(\mathbf{N}_{AB}\cdot\mathbf{v}_E)^2+2(\mathbf{n}_{EA}\cdot\mathbf{v}_E)(\mathbf{n}_{EB}\cdot\mathbf{v}_E)]\right\}. \quad (32)$$

From this equation, the first-order velocity contribution (v_E/c) to the delay is $-4.7\ \mu\mathrm{m}\cdot\cos\omega_s t$, where ω_s is the angular frequency of the spacecraft with respect to the Earth. (In this and subsequent subsections, the initial phase value in cosine/sine functions is set to 0.) The second-order velocity contribution $(v_E/c)^2$ is about $(4.1+0.2\cos2\omega_s t)\times10^{-10}$ m.

The last term in Eq. (26) is the acceleration correction to light delay, which contains the contributions from the Earth's acceleration and interactions with other bodies. From Eqs. (14) and (16), it is expressed as

$$\Delta_a^{1E} = \frac{GM_E}{c^4}\left[(\mathbf{N}_{AB}\cdot\mathbf{a}_E)(\mathbf{N}_{AB}\cdot\mathbf{r}_{EA})-(\mathbf{r}_{EA}\cdot\mathbf{a}_E) - \sum_{c\neq E}\frac{2GM_c}{r_{cE}^2}r_{cE}\right]\ln\frac{r_{EA}+r_{EB}+R_{AB}}{r_{EA}+r_{EB}-R_{AB}}. \quad (33)$$

Through rough estimations, the acceleration contribution to the delay is about 3.8×10^{-14} m (which is negligible), and the delay due to interactions is longer, reaching -2.3×10^{-10} m.

The third term in Eq. (25) represents the external gravitational contribution to the delay, which is mainly due to the Sun and other planets. Since these gravitational sources are remote, $\Delta^{1\mathrm{ext}}$ can be expressed as

$$\Delta^{1\mathrm{ext}} = \sum_{b\neq E,M}\left(1-\frac{2\mathbf{N}_{AB}\cdot\mathbf{v}_b}{c}\right)\frac{2GM_b}{c^2}\frac{R_{AB}}{d_b}, \quad (34)$$

with $d_b=(r_{bA}+r_{bB})/2$. Assuming that ω_E and e_E are the angular frequency and eccentricity of the Earth's orbit, respectively, a direct estimate gives that the Sun's contribution is $3.3\ \mathrm{m}\cdot(1+\zeta\cos\omega_s t+e_E\cos\omega_E t)$, whereas for Jupiter it is 0.5–0.8 mm. $\zeta=r_{EA}/(1\ \mathrm{AU})$ is a constant.

The delay contribution of tidal deformations is given by the fourth term in Eq. (25). Only taking the Earth into account, we obtain

$$\Delta_{\mathrm{tid}} = \sum_{b\neq E}\frac{\sqrt{3}(k_2)_E GM_b R_E^5}{c^2 r_{Eb}^3 d_E^2}(12\mathcal{B}_{b2}+30\mathcal{B}_{b3}-1), \quad (35)$$

where $(k_2)_E$ is Earth's Love number, and d_E is the impact parameter which has a value of about $r_{EA}/2$ or $r_{EB}/2$. The Moon and Sun are the main sources of deformations of the Earth. The calculation implies that their contribution reaches the picometer level and the contribution from the Moon's tidal force is about 2 times that from the Sun. This tidal-deformation delay is more significant when the impact parameter of light with respect to the Earth is smaller. In the BEACON mission, the tidal-deformation delay can even reach several nanometers, which suggests that BEACON could yield a test of this delay with an accuracy of order 10%.

Finally, we evaluate the last term in Eq. (25), which is the 2PN contribution of the Sun's mass. Equation (21) yields the Sun's 2PN delay, $2.8\times10^{-8}\ \mathrm{m}\cdot(1+2e_E\cos\omega_E t)$. Clearly, the Earth's contribution to this term is at the level of 0.1 pm, and the contributions from other bodies are much smaller. Therefore, we just keep the Sun's contribution in this term. To summarize, we give a list of the various gravitational delays in Table I.

III. GENERAL-RELATIVISTIC PHASE MODEL AND FREQUENCY SHIFT FOR TIANQIN

Arthur Schawlow advised to never measure anything but the frequency [45,46]. Essentially, the measurable laser ranging interferometric (LRI) quantity is the frequency difference, which is usually expressed in the form of the frequency shift. For the TianQin mission, the frequency shift between spacecraft A and B (connected by laser link) can be split into three parts: the light-trajectory-dependent part, the spacecraft-dependent (clock-dependent) part, the and coupling part between the light's trajectory and the spacecraft (Fig. 1). The light-trajectory-dependent part includes the first-order Doppler effect, propagation delay effects, the GW signal, and higher-order effects in the light trajectory. The spacecraft-dependent part concerns the gravitational redshift, gravitomagnetic clock effect, clock noise, etc., For the coupling part, the biggest term (of order c^{-3}) comes from the coupling between the gravitational redshift and first-order Doppler effect, while other terms are higher order than c^{-3}. From the theoretical point of view, the

TABLE I. Parametrized estimates of light propagation delays between TianQin spacecraft for various gravitational terms. For the sake of simplicity, we set the constant phase value of all cosine functions in Table I to zero, and use the function $f(x_1, x_2) = (5/2)(x_1^2 + x_2^2) + 4x_1x_2$, where $x_1 = \mathbf{I}_E \cdot \mathbf{n}_{EA}$ and $x_2 = \mathbf{I}_E \cdot \mathbf{n}_{EB}$. For the Moon, we assume that the orbits of the spacecraft and Moon are coplanar. $\mathcal{M}_b = GM_b c^{-2}/r$ describes the gravitational potential of body b.

Effect	Equation	Contribution Source	Parametrized value
Earth's monopole mass	Eq. (27)	$\mathcal{M}_E$	2.34 cm
Earth's velocity	Eq. (32)	$\mathbf{v}_E$	$-4.7\ \mu\text{m} \cdot \cos \omega_s t$
Earth's acceleration	Eq. (33)	$\mathbf{a}_E$	3.8×10^{-14} m
Interaction with Earth	Eq. (33)		-2.3×10^{-10} m
Moon's monopole mass	Eq. (29)	$\mathcal{M}_M$	$49\ \mu\text{m} + 6\ \mu\text{m} \cdot \cos \omega_{ms} t$
Earth's quadrupole moment J_2	Eq. (30)	J_2	$1.35 \times 10^{-7}\ \text{m} \cdot [1 - f(x_1, x_2)]$
Earth's angular momentum	Eq. (31)	$\mathbf{s}_E$	$1\ \text{nm} \cdot \mathbf{N}_{AB} \cdot [\mathbf{I}_E \times (\mathbf{n}_{EA} + \mathbf{n}_{EB})]$
Sun's monopole mass	Eq. (34)	$\mathcal{M}_S$	$3.3\ \text{m} \cdot (1 + \zeta \cos \omega_s t + e_E \cos \omega_E t)$
Deformation due to Moon	Eq. (35)		$2.1\ \text{pm}(12\mathcal{B}_{M2} + 30\mathcal{B}_{M3} - 1)$
Deformation due to Sun	Eq. (35)		$1\ \text{pm}(12\mathcal{B}_{S2} + 30\mathcal{B}_{S3} - 1)$
Sun's 2PN			$28\ \text{nm} \cdot (1 + 2e_E \cos \omega_E t)$
Jupiter's mass		$\mathcal{M}_J$	$(5 - 8) \times 10^{-4}$ m

one-way frequency shift between spacecraft is characterized as

$$\frac{\Delta f}{f}(\mathbf{v}_A, t_A, \mathbf{x}_A, t_B, \mathbf{x}_B) = \left(\frac{\Delta f}{f}\right)_{GW} + \left(\frac{\Delta f}{f}\right)_s + \left(\frac{\Delta f}{f}\right)_{GF} + \left(\frac{\Delta f}{f}\right)_n + \cdots, \tag{36}$$

which is dependent on the velocities and positions of the spacecraft. (For a more detailed description, the gravitational constants of the Earth and Moon, their angular momentum, and other parameters are needed.) The first term on the right-hand side of the equation is the gravitational-wave effect on the frequency shift, which is the target effect of TianQin. The second term represents the special-relativistic Doppler effect including the first-order and higher-order Doppler effects, and the third term contains all contributions from the gravitational field, such as gravitational redshift. The last term describes the noise term mentioned in Eq. (24). Laser frequency noise can be directly introduced into Eq. (36). For shot noise or phase noise, we can start the analysis by using Eq. (24). The ellipsis includes other possible observable effects in TianQin, such as effects due to the possible violation of local Lorentz invariance and local position invariance, which will be studied in future works.

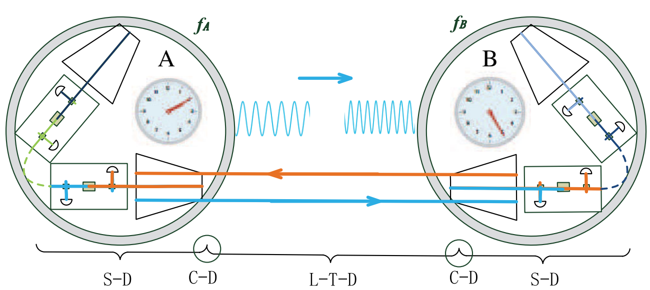

FIG. 1. Schematic diagram of the TianQin laser ranging interferometric instrument. A light signal with frequency f_A sent from spacecraft A is received by spacecraft B. The local oscillator or clock on the spacecraft compares its frequency f_B with the received signal frequency, which forms the frequency observable. Clearly, the frequency shift can be split into a spacecraft-dependent (S-D) part, light-trajectory-dependent (L-T-D) part, and their coupling (C-D) part.

In order to model the LRI observables of TianQin in detail, we consider that the spacecraft A and B move on their worldlines $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$, respectively. At coordinate time t_1, a laser signal with phase $\varphi(t_1, \mathbf{x}_{A1})$ is transmitted by the onboard oscillator of spacecraft A, where in the following $\mathbf{x}_{Ai}$ represents $\mathbf{x}_A(t_i)$, and similarly for the corresponding quantities $\mathbf{x}_{Bi}$. At coordinate time t_2, this signal is received at spacecraft B (t_2, $\mathbf{x}_{B2}$) with phase $\varphi(t_2, \mathbf{x}_{B2}) = \varphi(t_1, \mathbf{x}_{A1})$. The interferometer onboard spacecraft B compares the phase of the local laser oscillator at t_2 to the phase of the received signal at $\mathbf{x}_{B2}$ from spacecraft A. This comparison procedure produces the phase difference and frequency observables, from which the range and range rate between the two spacecraft are deduced. These phase and frequency data constitute the GW signal, gravitational field effects, Doppler effects, etc., For the two-way measurement (as shown in Fig. 2), this signal is coherently retransmitted at spacecraft B (t_2, $\mathbf{x}_{B2}$) with phase $\varphi(t_2, \mathbf{x}_{B2})$ and is subsequently received at spacecraft A (t_3, $\mathbf{x}_{A3}$) with phase $\varphi(t_3, \mathbf{x}_{A3}) = \varphi(t_2, \mathbf{x}_{B2})$. Similarly, the interferometer onboard spacecraft A compares the phase of the local laser oscillator at t_3 to the phase of the received signal at $\mathbf{x}_{A3}$ from the transponder on spacecraft B.

We start our discussion with the one-way measurement. At spacecraft B, the detectable quantity is the difference between the instantaneous local phase and the received phase. For the one-way measurement, an oscillator onboard

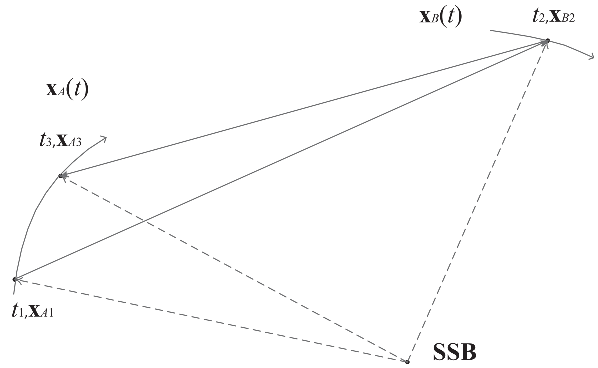

FIG. 2. Schematic diagram of the timing events on TianQin for signal propagation. $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$ are the trajectories of spacecraft A and B, respectively, which almost coincide with each other. The transmission of the signal is the event point $(t_1, \mathbf{x}_{A1})$, and it is received at $\mathbf{x}_{B2}$ by spacecraft B at time t_2. For the two-way measurement, the signal returns to spacecraft A at $\mathbf{x}_{A3}$.

spacecraft A with proper frequency f_A generates a signal with frequency $f_A(\tau_{A1})$ at proper time τ_{A1}. This signal is received by spacecraft B at proper time τ_{B2}, and the local oscillator's proper frequency is $f_B(\tau_{B2})$ at that instant. Then, the infinitesimal difference $\delta\varphi_{AB}(\tau_{B2})$ between the received phase $d\varphi_{AB}(\tau_{B2})$ and the locally generated phase $d\varphi_B(\tau_{B2})$ can be expressed by taking the difference between the two phase values as

$$\begin{aligned}\delta\varphi_{AB}(\tau_{B2}) &= d\varphi_B(\tau_{B2}) - d\varphi_{AB}(\tau_{B2}) \\ &= 2\pi(f_B(\tau_{B2}) - f_{AB}(\tau_{B2}))d\tau_{B2}, \end{aligned} \quad (37)$$

where $f_{AB}(\tau_{B2})$ is the frequency of the oscillator on spacecraft A measured at spacecraft B. The received phase $d\varphi_{AB}(\tau_{B2})$ is originally generated at spacecraft A at proper time τ_{A1}, and can be expressed in terms of the proper frequency $f_A(\tau_{A1})$ and infinitesimal proper time interval $d\tau_{A1}$ at spacecraft A as $d\varphi_{AB}(\tau_{B2}) = d\varphi_A(\tau_{A1}) = 2\pi f_A(\tau_{A1})d\tau_{A1}$. This relationship allows us to express the frequency $f_{AB}(\tau_{B2})$ in terms of the frequency $f_A(\tau_{A1})$ at the proper time τ_{A1} on spacecraft A,

$$f_{AB}(\tau_{B2}) = \frac{d\tau_{A1}}{d\tau_{B2}} f_A(\tau_{A1}). \quad (38)$$

Using Eq. (38), the infinitesimal difference $\delta\varphi_{AB}(\tau_2)$ in the phase can be rewritten in the form of proper frequencies generated locally at spacecraft A and B

$$\delta\varphi_{AB}(\tau_{B2}) = 2\pi\left(f_B(\tau_{B2}) - f_A(\tau_{A1})\frac{d\tau_{A1}}{d\tau_{B2}}\right)d\tau_{B2}. \quad (39)$$

In fact, spacecraft B will use a phase-locked detection scheme to confirm the presence of the detection signal with high sensitivity, in which a frequency offset $f_{\mathrm{Bo}}(\tau_{B2})$ is introduced into the locally generated signal. This frequency offset can respond to the received signal for its subsequent retransmission process. A coherent retransmission implies that the frequency offset satisfies the relationship

$$f_B(\tau_{B2}) + f_{\mathrm{Bo}}(\tau_{B2}) = f_A(\tau_{A1})\frac{d\tau_{A1}}{d\tau_{B2}}. \quad (40)$$

The retransmitted signal is received at proper time τ_{A3} and compared with the locally generated signal on spacecraft A. Thus, similar to the expression for $\delta\varphi_{AB}(\tau_2)$, the infinitesimal difference $\delta\varphi_{BA}(\tau_{A3})$ in phase measured on spacecraft A is given by

$$\begin{aligned}&\delta\varphi_{BA}(\tau_{A3}) \\ &\quad = 2\pi\left(f_A(\tau_{A3}) - (f_B(\tau_{B2}) + f_{\mathrm{Bo}}(\tau_{B2}))\frac{d\tau_{B2}}{d\tau_{A3}}\right)d\tau_{A3}.\end{aligned} \quad (41)$$

Equations (39) and (41) can be used to deduce the observational equations, which are needed to process the scientific data.

For more practical considerations, the LRI observables of TianQin are a continuous signal. The continuous changes in the phase difference generate time series data. To obtain the changes in the phase difference, the proper times should be treated as continuous variables, allowing to formally integrate Eqs. (39) and (41) as follows:

$$\Delta\varphi_{AB}(\tau_{B2}) = \int \delta\varphi_{AB}(\tau_{B2}), \quad (42)$$

$$\Delta\varphi_{BA}(\tau_{A3}) = \int \delta\varphi_{BA}(\tau_{A3}). \quad (43)$$

These two quantities are the LRI observables on spacecraft B and A, respectively, which are obtained by comparing the phase of the local oscillator with the phase of the received signal.

In order to develop Eq. (39) or Eq. (41), we should establish the differential equation between the spacecraft's proper times (τ_A and τ_B) and coordinate time in BCRS, which is given by

$$\frac{d\tau_{A/B}}{dt} = 1 - \frac{1}{c^2}\left[\frac{\mathbf{v}_{A/B}^2}{2} + w(\mathbf{x}_{A/B})\right] + O(c^{-4}). \quad (44)$$

We analyze $\delta\varphi_{AB}(\tau_{B2})$, but the same process can be used for $\delta\varphi_{BA}(\tau_{A3})$. Using Eq. (44), Eq. (39) is rewritten as follows:

$$\delta\varphi_{AB}(\tau_{B2}) = 2\pi\left(f_B(\tau_{B2}) - f_A(\tau_{A1})\left(\frac{d\tau_A}{dt}\right)_{t_1}\left(\frac{d\tau_B}{dt}\right)^{-1}_{t_2}\frac{dt_1}{dt_2}\right)d\tau_{B2}. \tag{45}$$

Considering light propagation in the gravitational field, the ratio of the coordinate times in this equation can be expressed as

$$\frac{dt_1}{dt_2} = 1 - \frac{1}{c}\frac{d}{dt_2}\mathcal{R}_{AB}(\mathbf{x}_{A1}, \mathbf{x}_{B2}). \tag{46}$$

By introducing the instantaneous coordinate distance $D_{AB} = |\mathbf{x}_{B2} - \mathbf{x}_{A2}|$ (see Appendix B), this equation is further written as

$$\frac{dt_1}{dt_2} = 1 - \frac{1}{c}\frac{d}{dt_2}\left\{D_{AB} + \Delta^{\mathrm{GF}}_{AB} + \frac{\mathbf{D}_{AB}\cdot\mathbf{v}_A}{c} + \frac{D_{AB}}{2c^2}\left[\mathbf{v}_A^2 - \mathbf{D}_{AB}\cdot\mathbf{a}_A + \frac{(\mathbf{D}_{AB}\cdot\mathbf{v}_A)^2}{D^2_{AB}}\right]\right\}. \tag{47}$$

Apparently, the light-trajectory-dependent parts of the frequency shift may be described by this equation, such as the first-order Doppler effect and Sagnac effect. The second term in the parentheses describes the frequency shift due to the gravitational delay. Using the Shapiro delay term gives

$$\frac{d\Delta^{\mathrm{GF}}_{AB}}{cdt} = \sum_b \frac{2GM_b}{c^3 r_{bA} r_{bB}}\left[\frac{(r_{bA}+r_{bB})\mathbf{N}_{AB}\cdot\mathbf{v}_{AB}}{1+\mathbf{n}_{bA}\cdot\mathbf{n}_{bB}} - \frac{(\mathbf{n}_{bA}\cdot\mathbf{v}_{bA} + \mathbf{n}_{bB}\cdot\mathbf{v}_{bB})R_{AB}}{1+\mathbf{n}_{bA}\cdot\mathbf{n}_{bB}}\right]. \tag{48}$$

The contribution of the Earth's gravitational delays is about $4\times 10^{-15}e$, where e is the eccentricity of the spacecraft with respect to the Earth.

Inserting Eq. (44) into Eq. (45), we have

$$\left(\frac{d\tau_A}{dt}\right)_{t_1}\left(\frac{d\tau_B}{dt}\right)^{-1}_{t_2} = 1 + \frac{1}{c^2}\left(\frac{\mathbf{v}_B^2 - \mathbf{v}_A^2}{2} + w_B - w_A\right) + O(c^{-4}). \tag{49}$$

This constitutes most of the clock-dependent part of the frequency shift. Estimating this equation, the second-order Doppler effect is about 1×10^{-9} and the gravitational redshift due to the Sun is about 1×10^{-11}. However, the observable physical quantities are much smaller.

Considering TianQin's constellation, the second-order Doppler term can be reexpressed as

$$\frac{1}{c^2}\left(\frac{\mathbf{v}_B^2 - \mathbf{v}_A^2}{2}\right) = \frac{1}{c^2}\frac{d}{dt}(\mathbf{R}_{AB}\cdot\mathbf{v}_E) - \frac{\mathbf{R}_{AB}\cdot\mathbf{a}_E}{c^2} + \frac{1}{2c^2}(\mathbf{v}_{EB}^2 - \mathbf{v}_{EA}^2). \tag{50}$$

The first term is essentially the coordinate effect, which will cancel with the third term in the parentheses of Eq. (47). This cancellation leads to measurable effects in the term of Eq. (47) that is only dependent on the spacecraft's velocity with respect to Earth. The magnitude of the second term is -1×10^{-11}, which is equivalent to the gravitational redshift but with opposite sign. Further, the gravitational redshift term can be rewritten as

$$\frac{w_B - w_A}{c^2} = \frac{\sqrt{3}GM_E e}{c^2 a}\left(1 + \frac{3R_E^2}{2a^2}J_2\right)\cos\omega_s t + \frac{1}{c^2}\sum_{b\neq E}\frac{GM_b}{r^3_{bE}}\mathbf{x}_{bE}\cdot\mathbf{R}_{BA} + O(e^2, r^{-3}_{bE}), \tag{51}$$

with semimajor axis $a = 10^5$ km. Clearly, the first term is the Earth's gravitational redshift, in which the contribution of the mass monopole is about $7.7e\times 10^{-11}$, whereas that of the quadrupole moment is $4.7e\times 10^{-16}$. The second term represents the gravitational redshift due to other body's gravitational field, which would cancel with the second term in Eq. (50) since the Earth's acceleration is given by $\nabla\sum_{b\neq E}(GM_b)/r_{bE}$. Thus, the influence of the Sun and the Moon's gravitational fields is only given in the form of a tidal potential. This is a characteristic of the geocentric orbit option. The influences of other bodies (except for the Earth) become

$$\frac{u_B^t - u_A^t}{c^2} \simeq \sum_{b\neq E}\frac{3GM_b a^2}{2c^2 r^3_{bE}}[(\mathbf{n}_{bE}\cdot\mathbf{n}_{EB})^2 - (\mathbf{n}_{bE}\cdot\mathbf{n}_{EA})^2], \tag{52}$$

where u^t is defined by the tidal potential. This equation is the ignored term in Eq. (51) involving r^{-3}_{bE}. The contributions due to the Moon and Sun are 1×10^{-14} and 6×10^{-15}, respectively. The other body's contributions are even smaller and can be omitted (e.g., of order 10^{-20} for Jupiter).

Subsequently, the Earth's tidal deformation is taken into account. From Eq. (11), its effect on the frequency shift $(\delta f/f)_{\mathrm{tid}}$ is obtained, which is given in the form of Love numbers,

$$\left(\frac{\delta f}{f}\right)_{\mathrm{tid}} = \sum_{c\neq E}\frac{3(k_2)_E GM_c R_E^5}{2c^2 r^3_{cE} a^3}[(\mathbf{n}_{cE}\cdot\mathbf{n}_{EB})^2 - (\mathbf{n}_{cE}\cdot\mathbf{n}_{EA})^2]. \tag{53}$$

The Sun and Moon are the main sources of tidal force deformations, which lead to a negligible frequency shift of order 10^{-20}. This effect grows for lower orbits. The estimate implies that the frequency shift due to tidal deformations may reach a measurable level in binary pulsar systems.

For the coupling terms, we consider a frequency shift due to the coupling between the Earth's gravitational redshift and the Doppler effect, which is about $5\times10^{-17}e^2$, whereas the coupling term involving the Sun is smaller. The coupling effects for TianQin are negligible.

After the above mathematical manipulations and using Eq. (B5), the infinitesimal difference $\delta\varphi_{AB}$ becomes

$$\delta\varphi_{AB}(\tau_{B2}) = 2\pi\Bigg\{f_B(\tau_{B2}) - f_A(\tau_{A1}) + f_A(\tau_{A1})\Bigg[\frac{\mathbf{N}_{AB}\cdot\mathbf{v}_{AB}}{c} - \frac{\mathbf{v}_{EB}^2-\mathbf{v}_{EA}^2}{2c^2} + \frac{1}{c^2}((\mathbf{N}_{AB}\cdot\mathbf{v}_{EA})(\mathbf{N}_{AB}\cdot\mathbf{v}_{EB}) - (\mathbf{N}_{AB}\cdot\mathbf{v}_{EA})^2 + \mathbf{D}_{AB}\cdot\mathbf{a}_{EA}) - \frac{\sqrt{3}GM_E e}{c^2 a}\left(1+\frac{3R_E^2}{2a^2}J_2\right)\cos\omega_s t - \sum_{b\neq E}\frac{3GM_b a^2}{2c^2 r_{bE}^3}[(\mathbf{n}_{bE}\cdot\mathbf{n}_{EB})^2 - (\mathbf{n}_{bE}\cdot\mathbf{n}_{EA})^2]\Bigg]\Bigg\}d\tau_{B2}. \quad (54)$$

The c^{-1} term is the first-order Doppler effect, which can be written in the orbit-parameter form

$$-\frac{\mathbf{N}_{AB}\cdot\mathbf{v}_{AB}}{c} = \frac{\sqrt{3}e}{2c}\sqrt{\frac{GM_E}{a}}\sin\omega_s t = 5.8\times10^{-6}\cdot e\sin\omega_s t. \quad (55)$$

For Keplerian orbits, the velocity is sufficiently described by the relation $\mathbf{v}^2 = 2U + K$, where U is the Newtonian gravitational potential and K is a constant. The clock's second-order Doppler effect can be written as

$$\frac{\mathbf{v}_{EB}^2-\mathbf{v}_{EA}^2}{2c^2} = \frac{\sqrt{3}GM_E e}{c^2 a}\cos\omega_s t. \quad (56)$$

This term has the same magnitude as the Earth's gravitational redshift, which is a characteristic of a Keplerian orbit. In fact, the practical orbit deviates from the Keplerian orbit, which also causes relativistic effects between the spacecraft's clocks. These effects may need to be considered after a long-time integral, which can be estimated by a perturbed Keplerian orbit [47]. After a complete orbital period (3.65 days), the relative frequency difference between the spacecraft's clocks may reach the level of $10^{-13}e$. To summarize, several important effects are listed in Table II.

IV. CONCLUSION

High-precision space missions require accurate modeling of the relativistic observations of laser and frequency data. By solving the eikonal equation in the BCRS within the post-Newtonian approximation, we studied light propagation in the gravitational field of an isolated, gravitationally bound N-body system. Based on the method of asymptotic perturbation theory, the various gravitational perturbations in phase were solved and the corresponding time delays were subsequently obtained. In addition to the conventional static fields, the solutions include the influence of motion, such as velocity and acceleration. At the same time, we treated the system not as N independent bodies, but rather as interaction-bound bodies. The gravitational interactions and tidal deformations were taken into account for realistic, nonrigid astronomic bodies. The condition of gravitationally bound N-body system is important in strong gravitational fields; for example, tidal effects are significant and must be considered in binary pulsar systems or some GW sources. Our solutions give a more precise description of physical N-body systems and are sufficient for use in modern space missions. We applied these solutions to a relativistic analysis of the TianQin mission, focusing on the gravitational influences of the Earth-Moon system. Based on the parameters of Keplerian orbits, we estimated the various terms in the light propagation delays between spacecraft (listed in Table I), which can be used in a numerical model of TianQin. Equation (24) or Eq. (36) can be used to discuss the various effects of noise on the TianQin sensitivity curve in future works. From the relativistic analysis of TianQin, we found that the Moon's gravitational effects on the onboard clocks and inter-spacecraft signal propagation are comparable to those of the Earth, and the contributions of the Earth-Moon system's gravitational delays are below the picometer level in the mHz

TABLE II. Parametrized estimates of the one-way frequency shift between spacecraft for the TianQin mission. We set the original phase in the sine and cosine functions to 0. The angles are given by $\cos\theta_{bA/B} = \mathbf{n}_{bE}\cdot\mathbf{n}_{EA/B}$, where b represents the Moon or Sun.

Effect	Parametrized value
First-order Doppler effect	$5.8\times10^{-6}\cdot e\sin\omega_s t$
Second-order Doppler effect	$7.7\times10^{-11}\cdot e\cos\omega_s t$
Earth's mass monopole	$7.7\times10^{-11}\cdot e\cos\omega_s t$
Earth's mass J_2 term	$4.7\times10^{-16}\cdot e\cos\omega_s t$
Moon's mass monopole	$1\times10^{-14}(\cos^2\theta_{MB}-\cos^2\theta_{MA})$
Sun's mass monopole	$6\times10^{-15}(\cos^2\theta_{SB}-\cos^2\theta_{SA})$

regime, and thus are negligible for GW detections. Furthermore, we found that TianQin may provide some classical tests of general relativity. For example, the uncertainty in the post-Newtonian parameter γ can be tested at the level of some scalar-tensor theories of gravity and the Earth's gravitomagnetic effect on light propagation can be tested.

We also computed the relativistic frequency shift between the spacecraft's clocks due to the various motion and gravitational field terms. Parametrized estimates are listed in Table II. We found the surprising result that the general-relativistic contribution to the frequency shift is much smaller than our *a priori* expectation. These smaller contributions to the frequency shift are mainly caused by the TianQin's configuration, which causes several effects to cancel with each other. At the same time, our results demonstrate that most general-relativistic effects are dependent on the orbital parameters. Thus, we can amplify or inhibit these effects by choosing an optimized orbit. This analytical formalism and parametrized estimates of the relativistic effects will provide support for the TianQin scientific mission.

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APPENDIX A: EIKONAL EQUATION

To solve the eikonal equation we use the asymptotic perturbation method. We choose a small parameter G (gravitational constant), and expand every function in the eikonal equation in corresponding power series. This method is similar to the approach developed by C. Le Poncin-Lafitte *et al.*, which was initially based on the Synge world function [7], and then on the TTF [8]. The asymptotic power series can provide a safe way to select the terms to keep at each order. The metric tensor $g_{\mu\nu}$ is represented by a series in ascending powers of G:

$$g_{\mu\nu}(x,G) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} G^n g_{\mu\nu}^{(n)}(x). \tag{A1}$$

Also, $g^{\mu\nu}$ can be given by a similar expansion with the relationships

$$g_{(1)}^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta} g_{\alpha\beta}^{(1)} \tag{A2}$$

and

$$g_{(n)}^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta} g_{\alpha\beta}^{(n)} - \sum_{p=1}^{n-1} \eta^{\mu\alpha} g_{\alpha\beta}^{(p)} g_{(n-p)}^{\beta\nu}. \tag{A3}$$

Further, the phase φ is expressed as a similar expansion,

$$\varphi(t,\mathbf{x}) = \varphi_0 + \int k_\mu dx^\mu + \sum_{n=1}^{\infty} G^n \varphi^{(n)}(t,\mathbf{x}), \tag{A4}$$

where φ_0 is a constant, and $\varphi^{(n)}(t,\mathbf{x})$ is the phase perturbation of the nth order in G.

Let us define a light ray connecting $x_A = (ct_A, \mathbf{x}_A)$ and $x_B = (ct_B, \mathbf{x}_B)$ with the point event $x = (|\mathbf{x} - \mathbf{x}_A| + \Delta, \mathbf{x})$, where Δ is the gravitational delay and $\mathbf{x}$ is defined by the parameter equation (8). Inserting Eqs. (A2)–(A4) into the eikonal equation, we have the Hamilton-Jacobi-like equation

$$g^{\mu\nu}(x)\partial_\mu \varphi(x_A, x)\partial_\nu \varphi(x_A, x) = 0. \tag{A5}$$

Using the above equations, the perturbation terms $\varphi^{(n)}$ can be determined by a recursive procedure. Each term $\varphi^{(n)}$ can be given by the integral along a straight line between transmission and reception. This avoids the calculation of the gravitational perturbation of the geodesic joining the given points. In contrast, the integrals for $\varphi^{(n)}$ have to contain products of the first-order derivatives of lower-order terms $\varphi^{(n-p)}$, $p = 1, \ldots n-1$. This means that the calculations of integrals along the null geodesic are replaced by calculations of the integrals of these derivatives. This method has been demonstrated by Poncin-Lafitte *et al.* [7,8]. When we only consider the case of $n \leq 2$, the solution of the eikonal equation is expressed as [8,29]

$$\varphi^{(1)} = -\frac{R_{AB}}{2k_0}\int_0^1 g_{(1)}^{\mu\nu} k_\mu k_\nu d\lambda \tag{A6}$$

and

$$\varphi^{(2)} = -\frac{R_{AB}}{2k_0}\int_0^1 (\eta^{\mu\nu}\partial_\mu\varphi^{(1)}\partial_\nu\varphi^{(1)} + 2g_{(1)}^{\mu\nu} k_\mu \partial_\nu \varphi^{(1)} + g_{(2)}^{\mu\nu} k_\mu k_\nu) d\lambda. \tag{A7}$$

All integrals are calculated along the straight line defined by Eq. (8). Obviously, Eq. (A6) are the terms of order G or c^{-2}, in which an integral along a straight line is valid. Equation (A7) is of order c^{-4}. For the calculation of $\varphi^{(2)}$, the recursive procedure with terms $\partial_\mu \varphi^{(1)}$ avoids the integral along the perturbed paths.

APPENDIX B: INSTANTANEOUS COORDINATE DISTANCE

In space missions, the arrival time of the signal is recorded at one of the satellites. Therefore, it is convenient to express the distance between the spacecraft at the time of reception t_B. We introduce an instantaneous coordinate distance $D_{AB} = |\mathbf{D}_{AB}| = |\mathbf{x}_B(t_B) - \mathbf{x}_A(t_B)|$. By using the Taylor expansion, the coordinate distance $\mathbf{R}_{AB} = \mathbf{x}_B(t_B) - \mathbf{x}_A(t_A)$ can be written as

$$\mathbf{R}_{AB} = \mathbf{D}_{AB} + \mathbf{v}_A(t_B)T_{AB} - \frac{1}{2}\mathbf{a}_A(t_B)T_{AB}^2 + O(c^{-3}), \tag{B1}$$

where $\mathbf{v}_A$ and $\mathbf{a}_A$ are the velocity and acceleration of A at the coordinate time t_B, respectively. By an iterative process, R_{AB} can be rewritten as

$$R_{AB} = D_{AB} + \frac{\mathbf{D}_{AB}\cdot\mathbf{v}_A}{c} + \frac{D_{AB}}{2c^2}\left[\mathbf{v}_A^2 - \mathbf{D}_{AB}\cdot\mathbf{a}_A + \frac{(\mathbf{D}_{AB}\cdot\mathbf{v}_A)^2}{D_{AB}^2}\right] + O(c^{-3}), \tag{B2}$$

where all of the quantities are measured at the reception time t_B. Then, we can obtain its derivative as

$$\frac{dR_{AB}}{cdt_B} = \frac{\mathbf{n}_{AB}\cdot\mathbf{v}_{AB}}{c} + \frac{1}{c^2}(\mathbf{v}_{AB}\cdot\mathbf{v}_A + \mathbf{D}_{AB}\cdot\mathbf{a}_A) + O(c^{-3}) \tag{B3}$$

with $\mathbf{n}_{AB} = \mathbf{D}_{AB}/D_{AB}$. Combining Eqs. (B1) and (B2), the unit vector $\mathbf{n}_{AB}$ can be expressed in terms of the unit vector $\mathbf{N}_{AB}$,

$$\mathbf{n}_{AB} = \mathbf{N}_{AB}\left[1 + \frac{\mathbf{N}_{AB}\cdot\mathbf{v}_A}{c} + \frac{1}{2c^2}(3(\mathbf{N}_{AB}\cdot\mathbf{v}_A)^2 - \mathbf{v}_A^2 - \mathbf{R}_{AB}\cdot\mathbf{a}_A)\right] - \frac{\mathbf{v}_A}{c}\left(1 + \frac{\mathbf{N}_{AB}\cdot\mathbf{v}_A}{c}\right) + \frac{\mathbf{a}_A}{2c^2}R_{AB} + O(c^{-3}). \tag{B4}$$

Using this, Eq. (B3) can be rewritten as

$$\frac{dR_{AB}}{cdt_B} = \frac{\mathbf{N}_{AB}\cdot\mathbf{v}_{AB}}{c} + \frac{1}{c^2}[(\mathbf{N}_{AB}\cdot\mathbf{v}_A)(\mathbf{N}_{AB}\cdot\mathbf{v}_B) - (\mathbf{N}_{AB}\cdot\mathbf{v}_A)^2 + \mathbf{R}_{AB}\cdot\mathbf{a}_A] + O(c^{-3}). \tag{B5}$$

Neglecting the acceleration term, this expression takes the usual Doppler-effect form, and this method allows us to obtain the higher-order Doppler terms.

APPENDIX C: USEFUL RELATIONSHIPS FOR KEPLERIAN ORBITS

We consider a Keplerian equation $r = a(1 - e\cos u)$ in the orbital plane, where a is the semimajor axis, e is the eccentricity, and u is the eccentric anomaly. In the orbital plane, the position vector is given by

$$\mathbf{r} = a(\cos u - e, \sqrt{1-e^2}\sin u). \tag{C1}$$

By this equation, its unit vector is given by

$$\mathbf{n} = \frac{\mathbf{r}}{r} = \left(\frac{\cos u - e}{1 - e\cos u}, \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}\right). \tag{C2}$$

Considering the time derivative of the eccentric anomaly $\dot{u} = \sqrt{GMa}/ar$, the velocity vector is

$$\mathbf{v} = \frac{\sqrt{GMa}}{r}\left(-\sin u, \sqrt{1-e^2}\cos u\right). \tag{C3}$$

We consider two spacecraft A and B with different eccentric anomalies u_A and u_B. Using Eqs. (C1)–(C3), we obtain several relationships in orders of e:

$$\frac{\mathbf{r}_{AB}\cdot\mathbf{v}_{AB}}{r_{AB}} = -2\sqrt{\frac{GM}{a}}e\sin\mathcal{K}_{AB}\cos\mathcal{K}_{AB}\sin\mathcal{L}_{AB}, \tag{C4}$$

$$\frac{\mathbf{r}_{AB}\cdot\mathbf{r}_A}{r_{AB}} = -a(\sin\mathcal{K}_{AB} - e\sin\mathcal{L}_{AB}), \tag{C5}$$

$$\frac{\mathbf{r}_{AB}\cdot\mathbf{v}_A}{r_{AB}} = \sqrt{\frac{GM}{a}}\cos\mathcal{K}_{AB}(1 + e\cos u_A), \tag{C6}$$

with $r_{AB} = |\mathbf{r}_{AB}| = |\mathbf{r}_B - \mathbf{r}_A|$, $\mathcal{K}_{AB} = (u_B - u_A)/2$, and $\mathcal{L}_{AB} = (u_B + u_A)/2$.

[1] S. Zschocke, Phys. Rev. D **92**, 063015 (2015).
[2] S. Zschocke, Phys. Rev. D **93**, 103010 (2016).
[3] S. Zschocke, Phys. Rev. D **94**, 124007 (2016).
[4] S. M. Kopeikin, J. Math. Phys. (N.Y.) **38**, 2587 (1997).
[5] S. M. Kopeikin, P. Korobkov, and A. Polnarev, Classical Quantum Gravity **23**, 4299 (2006).
[6] C. Le Poncin-Lafitte and P. Teyssandier, Phys. Rev. D **77**, 044029 (2008).
[7] C. Le Poncin-Lafitte, B. Linet, and P. Teyssandier, Classical Quantum Gravity **21**, 4463 (2004).
[8] P. Teyssandier and C. Le Poncin-Lafitte, Classical Quantum Gravity **25**, 145020 (2008).
[9] A. Hees, S. Bertone, and C.Le Poncin-Lafitte, Phys. Rev. D **89**, 064045 (2014).
[10] B. Linet and P. Teyssandier, Classical Quantum Gravity **30**, 175008 (2013).

[11] B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016).
[12] B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 241103 (2016).
[13] B. P. Abbott *et al.*, Phys. Rev. Lett. **119**, 141101 (2017).
[14] A. Nishizawa, Phys. Rev. D **97**, 104037 (2018).
[15] S. Arai and A. Nishizawa, Phys. Rev. D **97**, 104038 (2018).
[16] B. P. Abbott *et al.*, Phys. Rev. Lett. **119**, 161101 (2017).
[17] B. P. Abbott *et al.*, Phys. Rev. Lett. **118**, 221101 (2017).
[18] P. Bender *et al.*, *LISA (Laser Interferometer Space Antenna): An International Project in the Field of Fundamental Physics in Space* (Max-Planck-Institute fur Quantenoptic, Garching bei Munchen, 1998).
[19] R. Hellings, SAGITTARIUS: An ESA M3 Proposal[J], 1993.
[20] T. R. Boehly *et al.*, Opt. Commun. **133**, 495 (1997).
[21] J. Luo *et al.*, Classical Quantum Gravity **33**, 035010 (2016).
[22] N. J. Cornish and R. W. Hellings, Classical Quantum Gravity **20**, 4851 (2003).
[23] F. B. Estabrook, M. Tinto, and J. W. Armstrong, Phys. Rev. D **62**, 042002 (2000).
[24] S. V. Dhurandhar, K. R. Nayak, and J. Y. Vinet, Phys. Rev. D **65**, 102002 (2002).
[25] M. Tinto and O. Hartwig, Phys. Rev. D **98**, 042003 (2018).
[26] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **43**, 3273 (1991).
[27] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992).
[28] N. Ashby and B. Bertotti, Classical Quantum Gravity **27**, 145013 (2010).
[29] C. G. Qin and C. G. Shao, Phys. Rev. D **96**, 024003 (2017).
[30] B. Chauvineau, T. Regimbau, J. Y. Vinet, and S. Pireaux, Phys. Rev. D **72**, 122003 (2005).
[31] S. G. Turyshev, M. V. Sazhin, and V. T. Toth, Phys. Rev. D **89**, 105029 (2014).
[32] S. G. Turyshev, N. Yu, and V. T. Toth, Phys. Rev. D **93**, 045027 (2016).
[33] J. Sakstein, Phys. Rev. D **97**, 064028 (2018).
[34] R. Shaniv, R. Ozeri, M. S. Safronova, S. G. Porsev, V. A. Dzuba, V. V. Flambaum, and H. Haffner, Phys. Rev. Lett. **120**, 103202 (2018).
[35] C. G. Shao, Y. F. Chen, R. Sun, L. S. Cao, M. K. Zhou, Z. K. Hu, C. Yu, and H. Müller, Phys. Rev. D **97**, 024019 (2018).
[36] M. Soffel, S. A. Klioner, G. Petit *et al.*, Astrophys. J. **126**, 2687 (2003).
[37] S. G. Turyshev, V. T. Toth, and M. V. Sazhin, Phys. Rev. D **87**, 024020 (2013).
[38] V. Perlick, C. Lämmerzahl, and A. Macías, Phys. Rev. D **98**, 105014 (2018).
[39] E. Poisson and C. M. Will, *Gravity: Newonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014), pp. 119–134.
[40] N. K. Pavlis, S. A. Holmes, S. C. Kenyon, and J. K. Factor, J. Geophys. Res. **117**, n/a (2012).
[41] B. Bertotti, L. Iess, and P. Tortora, Nature (London) **425**, 374 (2003).
[42] T. Damour and K. Nordtvedt, Phys. Rev. Lett. **70**, 2217 (1993).
[43] J. E. Plowman and R. W. Hellings, Classical Quantum Gravity **23**, 309 (2006).
[44] S. G. Turyshev, M. Shao, A. Girerd, and B. Lane, Int. J. Mod. Phys. D **18**, 1025 (2009).
[45] F. Riehle, Physics **5**, 126 (2012).
[46] T. W. Hänsch, Rev. Mod. Phys. **78**, 1297 (2006).
[47] C. G. Qin, Y. J. Tan, and C. G. Shao, Classical Quantum Gravity **36**, 055008 (2019).