Reexamining $f(R,T)$ gravity

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We study $f(R,T)$ gravity, in which the curvature R appearing in the gravitational Lagrangian is replaced by an arbitrary function of the curvature and the trace T of the stress-energy tensor. We focus primarily on situations where f is separable, so that $f(R,T) = f_1(R) + f_2(T)$. We argue that the term $f_2(T)$ should be included in the matter Lagrangian $\mathcal{L}_m$, and therefore has no physical significance. We demonstrate explicitly how this can be done for the cases of free fields and for perfect fluids. We argue that all uses of $f_2(T)$ for cosmological modeling and all attempts to place limits on parameters describing $f_2(T)$ are misguided.

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I. INTRODUCTION

Cosmological observations have convincingly demonstrated that the expansion of the Universe is accelerating [1–3]. This observation is inconsistent with Einstein's general theory of relativity for a universe containing only ordinary matter and radiation. This suggests either the presence of novel matter with unusual properties or a breakdown of general relativity on cosmological scales [4].

One modification that has gained much attention to explain this expansion is $f(R)$ gravity [5], where the Lagrangian describing gravitational effects, normally proportional to the curvature scale R, is replaced by a function of that curvature. For example, the introduction of an R^2 term in $f(R)$ can lead to Starobinsky inflation [6]. The inclusion of a constant term, $f(R) = R + 2\Lambda$, corresponds to the introduction of a cosmological constant, and therefore leads to the standard ΛCDM cosmology. A further generalization of $f(R)$ gravity was proposed by Harko *et al.* in [7], where $f(R)$ is replaced by $f(R,T)$, an arbitrary function of the scalar curvature R and the trace of the stress-energy tensor T. Cosmological effects of $f(R,T)$ theories have been explored by choosing several functional forms of f. The separation $f(R,T) = f_1(R) + f_2(T)$ has received much attention [8–12]. In particular, for the special case $f_2(T) = -2\chi T$, limits on χ or observational predictions for nonzero χ have been applied to models of white dwarfs [10], strange stars [11], and Earth's atmosphere [12].

As we will argue below, when we can separate these theories in the form $f(R,T) = f_1(R) + f_2(T)$, the term $f_2(T)$ should not be treated as a new contribution to the gravitational action, but instead should be incorporated into the matter Lagrangian $\mathcal{L}_m$. In Sec. II, we will introduce the formalism and discuss general principles. In Sec. III, we will demonstrate how $f_2(T)$ can be incorporated into $\mathcal{L}_m$ for the trivial case of a free field. In Sec. IV, we will show how this can be implemented for a generalized perfect fluid. In Sec. V, we will summarize our conclusions and briefly discuss some of these ideas in generic $f(R,T)$ gravity. Throughout this paper we use units where $c = 1$, our metric signature is $(+---)$, and our curvature is given by $R_{\mu\nu} = R_{\mu\nu}^{\mathrm{MTW}}$, and $R = -R_{\mathrm{MTW}}$ where MTW refers to the conventions of Misner, Thorne, and Wheeler [13].

II. $f(R,T)$ FORMALISM

With our conventions, the conventional action takes the form

$$I = \int d^4x\sqrt{-g}\mathcal{L}, \tag{1a}$$

$$\mathcal{L} = \mathcal{L}_m - \frac{1}{2\kappa^2}R, \tag{1b}$$

where $\kappa^2 = 8\pi G$. A cosmological constant Λ can be included by adding $-\Lambda\kappa^{-2}$ to $\mathcal{L}$. This term can be thought of as either a modification to gravity, $R \to R + 2\Lambda$, or a modification of the matter Lagrangian $\mathcal{L}_m \to \mathcal{L}_m - \Lambda\kappa^{-2}$. The two interpretations are physically indistinguishable.

The stress-energy tensor is defined in general as[1]

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I_m}{\delta g_{\mu\nu}} \tag{2}$$

[1]Harko *et al.* [7] has this equation with the wrong sign. This error and consequences thereof were copied by other authors [9–11].

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where I_m is the contribution to I from $\mathcal{L}_m$. Where one incorporates the contribution from the cosmological constant affects the value of the stress-energy tensor: including it in $\mathcal{L}_m$ adds a term $\Lambda\kappa^{-2}g_{\mu\nu}$ to $T_{\mu\nu}$. Not surprisingly, Einstein's equations, which are derived by demanding that the full action Eq. (1a) remain stationary under changes of the metric $\delta g_{\mu\nu}$, are identical in both cases. Because the metric has no divergence, one can easily show that in either case $\nabla_\mu T^{\mu\nu} = 0$, so the stress-energy tensor will be conserved in both cases.

We discuss a cosmological constant here only to emphasize an important point: whether a term is inserted into the Lagrangian as matter or gravity is not physically meaningful. In this case, because the stress-energy is conserved either way, we cannot base a decision on naturalness either. When we generalize to $f(R,T)$ gravity, we will make the case that some apparently "new physics" results from artificial division of the Lagrangian, and is thus nonphysical. Because the cosmological constant is not really relevant to the subsequent discussion, we will dispense with it.

The premise of $f(R,T)$ gravity, as first suggested by Harko *et al.* [7], is to replace R with $f(R,T)$ in Eq. (1b), so

$$\mathcal{L} = \mathcal{L}_m - \frac{1}{2\kappa^2} f(R,T). \tag{3}$$

We will focus on the subcase of $f(R,T)$ gravity in which we can cleanly separate the effects of curvature and matter, namely

$$f(R,T) = f_1(R) + f_2(T). \tag{4}$$

The modified Einstein's equations in this theory, derived from demanding that the action remain invariant under changes of the metric, take the form

$$\begin{aligned} &f_1'(R)R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}f_1(R) - (\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)f_1'(R) \\ &\quad = \kappa^2 T^{\mu\nu} + \frac{1}{2}f_2(T)g^{\mu\nu} + f_2'(T)\frac{\partial T}{\partial g_{\mu\nu}}, \end{aligned} \tag{5}$$

where primes denote derivatives with respect to the argument. As has been noted in the literature, such a theory does not, in general, result in the conservation of the stress-energy tensor. Instead we find

$$\nabla_\mu T^{\mu\nu} = -\frac{1}{\kappa^2}\nabla_\mu\left[f_2'(T)\frac{\partial T}{\partial g_{\mu\nu}}\right] - \frac{1}{2\kappa^2}g^{\mu\nu}\nabla_\mu f_2(T). \tag{6}$$

III. FREE FIELDS

As a trivial example, consider a free scalar field, with matter Lagrangian

$$\mathcal{L}_m = \frac{1}{2}(\nabla_\mu\phi\nabla^\mu\phi - m^2\phi^2). \tag{7}$$

The stress-energy tensor computed from Eq. (2), and its trace, are then given by

$$T^{\mu\nu} = \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}(\nabla_\alpha\phi\nabla^\alpha\phi - m^2\phi^2), \tag{8a}$$

$$T = -\nabla_\alpha\phi\nabla^\alpha\phi + 2m^2\phi^2. \tag{8b}$$

Consider a simple linear term, where

$$f_2(T) = -\frac{\kappa^2\chi T}{4\pi}. \tag{9}$$

We note that, just like the cosmological constant, this contribution can be thought of as a modification of gravity or as a contribution to the matter Lagrangian. If we view it as gravity, we find, using Eq. (6), that stress-energy is not conserved. We can view it as matter by defining a modified matter Lagrangian

$$\begin{aligned} \mathcal{L}_m' &= \mathcal{L}_m + \frac{\chi T}{8\pi} \\ &= \frac{1}{2}\left(1 - \frac{\chi}{4\pi}\right)\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\left(1 - \frac{\chi}{2\pi}\right)m^2\phi^2. \end{aligned} \tag{10}$$

We can then define the rescaled field and mass as

$$\phi' = \phi\sqrt{1 - \frac{\chi}{4\pi}}, \tag{11a}$$

$$m' = m\sqrt{\frac{4\pi - 2\chi}{4\pi - \chi}}, \tag{11b}$$

and the resultant modified matter Lagrangian reduces to

$$\mathcal{L}_m' = \frac{1}{2}(\nabla_\mu\phi'\nabla^\mu\phi' - m'^2\phi'^2), \tag{12}$$

which has the same form as the original matter Lagrangian. Hence we see that Eq. (9) simply rescales the field and mass. The "bare" mass m and field ϕ cannot be found in the full Lagrangian, and thus have no physical meaning. The stress-energy tensor Eq. (8a) is similarly meaningless. A modified stress-energy tensor, derived from the modified Lagrangian Eq. (12), will be conserved. The same reasoning applies to a free fermion or vector field. For more complicated functions $f_2(T)$, the resulting terms will of course not be simply a rescaling of the field, but will change the free field into an interacting field. It will be the contention of the next section that the incorporation of $f_2(T)$ into $\mathcal{L}_m$ works more generally, and such incorporation should always be performed, rendering $f_2(T)$ irrelevant.

IV. PERFECT FLUIDS

Consider a perfect fluid that has a stress-energy tensor defined in terms of the number density of particles n, the entropy per particle s, and the fluid's local velocity vector u^μ normalized so that $u_\mu u^\mu = 1$. The particle number and entropy must be conserved, so that

$$0 = \nabla_\mu (n u^\mu), \tag{13a}$$

$$0 = \nabla_\mu (s n u^\mu). \tag{13b}$$

The stress-energy tensor is given by

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}, \tag{14}$$

where $\rho = \rho(n, s)$ is the energy density and $p = p(n, s)$ is the pressure. If stress-energy is conserved, then using the equation $u_\nu \nabla_\mu T^{\mu\nu} = 0$, one can show that the energy density and pressure are related by

$$n \frac{\partial}{\partial n} \rho = \rho + p. \tag{15}$$

When stress-energy is not conserved, we can still use Eq. (15) as a definition of p. We will henceforth use this equation throughout our work without referencing it.

It is not immediately obvious how to write the matter Lagrangian for a perfect fluid. The literature commonly assumes $\mathcal{L}_m = p$ [7,9–12][2] without explaining where this relation comes from. As we will demonstrate shortly, it is *not* true in general. In the absence of non-standard gravity, it is derived, as done in [14], by taking the starting Lagrangian

$$\mathcal{L}_m = -\rho(n, s) + J^\mu (\beta_A \nabla_\mu \alpha^A - s \nabla_\mu \theta - \nabla_\mu \phi), \tag{16}$$

where $J^\mu = n u^\mu$, α^A are a set of index functions used to label fluid flow lines, and β_A, θ and ϕ are Lagrange multipliers used respectively to ensure that current flows along flow lines, entropy is not transferred, and current is conserved. The number density n is now to be interpreted as an implicit function of J^μ, given by

$$n = \sqrt{g_{\mu\nu} J^\mu J^\nu}, \tag{17}$$

and not as an independent variable. The stress-energy tensor and its trace, computed using Eq. (2) will be

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - g^{\mu\nu} \mathcal{L}_m, \tag{18a}$$

$$T = \rho + p - 4\mathcal{L}_m. \tag{18b}$$

[2]Harko *et al.* [7] has $\mathcal{L}_m = -p$ due to his error in Eq. (2). This was copied by other authors [9–11].

The equations of motion resulting from demanding stationarity of the Lagrangian with respect to all the fields (other than the metric) are then

$$0 = \nabla_\mu \{[1 + 2\kappa^{-2} f_2'(T)] J^\mu\}, \tag{19a}$$

$$0 = \nabla_\mu \{s[1 + 2\kappa^{-2} f_2'(T)] J^\mu\}, \tag{19b}$$

$$0 = [1 + 2\kappa^{-2} f_2'(T)] J^\mu \nabla_\mu \alpha^A, \tag{19c}$$

$$0 = -\nabla_\mu \{\beta_A [1 + 2\kappa^{-2} f_2'(T)] J^\mu\}, \tag{19d}$$

$$0 = -\left[1 + \frac{2}{\kappa^2} f_2'(T)\right] \left[\frac{\partial \rho}{\partial s} - J^\mu \nabla_\mu \theta\right] - \frac{1}{2\kappa^2} f_2'(T) \frac{\partial}{\partial s} (\rho + p), \tag{19e}$$

$$0 = \left[1 + \frac{2}{\kappa^2} f_2'(T)\right] \left(\beta_A \nabla_\mu \alpha^A - s \nabla_\mu \theta - \nabla_\mu \phi - \frac{\partial \rho}{\partial n} u_\mu\right) - \frac{1}{2\kappa^2} f_2'(T) u_\mu \frac{\partial}{\partial n} (\rho + p). \tag{19f}$$

We can then use Eq. (19f) to show that the Lagrangian density Eq. (16), when evaluated on shell, can be rewritten as

$$\bar{\mathcal{L}}_m = p + \frac{f_2'(\bar{T})}{2\kappa^2 + 4 f_2'(\bar{T})} n \frac{\partial}{\partial n} (\rho + p), \tag{20}$$

where the bars on $\mathcal{L}_m$ and T will be used to mean "on shell" henceforward.

If $f_2 = 0$, then we have $\bar{\mathcal{L}}_m = p$, as is commonly assumed, and Eqs. (19a) and (19b) will conserve particles and entropy, corresponding to Eqs. (13a) and (13b), and the stress-energy tensor Eq. (18a) will match the desired form Eq. (14). But when $f_2(T)$ is anything other than a constant, this goal is not achieved, and the various terms that were included in Eq. (16) have not achieved their intended goals. It appears that the actual conserved current should be the rescaled current $[1 + 2\kappa^{-2} f_2'(T)] J^\mu$.

It turns out to be more convenient to define the actual current as this taken on shell:

$$J'^\mu = [1 + 2\kappa^{-2} f_2'(\bar{T})] J^\mu, \tag{21a}$$

$$n' = [1 + 2\kappa^{-2} f_2'(\bar{T})] n, \tag{21b}$$

where the primes on J and n denote corrected quantities. The change has no physical significance as all physics takes place on shell. The on-shell trace of the stress-energy tensor $\bar{T}$ can be found as an implicit function of n and s by substituting Eq. (20) into Eq. (18b) to yield

$$\bar{T} = \rho - 3p - \frac{2 f_2'(\bar{T})}{\kappa^2 + 2 f_2'(\bar{T})} n \frac{\partial}{\partial n} (\rho + p). \tag{22}$$

Equation (21a) guarantees that particle number and entropy will be conserved on shell, i.e., $\nabla_\mu J'^\mu = 0$ and $\nabla_\mu (sJ'^\mu) = 0$. It is worth noting that neither the four-velocity u^μ nor the entropy per particle s needs to be redefined.

By analogy with the scalar field, we contend that the "bare" stress-energy tensor of Eq. (18a) is not only not conserved, it is not physically meaningful, because the separation of $\mathcal{L}$ into a matter term $\mathcal{L}_m$ and the contribution $f_2(T)$ is not physically meaningful. Only the combination of the effects of these two quantities can be measured, and for this reason we define the physical stress-energy tensor as

$$\begin{aligned} T'^{\mu\nu} &= T^{\mu\nu} + \frac{1}{\kappa^2\sqrt{-g}}\frac{\partial}{\partial g_{\mu\nu}}[\sqrt{-g}f_2(T)] \\ &= T^{\mu\nu} + \frac{1}{\kappa^2}f_2'(T)\frac{\partial T}{\partial g_{\mu\nu}} + \frac{1}{2\kappa^2}f_2(T)g^{\mu\nu}. \end{aligned} \tag{23}$$

It is easy to see from Eq. (6) that this quantity will be conserved.

The stress-energy trace T, as given by Eq. (18b) depends on the metric only by the implicit dependence of ρ and p on the number density $n = \sqrt{g_{\mu\nu}J^\mu J^\nu}$, which works out to

$$\frac{\partial T}{\partial g_{\mu\nu}} = \frac{1}{2}u^\mu u^\nu \left(4 + n\frac{\partial}{\partial n}\right)(\rho + p). \tag{24}$$

Substituting Eqs. (24) and (18a) into Eq. (23), we find the true stress-energy tensor is

$$\begin{aligned} T'^{\mu\nu} = &\left[\rho + p + \frac{1}{2\kappa^2}f_2'(T)\left(4 + n\frac{\partial}{\partial n}\right)(\rho + p)\right]u^\mu u^\nu \\ &- g^{\mu\nu}\left[\mathcal{L}_m - \frac{1}{2\kappa^2}f_2(T)\right]. \end{aligned} \tag{25}$$

By comparison with Eq. (18a), we see that the true energy density and pressure will be given by

$$\rho' + p' = \rho + p + \frac{1}{2\kappa^2}f_2'(T)\left(4 + n\frac{\partial}{\partial n}\right)(\rho + p), \tag{26a}$$

$$p' = \mathcal{L}_m - \frac{1}{2\kappa^2}f_2(T). \tag{26b}$$

The true density ρ' is the difference of these two equations, which can be simplified by using Eq. (18b) to yield

$$\begin{aligned} \rho' = &\frac{1}{4}(3\rho + 3p + T) + \frac{1}{2\kappa^2}f_2'(T)\left(4 + n\frac{\partial}{\partial n}\right)(\rho + p) \\ &+ \frac{1}{2\kappa^2}f_2(T). \end{aligned} \tag{27}$$

This equation has the disadvantage that the density is a function of all the field variables, not just n and s. We can correct this deficiency by replacing all the T's by $\bar{T}$'s. The formula can be further simplified by using Eq. (22) to replace

$$\frac{1}{2\kappa^2}f_2'(\bar{T})n\frac{\partial}{\partial n}(\rho + p) = \frac{1}{4}\left[1 + \frac{2}{\kappa^2}f_2'(\bar{T})\right](\rho - 3p - \bar{T}), \tag{28}$$

so we find on shell that

$$\rho' = \rho + \frac{1}{2\kappa^2}\left[f_2'(\bar{T})\left(4 + n\frac{\partial}{\partial n}\right)\rho + f_2(\bar{T}) - \bar{T}f_2'(\bar{T})\right]. \tag{29}$$

We have written the stress-energy tensor on shell strictly in terms of n and s, but can we somehow incorporate $f_2(T)$ into $\mathcal{L}_m$, so as to eliminate the need for $f_2(T)$ entirely? Let us define a modified Lagrangian by analogy with Eq. (16), using the corrected current J'^μ and density ρ':

$$\mathcal{L}' = \mathcal{L}_m' - \frac{1}{2\kappa^2}f_1(R), \tag{30a}$$

$$\mathcal{L}_m' = -\rho'(n', s) + J'^\mu(\beta_A\nabla_\mu\alpha^A - s\nabla_\mu\theta - \nabla_\mu\phi). \tag{30b}$$

This Lagrangian is not identical to the original Lagrangian. But will it yield the same equations of motion? The difference between the two Lagrangians is given by

$$\begin{aligned} \mathcal{L}' - \mathcal{L} &= \mathcal{L}_m' - \mathcal{L}_m + \frac{1}{2\kappa^2}f_2(T) \\ &= \rho - \rho' + (J'^\mu - J^\mu)(\beta_A\nabla_\mu\alpha^A - s\nabla_\mu\theta - \nabla_\mu\phi) \\ &\quad + \frac{1}{2\kappa^2}f_2(T). \end{aligned} \tag{31}$$

This is then simplified using sequentially Eqs. (21a), (16), (18b), and (29) to yield

$$\mathcal{L}' - \mathcal{L} = \frac{1}{2\kappa^2}[f_2(T) - f_2(\bar{T}) + (\bar{T} - T)f_2'(\bar{T})]. \tag{32}$$

When we apply the equations of motion, so $T = \bar{T}$, this difference vanishes. In fact, for small variations near the stationary point, we see that

$$\delta\mathcal{L}' - \delta\mathcal{L} = \frac{1}{2\kappa^2}\{\delta T[f_2'(T) - f_2'(\bar{T})] + (\bar{T} - T)\delta f_2'(\bar{T})\}. \tag{33}$$

On shell, this vanishes as well. Hence $\mathcal{L}$ and $\mathcal{L}'$ will have identical equations of motion.

Do the physical pressure p' and energy density ρ' satisfy Eq. (15)? Starting with Eq. (29), we see that

$$n'\frac{\partial\rho'}{\partial n'} = n'\frac{\partial\rho}{\partial n'} + \frac{1}{2\kappa^2}f_2'(\bar{T})n'\frac{\partial}{\partial n'}\left(4\rho + n\frac{\partial\rho}{\partial n}\right) + \frac{1}{2\kappa^2}\left[n'\frac{\partial}{\partial n'}f_2'(\bar{T})\right]\left(4\rho + n\frac{\partial\rho}{\partial n} - \bar{T}\right). \tag{34}$$

We can now use Eq. (21b) to show that

$$n'\frac{\partial}{\partial n'} = \left[1 - \frac{2}{\kappa^2 + 2f_2'(\bar{T})}n'\frac{\partial}{\partial n'}f_2'(\bar{T})\right]n\frac{\partial}{\partial n}. \tag{35}$$

Applying this to the first two terms of Eq. (34) and substituting Eq. (22) for $\bar{T}$ in the final term, yields, after considerable simplification,

$$n'\frac{\partial\rho'}{\partial n'} = \rho + p + \frac{1}{2\kappa^2}f_2'(\bar{T})\left(4 + n\frac{\partial}{\partial n}\right)(\rho + p). \tag{36}$$

Comparison with Eq. (26a) shows that on shell we have

$$n'\frac{\partial\rho'}{\partial n'} = \rho' + p'. \tag{37}$$

Indeed, we would expect this relationship, since the true stress-energy tensor $T'^{\mu\nu}$ is conserved.

V. CONCLUSIONS AND GENERALIZATION

As we have demonstrated, when $f(R,T)$ gravity can be broken into a curvature term and a stress-energy term, $f(R,T) = f_1(R) + f_2(T)$, we can incorporate $f_2(T)$ into $\mathcal{L}_m$ so as to eliminate the need for $f_2(T)$ entirely, and therefore $f_2(T)$ is not physically meaningful. We demonstrated this explicitly for both a free scalar field and a generalized perfect fluid. But sources [10,12] have claimed to put limits on $f_2(T)$; specifically, on the parameter χ that appears in the linear case Eq. (9). If it is physically meaningless, how can such limits be obtained?

These papers have made two errors. First, they assume that $\mathcal{L}_m = p$, even though this formula does not generally apply. Secondly, they identify ρ and p with the physical energy density and pressure. This error is most clear in [10], which uses the equation of state for a degenerate electron gas. The presence of a term of the form Eq. (9) simply rescales the field and mass for a fermion, and hence the equation of state (once rescaled masses are used) for the electron gas will be unchanged. Similarly, in [12] the ideal gas law is used, but this ideal gas law should be applied to the physical energy density and pressure, not the "bare" energy density and pressure.

Can we generalize these conclusions to generic $f(R,T)$ gravity? Consider, for example, a perfect fluid, for which Eqs. (19a)–(19f) need to be modified by replacing $f_2'(T)$ with $f_T(R,T) = \frac{\partial}{\partial T}f(R,T)$. This would suggest that the physical current, analogous to Eq. (21a) should be defined as

$$J'^{\mu} = [1 + 2\kappa^{-2}f_T(R,\bar{T})]J^{\mu}. \tag{38}$$

We note that this would result in a number density depending on the curvature, and this in turn would result in an energy density and pressure that also depend on the curvature. Unlike $f_2(T)$, cross terms in $f(R,T)$ will yield new physics, and limits on such terms could be placed by comparison with observations.

At the least, it seems sensible that terms in $f(R,T)$ that do not depend on curvature should be incorporated into $\mathcal{L}_m$, so that we could define

$$\mathcal{L}_m' = \mathcal{L}_m - \frac{1}{2\kappa^2}f(0,T). \tag{39}$$

After all, $f(0,T)$ terms represent the behavior of matter in the absence of curvature, and hence should not be considered part of the gravitational Lagrangian. This is exactly what we did in Eq. (12) for a scalar field. For a perfect fluid, this was not exactly what we did, but Eq. (32) shows that it matches what we did on-shell and to first order nearly on-shell. This does not give us sufficient insight about how to deal with cross terms, so we do not expect that Eq. (39) would represent the correct "physical" Lagrangian. General $f(R,T)$ gravity is therefore a focus of our ongoing research.

[1] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiattia, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner, B. Leibundgut, M. M. Phillips, D. Reiss, B. P. Schmidt, R. A. Schommer, R. C. Smith, J. Spyromilio, C. Stubbs, N. B. Suntzeff, and J. Tonry, Astron. J. **116**, 1009 (1998).
[2] S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998).
[3] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999).
[4] J. A. Frieman, M. S. Turner, and D. Huterer, Annu. Rev. Astron. Astrophys. **46**, 385 (2008).
[5] H. A. Buchdahl, Mon. Not. R Astron. Soc. **150**, 1 (1970).
[6] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[7] T. Harko, F. S. N. Lobo, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **84**, 024020 (2011).
[8] R. Zaregonbadi, M. Farhoudi, and N. Riazi, Phys. Rev. D **94**, 084052 (2016).

[9] H. Velten and T. R. P. Carams, Phys. Rev. D **95**, 123536 (2017).
[10] G. A. Carvalho, R. V. Lobato, P. H. R. S. Moraes, J. D. V. Arbail, R. M. Marinho, Jr., E. Otoniel, and M. Malheiro, Eur. Phys. J. C **77**, 871 (2017).
[11] D. Deb, F. Rahaman, S. Ray, and B. K. Guha, J. Cosmol. Astropart. Phys. 03 (2018) 044.
[12] T. Ordines and E. Carlson, Phys. Rev. D **99**, 104052 (2019).
[13] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, CA, 1973).
[14] J. D. Brown, Classical Quantum Gravity **10**, 1579 (1993).