Charged Taub-NUT solution in Lovelock gravity with generalized Wheeler polynomials

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Wheeler's approach to finding exact solutions in Lovelock gravity has been predominantly applied to static spacetimes. This has led to a Birkhoff theorem for arbitrary base manifolds in dimensions higher than four. In this work, we generalize the method and apply it to a stationary metric. Using this perspective, we present a Taub-NUT solution in eight-dimensional Lovelock gravity coupled to Maxwell fields. We use the first-order formalism to integrate the equations of motion in the torsion-free sector. The Maxwell field is presented explicitly with general integration constants, while the background metric is given implicitly in terms of a cubic algebraic equation for the metric function. We display precisely how the NUT parameter generalizes Wheeler polynomials in a highly nontrivial manner.

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I. INTRODUCTION

Higher-dimensional theories appear in different contexts of theoretical physics. For instance, an important open problem is the question about the enormous difference between the Planck and the electroweak scales. An attempt to deal with this hierarchy problem consists in considering field theories with extra spatial dimensions [1,2]. For the additional dimensions, the corresponding field equations must be generalized by including higher-curvature terms in the action. These terms appear also in the renormalization approach of quantum field theory in curved spacetimes [3] or in the low-energy limit of string theory [4]. The AdS/CFT correspondence [5–7], on the other hand, is an additional motivation to study gravity in higher dimensions, since it provides a nonperturbative approach to strongly coupled systems by means of a weakly coupled gravitational dual within an extradimensional spacetime. This evidence indicates that gravitational theories with extra dimensions possessing higher-order curvature terms may have important applications in the context of quantum field theory and theoretical physics, in general.

In the case of gravity, the Lanczos-Lovelock theory is the natural generalization of general relativity (GR) in higher dimensions [8,9]. The corresponding action principle is endowed with higher-curvature terms, while sharing some of the main features of GR: namely, (i) it is invariant under local Lorentz transformations and diffeomorphisms, (ii) it is torsion free, and (iii) it yields second-order field equations for the metric. This theory is free of ghosts [10], and it has the same degrees of freedom as the Einstein-Hilbert action in any dimension [11]. The first nontrivial term of the Lovelock series, i.e., the Gauss-Bonnet term, appears as a low-energy correction of string theory [12], modifying the field equations in dimensions higher than four. In fact, several static exact solutions have been found in this scenario [13–23], some of which have not been studied from a thermodynamic viewpoint or any of its more recent extensions. Although in four dimensions the Gauss-Bonnet term does not contribute to the field equations since it is a topological invariant proportional to the Euler characteristic class, its inclusion becomes relevant in the regularization of conserved charges in asymptotically locally AdS spacetimes [24] and in the context of holographic renormalization [25]. Moreover, the AdS/CFT correspondence has been used to impose bounds on the shear-viscosity-to-entropy ratio for supersymmetric CFT by considering the Lanczos-Lovelock theory as its gravitational dual [26,27]. Quantum anomalies, on the other hand, have been computed from the holographic principle in Lanczos-Lovelock gravity, showing that the Weyl and a particular non-Abelian asymptotic symmetry are broken at the quantum level on the dual CFT [28]. Remarkably, when the theory has a unique AdS vacuum, there exists a gauge fixing that leads to a finite Fefferman-Graham expansion [29].

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One aspect of higher-dimensional gravity which is interesting for the present investigation is the (non)uniqueness of static black holes [30,31]. Indeed, consider the line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Sigma^2, \tag{1}$$

where $\mathrm{d}\Sigma^2$ is the metric of an arbitrarily chosen codimension-two submanifold, henceforth referred to as the base manifold. In fact, the only static black hole in higher-dimensional GR which is asymptotically flat is given by the Schwarzschild-Tangherlini metric, whose base manifold is a round hypersphere. However, nonasymptotically flat solutions are obtained for different base manifolds, although the field equations imply that it must be an Einstein manifold. In Eq. (1), the geometry of the Einstein manifold is parametrized so that its Ricci scalar coincides with that of a hypersphere with the same dimension; this fact is closely related to the higher-dimensional Birkhoff theorem. The Lovelock version of this result also imposes conditions on the base manifold. Nevertheless, they no longer need to be Einstein manifolds, and a number of new geometries come into the fold.

Returning to Eq. (1), when spherical symmetry is assumed, Wheeler devised an approach to determine the metric function f [32]. The differential equation for the metric function is integrated in an elementary way. Remarkably, this method yields an algebraic equation for $f = f(r)$. Moreover, defining an auxiliary function by $\mathcal{F} = (1-f)/r^2$, the general result is that

$$P(\mathcal{F}) \equiv \sum_{i=0}^{p} a_i \mathcal{F}(r)^i = \frac{M}{r^{D-1}}. \tag{2}$$

This polynomial in $\mathcal{F}$ has constant coefficients a_i determined by the Lovelock coupling constants, D is the spacetime dimension, $p = [(D-1)/2]$ is the highest-order curvature term contributing to the field equations, and the squared brackets denote the integer part. In Eq. (2), M is the integration constant, which is later related to the black hole mass. The function P is what has been dubbed the Wheeler polynomial of the solution. This showcases how incredibly restrictive spherical symmetry is. A family of p spacetimes are uniquely determined by the roots of Eq. (2). Of course, in higher dimensions, the exact solutions are increasingly more complex, and the lack of closed form begins in 11 dimensions for Lovelock gravity. Nonetheless, some general results have been proven to hold for this set of solutions. For instance, a solution always exists for at least one value of the sign of M [32]. Moreover, the extension of the asymptotic solution increases monotonically as r decreases, until it ends for small values of r in one of the following two possibilities: either a curvature singularity at the origin is surrounded by exactly one event horizon or the singularity happens at a finite value of r, where at most one event horizon is present. Notice that this includes the possibility of a naked singularity.

All maximally symmetric spaces are equally restrictive. The topological versions of these solutions are determined by Wheeler polynomials as well, but with the auxiliary function redefined as $\mathcal{F} = (\kappa - f)/r^2$, where κ is the spatial sectional curvature. However, the most general admissible base manifolds require the use of an analogue of Wheeler's polynomial defined by [31]

$$Q(U) \equiv \sum_{k=0}^{p} b_k r^{-2k} A_k(U) = \frac{M}{r^{D-1}}, \tag{3}$$

where the constants b_k depend on the geometry of the base manifold, and the auxiliary function is defined as $U = -f/r^2$. Wheeler polynomials [Eq. (2)] are rearrangeable as just above; e.g., spaces of constant sectional curvature κ have constants $b_s = \kappa^s$. The polynomials A_k are of order $p - k$ and are defined by

$$A_k(U) \equiv \sum_{i=0}^{p} \binom{i}{k} a_i U^{i-k}, \tag{4}$$

where the a_i's are, as before, the coefficients in Eq. (2). Notice that the highest-order polynomial is $A_0 = P(U)$ and that the polynomials comply with the recurrence relation $A'_k = (k+1)A_{k+1}$.

Even outside the context of determining exact solutions, Wheeler polynomials provide a remarkable theoretical tool to investigate gravitational physics. Equation (3), for instance, provides a way for black hole thermodynamics to be carried out even when a closed form for f is not available [33,34]. In essence, this can be carried out because black holes have event horizons characterized by the vanishing of the metric function. Hence, the Wheeler polynomial may be evaluated in the null hypersurface to yield an important algebraic relation. Taking the differentiation of the polynomial and restricting to the horizon determines the Hawking temperature. Of course, it is crucial to relate the integration constant with the physical parameters of the solution, especially the mass of the black hole. The relation between gravitational parameters and thermodynamical ones allows for a vast class of scenarios to be explored in this direction. However, staticity need not limit this line of research.

The original Taub and Newman-Tamburino-Unti metrics [35,36]—hereafter referred to as Taub-NUT—have motivated a plethora of investigations in gravitational physics. A particular research area is spacetime thermodynamics, where the similarities between Taub-NUT metrics and black holes have been studied through Euclidean techniques. Relying on the methods of finite-temperature quantum field theory, an analytical continuation of the metric is performed, and the period of the Euclidean time circle is chosen in such a way that no conical singularity is present. The action of the $U(1)$ isometry group, in general, has a set of fixed points which comes from the Killing horizon in the Lorentzian sheet. If the set is zero dimensional, the analytically continued sheet is called Taub-NUT; otherwise, it is dubbed Taub-Bolt. Possible observational signatures of this spacetimes have been studied in Ref. [37].

Higher-dimensional Taub-NUT and Taub-Bolt metrics are a special type of inhomogeneous geometry on complex line bundles over a Kähler manifold [38–40]. Thus, they exist only in even dimensions. These metrics have Lorentzian counterparts which in the static limit coincide with Eq. (1); in this case, the base manifold is Kähler. In fact, Taub-NUT geometries, in Boyer-Lindquist coordinates, quite resemble the line element (1). This, in turn, implies that the Wheeler approach is applicable to these stationary spacetimes as well. Of course, the method is blind to whether the metric is Lorentzian or not. These spacetimes carry a gravitational charge which in many ways is analogous to a magnetic monopole moment (for a recent discussion, see Ref. [41]). An important example are the famous Kaluza-Klein monopoles [42], where the Euclidean Taub-NUT space is used as a seed manifold. Both the Taub-NUT solution and the Kaluza-Klein monopole have rich geometric structures which have led to applications in GR [43,44] and string theory [45], as well as insights in differential geometry [46,47]. In a complementary manner, the Taub-Bolt space has a very interesting topological structure which closely resembles Euclidean black holes. This resemblance has allowed for the construction of holographic heat engines [48]. It also allows for the space to possess electromagnetic fields which generalize the Dirac monopole field [49]. Taub-NUT metrics have been found to exist in a wide range of vacuum and electrovacuum gravitational theories, which include, but are not limited to, the Lanczos-Lovelock-Maxwell theory [50–56].

In this work, we revisit the eight-dimensional Lovelock theory, where solutions in a closed form for arbitrary coupling constants are already intractable [57]. This framework is extended by considering arbitrary coefficients for the Lovelock series and by adding minimally coupled Maxwell fields with general integration constants. For the sake of comparison, we use the same *Ansätze* of Ref. [58] for the metric and Maxwell fields, which can be found in Eqs. (23) and (26), respectively. The charged Taub-NUT solution in cubic Lovelock gravity—the main result of this work—is presented as a root of the Wheeler polynomial in Eq. (3), given by $U(r) = -f(r)/r^2$, where $f(r)$ is the metric function appearing in Eq. (23). The latter is determined by a generalization of Eq. (3)—that is,

$$Q_n(U) \equiv \sum_{k=0}^{p} b_k r^{-2k} B_k(U) = \frac{M}{r^{D-1}}, \tag{5}$$

where $B_k(U)$ is a deformation of $A_k(U)$ by warping functions which depend on the NUT parameter n. When the latter vanishes, we recover Eq. (3), i.e., $Q_0 = Q$. Notice that in eight dimensions the polynomial is cubic—namely, $p = 3$. The field equations are solved by means of the first-order formalism, focusing on the torsion-free sector of the space of solutions. To the best of the authors' knowledge, this result represents the first Wheeler-like polynomial for Taub-NUT spacetimes in Lovelock gravity.

The article is organized as follows: In Sec. II, we present the eight-dimensional Lanczos-Lovelock theory coupled to Maxwell fields and their field equations. In Sec. II A, we restrict ourselves to lower orders in the Lovelock series and write the (analogue) Wheeler polynomials for a spherical and complex projective base manifold. This explicitly shows that, although one may freely parametrize the base manifold to set $b_1 = 1$, other b_k coefficients cannot be arbitrarily fixed, in contrast to the Einstein case for higher-order theories. In Sec. III, the higher-dimensional *Ansatz* is presented together with lower-dimensional Taub-NUT Wheeler polynomials which represent a generalization relative to Eq. (3). In Sec. IV, we report and discuss the charged Taub-NUT solution with arbitrary coefficients of the Lovelock series. Finally, conclusions and further discussions are given in Sec. V. The Appendix has been included for additional details of the computation. In our notation, greek and latin characters denote spacetime and Lorentz indices, respectively; the Minkowski metric is $\eta_{ab} = \mathrm{diag}(-, +, \ldots, +)$; and the language of differential forms will be used from here onwards.

II. EIGHT-DIMENSIONAL LANCZOS-LOVELOCK GRAVITY

In this work, we use the first-order formalism to treat Lovelock's gravity [59]. This is done by considering the vielbein $e^a = e^a{}_\mu \mathrm{d}x^\mu$ and the Lorentz connection $\omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu$ 1-forms as independent gravitational fields. The former is related to the spacetime metric through $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where η_{ab} is the Minkowski metric, while the latter allows us to perform the parallel transport of Lorentz-valued p-forms over the spacetime manifold. The curvature and torsion 2-forms are defined through the Cartan structure equations

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \frac{1}{2} R^{ab}{}_{cd} e^c \wedge e^d, \tag{6}$$

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c, \tag{7}$$

where $\wedge$ is the wedge product, d is the exterior derivative, and D is the Lorentz-covariant exterior derivative with respect to $\omega^a{}_b$. These fields satisfy the Bianchi identities $\mathrm{D}T^a = R^a{}_b \wedge e^b$ and $\mathrm{D}R^{ab} = 0$.

The eight-dimensional Lovelock theory coupled to $U(1)$ gauge fields $A = A_\mu \mathrm{d}x^\mu$—the theory we are interested in throughout this work—is described by the action principle

$$S[e^a, \omega^{ab}, A] = S_g + S_m, \tag{8}$$

where the gravity and matter action are denoted by S_g and S_m, respectively, and they are considered to be minimally coupled. The Lovelock action functional is given by

$$\begin{aligned} S_g = \int \epsilon_{abcdefgh}\Big(&\frac{\alpha_0}{8!} e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^e \wedge e^f \\ &+ \frac{\alpha_1}{6!} R^{ab} \wedge e^c \wedge e^d \wedge e^e \wedge e^f \\ &+ \frac{\alpha_2}{4!} R^{ab} \wedge R^{cd} \wedge e^e \wedge e^f \\ &+ \frac{\alpha_3}{2!} R^{ab} \wedge R^{cd} \wedge R^{ef}\Big) \wedge e^g \wedge e^h, \end{aligned} \tag{9}$$

with $\epsilon_{01234567} = 1$ (for a discussion of Lovelock gravity in terms of spacetime components, see Ref. [9]). Notice that in eight dimensions, this theory admits a quartic term in the curvature 2-form. However, it represents the dimensional continuation of the Euler density, and it does not contribute to the vielbein dynamics on the bulk. The Lovelock action is conformed by a series of dimensionally continued Euler densities. For a given dimension, the series terminates according to the differential form of maximum degree. In addition to the gravitational sector, we write the Maxwell action functional as

$$S_m = -\frac{1}{2}\int F \wedge \star F. \tag{10}$$

Here, $\star$ denotes the Hodge dual and $F = \mathrm{d}A$ is the field strength of the $U(1)$ gauge fields.

The field equations of this theory are obtained by performing stationary variations with respect to the vielbein, Lorentz connection, and $U(1)$ gauge fields, leading to

$$\begin{aligned} 0 = \epsilon_{abcdefgh}\Big(&\frac{\alpha_0}{7!} e^b \wedge e^c \wedge e^d \wedge e^e \wedge e^f \wedge e^g \\ &+ \frac{\alpha_1}{5!} R^{bc} \wedge e^d \wedge e^e \wedge e^f \wedge e^g \\ &+ \frac{\alpha_2}{3!} R^{bc} \wedge R^{de} \wedge e^f \wedge e^g \\ &+ \alpha_3 R^{bc} \wedge R^{de} \wedge R^{fg}\Big) \wedge e^h - \tau_a, \end{aligned} \tag{11}$$

$$\begin{aligned} 0 = \epsilon_{abcdefgh}\Big(&\frac{\alpha_1}{5!} e^c \wedge e^d \wedge e^e \wedge e^f + \frac{\alpha_2}{3} R^{cd} \wedge e^e \wedge e^f \\ &+ 3\alpha_3 R^{cd} \wedge R^{ef}\Big) \wedge T^g \wedge e^h, \end{aligned} \tag{12}$$

$$0 = \mathrm{d}\star F, \tag{13}$$

respectively, where we have defined the energy-momentum 7-form of the gauge fields as

$$\tau_a = \frac{1}{2}(F \wedge \star(e_a \wedge F) - i_a F \wedge \star F), \tag{14}$$

with i_a being the inner contraction along the vector field $E_a = E^\mu{}_a \partial_\mu$ such that $e^a{}_\mu E^\nu{}_a = \delta^\nu_\mu$ and $e^a{}_\mu E^\mu{}_b = \delta^a_b$. The Noether theorem associated with the invariance under diffeomorphisms [60–62] implies that the energy-momentum 7-form in Eq. (14) satisfies the conservation law:

$$\mathrm{D}\tau_a = i_a T^b \wedge \tau_b. \tag{15}$$

Invariance under local Lorentz transformations, on the other hand, implies a conservation law that is trivially satisfied for Maxwell fields. It is worth mentioning that the Bianchi identities impose severe restrictions on the torsion components when arbitrary coefficients of the Lovelock series are considered in vacuum [63]. These restrictions can be avoided if the coefficients are chosen in such a way that the action principle can be written as the Chern-Simons form for the (A)dS group or as Born-Infeld gravity in odd and even dimensions, respectively. This implies that the theory has the maximum number of degrees of freedom [63].[1] Here, we consider arbitrary coefficients of the Lovelock series and focus our attention on the torsion-free sector of the space of solutions, namely $T^a = 0$, which automatically solves Eq. (12). This condition allows one to solve the Lorentz connection in terms of the vielbein, reducing its form to the standard Levi-Civita connection. Thus, the solution presented here belongs to the Riemannian branch of the Lovelock theory, even though vacuum solutions with nontrivial torsion have been reported for different isometry groups in Refs. [64–69].

A. Lower-order Wheeler polynomials

Before going on to compute the Wheeler polynomial for the Taub-NUT solution in eight-dimensional Lovelock-Maxwell theory, it is useful to summarize the lower-order solutions in the static limit. They portray how the original Wheeler polynomials [32], which consider spherical symmetry, are generalized to the Kähler case. In the next section, we explain why we specialize to the case where the base space is complex projective.

Let us focus on vacuum Einstein-Gauss-Bonnet theory with a cosmological constant Λ and a Gauss-Bonnet coupling constant α_{GB}. This fixes the coupling constants in Eq. (9) in terms of these last two parameters, and in particular sets $\alpha_3 = 0$. In arbitrary spacetime dimension D, the Wheeler polynomial (2) is

[1] In fact, the choice of the coefficients such that the action can be written in a Born-Infeld form has been used in Ref. [57] to obtain the uncharged Taub-NUT solution in third-order Lovelock theory. For the sake of generality, the analysis presented in this work does not assume any relation on the parameters whatsoever.

$$-\frac{2\Lambda}{(D-1)(D-2)}+\mathcal{F}+(D-3)(D-4)\alpha_{\mathrm{GB}}\mathcal{F}^2=\frac{M}{r^{D-1}}. \tag{16}$$

This equation yields the Boulware-Deser solution [13] and, setting $\alpha_{\mathrm{GB}}=0$, leads to the familiar Schwarzschild-Tangherlini result

$$f(r)=1-\frac{M}{r^{D-3}}-\frac{2\Lambda r^2}{(D-1)(D-2)}. \tag{17}$$

For comparative reasons, we rewrite this result in the form of Eq. (3), which in eight dimensions is

$$A_0+r^{-2}A_1+r^{-4}A_2=\frac{M}{r^7}, \tag{18}$$

with polynomials $A_k(U)$ given by

$$A_0(U)=-\frac{\Lambda}{21}+U+20\alpha_{\mathrm{GB}}U^2, \tag{19}$$

$$A_1(U)=1+40\alpha_{\mathrm{GB}}U, \tag{20}$$

$$A_2(U)=20\alpha_{\mathrm{GB}}, \tag{21}$$

recalling that $U=-f/r^2$. If we now substitute the base manifold from a hypersphere S^6 to a complex projective space $\mathbb{CP}^3$, then the previous polynomials remain unchanged, but the equivalent to Eq. (18) is

$$A_0+r^{-2}A_1+2r^{-4}A_2=\frac{M}{r^7}. \tag{22}$$

Recall that the coefficients b_k in Eq. (3) depend on the geometry of the base manifold. Since the complex projective spaces with the Fubini-Study metric are Einstein manifolds, the results of Ref. [30] imply that Eqs. (18) and (22) only differ in the coefficient b_2, once the parametrization convention of this reference is adopted. Additionally, we mention that the Taub-NUT solution found in Ref. [70] has as its static limit the black hole determined by Eq. (22). In the next section, we discuss how the NUT parameter generalizes polynomials such as the ones presented above.

III. HIGHER-DIMENSIONAL TAUB-NUT GEOMETRY

The definition of a higher-dimensional Taub-NUT space we consider here is given by the family of inhomogeneous geometries built over complex line bundles presented in Ref. [39]; this is

$$\mathrm{d}s^2=f(r)(\mathrm{d}\tau+2nB)^2+\frac{\mathrm{d}r^2}{f(r)}+(r^2-n^2)\mathrm{d}\Sigma^2, \tag{23}$$

where τ is the Euclidean time coordinate and n is the NUT parameter. This parameter sources the magnetic part of the Weyl tensor, and it is, in general, related to the magnetic mass of the geometry [71,72]. Notice that, for $n\to 0$, we recover a metric equivalent to Eq. (1), which is a static metric modulo a Wick rotation. The line element $\mathrm{d}\Sigma^2$ is Kähler, and its associated symplectic form is given by $\omega=\mathrm{d}B$. The original Taub-NUT solutions are the special case where the base manifold is a sphere S^2, which coincides with the complex projective line $\mathbb{CP}^1$. Thus, the static limit leads to a spherically symmetric spacetime. This is the particular case in four dimensions, since no hypersphere admits a Kähler structure [73]. We will specialize to higher-dimensional Taub-NUT solutions with hyperspherical boundary conditions. These are the only ones which admit non-singular Euclidean sheets with nuts [38]. This, in turn, implies that they are the only conditions under which Hawking-Page-like [74] phase transitions are possible [75]. As for Eq. (3), there is no greater loss of generality than variation of its coefficients. These boundary conditions imply a Hopf fibration of the Euclidean time direction over a complex projective space. Hence, we fix the geometry of the base manifold to that of Fubini-Study. For the complex projective space of real dimension $2k$, our notation is $B=\mathcal{A}_k$, and we add a subscript k to the line element in Eq. (23) to indicate that it is the Fubini-Study metric on $\mathbb{CP}^k$.

An iterative construction of the Fubini-Study metric using explicitly real expressions is useful [76]. We write the recursion relation as

$$B=\mathcal{A}_k=(k+1)\sin^2\psi_k\left(\mathrm{d}\phi_k+\frac{1}{k}\mathcal{A}_{k-1}\right), \tag{24}$$

$$\mathrm{d}\Sigma_k^2=2(k+1)\left[\mathrm{d}\psi_k^2+\sin^2\psi_k\cos^2\psi_k\left(\mathrm{d}\phi_k+\frac{1}{k}\mathcal{A}_{k-1}\right)^2+\frac{1}{2k}\sin^2\psi_k\mathrm{d}\Sigma_{k-1}^2\right]. \tag{25}$$

Notice how the metric on the $\mathbb{CP}^k$ manifold is built on top of the one on the $\mathbb{CP}^{k-1}$ submanifold. This submanifold is in fact totally geodesic, or extrinsically flat. In these coordinates, $\psi_k=\pi/2$ corresponds exactly to this special submanifold. This fact is commented on further below.

The four-dimensional charged Taub-NUT solution [50,51] possesses a Maxwell field whose null directions are aligned with the repeated principal null directions of the Weyl tensor. In this spirit, we choose

$$A=h(r)(\mathrm{d}\tau+2nB), \tag{26}$$

as the *Ansatz* for the gauge potential. This form of the gauge field was used in a higher-dimensional setting for the first time in Ref. [55]. Moreover, even without an explicit form of the metric function f in Eq. (23), we notice that the Maxwell equation [Eq. (13)] can be solved independently. In other words, Maxwell's equations together with the *Ansatz* (26) yield a differential equation for h, namely

$$h''(r^2-n^2)^2+(D-2)[r(r^2-n^2)h'-2n^2h]=0, \tag{27}$$

where the prime denotes a derivative with respect to the coordinate r. This equation admits the general solution

$$h(r)=\frac{qr}{(r^2-n^2)^k}+\frac{vW_k}{(1-n^2/r^2)^k}, \tag{28}$$

where q and v are integration constants and W_k denotes the series

$$W_k\equiv\sum_{i=0}^{k}\binom{k}{i}\frac{2k-1}{2i-1}\left(-\frac{n^2}{r^2}\right)^{k-i}. \tag{29}$$

Notice that it resembles the binomial expansion

$$\left(1-\frac{n^2}{r^2}\right)^k=\sum_{i=0}^{k}\binom{k}{i}\left(-\frac{n^2}{r^2}\right)^{k-i}. \tag{30}$$

The function W_k may be generated, if so desired, by an integral formula. It may also be written in terms of Legendre polynomials or a hypergeometric function by setting the appropriate parameters. To illustrate how Wheeler polynomials are generalized by NUT parameters, we present the special cases of Lovelock Taub-NUTs in four dimensions given by

$$Q_n(U)=-\frac{\Lambda}{3}W_2+\left(1-\frac{n^2}{r^2}\right)U+r^{-2}W_1, \tag{31}$$

and in six dimensions by

$$\begin{aligned}Q_n(U)=&-\frac{\Lambda}{10}W_3+\left(1-\frac{n^2}{r^2}\right)^2U+6\alpha_{\mathrm{GB}}W_1U^2\\&+r^{-2}\left[W_2+12\alpha_{\mathrm{GB}}\left(1-\frac{n^2}{r^2}\right)U\right]\\&+2r^{-4}[6\alpha_{\mathrm{GB}}W_1].\end{aligned} \tag{32}$$

Recall that Q_n has been defined in Eq. (5). Equations (29) and (30) are, in fact, the deformation elements of the Wheeler polynomials [Eq. (3)] when the NUT parameter is turned on. It should be noted that, when $n\to 0$, both series become unity. Recall that in four dimensions, the base manifold is the complex projective line, while in six dimensions, it is the complex projective plane.

IV. CHARGED EIGHT-DIMENSIONAL SOLUTION

We are now in a position to present the charged eight-dimensional solution which fits within *Ansatz* (23). To this end, we use a generalized Wheeler polynomial. The base manifold is the complex projective space $\mathbb{CP}^3$. The Euclidean time direction is Hopf-fibered over this base space, resulting in $r=$ constant hypersurfaces wich are hyperspheres S^7. The isometry algebra of the total space is $\mathfrak{su}(4)\oplus\mathfrak{u}(1)$, and the topology will either be Euclidean, if it has a nut, or complex projective minus a point, if it possesses a bolt.

The explicitly real Fubini-Study metric on the base manifold may be found by Eqs. (24) and (25); thus,

$$\mathcal{A}_1=2\sin^2\psi_1\mathrm{d}\phi_1, \tag{33}$$

$$\mathrm{d}\Sigma_1^2=4[\mathrm{d}\psi_1^2+\sin^2\psi_1\cos^2\psi_1\mathrm{d}\phi_1^2], \tag{34}$$

$$\mathcal{A}_2=3\sin^2\psi_2\left(\mathrm{d}\phi_2+\frac{\mathcal{A}_1}{2}\right), \tag{35}$$

$$\mathrm{d}\Sigma_2^2=6\left[\mathrm{d}\psi_2^2+\sin^2\psi_2\cos^2\psi_2\left(\mathrm{d}\phi_2+\frac{\mathcal{A}_1}{2}\right)^2+\frac{1}{4}\sin^2\psi_2\mathrm{d}\Sigma_1^2\right], \tag{36}$$

$$\mathcal{A}_3=4\sin^2\psi_3\left(\mathrm{d}\phi_3+\frac{\mathcal{A}_2}{3}\right), \tag{37}$$

$$\mathrm{d}\Sigma_3^2=8\left[\mathrm{d}\psi_3^2+\sin^2\psi_3\cos^2\psi_3\left(\mathrm{d}\phi_3+\frac{\mathcal{A}_2}{3}\right)^2+\frac{1}{6}\sin^2\psi_3\mathrm{d}\Sigma_2^2\right]. \tag{38}$$

We choose the vielbein basis, as shown in the Appendix. Since we are looking for torsion-free solutions, the Lorentz connection can be solved in terms of the vielbein by solving $\mathrm{d}e^a+\omega^a{}_b\wedge e^b=0$. The 2-form curvature associated with this connection can be computed from the first Cartan equation (6). However, due to the cumbersome nature of its components, we report them in the Appendix. Moreover, we write the field strength in the following manner:

$$F=\mathrm{d}A=F_Ie^0\wedge e^1+F_{II}(e^2\wedge e^3+e^4\wedge e^5+e^6\wedge e^7), \tag{39}$$

with

$$F_I=-h' \quad \text{and} \quad F_{II}=\frac{2nh}{r^2-n^2}. \tag{40}$$

Here, $h(r)$ is the function defined in Eq. (26). The $\mathfrak{so}(1,7)$-valued energy-momentum 7-form [Eq. (14)] for this *Ansatz* yields

$$\tau_0 = -\rho e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7, \qquad (41)$$

$$\tau_1 = \rho e^0 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7, \qquad (42)$$

$$\tau_{\bar a} = p\epsilon_{\bar a \bar b \bar c \bar d \bar e \bar f} e^0 \wedge e^1 \wedge e^{\bar b} \wedge e^{\bar c} \wedge e^{\bar d} \wedge e^{\bar e} \wedge e^{\bar f}, \qquad (43)$$

where $\bar a = 2, \ldots, 7$ are indices of $\mathrm{d}\Sigma_3^2$ such that $\epsilon_{234567} = 1$, and

$$\rho_M = \frac{F_I^2 - 3F_{II}^2}{2} \quad \text{and} \quad p_M = \frac{F_I^2 + F_{II}^2}{2}. \qquad (44)$$

Although we know the solution for the Maxwell field beforehand, we mention that the Maxwell equation takes the form

$$F_I'(r^2 - n^2) + 6(rF_I + nF_{II}) = 0, \qquad (45)$$

whose explicit solution is [cf. Eq. (28)]

$$h(r) = \frac{1}{(r^2 - n^2)^3}[qr + v(r^6 - 5n^2r^4 + 15n^4r^2 + 5n^6)]. \qquad (46)$$

In our notation, this corresponds to

$$F_I = \frac{v(60n^6r + 40n^4r^3 - 4n^2r^5) + q(5r^2 + n^2)}{(r^2 - n^2)^4}, \qquad (47)$$

$$F_{II} = \frac{2n[v(5n^6 + 15n^4r^2 - 5n^2r^4 + r^6) + qr]}{(r^2 - n^2)^4}. \qquad (48)$$

Examining the asymptotic behavior of the field strength reveals q to be the electric charge up to some rescaling. The other integration constant v can be interpreted as the value of the electric potential at infinity [55]. For the gauge potential to be regular at the nut (or bolt, respectively), where the Euclidean time direction degenerates, it must be null there. So v is, in fact, a potential difference across the entire manifold. Furthermore, there is a topological interpretation of v which endows it with a magnetic flavor [41].

This Maxwell field naturally lives in a principal $U(1)$ bundle over the Euclidean background. The bundle's connection is locally represented by the gauge potential. This circle bundle is classified by a single topological index, which can be calculated by integrating over the background. If the background has a nut, then the index vanishes. In the complementary case, we have

$$c = \frac{1}{16\pi^4}\int F \wedge F \wedge F \wedge F = (8nv)^4. \qquad (49)$$

So, we see that v and n are related to a topological invariant of the underlying bundle space. However, the circle bundle just described possesses a principal $U(1)$ sub-bundle defined over the unique totally geodesic sphere that lies at the asymptotic boundary. This sub-bundle is isomorphic to the Dirac monopole bundle and has Chern number $8nv$, which must be an integer. In the Dirac monopole, the Chern number is twice the magnetic charge; this can be carried over to this eight-dimensional Taub-Bolt. For the gauge potential [Eq. (46)], this means that the magnetic charge, p, is given by $4nv$. This is also consistent with an asymptotic examination such as the one carried out for the electric charge.

On the other hand, the functions ρ_M and p_M can be read off from Eqs. (47) and (48) by using their definition in Eq. (44). Then, the field equation (11) reads

$$\begin{aligned}-\rho_M = {}& \alpha_0 + 6\alpha_1(2R_{III} + R_{IV} + 4R_V) + 24\alpha_2(4R_{III}R_{IV} + 16R_{III}R_V + R_{IV}^2 + 4R_{IV}R_V + 10R_V^2 + 6R_{VI}^2) + 48\alpha_3(6R_{III}R_{IV}^2 \\ & + 24R_{III}R_{IV}R_V + 60R_{III}R_V^2 + 36R_{III}R_{VI}^2 + R_{IV}^3 + 6R_{IV}R_V^2 + 18R_{IV}R_{VI}^2 + 8R_V^3 + 24R_{VI}^3), \qquad (50)\end{aligned}$$

$$\begin{aligned}p_M = {}& \alpha_0 + 2\alpha_1(R_I + 10R_{III} + 2R_{IV} + 8R_V) + 8\alpha_2(2R_IR_{IV} + 8R_IR_V + 12R_{II}^2 + 20R_{III}^2 + 12R_{III}R_{IV} + 48R_{III}R_V + R_{IV}^2 \\ & + 4R_{IV}R_V + 10R_V^2 + 6R_{VI}^2) + 48\alpha_3(R_IR_{IV}^2 + 4R_IR_{IV}R_V + 10R_IR_V^2 + 6R_IR_{VI}^2 + 12R_{II}^2R_{IV} + 24R_{II}^2R_V + 24R_{II}^2R_{VI} \\ & + 12R_{III}^2R_{IV} + 48R_{III}^2R_V + 2R_{III}R_{IV}^2 + 8R_{III}R_{IV}R_V + 20R_{III}R_V^2 + 12R_{III}R_{VI}^2), \qquad (51)\end{aligned}$$

where $R_I, \ldots, R_{VI}$ have been defined in the Appendix. It is worth mentioning that these equations are not linearly independent, since differentiating the former results in the latter, after some algebraic manipulation. Thus, the equation of motion admits the following solution given in terms of a generalized Wheeler polynomial:

$$\sum_{k=0}^{3} b_k r^{-2k} B_k(U) = \frac{M}{r^7} + \frac{P(r)}{r^7}, \qquad (52)$$

where M is an integration constant and

$$B_0(U) = \frac{\alpha_0}{42}W_4 + \alpha_1 U\left(1-\frac{n^2}{r^2}\right)^3 + 20\alpha_2 U^2\left(1-\frac{2n^2}{5r^2}-\frac{3n^4}{5r^4}\right) + 120\alpha_3 U^3\left(1-\frac{n^2}{r^2}+\frac{16n^2}{5(r^2-n^2)}\right), \quad (53)$$

$$B_1(U) = \alpha_1 W_3 + 40\alpha_2 U\left(1-\frac{n^2}{r^2}\right)^2 + 360\alpha_3 U^2 W_1, \quad (54)$$

$$B_2(U) = 20\alpha_2 W_2 + 360\alpha_3 U\left(1-\frac{n^2}{r^2}\right), \quad (55)$$

$$B_3(U) = 120\alpha_3 W_1. \quad (56)$$

In Eq. (52), the coefficients are $b_0 = 1$, $b_1 = 1/5$, $b_2 = 1/20$, and $b_3 = 1/40$. Notice that we have not set $b_1 = 1$, which is convenient in the setting of Ref. [30]. However, this may be done by a reparametrization of r. The left-hand side of Eq. (52) is completely invariant under this change except in the b_k coefficients. Moreover, $P(r)$ is the Maxwell contribution, and it is a shorthand for

$$P(r) \equiv \frac{-1}{12r(r^2-n^2)^3}\Big[300v^2n^{10}(r^2-n^2) + 280n^6v^2r^4(r^2-5n^2) - 4v^2r^8n^2(r^2-25n^2) + 32qn^2vr^3(r^2-5n^2) - 5q^2\left(r^2-\frac{n^2}{5}\right)\Big]. \quad (57)$$

To evaluate the static limit, we first interchange v with its equivalent $p/4n$ and then take $n \to 0$. After this limit has been taken, the vielbein component e^0 has only the Euclidean time direction. Careful evaluation yields two parts of the gauge potential, that we write in the following manner:

$$A = \frac{q}{r^5}d\tau + 2p\sin^2\psi_3[d\phi_3 + \sin^2\psi_2(d\phi_2 + \sin^2\psi_1 d\phi_1)]. \quad (58)$$

The Wheeler polynomial (52) reduces to

$$\sum_{k=0}^{3} b_k r^{-2k} A_k(U) = \frac{M}{r^7} + \frac{5q^2}{12r^{12}} + \frac{p^2}{48r^4}, \quad (59)$$

with polynomials $A_k(U)$, given by

$$A_0(U) = \frac{\alpha_0}{42} + \alpha_1 U + 20\alpha_2 U^2 + 120\alpha_3 U^3, \quad (60)$$

$$A_1(U) = \alpha_1 + 40\alpha_2 U + 360\alpha_3 U^2, \quad (61)$$

$$A_2(U) = 20\alpha_2 + 360\alpha_3 U, \quad (62)$$

$$A_3(U) = 120\alpha_3. \quad (63)$$

As far as the Wheeler polynomial is concerned, the static limit amounts to setting the warping functions in the Taub-NUT solution to unity. Moreover, the appearance of warping functions (29) and (30) is recurrent. The Gauss-Bonnet case ($\alpha_3 = 0$) shows a change of warping function from six dimensions to eight, cf. Eqs. (32) and (56). Notice that, in eight dimensions, the coefficients that appear in the polynomials just above are recurrent in Lovelock gravity. The cosmological constant $\alpha_0/2$ is divided by $(D-1)(D-2)/2 = 21$, and the Gauss-Bonnet parameter is multiplied by $(D-3)(D-4) = 20$. The factor $(D-3)(D-4)(D-5)(D-6) = 120$ accompanies the cubic order coupling.

V. CONCLUSIONS

In this work, we considered the eight-dimensional Lanczos-Lovelock-Maxwell theory in the realm of the first-order formalism of gravity. By focusing on the torsion-free sector of the space of solutions, we have generalized the Wheeler approach of integrating the equations of motion of Lovelock theory, reducing them to an algebraic equation. This new generalization allows us to investigate stationary spacetimes, in addition to the static case which has been considered so far in the literature. In particular, we focus on Taub-NUT geometries with different higher-curvature terms of the Lovelock series to pave the way towards most general situations. The application of the method is novel, since previous cases were limited only to static manifolds. Taub-NUT spacetimes are stationary and are considerably more tractable than rotating spacetimes such as the Kerr solution. Considering inhomogeneous geometries on complex line bundles over Kähler manifolds has proven to be a nontrivial generalization of the approaches used for static manifolds [31,32]. However, the geometries resemble static metrics in such a way that the generalization is straightforward.

Using the extended version of Wheeler's methodology, we presented a new solution to Lanczos-Lovelock theory supplemented by Maxwell sources in a rather compact form. Arbitrary parameters of the Lovelock series are used, allowing us to analyze gravity theories such as Born-Infeld or pure Lovelock for the corresponding values of the couplings. The warping functions in the Wheeler polynomial are independent of the rescalings of the base manifold, except in the coefficients which encode its geometry. The Taub-Bolt branch of the solution presented here is a generalization of the Dirac monopole which includes self-gravity [41]. It has a unique Chern index [cf. Eq. (49)] which completely classifies all possible configurations and results in an electromagnetic parameter being a topological charge.

Interesting questions remain open. For instance, given the recent development of Lorentzian thermodynamics for Taub-NUT spacetimes [77–79], a higher-dimensional treatment including the example presented here is certainly desirable. The Euclidean method can be applied to the generalized Wheeler polynomial we provide in Eqs. (52) and (57). We stress that this thermodynamic exploration does not require the explicit solution of the metric function, as the Wheeler polynomial suffices. The black hole limit may also deserve a thermodynamic study in the extended black hole mechanics by considering the Lovelock coupling constants as thermodynamic entities. Interpreting them as thermodynamic variables which are held fixed in the action—and thus in the ensemble associated to them as well—naturally leads to their variation in the associated thermodynamic potential. We expect to consider this task in future works.

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APPENDIX: VIELBEINS AND CURVATURE ASSOCIATED WITH AN EIGHT-DIMENSIONAL TAUB-NUT SPACE

For the eight-dimensional geometry we focus on, the vielbein basis has been chosen as follows:

$$e^0 = \sqrt{f(r)}[\mathrm{d}\tau + 8n\sin^2\psi_3\{\mathrm{d}\phi_3 + \sin^2\psi_2(\mathrm{d}\phi_2 + \sin^2\psi_1\mathrm{d}\phi_1)\}], \tag{A1a}$$

$$e^1 = \frac{\mathrm{d}r}{\sqrt{f(r)}}, \tag{A1b}$$

$$e^2 = \sqrt{8(r^2-n^2)}\mathrm{d}\psi_3, \tag{A1c}$$

$$e^3 = \sqrt{8(r^2-n^2)}\sin\psi_3\cos\psi_3(\mathrm{d}\phi_3 + \sin^2\psi_2[\mathrm{d}\phi_2 + \sin^2\psi_1\mathrm{d}\phi_1]), \tag{A1d}$$

$$e^4 = \sqrt{8(r^2-n^2)}\sin\psi_3\mathrm{d}\psi_2, \tag{A1e}$$

$$e^5 = \sqrt{8(r^2-n^2)}\sin\psi_3\sin\psi_2\cos\psi_2(\mathrm{d}\phi_2 + \sin^2\psi_1\mathrm{d}\phi_1), \tag{A1f}$$

$$e^6 = \sqrt{8(r^2-n^2)}\sin\psi_3\sin\psi_2\mathrm{d}\psi_1, \tag{A1g}$$

$$e^7 = \sqrt{8(r^2-n^2)}\sin\psi_3\sin\psi_2\sin\psi_1\cos\psi_1\mathrm{d}\phi_1. \tag{A1h}$$

We write them here explicitly to complement recursive definitions in the main text. These recursive relations appear because the base manifold of the complex line bundle, where the metric is supported, has as base manifold a complex projective space of six real dimensions. As is probably anticipated by the reader, the geometry is that of Fubini-Study, up to a rescaling.

The components of the curvature 2-form are

$$R^{01} = R_I e^0\wedge e^1 + 2R_{II}(e^2\wedge e^3 + e^4\wedge e^5 + e^6\wedge e^7), \tag{A2}$$

$$R^{02} = R_{III}e^0\wedge e^2 + R_{II}e^1\wedge e^3,\quad R^{03} = R_{III}e^0\wedge e^3 - R_{II}e^1\wedge e^2, \tag{A3}$$

$$R^{04} = R_{III}e^0\wedge e^4 + R_{II}e^1\wedge e^5,\quad R^{05} = R_{III}e^0\wedge e^5 - R_{II}e^1\wedge e^4, \tag{A4}$$

$$R^{06} = R_{III}e^0\wedge e^6 + R_{II}e^1\wedge e^7,\quad R^{07} = R_{III}e^0\wedge e^7 - R_{II}e^1\wedge e^6, \tag{A5}$$

$$R^{12} = R_{III}e^1\wedge e^2 - R_{II}e^0\wedge e^3,\quad R^{13} = R_{III}e^1\wedge e^3 + R_{II}e^0\wedge e^2, \tag{A6}$$

$$R^{14} = R_{III}e^1\wedge e^4 - R_{II}e^0\wedge e^5,\quad R^{15} = R_{III}e^1\wedge e^5 + R_{II}e^0\wedge e^4, \tag{A7}$$

$$R^{16} = R_{III}e^1\wedge e^6 - R_{II}e^0\wedge e^7,\quad R^{17} = R_{III}e^1\wedge e^7 + R_{II}e^0\wedge e^6, \tag{A8}$$

$$R^{23} = 2R_{II}e^0\wedge e^1 + R_{IV}e^2\wedge e^3 + 2R_{VI}(e^4\wedge e^5 + e^6\wedge e^7), \tag{A9}$$

$$R^{24} = R_V e^2\wedge e^4 + R_{VI}e^3\wedge e^5,\quad R^{25} = R_V e^2\wedge e^5 - R_{VI}e^3\wedge e^4, \tag{A10}$$

$$R^{26} = R_V e^2\wedge e^6 + R_{VI}e^3\wedge e^7,\quad R^{27} = R_V e^2\wedge e^7 - R_{VI}e^3\wedge e^6, \tag{A11}$$

$$R^{34} = R_V e^3\wedge e^4 - R_{VI}e^2\wedge e^5,\quad R^{35} = R_V e^3\wedge e^5 + R_{VI}e^2\wedge e^4, \tag{A12}$$

$$R^{36} = R_V e^3\wedge e^6 - R_{VI}e^2\wedge e^7,\quad R^{37} = R_V e^3\wedge e^7 + R_{VI}e^2\wedge e^6, \tag{A13}$$

$$R^{45} = 2R_{II}e^0 \wedge e^1 + 2R_{VI}e^2 \wedge e^3 + R_{IV}e^4 \wedge e^5 + 2R_{VI}e^6 \wedge e^7, \tag{A14}$$

$$R^{46} = R_V e^4 \wedge e^6 + R_{VI}e^5 \wedge e^7,$$
$$R^{47} = R_V e^4 \wedge e^7 - R_{VI}e^5 \wedge e^6, \tag{A15}$$

$$R^{56} = R_V e^5 \wedge e^6 - R_{VI}e^4 \wedge e^7,$$
$$R^{57} = R_V e^5 \wedge e^7 + R_{VI}e^4 \wedge e^6, \tag{A16}$$

$$R^{67} = 2R_{II}e^0 \wedge e^1 + 2R_{VI}e^2 \wedge e^3 + 2R_{VI}e^4 \wedge e^5 + R_{IV}e^6 \wedge e^7. \tag{A17}$$

Here we have introduced various shorthands, $R_I \ldots R_{VI}$, which are detailed below:

$$R_I = -\frac{f''}{2}, \qquad R_{II} = \frac{n}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{f}{(r^2 - n^2)}\right],$$
$$R_{III} = -\frac{f' r}{2(r^2 - n^2)} + \frac{f n^2}{(r^2 - n^2)^2}, \tag{A18}$$

$$R_{IV} = \frac{1}{2}\frac{1}{r^2 - n^2} - f\frac{r^2 + 3n^2}{(r^2 - n^2)^2},$$
$$R_V = \frac{1}{8}\frac{1}{r^2 - n^2} - \frac{f r^2}{(r^2 - n^2)^2},$$
$$R_{VI} = \frac{1}{8}\frac{1}{r^2 - n^2} - \frac{f n^2}{(r^2 - n^2)^2}. \tag{A19}$$

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