Lukewarm black holes in the Nash-Greene framework

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In this work, we investigate the embedding of a four-dimensional spherically symmetric metric in a six-dimensional bulk. By using the Nash-Greene embedding theorem, the additional $SO(2)$ symmetry of the two spacelike extra dimensions induces the appearance of horizons of a lukewarm charged black hole. In addition, a mass-dependent cosmological constant is obtained with a prediction of a bound for the mass and minimal charge.

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I. INTRODUCTION

The possibility that the Universe might be embedded and evolves in a higher-dimensional space-time has been explored in the last decades as a tentative way to explain the hierarchy of the fundamental interactions (the large disparity of the weakness of the gravitational interaction as compared to the gauge fields) in the hope that such hierarchy may be broken in a higher space-time and the ordinary matter, standard gauge interactions, and gravity should be unified somehow. Most of these models have been Kaluza-Klein and/or string inspired, such as the seminal works of Arkani-Hamed, Dvali, and Dimopolous (ADD) [1] who predicted a unification of the fundamental interactions in a six-dimensional bulk for large extra dimensions, later named braneworlds, suggesting a sub-millimeter gravity. Another successful model is the Randall-Sundrum model [2,3] in which the fixed three-brane is embedded in a five-dimensional anti–de Sitter (AdS_5) space-time where the Israel condition applies [4]. The Dvali-Gabadadze-Porrati model is another interesting approach where the $3+1$ Minkowski space-time is fixed and embedded in a flat five-dimensional bulk, predicting no need for a small but nonzero vacuum energy density [5]. In all those models and variants, the embedding geometry was not completely regarded as a theoretical background since it is generally fixed to a boundary and specific conditions are needed to obtain its dynamics.

In this paper, we take a different path and consider the geometric embedding as a fundamental cornerstone for a gravitational theory, and neither a brane nor string related framework is proposed since we start with different propositions. In the last decades, several authors investigated the geometries of embedding as a prior mathematical structure for a physical theory [6–18]. The fundamentals of the model presented in this paper were originally proposed in [7–9,19,20]. We investigate some consequences of a spherically symmetric metric in a six-dimensional bulk in the context of a dynamical embedding. For instance, in the case of a Schwarzschild solution the embedding would be compromised in the Randall-Sundrum scheme, and this problem evinces the necessity for a more general framework, once the Schwarzschild geometry is completely embedded in six dimensions [17,21–23]. Moreover, due to the impact of the direct evidence of gravitational waves [24], recent black hole studies have turned to a huge area of investigation that impacts the problems of the standard model of particles, the unification of the standard interactions, and the early Universe. Thus, the study of higher-dimensional space-times has been the focus of active research, as in [25–28], such as analytical solutions of localized black branes in $2+1$ dimensions [29–31] or several numerical works providing solutions in $3+1$ and also higher dimensions [32,33]. A review on black holes in the context of several theories in higher dimensions can be found in [25,34].

We study the consequences of the dynamical embedding of a four-dimensional space-time into a six-dimensional bulk and find solutions for an electrically and magnetically charged nonrotating black hole. The paper is organized as follows: in the second section, we give a brief mathematical review on the geometry of embeddings. The third section presents the development of a calculation for a four-dimensional metric embedded in a six-dimensional bulk, and black hole horizons are also determined. In the fourth section, we present the emergent cosmological constant related to a thick embedded space-time and related quantities. In the final section, we present our remarks.

II. INDUCED EMBEDDED DIMENSIONAL EQUATIONS

The gravitational action functional in the presence of a confined matter field on a four-dimensional embedded

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space with thickness l embedded in a D-dimensional ambient space (bulk) has the form

$$S = -\frac{1}{2\kappa_D^2}\int \sqrt{|\mathcal{G}|}\mathcal{R}d^Dx - \int \sqrt{|\mathcal{G}|}\mathcal{L}_m^* d^Dx, \qquad (1)$$

where κ_D^2 is the fundamental energy scale on the embedded space, $\mathcal{R}$ denotes the Ricci scalar of the bulk, and $\mathcal{L}_m^*$ is the confined matter Lagrangian (gauge fields may also be included). In this model, the matter energy-momentum tensor occupies a finite hypervolume with constant radius l along the extra dimensions. The variation of Einstein-Hilbert action in Eq. (1) with respect to the bulk metric $\mathcal{G}_{AB}$ leads to the Einstein equations for the bulk,

$$\mathcal{R}_{AB} - \frac{1}{2}\mathcal{G}_{AB} = \alpha^\star \mathcal{T}_{AB}, \qquad (2)$$

where $\alpha^\star = 8\pi G^*$ is the energy scale parameter and G^* is the bulk "gravitational constant." The tensor $\mathcal{T}_{AB}$ is the energy-momentum tensor for the bulk [8,9,12]. Generating a thick embedded space-time is important to perturb the related background. It can be done using the confinement hypothesis that depends only on the four dimensionality of the space-time [35,36]. Even though any gauge theory can be mathematically constructed in a higher-dimensional space, the observed phenomenology still imposes the fourth dimensionality of space-time [37].

Looking for a more general framework as a basis for a physical model, we adopt the embedding theorem of differentiable functions as our main mathematically-oriented guide. In this sense, Nash's original embedding theorem [38] used a flat D-dimensional Euclidean space, later generalized to any Riemannian manifold including nonpositive signatures by Greene [39] with independent orthogonal perturbations. This choice of perturbation facilitates obtaining a differentiable smoothness of the embedding between the manifolds, which is a primary concern of Nash's theorem and satisfies the Einstein-Hilbert principle, where the variation of the Ricci scalar is the minimum possible. Hence, it guarantees that the embedded geometry remains smooth (differentiable) after smooth (differentiable) perturbations. With all these concepts, let us consider a Riemannian manifold V_4 with a nonperturbed metric $\bar{g}_{\mu\nu}$ being locally and isometrically embedded in a D-dimensional Riemannian manifold V_n. The embedded space-time V_4 is endowed with the local coordinates $x^\mu = \{x^0, \ldots, x^3\}$, whereas the extra dimensions in the bulk space can be defined with the coordinates $x^a = \{x^4, \ldots, x^{D-1}\}$ and $D = 4 + n$. Hence, the bulk local coordinates can be denoted by the set $\{x^\mu, x^a\}$. All these definitions allow us to construct a differentiable and regular map $\mathcal{X}: V_4 \to V_n$ satisfying the embedding equations

$$\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu}\mathcal{G}_{AB} = \bar{g}_{\mu\nu}, \qquad (3)$$

$$\mathcal{X}^A_{,\mu}\bar{\eta}^B_a\mathcal{G}_{AB} = 0, \qquad (4)$$

$$\bar{\eta}^A_a\bar{\eta}^B_b\mathcal{G}_{AB} = \bar{g}_{ab}, \qquad (5)$$

where the set of $\mathcal{X}^A(x^\mu, x^a): \mathcal{X}^A = \{\mathcal{X}^0 \ldots \mathcal{X}^{D-1}\}$ denotes the nonperturbed embedding function coordinates, the metric $\mathcal{G}_{AB}$ denotes the metric components of V_D in arbitrary coordinates, and $\bar{\eta}^A_a$ denotes a nonperturbed unit vector field orthogonal to V_4. Concerning the notation, capital Latin indices run from 1 to n. Lowercase Latin indices refer to the extra dimension considered. All Greek indices refer to the embedded space-time from 1 to 4. Those sets of equations represent, respectively, the isometry condition in Eq. (3), the orthogonality between the embedding coordinates $\mathcal{X}$ and $\bar{\eta}$ in Eq. (4), and the vector normalization $\bar{\eta}^A_a$ and $\bar{g}_{ab} = \epsilon_a\delta_{ab}$ with $\epsilon_a = \pm 1$ in which the signs represent the signatures of the extra dimensions. Hence, the integration of the system of equations (3)–(5) assures the configuration of the embedding map $\mathcal{X}$.

The second fundamental form or, more commonly, the nonperturbed extrinsic curvature $\bar{k}_{\mu\nu}$ of V_4 is, by definition, the projection of the variation of $\bar{\eta}$ onto the tangent plane:

$$\bar{k}_{\mu\nu} = -X^A_{,\mu}\bar{\eta}^B_{,\nu}\mathcal{G}_{AB} = X^A_{,\mu\nu}\bar{\eta}^B\mathcal{G}_{AB}, \qquad (6)$$

where the comma denotes the ordinary derivative.

If one defines a geometric object $\bar{\omega}$ in V_4, its Lie transport along the flow for a small distance δy is given by $\Omega = \bar{\Omega} + \delta y \pounds_{\bar{\eta}}\bar{\Omega}$, where $\pounds_{\bar{\eta}}$ denotes the Lie derivative with respect to $\bar{\eta}$. In particular, the Lie transport of the Gaussian vielbein $\{X^A_\mu, \bar{\eta}^A_a\}$ defined on V_4 straightforwardly gives the perturbed coordinate $\mathcal{Z}^A(x^\mu, y^a) \coloneqq \mathcal{Z}^A$ such as

$$\mathcal{Z}^A_{,\mu} = X^A_{,\mu} + \delta y^a \pounds_{\bar{\eta}} X^A_{,\mu} = X^A_{,\mu} + \delta y^a \bar{\eta}^A_{a,\mu}, \qquad (7)$$

$$\eta^A_a = \bar{\eta}^A_a + \delta y^b[\bar{\eta}_a, \bar{\eta}_b]^A = \bar{\eta}^A_a. \qquad (8)$$

It is worth mentioning that Eq. (8) shows that the normal vector η^A does not change under orthogonal perturbations. However, from Eq. (6), we note that, in general, $\eta_{,\mu} \neq \bar{\eta}_{,\mu}$. Likewise, it occurs that the so-called third geometrical form or, more commonly, the torsion vector $A_{\mu ab}$ does not change under orthogonal perturbations. To see how it works, we take Eq. (12) and rewrite Eq. (4) as

$$g_{\mu b} = \mathcal{Z}^A_{,\mu}\eta^B_b\mathcal{G}_{AB} = \delta y^a A_{\mu ab}, \qquad (9)$$

where $\mathcal{Z}^A$ is a set of perturbed coordinates. Equation (9) results from a generalization of the Gauss-Weingarten equations

$$\eta^A_{a,\mu} = A_{\mu ac}g^{cb}\eta^A_b - \bar{k}_{\mu\rho a}\bar{g}^{\rho\nu}\mathcal{Z}^A_{,\nu}. \qquad (10)$$

Then,

$$A_{\mu ab} = \eta^A_{a,\mu}\eta^B_b\mathcal{G}_{AB} = \bar{\eta}^A_{a,\mu}\bar{\eta}^B_b\mathcal{G}_{AB} = \bar{A}_{\mu ab}, \tag{11}$$

which ratifies that the torsion vector is not altered under perturbations. In the geometric language, the presence of a torsion potential tilts the embedded family of submanifolds with respect to the normal vector η^A_a. If the bulk has certain Killing vectors, then $A_{\mu ab}$ transforms as the component of a gauge field under the group of isometries of the bulk [7,40,41]. It is worth noting that the gauge potential can only be present if the dimension of the bulk space is equal to or greater than 6 ($n \geq 2$) in accordance with Eq. (11) since the torsion vector fields are antisymmetric under the exchange of extra coordinates a and b.

To describe the would-be perturbed embedded geometry, we set a perturbed coordinate $\mathcal{Z}^A$ needed to satisfy the embedding equations similar to Eqs. (3)–(5) as

$$\mathcal{Z}^A_{,\mu}Z^B_{,\nu}\mathcal{G}_{AB} = g_{\mu\nu}, \quad \mathcal{Z}^A_{,\mu}\eta^B_b\mathcal{G}_{AB} = g_{\mu b}, \quad \eta^A_a\eta^B_b\mathcal{G}_{AB} = g_{ab}, \tag{12}$$

where $g_{ab} = \epsilon_a\delta_{ab}$ with $\epsilon_a = \pm 1$. Thus, with Eq. (12) and using the definition from Eq. (6), one obtains the perturbed metric and extrinsic curvature of the new manifold as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - 2y^a\bar{k}_{\mu\nu a} + \delta y^a\delta y^b[\bar{g}^{\sigma\rho}\bar{k}_{\mu\sigma a}\bar{k}_{\nu\rho b} + g^{cd}A_{\mu ca}A_{\nu db}], \tag{13}$$

and the related perturbed extrinsic curvature

$$k_{\mu\nu a} = \bar{k}_{\mu\nu a} - \delta y^b(g^{cd}A_{\mu ca}A_{\nu db} + \bar{g}^{\sigma\rho}\bar{k}_{\mu\sigma a}\bar{k}_{\nu\rho b}). \tag{14}$$

Taking the derivative of Eq. (13) with respect to the y coordinate, one obtains Nash's deformation condition

$$k_{\mu\nu a} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y^a}. \tag{15}$$

The meaning of this expression can be realized in a pictorial view under the basic theory of curves. For instance, one can construct a one-parameter group of diffeomorphisms defined by a map $h_y(p)\colon V_D \to V_D$, describing a continuous curve $\alpha(y) = h_y(p)$ that passes through the point $p \in V_4$, with unit normal vector $\alpha'(p) = \eta(p)$. The group is characterized by the composition $h_y \circ h_{\pm y'}(p) \overset{\text{def}}{=} h_{y\pm y'}(p)$, $h_0(p) \overset{\text{def}}{=} p$. With the diffeomorphism mapping all points of a small neighborhood of p, one gets a congruence of curves (or orbits) orthogonal to V_4 [42] which consists of the action of the extrinsic curvature. Thus, it is not important if the parameter y is timelike or not, nor is the sign of its signature. A similar expression was obtained years later in the ADM formulation by Choquet-Bruhat and York [43]. From the physical point of view, the expression in Eq. (15) localizes the matter in the embedded space-time, imposing on it a geometric confinement. In other words, it holds true for any perturbations resulting from n-parameter families of embedded submanifolds denoted by y^a, and the matter remains confined to the resulting perturbed metric, which can bend and/or stretch without ripping the manifold (embedded space-time), which can be a valuable feature for a quantization process.

In addition, the integrability conditions for Eq. (12) are given by the nontrivial components of the Riemann tensor of the embedding space expressed in the Gaussian frame $\{\mathcal{Z}^A_\mu, \eta^A_a\}$ known as the Gauss-Codazzi-Ricci equations. This guarantees reconstructing the embedded geometry and understanding its properties from the dynamics of the four-dimensional embedded space-time. Consequently, we can define a Gaussian coordinate system $\{\mathcal{Z}^A_{,\mu}, \eta^A_a\}$ for the bulk in the vicinity of V_4 in such a way,

$$\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu} + g^{ab}A_{\mu a}A_{\nu b} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix} \tag{16}$$

where the perturbed metric $g_{\mu\nu}$ is given by Eq. (13).

The expression in Eq. (16) is the metric of the bulk with $D \geq 6$ or at least two extra dimensions. This resembles the non-Abelian Kaluza-Klein metric, and the quantity $A_{\mu a}$ plays the role of the Yang-Mills potentials where $A_{\mu a} = x^b A_{\mu ab}$. We emphasize that for just one extra dimension, the torsion vector does not exist, and for two extra dimensions it turns out to be the usual Maxwell field, which means that the non-Abelian part of $A_{\mu a}$ is lost in a six-dimensional bulk. This means that the resulting force is the ordinary electromagnetic one in the case of two extra dimensions [10,11,19,41].

As proposed in [7–9,20], one obtains the induced field covariant equations of motion, taking Eq. (2) in the frame defined in Eq. (16). In the background for a 4D observer in the embedded space, we have the following set of equations denoted by Eqs. (17), (21), and (22):

$$G_{\mu\nu} + Q_{\mu\nu} = 8\pi G_N(T_{\mu\nu} + T^{(\mathrm{YM})}_{\mu\nu}), \tag{17}$$

where the quantities $T_{\mu\nu}$ and $T^{(\mathrm{YM})}_{\mu\nu}$ denote the stress energy tensors for ordinary matter and Maxwell-Yang-Mills fields. In this sense, the tensor $T^{(\mathrm{YM})}_{\alpha\beta}$ is written as

$$T^{(\mathrm{YM})}_{\mu\nu} = \frac{1}{4\pi\beta g_i^2}\left(F^\sigma_{\mu lm}F^{lm}_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\mu\nu lm}F^{\mu\nu lm}\right), \tag{18}$$

where the quantity $F^{\mu\nu ab}$ in terms of $A_{\mu a}$ is denoted by $F^{\mu\nu ab} = A_{\mu ab,\nu} - A_{\nu ab,\mu} - A^c_{\nu a}A_{\mu cb} + A^c_{\mu a}A_{\nu cb}$, which is the curvature associated with the torsion vector $A_{\mu ab}$ that obeys

$$\nabla_\nu F^{\mu\nu ab} = 0. \tag{19}$$

The structure constant β originates from the adjoint representation of the Lie algebra L^{ab} of the group of rotations [19,20] such that $[L^{ab}, L^{cd}] = f^{abcd}_{mn} L^{mn}$. Moreover, the structure constants are given by $f^{abcd}_{mn} = 2\delta^m_{[b}\eta_{a][c}\delta^n_{d]}$, and we calculate the following relation [44]:

$$f^{abcd}_{mn} f^{a'b'mn}_{cd} = \beta^{(ad)}[\delta_{ab'}\delta_{a'b} - \delta_{aa'}\delta_{bb'}]. \quad (20)$$

Hence, using the former relation, one determines the resulting constant β, which can be written as

$$\beta = \begin{cases} 2(n-2) & n \neq 2 \\ 2 & n = 2 \end{cases},$$

and the terms g_i are the coupling constants that indicate the strength of the fundamental gauge interaction.

The second equation involves relations with extrinsic terms $\bar{k}_{\alpha\beta a}$ and $A_{\mu ab}$,

$$\nabla^*_\nu \bar{k}_a - \nabla^*_\mu \bar{k}^\mu_{a\nu} = 8\pi G_N T_{a\nu}, \quad (21)$$

where the term $\nabla^*_\mu \bar{k}_{\alpha\beta a}$ denotes $\nabla^*_\mu \bar{k}_{\alpha\beta a} \coloneqq \bar{k}_{\alpha\beta a;\mu} - A_{\mu ab}\bar{k}^b_{\alpha\beta}$ and the semicolon denotes the covariant derivative.

The third equation is denoted as

$$\frac{G_N}{\beta}\left(F^{\mu\nu}_{am}F^m_{\mu\nu b} + \frac{1}{2}\eta_{ab}F^{lm}_{\mu\nu}F^{\mu\nu}_{lm}\right) - \frac{1}{2}\eta_{ab}(R + \bar{k}_{\mu\nu m}\bar{k}^{\mu\nu m} - \bar{k}_a\bar{k}^a) = 8\pi G_N T_{ab}, \quad (22)$$

where $\eta_{ab} = \epsilon_a\delta_{ab}$ with $\epsilon_a = \pm 1$. The quantities G_N, $T_{a\nu}$, T_{ab} denote the induced gravitational Newton's constant, and the stress energy tensor projections of T_{AB} on the cross and normal directions of the space-time, respectively.

Those sets of equations are the result of the integrability conditions of the embedding given by the Gauss-Codazzi-Ricci equations. From the Nash-Green theorem, the solutions of these equations were obtained by a differentiable process [7]. The first two equations are known, respectively, by the gravitational tensorial equation (a modified Einstein's equation by the appearance of the extrinsic curvature) as in Eq. (17) and the gravitational vectorial equation as in Eq. (21). In summary, they reflect the meaning of a dynamical embedding: the pseudo-Riemann curvature of the embedding space acts as a reference for the pseudo-Riemann curvature of the embedded space-time. Moreover, the projection of the Riemann tensor of the embedding space along the normal direction is given by the tangent variation of the extrinsic curvature as shown by Eq. (21), which is the trace of the Codazzi equation composed of the extrinsic terms $\bar{k}_{\alpha\beta a}, A_{\mu ab}$. The last equation is known as the gravitational scalar equation and serves as a constraint on the torsion vector fields $A_{\mu ab}$.

The quantity $Q_{\mu\nu}$ is denoted by

$$Q_{\mu\nu} = \bar{g}^{cd}(\bar{g}^{\rho\sigma}\bar{k}_{\mu\rho c}\bar{k}_{\nu\sigma d} - \bar{k}_{\mu\nu d}\bar{g}^{\alpha\beta}\bar{k}_{\alpha\beta c}) \quad (23)$$

$$-\frac{1}{2}(\bar{k}_{\lambda\phi c}\bar{k}^{\lambda\phi}_d - \bar{g}^{\alpha\beta}\bar{k}_{\alpha\beta d}\bar{g}^{\gamma\delta}\bar{k}_{\gamma\delta c})\bar{g}_{\mu\nu}, \quad (24)$$

and it is an independently conserved quantity in the sense that $Q_{\mu\nu;\nu} = 0$, which means that this geometric new term does not exchange gravitational energy with ordinary matter resembling the quintessence in the dark energy problem. The conservation of $Q_{\mu\nu}$ holds true for perturbed quantities of $g_{\mu\nu}$ and $k_{\mu\nu a}$. Implications of this term in cosmology have been investigated in [8–13,15–18].

From a relation between normal curvature radii L with a thickness l of the embedded space-time and the gauged coupling g_i, one can write the four-dimensional Planck mass applied in such a way [20],

$$L = \frac{n\beta}{4(n+2)} l^3 M^2_{\rm pl}\frac{g_i^2}{4\pi},$$

where normal radii are the smallest value of the curvature radii obtained from

$$\det(g_{\mu\nu} - l^a k_{\mu\nu a}) = 0. \quad (25)$$

In a geometrical sense, the term $y^a = l^a$ represents displacement of the embedded space along the extra dimensions. To interpret this solution, consider a small displacement of the foot of the normal vector η_a along a tangent direction dx^μ to the embedded space-time and look for a fixed point of the bulk in line with the same normal direction. The point is the local center of curvature, and the local radii of curvature are the values y^a of the extra dimensional coordinate satisfying the condition $\det(g_{\mu\nu} - y^a k_{\mu\nu}) = 0$ [7,44,45]. For a given $k_{\mu\nu}$, there are at most four distinct solutions $y^{a(\alpha)}, (\alpha) = 1, \ldots, 4$ of such an equation. Consequently, all points of the embedded space-time solutions must have all directions dx^μ as principal directions. The single curvature radius l^a is the smallest of these solutions, corresponding to the direction in which the embedded space deviates more sharply from the tangent plane. Considering all contributions of l^a, in such a way that the smaller solution prevails, the curvature radius may also be expressed as

$$\frac{1}{l} = \sqrt{g_{\mu\nu}g_{ab}\frac{1}{l^\mu_a}\frac{1}{l^\nu_b}}.$$

III. CHARGED BLACK HOLE IN A SIX-DIMENSIONAL BULK

In this section, we consider a static and symmetric solution of induced field equations and the consequences of an electromagnetic energy-momentum tensor $T^{(\mathrm{EM})}_{\mu\nu}$ as a source. In six dimensions, there is an additional $SO(2)$ symmetry generated by two spacelike Killing vectors of the two extra dimensions ($n = 2$). Moreover, the gauge fields, the gauge fields result in the ordinary electromagnetic force, which can be realized if one assumes that $A_{ab\mu} = \epsilon_{ab}A_{\mu}$, where the antisymmetric symbols are defined as $\epsilon_{12} = -\epsilon_{21} = 1$. Without any further ado, we determine the induced field equations (2), which are given by the following induced metric:

$$ds^2 = -e^{-2a(r)}dt^2 + e^{2b(r)}dt^2 + r^2 d\Omega^2, \tag{26}$$

where $d\Omega$ is the ordinary two-sphere element. We stress that the y coordinate is not necessary on the induced metric once the embedding equations are correctly applied. Since Nash's idea on embedding of manifolds using smooth deformations is applied to the embedding, the y coordinate, commonly used in rigid embedded models, e.g., Randall-Sundrum and variants [2,3], can be omitted in the line element [7–14].

In six dimensions, Eq. (19) reduces to the Maxwell equations $\nabla_{\nu}F^{\mu\nu} = 0$, and $T^{(\mathrm{YM})}$ reduces to the energy-momentum tensor $T^{(\mathrm{EM})}_{\mu\nu}$. Therefore, the field strength in the Lorentz gauge $\nabla_{\mu}A^{\mu} = 0$ is given by

$$F = -\frac{q}{\sqrt{4\pi}}\frac{e^{a+b}}{r^2}dt \wedge dr - \frac{g}{\sqrt{4\pi}}\sin(\theta)d\theta \wedge d\varphi, \tag{27}$$

where q and g denote electric and magnetic charges, respectively. Because of the embedding, they are located in the center of Kruskal space-time. The related energy-momentum tensor components of Eq. (18) are found to be

$$T^{\mathrm{EM}}_{00} = \frac{Z^2 e^{2a}}{8\pi r^4}, \qquad T^{\mathrm{EM}}_{11} = -\frac{Z^2 e^{2b}}{8\pi r^4},$$
$$T^{\mathrm{EM}}_{22} = \frac{Z^2}{8\pi r^2}, \qquad T^{\mathrm{EM}}_{33} = \sin^2(\theta)T^{\mathrm{EM}}_{22}, \tag{28}$$

where all other $T^{(\mathrm{EM})}_{\mu\nu} = 0$, and $Z^2 = \frac{1}{4\pi}(q^2 + g^2)$. Equations (22) give the restriction on the electric and magnetic charges $q^2 = g^2$, and $\frac{g_i^2}{4\pi} = Z^2$. Moreover, Eq. (21) gives the components of extrinsic curvature as

$$k_{\mu\nu a} = \phi_a g_{\mu\nu}, \tag{29}$$

where the set ϕ_a are constants. Hence, Eq. (25) leads to $L^{-2} = \eta_{ab}\phi^a\phi^b$. Therefore, there is not a direction in which any normal curvature has an extreme value, and consequently, the four-dimensional submanifold is umbilic [44,45], which leads to

$$Q_{\mu\nu} = \frac{3}{L}g_{\mu\nu}. \tag{30}$$

Hence, using Eqs. (28) and (30) the components of G_{00} and G_{11} in Eq. (17) become

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = \frac{3}{L^2} + \frac{Z^2 G}{r^4}, \tag{31}$$

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = \frac{3}{L^2} + \frac{Z^2 G}{r^4}. \tag{32}$$

By substraction, we see that $a + b = 0$, and consequently

$$(re^{-2b})' = 1 - \frac{3}{L^2}r^2 - \frac{Z^2 G}{r^2}.$$

The metric functions are thus given by

$$e^{2a} = e^{-2b} = 1 - \frac{2GM}{r} + \frac{Z^2 G}{r^2} - \frac{r^2}{L^2}. \tag{33}$$

The other components of the Einstein equations are also satisfied. Hence, in a general sense, the D-dimensional Einstein vacuum equations (2) induce a four-dimensional Reissner-Nordström-de Sitter (RNdS) space-time, where the induced charge is a consequence of a noncompact spacelike extra dimensions and an induced cosmological constant $\Lambda_{\mathrm{ind}} = \frac{3}{L^2}$ is a consequence of the extrinsic shape of the black hole.

As is largely known, according to the Einstein field equations, the distribution of matter determines the intrinsic geometric properties. On the other hand, in a geometry of embeddings with the corresponding field equations, Eqs. (17) and (21) determine both intrinsic $(G_{\mu\nu})$ and extrinsic $(Q_{\mu\nu})$ geometric properties of the space-time. Consequently, in the absence of any matter fields, the embedded space-time will be a trivial flat space-time (with both intrinsic and extrinsic features). As a result, the extrinsic radii of the embedded space-time will be globally infinite.

IV. AN EMERGENT COSMOLOGICAL CONSTANT

The minimum measurable length over which the masses can be localized is about of the order of their Compton wavelengths. Hence, let us assume that the width of the four-dimensional embedded space-time has the same order of the Compton wavelength of the black hole, which we set as $l = M^{-1}$. Therefore, the curvature radii defined in Eq. (25) become

$$L = \frac{1}{4}\frac{M_{\mathrm{pl}}^2}{M^3}Z^2. \tag{34}$$

Hence, taking Eqs. (33) and (34), we find an emergent cosmological constant Λ_e as

$$\Lambda_e = 48G^2 \frac{M^6}{Z^4}. \tag{35}$$

Interestingly, the "cosmological constant" as a mass-dependent quantity was previously conjectured by Zeldovich [46,47].

Eventually, the horizons can be found using the algebraic equation $e^{2a} = 0$, which has three positive roots. The outer horizon is located at r_{++}, the black hole horizon at r_+, and the Cauchy horizon at r_i. If we set $M^2 G = Z^2$, then the RNdS space-time is called a lukewarm black hole [48,49]. In the naive picture of black hole evaporation, in the lukewarm solution, the cosmological constant Λ comes from the background universe, and consequently this solution is thermodynamically stable and is the endpoint of the evaporation process: if $M/M_{\rm pl} > |Z|$ then the black hole is hotter than the de Sitter horizon and will evaporate until it reaches $M/M_{\rm pl} = |Z|$. If $M/M_{\rm pl} < |Z|$ then the de Sitter horizon is hotter, and the black hole will accrete radiation until it reaches $M/M_{\rm pl} = |Z|$. A similar process applies to our model since the black hole will evaporate until it reaches $M/M_{\rm pl} = |Z|$. Then, an emergent Λ_e will rise in the black hole with a value

$$\Lambda_e = 48Z^2 M_{\rm pl}^2, \tag{36}$$

and the three related horizons will have the form

$$r_i = \frac{1}{8M_{\rm pl}|Z|}\left(-1 + \sqrt{1+16Z^2}\right), \tag{37}$$

$$r_+ = \frac{1}{8M_{\rm pl}|Z|}\left(1 - \sqrt{1-16Z^2}\right), \tag{38}$$

$$r_{++} = \frac{1}{8M_{\rm pl}|Z|}\left(1 + \sqrt{1-16Z^2}\right), \tag{39}$$

which shows that the lukewarm black holes are possible if $|Z| < \frac{1}{4}$ or $M < \frac{1}{4}M_{\rm pl}$. In this case, both event and outer horizons have the same surface gravity,

$$\kappa_+ = \kappa_{++} = 4|Z|M_{\rm pl}\sqrt{1-16Z^2}, \tag{40}$$

which means they have the same temperature, $T = \frac{|\kappa_+|}{2\pi}$. Hence, for $Z = \mathcal{O}(1)$ there will be an unstable black hole. Therefore, the lukewarm black holes in this model will reach a maximum temperature at

$$Z = \frac{1}{2\sqrt{2}}, \tag{41}$$

$$M = \frac{1}{2\sqrt{2}} M_{\rm pl}, \tag{42}$$

$$\Lambda_e = 6M_{\rm pl}^2 \sim \Lambda_{\rm QFT}, \tag{43}$$

$$T = \frac{M_{\rm pl}}{2\pi} \sim T_{\rm pl} = 3.5\times 10^{32} K. \tag{44}$$

The emergent Λ_e is of the order of the vacuum energy of quantum field theory; consequently, it is hotter than the cosmological horizon, and these solutions are not thermodynamically stable. Since the total gravitational entropy is given by the sum of the area of the black holes and cosmological event horizons, the entropy is extremized for $\kappa_{\rm bh} = -\kappa_{\rm universe}$, which coincides with the condition that the black hole and de Sitter temperature must be equal. Thus, the final stable state of the black hole will be determined by the following quantities:

$$Z = \frac{1}{4M_{\rm pl}}\sqrt{\frac{\Lambda}{3}} \sim 10^{-60}, \tag{45}$$

$$M = \frac{1}{4}\sqrt{\frac{\Lambda}{3}} \sim 1.5\times 10^{-66} g, \tag{46}$$

$$\Lambda_e = \Lambda_{\rm universe} \sim 3\times 10^{-56}\ {\rm cm}^{-2}, \tag{47}$$

$$T = \frac{2}{\pi}M = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}} \sim 10^{-60}, \tag{48}$$

with the following horizons: $r_i \sim r_+ \sim \frac{1}{4M_{\rm pl}}\sqrt{\frac{\Lambda}{3}} \sim 10^{-93}$ cm and $r_{++} \sim \sqrt{\frac{3}{\Lambda}} \sim 10^{28}$ cm.

Hence, this means that the black hole reaches the universal lower bound of mass $M \sim 10^{-66}$ g and minimum charge $q \sim 10^{-60} e$. This bound for the mass has been verified by several authors [50–52].

V. REMARKS

In this paper, we have discussed the embedding of a four-dimensional symmetric metric in a bulk of six dimensions. Applying the Nash-Greene embedding theorem to a static spherical symmetric metric, we have found a modification induced by the extrinsic curvature using a dynamical embedding with the appearance of the torsion vector $A_{\mu ab}$ as a gauge group of rotations $SO(2)$ in the extra dimensions. The conserved quantity $Q_{\mu\nu}$ is a new component of curvature that may be interpreted, in a cosmological sense, as a component of some mechanical energy responsible for the observed acceleration of the Universe. However, from the point of view of geometry, it may also be interpreted as a necessary observational quantity, reintroducing some topological qualities to a theory of gravitation. This latter interpretation gives full support to the Gauss and Riemann views that geometry is determined essentially by the observations, regardless of how small and near or how large and distant they may be.

Since a spherically symmetric metric, just like Schwarzschild geometry, is completely embedded in a six-dimensional bulk, we have calculated the related horizons for a charged nonrotating black hole, and we have found a lukewarm black hole plus the appearance of new elements like an emergent cosmological constant Λ_e, which has turned this model into a worthy framework to investigate further. It is important to mention that even in the vacuum case, this is not a solution of the Einstein-Maxwell equations in four dimensions due to the presence of the tensor $Q_{\mu\nu}$ representing the conserved energy of the extrinsic part of the gravitational field at the TeV energy scale. These results are quite different from those we obtained in a five-dimensional bulk, where the restrictions to embedding induced a serious constraint on the classical black hole thermodynamic stability [17]. The present results may suggest a new way of thinking about the cosmological constant problem and its variants [53–55]. Hopefully, the understanding of an emergent Λ_e will explain why the measured cosmological constant is not precisely zero and has a nonzero but very small value. Future prospects include the study of the related quasi-normal modes applied to the same framework in the analysis of the signal-to-noise ratio, which will be reported elsewhere.

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