New results on the physical interpretation of black-brane gravitational perturbations

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The linear perturbation theory applied to the study of black holes is a traditional and powerful tool to investigate some of the basic properties of these objects, such as the stability of the event horizon, the spectra of quasinormal modes, the scattering and the production of waves in a process of gravitational collapse. Since long ago, the physical interpretation of the linear fluctuations in the metric of spherically symmetric black holes has been established. In a multipolar expansion, it is known that polar perturbations of a monopole type ($l = 0$) can only increase the black-hole mass, axial perturbations of a dipole type ($l = 1$) induce a slow rotation in the system, and perturbations with $l \geq 2$ always lead to the production of gravitational waves. However, in relation to the planar Schwarzschild anti–de Sitter black holes (or black branes, for short), there is still no conclusive study on some aspects of the physical meaning of these perturbations. In particular, there is some controversy concerning the polar sector of fluctuations with zero wave numbers ($k = 0$). Some authors claim that this kind of perturbation causes only a small variation in the black-brane mass parameter, while others obtained also evidence for the existence of gravitational waves associated to such modes. The present study aims to contribute to the resolution of this controversy by revealing the physical meaning of the gravitational perturbations of anti–de Sitter black branes. In this work we use the Chandrasekhar's gauge formalism to evaluate the linear variations in the complex Weyl scalars in terms of the Regge-Wheeler-Zerilli gauge-invariant quantities. Then we use the Szekeres' proposal for the meaning of the Weyl scalars and the Pirani's criterion for the existence of gravitational radiation in order to give a physical interpretation of the black-brane perturbations with arbitrary wave number values.

I. INTRODUCTION

The perturbation theory is an important tool in studying some of the properties of black holes, for instance, in investigating the stability of event horizons under metric fluctuations, as well as in the study of generation, absorption and scattering of gravitational waves by such objects (see, e.g., Refs. [1–3]). Furthermore, from the perspective of the AdS/CFT correspondence [4–7], black holes in asymptotically anti–de Sitter (AdS) spacetimes are dual to a thermal equilibrium state of a boundary conformal field theory (CFT), and the first-order gravitational perturbations of these black holes correspond to linear fluctuations in the energy-momentum tensor of the dual thermal state [8].

There are two main methods to study gravitational perturbations of black holes. The first one consists in considering perturbations of the black-hole metric and then by linearizing the Einstein equations around the given background spacetime. The second method consists in the linearization of the equations resulting from the use of the Newman-Penrose formalism [9]. In both cases, a major

difficulty is writing the physically interesting variables in terms of gauge-invariant quantities. The gauge freedom arises from the identification of events in the background with events in the physical (perturbed) spacetime. For this reason, the gravitational perturbations are usually described in terms of gauge-dependent variables, whose gauge freedom may be explored to simplify the analysis, and only the final results are written in terms of gauge-invariant quantities.

As far as we know, the first work to present a complete and clear formulation of the gravitational perturbation theory was published in 1974 by Stewart and Walker [10]. In the meantime, important progress on the study of metric perturbations of black-hole spacetimes was made in the work by Regge and Wheeler [11] published in 1957, where the stability of a Schwarzschild black hole under axial perturbations was studied. The gauge used in this paper is now known as the Regge-Wheeler gauge. Based on the same idea, Zerilli [12,13] analyzed the gravitational radiation that arises when stars fall into a static black hole and extended the Regge-Wheeler study for polar perturbations, completing the

fundamental equations for the gravitational perturbations of Schwarzschild black holes. Later on, Chandrasekhar [14] obtained the same equations as Regge-Wheeler and Zerilli by using a different gauge, that is currently known as the Chandrasekhar gauge. Following different routes, Teukolsky [15,16] and Moncrief [17,18] developed studies on the gravitational perturbations of the Kerr and Reissner-Nordström black holes, respectively, with the aim of investigate the stability of these black-hole spacetimes.

In more recent years, Kodama, Ishibashi, and Seto [19] developed an interesting strategy to investigate the gravitational perturbations in static higher-dimensional spacetimes with some specified spatial symmetry. In such a strategy, the spatial symmetry of the background spacetime is used from the beginning to decompose the perturbations in three independent sectors, namely, tensor, vector, and scalar sectors, and to construct gauge-invariant quantities for each one of the sectors. This construction has been used to study several properties of gravitational perturbations in black-hole and black-brane spacetimes, as for instance in the work by Dias and Reall [20], where the algebraically special modes of higher dimensional black holes are investigated. The vector and scalar sectors of the Kodama-Ishibashi-Seto gauge-invariant formulation correspond, respectively, to the axial and polar perturbations in the language of the Regge-Wheeler-Zerilli formalism, while the tensor perturbations are not present in 4-dimensional spacetimes.

An interesting approach regarding the physical interpretation of a given background metric and its gravitational perturbations is based on the effects of the curvature tensor on the relative motion of free test particles through the geodesic deviation equation. Using a tetrad of null vectors, Szekeres [21] wrote the geodesic deviation equations for empty spacetimes in terms of the Weyl scalars and obtained a physical interpretation for these scalars. Motivated by this technique, Podolský and Švarc [22] arrived at a similar interpretation for the Weyl scalars in higher-dimensional spacetimes. They went beyond the free part of the gravitational field, and took into account the isotropic action of the cosmological constant and the influence of matter in the spacetime curvature. From the Szekeres' and Podolský-Švarc's interpretation, it is possible to extract the effects of gravitational perturbations on freely falling test particles. In particular, it is possible to verify whether a given class of metric perturbations are really associated to gravitational waves or not. The aforementioned approaches allowed to establish a classification and an interpretation of gravitational perturbations of spherically symmetric black holes, even in the presence of a cosmological constant. On the other hand, in relation to the case of AdS black-brane perturbations, there still exist some open questions regarding the physical interpretation of the polar-sector perturbations with zero wave numbers. For instance, Kodama and Ishibashi [23] argue that these perturbations produce only a small change in the mass parameter of the black brane, in

complete agreement with the case of a monopole-type ($l = 0$) perturbation of a Schwarzschild black hole. However, in a more detailed study with the use of the Chandrasekhar diagonal gauge, it was shown in Ref. [24] that the zero wave number polar perturbations may also represent gravitational waves along the radial direction, i.e., they describe also gravitational waves propagating in the perpendicular direction to the black-brane horizon.

Therefore, motivated by the absence of a conclusive study on this theme, the present paper aims to investigate the physical meaning of the gravitational perturbations of black branes in asymptotically anti–de Sitter spacetimes and, consequently, to analyze the possibility of obtaining solutions with gravitational waves for polar perturbations with zero wave numbers. For this, the perturbations in the Weyl scalars are calculated and, from the Szekeres [21] approach for the analysis of the geodesic deviation equations, the physical meaning of the perturbations shall be obtained in an invariant way. As an additional investigation, the canonical form of the Riemann tensor is calculated for the perturbed black brane, and the Pirani's criterion [25] is used to study the physical meaning of the polar perturbation sector. In the studies of the metric perturbations related to this problem, the Chandrasekhar gauge is employed. Finally, let us stress that the main interest here is in the physical interpretation of the perturbation functions and, in particular, in the zero wave number perturbations. We will not focus on the calculation of the complete black-brane quasinormal mode spectrum because it is a well studied subject in the literature. See, e.g., Refs. [26–29] for more details.

The structure of this paper is as follows. We present in Sec. II the black-brane spacetime and define the basic quantities of the Newman-Penrose formalism for such a background. Section III is devoted to review the gravitational perturbation theory in the Chandrasekhar gauge, and to present the fundamental equations for the axial and polar metric variations. In Sec. IV we use the Szekeres' proposal to extract the physical meaning of the gravitational fluctuations with both vanishing and nonvanishing wave numbers. We use in Sec. V the Pirani's criterion [25] as an alternative way to interpret the polar-sector perturbations, in particular the perturbations with zero wave numbers. In Sec. VI we show how the gravitational waves associated to the zero wave number polar fluctuations arise in the Kodama-Ischibashi-Seto approach. We conclude in Sec. VII by discussing the main results of this paper.

Geometric units are used throughout this text, so that the speed of light c and the gravitational constant G are set to unity, $c = 1 = G$. The signal convention for the Riemann, Ricci, and Einstein tensors is that of Ref. [30]. For the Newman-Penrose quantities, the signal convention of Ref. [31] is adopted.

II. THE BACKGROUND SPACETIME

The Einstein equations with a negative cosmological constant admit an asymptotically AdS solution, whose associated metric can be written in the form [32–34]

$$ds^2 = -f(r,M)dt^2 + f^{-1}(r,M)dr^2 + r^2(d\varphi^2 + dz^2), \quad (1)$$

with

$$f(r,M) = \frac{r^2}{\ell^2} - \frac{2M}{r}, \quad (2)$$

where M represents the mass parameter, $\ell^2 = -3/\Lambda_c$ is the AdS radius, and Λ_c is the negative cosmological constant.

The local geometry of such a solution is Euclidean in the sense that the surfaces of constant t and r are locally flat, but the topology can be planar ($\varphi, z \in \mathbb{R}$), cylindrical ($\varphi \in \mathbb{S}^1$, $z \in \mathbb{R}$), or toroidal ($\varphi, z \in \mathbb{S}^1$). In short, in this text we stick to the geometry and refer to this solution as an AdS plane-symmetric black hole, or simply as a black brane.

The zeros of the function $f(r,M)$, given by Eq. (2), determine the horizons of the background spacetime (1). The only real root of the equation $f(r,M) = 0$ gives the location of the event horizon of the black brane,

$$r_h = (2M\ell^2)^{1/3}. \quad (3)$$

The present work is partly performed by using the Newman-Penrose formalism [9]. For this, we consider a null tetrad basis $(l^\mu, n^\mu, m^\mu, m^{*\mu})$, where the real null vectors l^μ and n^μ are, respectively, tangent to the ingoing and outgoing radial null geodesics of the background solution (1), i.e.,

$$l^\mu \partial_\mu = \frac{1}{f}(\partial_t + f\partial_r), \qquad n^\mu \partial_\mu = \frac{1}{2}(\partial_t - f\partial_r), \quad (4)$$

while the complex null vector m^μ is defined by

$$m^\mu \partial_\mu = \frac{1}{\sqrt{2}r}(\partial_z + i\partial_\varphi), \quad (5)$$

with the vector $m^{*\mu}$ being the complex conjugate of m^μ.

For the above null tetrad, the only nonvanishing Weyl scalar of the background metric (1) is

$$\Psi_2 = C_{\rho\sigma\mu\nu}l^\rho m^\sigma m^{*\mu}n^\nu = -\frac{M}{r^3}. \quad (6)$$

Hence, the black-brane spacetime is type D in the Petrov classification.

III. GRAVITATIONAL PERTURBATIONS OF BLACK BRANES

A. General remarks

Let us review here the basic properties of the axial (odd) and polar (even) sectors of the black-brane gravitational perturbations, since both are important for the present analysis. The perturbations of the black-brane metric (1)

and its quasinormal modes have been extensively investigated in the literature (see, e.g., Refs. [26–29]). However, the interpretation of the resulting perturbation fields has generated some controversy. In order to investigate this problem in more detail, we start revisiting the gravitational perturbation theory in the Chandrasekhar gauge formalism [14,35].

We denote the components of the background metric (1) by $g_{\mu\nu}^{(0)}$. In a first-order theory, the gravitational perturbations are defined as the linear variations $\delta g_{\mu\nu} \equiv h_{\mu\nu}$ on the background metric $g_{\mu\nu}^{(0)}$, i.e., the perturbed metric is given by $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$. In the present case, the nonzero components of the perturbation $h_{\mu\nu}$ may be written as

$$\begin{aligned} h_{tt} &= -2f\mu_0, & h_{t\varphi} &= r^2 q_0, \\ h_{rr} &= 2f^{-1}\mu_2, & h_{r\varphi} &= r^2 q_2, \\ h_{zz} &= 2r^2\mu_3, & h_{z\varphi} &= r^2 q_3, \\ h_{\varphi\varphi} &= 2r^2\psi, \end{aligned} \quad (7)$$

where the other components of $h_{\mu\nu}$ are set to zero by an appropriate gauge choice as done by Chandrasekhar [14]. The functions μ_0, μ_2, μ_3 q_0, q_2, and q_3 are all small quantities when compared to unity.

Under the substitution $\varphi \to -\varphi$, the variables q_0, q_2, and q_3 induce odd-parity variations in the metric, and so they are called axial (or odd) perturbations. On the other hand, the variables μ_0, μ_2, μ_3, and ψ induce even-parity variations under sign change of φ, and so they represent polar (even) metric perturbations.

In the Chandrasekhar gauge, the Einstein field equations for the gravitational perturbations result in a set of coupled differential equations. However, in the case of axially symmetric perturbations, i.e., in the case the perturbation functions do not depend on the variable φ, the system of equations can be decoupled into two independent sets of equations, one for each perturbation sector. In fact, due to the plane-symmetric nature of the spacetime, it is always possible to choose the orientation of the frame (in the φ, z plane) in such a way that the perturbation functions result independent of φ. Hence, without loss of generality, from now on we consider only the axially symmetric perturbations, since this choice allows us to study independently each one of the perturbation sectors.

Since metric (1) does not depend on the coordinates t, z, and φ, any perturbation function $F(t, r, z, \varphi)$ in (7) can be conveniently represented in terms of Fourier modes as

$$F(t, r, z, \varphi) \sim \tilde{F}(r)e^{i(m\varphi + kz - \omega t)}, \quad (8)$$

with $\tilde{F}(r)$ being a function of r only, and m and k being real numbers. The wave numbers m and k may be quantized on not, depending on the topology of the $t, r = $ constant

subspace. Notice that, since we are interested in axis-symmetric perturbations, we put m to zero.

B. Fundamental perturbation equations

1. Axial perturbations

For nonvanishing wave numbers, it is possible to describe the axial perturbations by a single gauge-invariant quantity $Z^{(-)}$, defined in terms of the quantities $\tilde{q}_2$ and $\tilde{q}_3$ (see, e.g., Ref. [26]). In the Fourier space, this master variable satisfies the differential equation

$$\Lambda^2 \tilde{Z}^{(-)} = V^{(-)} \tilde{Z}^{(-)}, \tag{9}$$

where the function $\tilde{Z}^{(-)} = \tilde{Z}^{(-)}(r)$ is defined by

$$\tilde{Z}^{(-)} = -rf\left[\frac{d}{dr}\tilde{q}_3(r) - ik\tilde{q}_2(r)\right], \tag{10}$$

the effective potential $V^{(-)}$ is given by

$$V^{(-)} = \frac{f}{r^2}\left(k^2 - \frac{6M}{r}\right), \tag{11}$$

and Λ^2 represents the differential operator

$$\Lambda^2 = \frac{d^2}{dr_*^2} + \omega^2, \tag{12}$$

with r_* being the Regge-Wheeler tortoise coordinate, defined in such a way that $dr_* = f^{-1}dr$.

For future reference, we introduce here the operators

$$\Lambda_\pm = \frac{d}{dr_*} \pm i\omega, \tag{13}$$

which satisfy the relations $\Lambda^2 = \Lambda_+\Lambda_- = \Lambda_-\Lambda_+$.

2. Polar perturbations

Similarly to the axial sector, the polar metric perturbations may be combined to construct a gauge-invariant function $Z^{(+)}$, which satisfy the Fourier-transformed differential equation

$$\Lambda^2 \tilde{Z}^{(+)} = V^{(+)} \tilde{Z}^{(+)}, \tag{14}$$

where, for nonvanishing wave numbers, $\tilde{Z}^{(+)} = \tilde{Z}^{(+)}(r)$ is defined in terms of the quantities $\tilde{\psi}$ and $\tilde{\mu}_3$ by

$$\tilde{Z}^{(+)} = \frac{3Mr}{rk^2 + 6M}\left[\left(1 + \frac{rk^2}{3M}\right)\tilde{\psi}(r) - \tilde{\mu}_3(r)\right], \tag{15}$$

and the effective potential $V^{(+)}$ is given by

$$V^{(+)} = \frac{f}{r^2}\left[k^2 + \frac{72M^2(M\ell^2 + r^3) - 6k^4M\ell^2 r}{r\ell^2(rk^2 + 6M)^2}\right]. \tag{16}$$

In what follows, we consider the particular case of perturbations with zero wave numbers.

C. Perturbations with zero wave numbers

1. General remarks

The perturbations characterized by zero wave numbers $(m, k) = (0, 0)$ do not propagate along the directions parallel to the brane. However, they could be associated to waves propagating along the radial direction. In fact, in this case there are additional gauge degrees of freedom and the physical interpretation of the metric perturbations is not straightforward. Using the Chandrasekhar gauge formalism, the authors of Ref. [24] have presented a set of solutions for black-brane perturbations with zero wave numbers. For completeness, and for future reference, we rewrite those solutions here.

2. Axial perturbations

There exists extra gauge freedom that arises in the zero wave number case, i.e., there is no relation among q_0, q_2 and q_3, and so q_0 cannot be eliminated. Therefore, it is possible to reduce the metric perturbations of the axial sector to a single nonzero component, namely,

$$h_{t\varphi} = q_0 r^2 = -\frac{J}{r}, \tag{17}$$

where J is a constant. As a consequence, the line element of the perturbed spacetime reads

$$ds^2 = -f(r,M)dt^2 + \frac{dr^2}{f(r,M)} - \frac{2J}{r}d\varphi dt + r^2(d\varphi^2 + dz^2). \tag{18}$$

Depending on the compactness of φ and z, the foregoing metric may represent a slowly (J is small) rotating black string or black torus [32,33]. In the case of a planar topology ($\mathbb{R}^2$), a further coordinate transformation (an infinitesimal Lorentz boost) in the t, φ plane puts the metric back to the unperturbed form (1), and so the black brane does not rotate. For the cylindrical and toroidal topologies, this Lorentz boost is a globally "forbidden" transformation, and then it results in a slowly rotating geometry, as just mentioned.

3. Polar perturbations

There is a larger variety of phenomena associated to the zero wave number polar perturbations than the ones associated to the axial modes. In particular, the gauge freedom of the zero wave number case can be used to write the perturbed metric in the form [24]

$$ds^2 = -f(r, M + \delta M)dt^2 + f^{-1}(r, M + \delta M)dr^2$$
$$+ r^2(e^{2\psi}d\varphi^2 + e^{-2\psi}dz^2), \tag{19}$$

where δM stands for an increment in the mass parameter, and $\psi(t, r) = \tilde{\psi}(r)e^{-i\omega t}$ is a perturbation function, whose Fourier transform satisfies a wave equation of the same form as Eq. (14), with $\tilde{Z}^{(+)} = r\tilde{\psi}$ and the effective potential

$$V^{(+)} = \frac{2f}{\ell^2 r^2}\left(r^2 + \frac{M\ell^2}{r}\right). \tag{20}$$

For the sake of comparison, it is important at this point to investigate the limit $k \to 0$ of the results presented in Sec. III B 2. In such a limit, Eq. (15) results in $\tilde{Z}^{(+)} = r(\tilde{\psi} - \tilde{\mu}_3)/2$. On the other hand, by comparing metric (19) with the corresponding relations in (7), we obtain $\mu_3 = -\psi$ at first order in a perturbative expansion. Hence, Eq. (15) reduces to

$$\tilde{Z}^{(+)}(r) = r\tilde{\psi}(r). \tag{21}$$

This is the same relation between the wave function $Z^{(+)}$ and the metric perturbation $\psi(t, r)$, as it was found in Ref. [24] for the metric (19). It is also straightforward verifying that, for a vanishing value of k, the potential $V^{(+)}$ given by Eq. (16) reduces to the expression given in Eq. (20). Therefore, we conclude that the set of equations (14), (15) and (16) describes the polar perturbations for all wave number values, and hence the zero wave number polar perturbations represent also gravitational waves.

Here it is interesting to compare the zero wave number polar perturbations of planar geometries to the special modes of polar perturbations of spherically symmetric geometries. For spherically symmetric black holes, the physical interpretation of the special polar perturbations with azimuthal number $l = 0$ and $l = 1$ cannot be obtained from the equations of motion for perturbations with $l \geq 2$. In comparison, positive result here introduces a new way to investigate the polar perturbations of black branes with zero wave numbers, just by taking $k = 0$ directly into the general equations (14), (15) and (16).

To conclude this section let us mention once again that there is a dispute regarding the physical interpretation of the resulting perturbations in the plane symmetric case, since some authors conclude that there exist no waves in the zero wave number case of the gravitational perturbations of plane-symmetric AdS black holes (see Refs. [19,23]). In order to investigate this problem more closely, we shall analyze other geometric quantities from which the physical interpretation of the perturbations can be obtained. The Weyl curvature scalars appearing in the Newman-Penrose formalism are the best candidates for such an analysis, and so we calculate them below.

IV. THE PHYSICAL MEANING OF THE BLACK-BRANE PERTURBATIONS ACCORDING TO THE SZEKERES' PROPOSAL

A. The physical interpretation of the spacetime curvature

Here we follow the strategy suggested by Szekeres [21], according to which the physical meaning of the Weyl scalars is derived from the geodesic deviation equations. In this proposal, the geodesic deviation equations is projected onto an orthonormal basis $(u^\mu, s^\mu, e^\mu_{(2)}, e^\mu_{(3)})$, where u^μ is the four-velocity of the observer and s^μ, $e^\mu_{(2)}$ and $e^\mu_{(3)}$ are orthogonal spacelike four-vectors. The null tetrad defined in Eqs. (4) and (5) are related to this new tetrad by

$$l^\mu = (u^\mu + s^\mu), \qquad m^\mu = \frac{1}{\sqrt{2}}(e^\mu_{(2)} + ie^\mu_{(3)}),$$
$$n^\mu = \frac{1}{2}(u^\mu - s^\mu), \qquad m^{*\mu} = \frac{1}{\sqrt{2}}(e^\mu_{(2)} - ie^\mu_{(3)}). \tag{22}$$

On the basis of these relations, Szekeres wrote the geodesic deviation equation in terms of the Weyl scalars and showed that the scalar $\Psi_4(\Psi_0)$ describes a gravitational wave propagating in the direction of s^μ $(-s^\mu)$. In turn, the scalar Ψ_3 (Ψ_1) corresponds to a longitudinal component of the gravitational field in the direction of s^μ $(-s^\mu)$. Finally, the real part of the scalar Ψ_2 is associated with Newton-Coulombian effects of the gravitational field with a principal direction s^μ.

It is worth emphasizing that the real and imaginary parts of the Weyl scalar Ψ_4 are associated, respectively, with the "+" and "×" polarization modes of the gravitational waves propagating in the s^μ direction (see Fig. 1). The two parts of the Weyl scalar Ψ_0 produce the same effect that Ψ_4 but with gravitational waves propagating in the $-s^\mu$ direction. Besides that, just the real part of Ψ_2 appears in the geodesic deviation equations. This part of Ψ_2 is associated with a force that deforms a sphere of free particles around an observer, turning it into an ellipsoid with principal axis in the s^μ direction (see Fig. 2), which is typical of bodies in a central field.

B. Perturbations in a Petrov type-D spacetime

For a general Petrov type-D spacetime, a perturbation $\delta\Psi_j$ in an arbitrary Weyl scalar can be written as
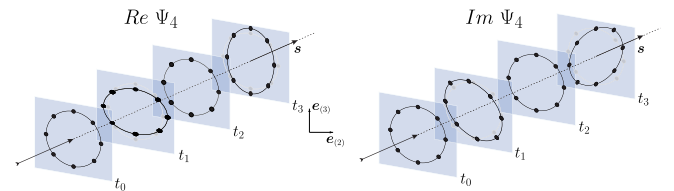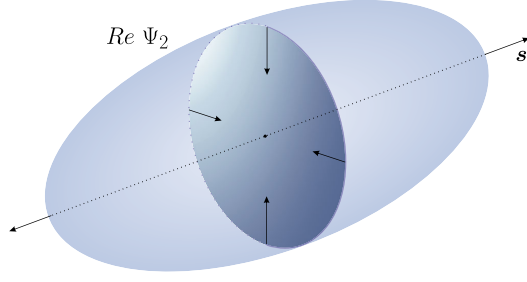

FIG. 1. The two polarization modes of gravitational waves and the Weyl scalar Ψ_4.

FIG. 2. Newton-Coulombian effect of Ψ_2.

$$\delta\Psi_j = \delta\Psi_j^P + i\delta\Psi_j^A, \quad j = 0, \ldots, 4, \qquad (23)$$

where $\delta\Psi_j^P$ is the part of the Weyl scalar given in terms of the polar metric perturbations, while $\delta\Psi_j^A$ is the part of the Weyl scalar given in terms of the axial metric perturbations.

Moreover, it is important to emphasize that perturbations in the Weyl scalars are subject to two different kinds of gauge freedom. The first gauge freedom is connected with the infinitesimal transformations on the tetrad vectors. When a scalar is invariant under these tetrad transformations, it is said to be tetrad-gauge invariant. The second gauge freedom is associated to infinitesimal coordinate transformations, $x^\mu \to x^\mu + \xi^\mu$, and a quantity that is invariant under this kind of transformation is said to be coordinate-gauge invariant.

In the case of gravitational perturbations on a Petrov type-D spacetime, it is possible to show that the perturbations $\delta\Psi_0$ and $\delta\Psi_4$ are both coordinate and tetrad-gauge invariant quantities. However, the perturbations $\delta\Psi_1$ and $\delta\Psi_3$ are only coordinate-gauge invariant quantities and, therefore, we can choose a tetrad orientation such that $\delta\Psi_1$ and $\delta\Psi_3$ vanish (see, e.g., [36] for more details). Finally, for a Petrov type-D background, $\delta\Psi_2$ is tetrad-gauge invariant, but just the imaginary part $\delta\Psi_2^A$ is also coordinate-gauge invariant. For this reason, we are allowed to choose a particular gauge such that $\delta\Psi_2^P = 0$.

C. Axial sector

1. Perturbations with nonzero wave numbers

The aim of this subsection is to write the linear variations in the Weyl scalars in terms of the wave function $\tilde{Z}^{(-)}$ and, from the physical interpretation of the Weyl scalars, to extract the meaning of the axial gravitational fluctuations of the black branes. In order to obtain an expression for the Weyl-scalar perturbations in terms of the metric variations, it is necessary to know the relations between the perturbations in the Weyl scalars and the components of the Weyl tensor (see, e.g., [37] for details on such relationships). For the axial perturbations, we write the linear variations of the Weyl scalar $\delta\tilde{\Psi}_j^A$ in terms of the metric perturbations (7) and, after that, we use Eq. (10) to express the result in terms

of the master variable $\tilde{Z}^{(-)}$. After some algebra, we can then cast the perturbation $\delta\tilde{\Psi}_0^A$ into the form

$$2i\omega\delta\tilde{\Psi}_0^A = \frac{f^{-2}}{r}[V^{(-)} + (W^{(-)} - 2i\omega)\Lambda_-]\tilde{Z}^{(-)}, \qquad (24)$$

where the potential $V^{(-)}$ and the operator Λ_- are defined respectively by Eqs. (11) and (13), and the function $W^{(-)}$ is given by

$$W^{(-)} = -\frac{6M}{r^2}. \qquad (25)$$

Similarly, we can express the perturbation $\delta\tilde{\Psi}_4^A$ as

$$2i\omega\delta\tilde{\Psi}_4^A = -\frac{1}{4r}[V^{(-)} + (W^{(-)} + 2i\omega)\Lambda_+]\tilde{Z}^{(-)}, \qquad (26)$$

where the notation here is the same as in Eq. (24) and the operator Λ_+ is defined by Eq. (13).

A necessary and important further step is to write Eqs. (24) and (26) in terms of the fundamental variables $Y_{\pm 2}$, which arise in the study of the black-brane gravitational perturbations via the Newman-Penrose formalism (see, e.g., Ref. [27]). We have that $Y_{+2} = rf^2\delta\Psi_0$ and $Y_{-2} = 4r\delta\Psi_4$, and hence, combining these relations to Eqs. (24) and (26), we obtain

$$2i\omega\tilde{Y}_{+2}^A = [V^{(-)} + (W^{(-)} - 2i\omega)\Lambda_-]\tilde{Z}^{(-)},$$
$$-2i\omega\tilde{Y}_{-2}^A = [V^{(-)} + (W^{(-)} + 2i\omega)\Lambda_+]\tilde{Z}^{(-)}. \qquad (27)$$

From the Szekeres interpretation, we know that the Weyl scalars shown in Eqs. (24) and (26) are associated with gravitational waves propagating in opposite directions. Therefore, we conclude that axial perturbations with $k > 0$ generate ingoing and outgoing gravitational waves.

Lastly, it is possible to write the perturbation $\delta\tilde{\Psi}_2^A(r)$ in terms of the variable $\tilde{Z}^{(-)}$ as

$$\delta\tilde{\Psi}_2^A = -\frac{k^2}{4r^3}\tilde{Z}^{(-)}. \qquad (28)$$

2. Perturbations with zero wave numbers

In the case of zero wave numbers, Eq. (27) does not hold. Because of extra gauge freedom, the master variable $Z^{(-)}$ can be set to zero. Moreover the scalars Ψ_0^A and Ψ_4^A are zero and no gravitational wave is detected. As discussed in Sec. III C 2, the axial perturbations with $k = 0$ generate at most a slow rotation on the topologically compact black brane spacetimes. The result is the perturbed metric (18). In fact, for this metric the Weyl scalars $\delta\Psi_1$ and $\delta\Psi_3$ are proportional to the angular momentum J, but as discussed in Sec. IV B the tetrad-gauge freedom may be used to set them to zero. Therefore, axial perturbations with zero wave numbers preserve the type D structure of the background.

D. Polar sector

1. Perturbations with nonzero wave numbers

In the case of polar perturbations with $k \neq 0$, the Weyl scalar $\delta \Psi_0^P$ can be cast into the following form

$$\delta \tilde{\Psi}_0^P = -\frac{f^{-2}}{r} [V^{(+)} + (W^{(+)} - 2i\omega)\Lambda_-] \tilde{Z}^{(+)}, \quad (29)$$

while the scalar $\delta \tilde{\Psi}_4^P$ can be written as

$$\delta \tilde{\Psi}_4^P = -\frac{1}{4r} [V^{(+)} + (W^{(+)} + 2i\omega)\Lambda_+] \tilde{Z}^{(+)}. \quad (30)$$

The function $W^{(+)}$ that appears in Eqs. (29) and (30) is given by

$$W^{(+)} = -\frac{6M(2r^3 + k^2\ell^2 r + 2M\ell^2)}{\ell^2 r^2 (k^2 r + 6M)}, \quad (31)$$

and the operators $\Lambda_\pm$ are defined in Eq. (13).

The polar perturbation variables $\tilde{Y}_{\pm 2}$ can be written in terms of the master variable $\tilde{Z}^{(+)}$ as

$$\tilde{Y}_{+2}^P = -[V^{(+)} + (W^{(+)} - 2i\omega)\Lambda_-] \tilde{Z}^{(+)},$$
$$\tilde{Y}_{-2}^P = -[V^{(+)} + (W^{(+)} + 2i\omega)\Lambda_+] \tilde{Z}^{(+)}. \quad (32)$$

Equations (27) and (32) are identical to the results found in Ref. [27]. It worth mentioning that the authors of Ref. [27] followed a different approach, by using the Newman-Penrose formalism [9] and the Chandrasekhar transformation theory [36], so to get the fundamental Eqs. (9) and (14). Here, however, relations (27) and (32) were obtained straightforwardly from the perturbations of the Weyl scalars. This fact indicates that both approaches are consistent in the case of a planar geometry, just as it happens with gravitational perturbations of spherically symmetric black-hole spacetimes.

At last, we conclude from the Szekeres approach that the zero wave number polar perturbations of black branes represent gravitational waves that propagate in two different null directions.

2. Perturbations with zero wave numbers

In the case of polar perturbations with $k = 0$, the linear variations $\delta \tilde{\Psi}_0^P$ and $\delta \tilde{\Psi}_4^P$ may be reduced to the same general expressions as given by Eqs. (29) and (30), now with $V^{(+)}$ and $\tilde{Z}^{(+)}$ given, respectively, by Eqs. (20) and (21), while the $W^{(+)}$ function now reads

$$W^{(+)} = -\frac{2}{r}\left(\frac{r^2}{\ell^2} + \frac{M}{r}\right). \quad (33)$$

Therefore, as $\delta \tilde{\Psi}_0^P$ and $\delta \tilde{\Psi}_4^P$ are nonvanishing, from the Szekeres interpretation we conclude that polar perturbations with zero wave numbers are also associated with radial gravitational waves.

It is worth noticing that the wave character of the complete functions $Y_{\pm 2}(t, r)$ may be more easily exhibited through their behavior close to the event horizon $r \to r_h = (2M\ell^2)^{1/3} (r_* \to -\infty)$. In such a region, the potential $V^{(+)}$ vanishes [cf. Eq. (20)] and then the wave equation (14) leads to a solution (in the physical space) of the form $Z^{(+)}(t, r) \to e^{i\omega(\pm r_* - t)}$. Moreover, considering the condition of having just ingoing waves at the horizon, the only allowed solution is $Z^{(+)} \to e^{-i\omega(r_* + t)}$. As a consequence, we get

$$\Lambda_+ Z^{(+)} \to 0, \qquad \Lambda_- Z^{(+)} \to -2i\omega e^{-i\omega(r_* + t)},$$
$$W^{(+)} \to -\frac{6M}{(2M\ell^2)^{2/3}}. \quad (34)$$

Hence, by using the relations (32), it is possible to express the asymptotic form of the functions $Y_{\pm 2}(t, r)$ close to the horizon (for $k = 0$) as

$$Y_{+2} \to 4\omega \left[\omega - \frac{3Mi}{(2M\ell^2)^{2/3}}\right] e^{-i\omega(r_* + t)},$$
$$Y_{-2} \to 0. \quad (35)$$

It is then seen that the function Y_{+2} reduces to a plane wave traveling radially inwards the black-brane horizon. Additionally, the function Y_{-2}, which is related to the $\delta \Psi_4$ scalar, tends to zero in such a region. This behavior is expected since, as it was first discussed in Ref. [21], the scalar Ψ_4 represents an outgoing wave and, because of the imposed boundary condition, it is not possible to observe such a wave close to the horizon. Moreover, for stationary waves ($\omega = 0$), it also follows that $Y_{+2} = 0$, showing no gravitational radiation at all (as expected). Furthermore, following the same procedure and considering the outgoing solution $Z^{(+)} \to e^{i\omega(r_* - t)}$, it is possible to show that this solution corresponds to a plane wave traveling radially outwards.

In short, the conclusion of the analysis presented in this section is that the zero wave number polar perturbations correspond to gravitational waves traveling in the radial direction. This, in turn, answers affirmatively the original question on whether the zero wave number gravitational perturbations of black branes can be associated to the production of gravitational waves, or not.

At last, the computation of the scalar $\delta \Psi_2$ results in

$$\delta \Psi_2 = -\frac{\delta M}{r^3}, \quad (36)$$

which describes a perturbation in the "Coulomb" gravitational field of the black brane. Such a result is expected from the discussion presented in Sec. III C 3. From

Eq. (19), δM is a small variation in the mass parameter of the black brane and, therefore, it corresponds only to a perturbation in the Coulomb-type term of the Weyl scalar Ψ_2.

V. THE PHYSICAL INTERPRETATION OF THE POLAR PERTURBATIONS FROM PIRANI'S CRITERION

A. General remarks

In accordance with Pirani [25]: *"At any event in empty spacetime, gravitational radiation is present if the Riemann tensor is of Type* II *or Type* III, *but not if it is of Type* I.*"*

In the current nomenclature of the Petrov classification, a Riemann tensor of type I represents a spacetime of type I (non-degenerated Riemann tensor of type I) or a spacetime of type D (degenerated Riemann tensor of type I), while a Riemann tensor of type II represents a spacetime of type II (non-degenerated Riemann tensor of type II) or a spacetime of type N (degenerated Riemann tensor of type II). Finally, a Riemann tensor of type III corresponds to a spacetime of type III in the Petrov classification.

Another important point to be mentioned here concerns the application of the Pirani's criterion in asymptotically (anti-)de Sitter spacetimes. Although proposed for asymptotically flat spacetimes, such a criterion is based in the Petrov classification scheme [38] for the canonical form of the Riemann tensor. In this classification, an Einstein manifold is assumed, i.e., the corresponding Ricci and metric tensors satisfy the equation $R_{\mu\nu} = \kappa g_{\mu\nu}$ with constant κ. This means that the Pirani's criterion can be used to certify the existence of gravitation waves also in the case of asymptotically (anti–)de Sitter spacetimes.

B. The canonical form of the Riemann tensor for the background spacetime

Using the Petrov technique of Ref. [38], it is possible to put the Riemann tensor for the background metric (1) in the following canonical matrix form

$$
R_{AB} = \left(
\begin{array}{ccc|ccc}
\alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \alpha_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\alpha_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\alpha_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\alpha_3
\end{array}
\right), \quad (37)
$$

where the components of the Riemann tensor are represented in a 6-dimensional pseudo-Euclidean space and the capital indices A and B assume values from 1 to 6. The scalar quantities α_i are given by

$$
\alpha_1 = \frac{\Lambda_c}{3} - 2\Psi_2, \qquad \alpha_2 = \alpha_3 = \frac{\Lambda_c}{3} + \Psi_2, \quad (38)
$$

where Λ_c is the cosmological constant and Ψ_2 is only nonzero Weyl scalar given by Eq. (6). Since the black brane is an asymptotically AdS spacetime, it follows that $\sum_i \alpha_i = \Lambda_c$. As expected, the canonical form (37) represents a Petrov type-D spacetime and, according to the Pirani's criterion, there are no gravitational waves in such a background.

C. Riemann canonical form for nonzero wave number polar perturbations

In this section we write the canonical Riemann tensor for the polar perturbations of the metric (1) with an arbitrary wave number. Again, the tetrad-gauge freedom is used to make $\delta\Psi_1 = \delta\Psi_3 = 0$. In this case, the canonical form of the Riemann tensor is

$$
R_{AB} = \left(
\begin{array}{ccc|ccc}
\alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \alpha_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\alpha_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\alpha_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\alpha_3
\end{array}
\right), \quad (39)
$$

where the eigenvalues α_1, α_2, and α_3 can be written as

$$
\alpha_1 = \frac{\Lambda_c}{3} - 2\Psi_2 - 2\delta\Psi_2,
$$
$$
\alpha_2 = \frac{\Lambda_c}{3} + \Psi_2 + \delta\Psi_2 + \sqrt{\delta\Psi_0 \delta\Psi_4},
$$
$$
\alpha_3 = \frac{\Lambda_c}{3} + \Psi_2 + \delta\Psi_2 - \sqrt{\delta\Psi_0 \delta\Psi_4}. \quad (40)
$$

The Fourier transforms of the perturbations $\delta\Psi_0$ and $\delta\Psi_4$ are given respectively by Eqs. (29) and (30), and $\delta\Psi_2$ can be made zero by an appropriate coordinate-gauge choice.

The canonical matrix form (39) represents a Petrov type-I spacetime and, from the Pirani's criterion, there is no gravitational-wave propagation in this spacetime. This finding seems to be in opposition to the results of the analysis presented in Sec. IV D 2. However, as discussed in Ref. [39], any kind of gravitational perturbation in a Petrov type-D background leads, in general, to a type-I spacetime. It was also shown in the same work that metric perturbations of a Schwarzschild black hole (with multipole index $l \geq 2$) also lead to a Petrov type-I spacetime. Such a result contradicts the Pirani's criterion, in view of the well-established fact that gravitational perturbations of a Schwarzschild black hole with multipole number higher than unity describe the propagation of gravitational waves.

The origin of the contradiction is the Pirani's assumption that the gravitational radiation propagates along a single direction. Nevertheless, in a spacetime with nonvanishing scalars Ψ_4 and Ψ_0, both the ingoing and outgoing waves are present. In this case, the spacetime is more general than the ones described by the canonical forms of type II (or type N) and type III. Therefore, the only option in the Petrov classification scheme that the problem fits in is the type I. In the next section the Pirani's criterion for polar perturbations with $k = 0$ is discussed and, by imposing the ingoing-wave condition at the event horizon, it is shown that the Riemann tensor reduces to the typical form characterizing a type-II Petrov spacetime.

D. Riemann canonical forms for zero wave number polar perturbations

For the polar gravitational perturbations with zero wave numbers, the canonical form of the Riemann tensor and its eigenvalues are the same as presented, respectively, in Eqs. (39) and (40). However, adopting the coordinate gauge of Ref [24], the perturbation $\delta\Psi_2$ is now given by Eq. (36), and it is necessary to take $k = 0$ in the expressions for the Fourier transforms of $\delta\Psi_0$ and $\delta\Psi_4$ given by Eqs. (29) and (30), respectively.

As discussed in Sec. IV D 2, it is interesting to consider an ingoing-wave condition in the region close to the horizon and to investigate the canonical form of the Riemann tensor in that limit, i.e., for $r_* \to -\infty$. Once again, it is more convenient to work with the variables Y_{+2} and Y_{-2}, instead of the corresponding Weyl scalars $\delta\Psi_0$ and $\delta\Psi_4$, respectively. As shown above, Y_{-2} vanishes at the horizon while Y_{+2} is given by Eq. (35). So, the canonical form of the Riemann tensor for $r_* \to -\infty$ is given by

$$R_{AB} = \left(\begin{array}{ccc|ccc} \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha_2 - \sigma & \cdot & \cdot & \cdot & -\sigma \\ \cdot & \cdot & \alpha_2 + \sigma & \cdot & -\sigma & \cdot \\ \hline \cdot & \cdot & \cdot & -\alpha_1 & \cdot & \cdot \\ \cdot & \cdot & -\sigma & \cdot & -\alpha_2 + \sigma & \cdot \\ \cdot & -\sigma & \cdot & \cdot & \cdot & -\alpha_2 - \sigma \end{array}\right),$$

(41)

where the eigenvalues are

$$\alpha_1 = \frac{\Lambda_c}{3} - 2\Psi_2 - 2\delta\Psi_2,$$

$$\alpha_2 = \alpha_3 = \frac{\Lambda_c}{3} + \Psi_2 + \delta\Psi_2,$$

(42)

and

$$\sigma = \frac{f^{-1}}{4r} Y_{+2}$$

(43)

is the contribution of the ingoing gravitational wave. Hence, close to the horizon, the Riemann tensor assumes the canonical form of a type-II gravitational field, which, according to the Pirani's criterion, characterizes the presence of the gravitational radiation.

Thus, the conclusion here is that gravitational perturbations whose modes describe simultaneously ingoing and outgoing gravitational radiation, i.e., perturbations with both $\delta\Psi_0$ and $\delta\Psi_4$ different from zero, generate a Riemann tensor of type I in the Petrov classification. In this case, the Pirani criterion fails to identify the gravitational waves. On the other hand, as we have just shown, for the particular case when $\delta\Psi_4 = 0$, the Riemann tensor takes the type II form and then one concludes from the Pirani criterion that it describes ingoing gravitational waves. This result suggests that such a criterion is conclusive as long as the radiation propagates along a specific direction in the spacetime.

VI. ZERO WAVE NUMBER POLAR GRAVITATIONAL PERTURBATIONS IN THE KODAMA-ISHIBASHI-SETO FORMALISM

As a final analysis, we explore here the formalism of Kodama, Ishibashi, and Seto [19] to show explicitly that zero wave number polar perturbations represent gravitational waves propagating in the radial direction. We start by setting up notation. The metric is split into the form

$$ds^2 = g_{\mu\nu}dz^\mu dz^\nu = g_{ab}(y)dy^a dy^b + r^2(y)d\Omega_2^2, \quad (44)$$

where $y^a = \{t, r\}$ and $d\Omega_2^2 = \gamma_{ij}(x)dx^i dx^j$ is the metric of the two-dimensional maximally symmetric space with coordinates $x^i = \{\varphi, z\}$. The background metric is given by $g_{ab} = \text{diag}(-f, 1/f)$ and $\gamma_{ij} = \delta_{ij}$ with δ_{ij} being the Kronecker delta.

For the case of polar gravitational perturbations, the metric perturbations $h_{\mu\nu}$ may be decomposed in terms of the scalar harmonic function $\mathbb{S}$ as [19]

$$h_{ab} = f_{ab}\mathbb{S},$$
$$h_{ai} = rf_a\mathbb{S}_i,$$
$$h_{ij} = 2r^2(H_L\gamma_{ij}\mathbb{S} + H_T\mathbb{S}_{ij}), \quad (45)$$

where $\mathbb{S}$ represents the solutions of the harmonic equation $(\hat{D}_i\hat{D}^i + k^2)\mathbb{S} = 0$, and quantities $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ introduced above are respectively the vectorial and tensorial harmonic functions defined by

$$\mathbb{S}_i = -\frac{1}{k}\hat{D}_i\mathbb{S},$$

$$\mathbb{S}_{ij} = \frac{1}{k^2}\hat{D}_i\hat{D}_j\mathbb{S} + \frac{1}{2}\gamma_{ij}\mathbb{S}, \quad (46)$$

with $\hat{D}_i$ standing for the covariant derivative with respect to the metric γ_{ij}. Again, for black-brane

perturbations, the wave number k may assume any real non-negative value.

The coefficients of the decompositions (45) are not invariant under the infinitesimal gauge transformation $x^\mu \to x^\mu + \xi^\mu$, i.e., they are dependent on the identification map between points of the background spacetime and the physical spacetime. As well as the perturbations in the metric, the vector ξ^μ can be decomposed in terms of the harmonic functions as

$$\xi_a = T_a \mathbb{S}, \qquad \xi_i = rL \mathbb{S}_i. \qquad (47)$$

As discussed in Refs. [19,23] for $k^2 > 0$, it is possible to combine the coefficients in Eqs. (45) to build the following gauge-invariant quantities:

$$F = H_L + \frac{1}{2}H_T + \frac{1}{r}D^a r X_a,$$
$$F_{ab} = f_{ab} + D_a X_b + D_b X_a, \qquad (48)$$

where $X_a = \frac{r}{k}f_a + D_a H_T$ and D_a is the covariant derivative with respect to the metric g_{ab}. However, these relations do not hold for $k^2 = 0$ and, moreover, as shown in Ref. [40], in such a case the functions F and F_{ab} are no longer gauge-invariant quantities. Thus, once again a detailed study of the gauge freedom of the polar perturbations with zero wave numbers is mandatory.

The first step is then to write the transformations of the quantities H_T, H_L, f_a, and f_{ab}, defined in (45) under the infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, with ξ^μ given by (47). For a vanishing wave number $k = 0$ (besides having put $m = 0$ since the beginning), it follows

$$f_{ab} \to f_{ab} - D_a T_b - D_b T_a, \qquad H_T \to H_T,$$
$$f_a \to f_a - rD_a\left(\frac{L}{r}\right), \qquad H_L \to H_L - \frac{D^a r}{r}T_a. \qquad (49)$$

It is promptly seen that the function H_T is now gauge invariant. Additionally, convenient choices of L in two successive gauge transformations can lead f_a to zero, i.e., $f_t = f_r = 0$. Another gauge degree of freedom is fixed by choosing T_r so as to get rid of the perturbation H_L, i.e., we also may choose $H_L = 0$. At last, the function T_t helps to take the 2×2 matrix f_{ab} into a diagonal form, i.e., with $f_{rt} = f_{tr} = 0$. These choices fix all the gauge degrees of freedom, reducing the perturbed metric to

$$ds^2 = (g_{ab} + f_{ab}\mathbb{S})dy^a dy^b$$
$$+ r^2(\gamma_{ij} + 2H_T\mathbb{S}_{ij})dx^i dx^j, \qquad (50)$$

where g_{ab} and γ_{ij} stand for the background values of the metric, cf. Eq. (44).

Now let us examine the harmonic functions. According to Kodama and Ishibashi [23], for $k = 0$ the scalar

harmonic $\mathbb{S}$ is a constant and the vectorial and tensorial harmonic functions in (46) are not defined. On the other hand, Dias and Reall assume in Ref. [20] that, for vanishing wave numbers, $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ are zero by definition. Therefore, in both papers, $\mathbb{S}_{ij}$ is neglected and it is argued that the metric (50) preserves the symmetry of the background and, on the basis of the Birkhoff theorem, the perturbation f_{ab} represents just a variation in the mass parameter of the black brane. This is clearly true for spherically symmetric black-hole spacetimes, but it contradicts the results presented in the preceding section regarding the zero wave number polar perturbations of black branes.

To solve this seeming inconsistency, we first notice that the harmonic functions $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ in Eq. (46) are not defined for $k = 0$. In order to raise the indeterminacy, we follow the standard procedure by calculating explicitly such functions for arbitrary k and by taking the limit of each function as k goes to zero. The scalar harmonic $\mathbb{S}$ for a black-brane spacetime is of the following form

$$\mathbb{S} = e^{\pm ik_j x^j}, \qquad (51)$$

with $x^i = (\varphi, z)$ and $k^i = (0, k)$, and where we have taken $k^\varphi = 0$ because we are dealing with axisymmetric perturbations. For the planar geometry of the black-brane spacetimes, the covariant derivatives in Eqs. (46) reduce to partial derivatives, and we get

$$\mathbb{S}_{zz} = \frac{1}{k^2}\hat{\partial}_z\hat{\partial}_z(e^{\pm ikz}) + \frac{1}{2}\gamma_{zz}e^{\pm ikz} = -\frac{1}{2}e^{\pm ikz},$$
$$\mathbb{S}_{\varphi\varphi} = \frac{1}{2}\gamma_{\varphi\varphi}e^{\pm ikz} = \frac{1}{2}e^{\pm ikz},$$
$$\mathbb{S}_{z\varphi} = \mathbb{S}_{\varphi z} = 0. \qquad (52)$$

Notice that we have neglected the vectorial harmonic functions $\mathbb{S}_i$ because they do not appear in the final perturbed metric (50). Finally, the conclusion is that the limit $k \to 0$ of Eqs. (52) is well defined and gives

$$\mathbb{S} \to 1, \qquad \mathbb{S}_{\varphi\varphi} \to \frac{1}{2}, \qquad \mathbb{S}_{zz} \to -\frac{1}{2}. \qquad (53)$$

Collecting the foregoing results and substituting into the metric (50), it follows

$$ds^2 = (g_{ab} + f_{ab})dy^a dy^b + r^2(e^{H_T}d\varphi^2 + e^{-H_T}dz^2), \qquad (54)$$

where we have used the identity $e^{\pm H_T} = 1 \pm H_T$, which holds for first-order perturbations. Furthermore, using the Einstein equations we find that

$$-(g_{tt} + f_{tt}) = (g_{rr} + f_{rr})^{-1} = f(r, M + \delta M) \qquad (55)$$

Comparing the present approach to the formulation of Sec. III C 3, we find the relation $H_T = 2\psi$. This shows that

metric (54) is identical to that shown in Eq. (19) and obtained in Ref. [24] by means of the Chandrasekhar gauge formalism. This result confirms that zero wave number polar metric perturbations represent also gravitational waves.

VII. DISCUSSION

We have reviewed the physical interpretation of the gravitational perturbations of AdS black branes. The focus was on the identification of a given metric variation with the presence of gravitational waves. The chief motivation is the conflict on the interpretation of the zero wave number polar perturbations of black branes (cf. Refs. [23,24]). In particular, we have just shown that such perturbations in fact represent gravitational waves propagating in the radial direction, a situation that happens for black branes but not for spherically symmetric black holes.

We started by setting up the necessary basic formulation and by showing that the equations in the Chandrasekhar gauge formalism accommodate both the nonzero and the zero wave number gravitational fluctuations. The explicit form of the perturbation equations for the zero wave number cases has been needed for such an analysis. For the polar sector, this task was accomplished by writing the curvature perturbations in terms of the quantities $V^{(+)}$, $W^{(+)}$ and $\tilde{Z}^{(+)}$ [cf. Eqs. (16), (15), (31)], and then by taking the limit of vanishing wave numbers. This approach for polar perturbations of planar black holes with $k = 0$ is quite different from the approach for spherically symmetric black holes, since it is not possible to investigate the special modes $l = 0$ and $l = 1$ from the general equations of perturbations with $l \geq 2$ in the spherically symmetric case.

In the analysis of the gravitational perturbations we performed a direct evaluation of the complex Weyl scalars in terms of the perturbations in the metric. The resulting expressions are in complete accord with those obtained for the vanishing wave number case in Ref. [27] via the Newman-Penrose formalism and the Chandrasekhar transformation theory. Thereby, the equivalence between both procedures, which was known to hold for spherical Schwarzschild black holes, was extended for black branes.

As an additional technique, we relied on the work by Pirani [25] to study the physical meaning of the polar-sector fluctuations. Using this technique, we showed that perturbations with both vanishing and nonvanishing wave numbers lead to Petrov type-I spacetimes. However, according to the Pirani's criterion, this type of spacetime would not be associated to the propagation of gravitational waves. Obviously, this criterion presents problems in some cases. For instance, the ondulatory character of nonvanishing wave number perturbations of black branes is well established in the literature. Moreover, as shown in Ref. [39], non-stationary black-hole perturbations of the Kerr-Newman family lead to Petrov type-I spacetimes, a result which confirms that Pirani's criterion is not conclusive when dealing with Petrov type-I spacetimes.

In the analysis of perturbations with vanishing wave numbers by using the approach by Pirani, we performed an additional investigation. Our results show that such perturbations correspond to type-II gravitational fields in the region very close to the horizon just after imposing the condition of no outgoing waves in that neighborhood. According to the Pirani's criterion, this kind of gravitational field characterizes the presence of gravitational waves. Again, this outcome is in agreement with the study of Ref. [39], where it was shown that in the particular case of vanishing $\delta\Psi_0$ or $\delta\Psi_4$, gravitational perturbations in a type-D background lead to type-II spacetimes. From the analysis of these results we conclude that the Pirani criterion is not conclusive for general perturbations of black branes. The reason is that such a criterion neglects the possibility of coexisting ingoing and outgoing waves and, therefore, the Szerekes proposal is a more efficient invariant technique to investigate the existence of gravitational waves that propagate in different principal null directions.

In regard to the Kodama-Ishibashi-Seto [19] gauge-invariant formalism, we have verified that polar gravitational perturbations of the black branes have a well-defined behavior in the limit of vanishing wave numbers, also in complete agreement with the presence of gravitational waves propagating along the radial direction. It is worth mentioning here that the main aim of the work of Ref. [23] was to investigate the metric linear fluctuations in higher-dimensional spacetimes and the analysis of the vanishing wave number perturbations of black branes was made *en passant*, in a short comment within a long paper presenting many interesting results.

As a possible extension of the present work, it would be interesting to use the physical interpretation of the higher-dimensional Weyl scalars [22] to extract the meaning of the gravitational perturbations of a black brane in a general d–dimensional spacetime. This is a work in progress by ourselves.

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