Accelerating universe with a stable extra dimension in cuscuton gravity

Asuka Ito,[1,*] Yuki Sakakihara,[1,2,†] and Jiro Soda[1,‡]
[1]*Department of Physics, Kobe University, Kobe 657-8501, Japan*
[2]*Department of Mathematics and Physics, Osaka City University, Osaka 558-8585, Japan*

(Received 1 July 2019; published 23 September 2019)

We study Kaluza-Klein cosmology in cuscuton gravity and find an exact solution describing an accelerating four-dimensional universe with a stable extra dimension. A cuscuton which is a nondynamical scalar field is responsible for the accelerating expansion, and a vector field makes the extra dimensional space stable. Remarkably, the accelerating universe in our model is not exactly de Sitter.

DOI: 10.1103/PhysRevD.100.063531

I. INTRODUCTION

As is well known, superstring theory predicts our spacetime has ten dimensions. However, the idea of higher dimensional spacetime itself is not new, in fact, goes back to Kaluza-Klein theory [1,2] which unifies electromagnetic force and gravity in four-dimensional spacetime as pure gravity in five-dimensional spacetime. Of course, since our real world has four dimensions, the extra dimensions must be invisible. This can be realized if the extra dimensions are compacified into a small size. A natural mechanism for compactification of the extra dimensions has been proposed in the context of cosmology. In the original proposal [3], the four-dimensional universe is expanding and the extra dimensions are contracting. Since then, Kaluza-Klein cosmology has been intensively investigated [4–7]. Now, the main issue is how to stabilize the contracting extra dimensions. Actually, more natural scenario is as follows. Initially, all of spatial dimensions were compact and small. Subsequently, in the course of cosmological evolution, only four-dimensional universe has expanded up to the present scale. In this paper, we call this particular scenario "Kaluza-Klein scenario." To our best knowledge, there seems no concrete model to realize this Kaluza-Klein scenario. In particular, it is difficult to construct four-dimensional inflationary universe with stable extra dimensions. Indeed, when we put cosmological constant in higher dimensions, the extra dimensions can be easily decompactified. Recently, the mechanism for compactification has been also discussed in the context of string theory and then most discussions rely on the four-dimensional effective action method. First, the stability of extra dimensions is realized using a four-dimensional nonperturbative mechanism. Next, matters are considered in four dimensions to realize inflation. However, the higher dimensional picture of this stabilization procedure is not obvious. Thus, it is still worth seeking the Kaluza-Klein scenario.

For resolving difficulties in cosmology such as the dark energy problem, modified theories of gravity have been extensively utilized. We can expect that modified gravity also plays a role for realizing the Kaluza-Klein scenario. Indeed, a compactification mechanism in Einstein-aether gravity [8] has been proposed [9]. The aether field defines a preferred spacelike direction and violates the rotational invariance of the five-dimensional space. It is shown that an attractive force of the aether field can stabilize the extra dimension. Unfortunately, the above model suffers from instabilities due to ghosts or tachyons [10–13]. Notice that, in four dimensions, this fact was known as the difficulty of constructing anisotropic inflation models (see review papers [14,15]). Remarkably, in 2009, there appeared a healthy model realizing anisotropic inflationary expansion with a vector field which induces a preferred direction due to a gauge kinetic function [16]. Hence, it is legitimate to apply the anisotropic inflation model in modified gravity to the Kaluza-Klein scenario. As to the modified gravity, in this paper, we focus on cuscuton gravity containing a nondynamical scalar field, the so-called cuscuton [17].

The cuscuton gravity is a minimally modified theory of gravity in the sense that there are only two physical degrees of freedom of a tensor field. It belongs to a subclass of scalar-tensor theories where the Lorentz invariance is broken [18–21] and can be extended into more general class [22]. Aspects of symmetry in cuscuton gravity are studied in [23]. In particular, it has been shown that cuscuton gravity is useful in cosmology [24,25]; for instance, a healthy bouncing solution without instabilities was realized [26]. Moreover, any inflation models could be reconciled with observations by virtue of cuscutons [27]. Hence, it is interesting to apply cuscuton gravity to the Kaluza-Klein scenario.

*asuka-ito@stu.kobe-u.ac.jp
†sakakihara@port.kobe-u.ac.jp
‡jiro@phys.sci.kobe-u.ac.jp

In our model of the Kaluza-Klein scenario, there is a vector field coupled with a cuscuton. We find the first exact solution describing an accelerating universe with a static extra dimension [28,29]. The vector field is responsible for the stability of the extra dimension, and the cuscuton drives the accelerating expansion of our three-dimensional space. Note that the universe is not a de Sitter spacetime as a consequence of cuscuton gravity. This is in contrast to the conventional compactification models. Indeed, usually, we use four-dimensional effective potentials for a radion, the radius of the extra dimension, to describe the dynamics of the extra dimensions during inflation or the late time acceleration. Then, the radion is constant only when it is at the minimum of the potential. Consequently, the four-dimensional spacetime has to be the de Sitter if the extra dimension is static. Even if one puts additional matters into the effective theory, they will vanish eventually in light of the Wald's no-hair theorem [30]. It implies that we always obtain the four-dimensional de Sitter spacetime if the extra dimension is static. To avoid this consequence, one can tune the potential so that the minimum point is Minkowski. In this case, we can add the four-dimensional inflaton by hand to get a quasi-de Sitter universe. However, this procedure has no clear meaning from the higher dimensional point of view. The reason why we obtain nonconventional results can be attributed to the followings. First, the presence of cuscutons can violate energy conditions assumed in the no-hair theorem [30,31]. Remarkably, even if the energy condition is violated, there are no instabilities in cuscuton gravity because the cuscuton is nondynamical [25–27]. Second, the cuscuton has a nontrivial coupling with the vector field, which also violates the assumption in the proof of no-hair theorem [14,15].

The paper is organized as follows: in Sec. II, we explain our setup, the action including a cuscuton and a vector field in $(4+1)$-dimensional spacetime, and derive the equations of motion. We then find exact power-law solutions. In Sec. III, we give accelerating universe solutions with a static extra space dimension. In Sec. IV, we investigate the stability of the solutions we found in Sec. III. It turns out that the solutions are attractors in phase space. Section V is devoted to the conclusion. The discussion is extended into general $(n+1)$-dimensional spacetime in the Appendix.

II. ANISOTROPIC (4+1)-DIMENSIONAL SPACETIME WITH CUSCUTON

For simplicity, we start from $(4+1)$-dimensional spacetime with the action including a cuscuton. The case of general $(n+1)$-dimensional spacetime will be studied in Appendix.

In order to violate rotational invariance and realize the compactification, we also include a vector field coupled with the cuscuton. Then, we have

$$\mathcal{S} = \int d^5x\sqrt{-g}\bigg[\frac{1}{2\kappa}R \pm \mu^2\sqrt{-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} - V(\phi) - \frac{1}{4}f(\phi)F_{\mu\nu}F^{\mu\nu}\bigg], \tag{1}$$

where we have defined $\kappa = 1/M_{\mathrm{pl}5}^3$ using five-dimensional Planck scale $M_{\mathrm{pl}5}$. μ is a parameter associated with the kinetic term of the cuscuton, which has the mass dimension $[\mu^2] = 5/2$. The field strength of the $U(1)$ vector field A_μ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

The cuscuton is coupled with the kinetic term of the vector field though a function $f(\phi)$. Note that the mass dimensions of the fields are $[\phi] = 3/2$ and $[A_\mu] = 3/2$. We assume that homogeneous electric fields exist along with the extra space dimension and the cuscuton is supposed to depend only on time,

$$A_\mu = (0,0,0,0,v(t)), \qquad \phi = \phi(t). \tag{3}$$

With these ansatzes, we take a homogeneous but anisotropic metric ansatz discriminating the extra dimension

$$ds^2 = -dt^2 + e^{2\alpha(t)}(e^{-6\sigma(t)}dx_4^2 + e^{2\sigma(t)}(dx^2+dy^2+dz^2)). \tag{4}$$

We expect that the expansion of the extra dimension slows down due to the vector field. Substituting (2)–(4) into (1), we obtain the action,

$$\mathcal{S} = \int d^5x e^{4\alpha}\bigg[\frac{1}{\kappa}(-6\dot\alpha^2 + 6\dot\sigma^2) \pm \mu^2 \mathrm{sgn}(\dot\phi)\dot\phi - V(\phi) + \frac{1}{2}f(\phi)e^{-2\alpha+6\sigma}\dot v^2\bigg], \tag{5}$$

and the equations of motion derived from this action are

$$\ddot\alpha = -4\dot\alpha^2 \mp \frac{\kappa\mu^2}{3}\mathrm{sgn}(\dot\phi)\dot\phi + \frac{2\kappa}{3}V(\phi) + \frac{\kappa}{12}f(\phi)e^{-2\alpha+6\sigma}\dot v^2, \tag{6}$$

$$\dot\alpha^2 = \dot\sigma^2 + \frac{\kappa}{6}\bigg[V(\phi) + \frac{1}{2}f(\phi)e^{-2\alpha+6\sigma}\dot v^2\bigg], \tag{7}$$

$$\ddot\sigma = -4\dot\alpha\dot\sigma + \frac{\kappa}{4}fe^{-2\alpha+6\sigma}\dot v^2, \tag{8}$$

$$\pm 4\mu^2\dot\alpha\,\mathrm{sgn}(\dot\phi) + V_\phi - \frac{1}{2}f_\phi e^{-2\alpha+6\sigma}\dot v^2 = 0, \tag{9}$$

$$(e^{2\alpha+6\sigma}f\dot v)^{\cdot} = 0. \tag{10}$$

The second equation is the Friedmann equation in Einstein's gravity, which is not independent from the others. We integrate (10) and have

$$\dot{v} = f^{-1} e^{-2\alpha - 6\sigma} C, \tag{11}$$

where C is a constant of integration with the mass dimension $[C] = 5/2$. After substituting the solution into (6)–(9), we obtain the four equations as follows:

$$\ddot{\alpha} = -4\dot{\alpha}^2 \mp \frac{\kappa\mu^2}{3}\mathrm{sgn}(\dot{\phi})\dot{\phi} + \frac{2\kappa}{3}V + \frac{\kappa}{12} f^{-1} e^{-6\alpha-6\sigma} C^2, \tag{12}$$

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{\kappa}{6}\left[V + \frac{1}{2} f^{-1} e^{-6\alpha-6\sigma} C^2\right], \tag{13}$$

$$\ddot{\sigma} = -4\dot{\alpha}\dot{\sigma} + \frac{\kappa}{4} f^{-1} e^{-6\alpha-6\sigma} C^2, \tag{14}$$

$$\pm 4\mu^2 \dot{\alpha}\mathrm{sgn}(\dot{\phi}) + V_{\phi} - \frac{1}{2} f_{\phi} f^{-2} e^{-6\alpha-6\sigma} C^2 = 0. \tag{15}$$

From now on, to seek for exact power-law solutions of above equations, we choose the potential of the cuscuton and the gauge kinetic function as [22,32–34]

$$V = \frac{1}{2} m^2 \phi^2, \qquad f = f_0 \left(\frac{\phi}{M_{\mathrm{pl}5}^{3/2}}\right)^{2w}, \tag{16}$$

respectively, where m is the mass of the cuscuton, f_0 is a positive constant, and w is an integer. Then, we take the ansatz,

$$\alpha = p_1 \log M_{\mathrm{pl}5} t, \quad \sigma = p_2 \log M_{\mathrm{pl}5} t, \quad \phi = \frac{q}{t} M_{\mathrm{pl}5}^{1/2}, \tag{17}$$

where p_1, p_2, q are parameters. Equations (12)–(15) accommodate power-law solutions with the ansatz (17). First of all, we need a relation,

$$w - 3p_1 - 3p_2 + 1 = 0. \tag{18}$$

We also find the following set of the algebraic equations relating the parameters:

$$-p_1 + 4p_1^2 = -4\xi|q| + 4\lambda|q|^2 + \frac{\gamma}{|q|^{2w}}, \tag{19}$$

$$p_1^2 - p_2^2 = \lambda|q|^2 + \frac{\gamma}{|q|^{2w}}, \tag{20}$$

$$-p_2 + 4p_1 p_2 = 3\frac{\gamma}{|q|^{2w}}, \tag{21}$$

$$4|q|\xi p_1 = \lambda|q|^2 - w\frac{\gamma}{|q|^{2w}}, \tag{22}$$

where we have defined new parameters as

$$\gamma = \frac{(C^2/M_{\mathrm{pl}5}^{5})}{12 f_0}, \qquad \lambda = \frac{(m^2/M_{\mathrm{pl}5}^{2})}{12}, \qquad \xi = \pm\frac{(\mu^2/M_{\mathrm{pl}5}^{\frac{5}{2}})}{12}. \tag{23}$$

We have seven parameters in the equations: $\{p_1, p_2, |q|, \gamma, w, \lambda, \xi\}$ and four-independent equations (18), (19), (21), and (22). Let us solve the algebraic equations about $\{\gamma, w, \lambda, \xi\}$ and express them by $\{p_1, p_2, |q|\}$. One can solve (18) for w, and (21) for γ,

$$w = 3(p_1 + p_2) - 1, \tag{24}$$

$$\gamma = \frac{1}{3} p_2(-1 + 4p_1)|q|^{2w}, \tag{25}$$

respectively. Furthermore, from (19), (22), (24), and (25), we obtain

$$\lambda = \frac{3(p_1^2 - p_2^2) + p_2(1 - 4p_1)}{3|q|^2} \tag{26}$$

and

$$\xi = \frac{(p_1 + p_2)(1 - 4p_2)}{4|q|}. \tag{27}$$

Apparently, there exists power-law solutions for an arbitrary set of parameters $\{p_1, p_2, |q|\}$. However, the parameter region is restricted because of the positivity of γ and λ. In the next section, we focus on power-law solutions corresponding to expanding four-dimensional spacetime with a static extra dimension and reveal the allowed parameter region.

III. KALUZA-KLEIN SCENARIO IN CUSCUTON GRAVITY

In the previous section, we have obtained power-law solutions for the system with a cuscuton and a vector field. Depending on the parameters of the solution, $\{p_1, p_2, |q|\}$, various situations can be realized. Hence, we are now in a position to realize the Kaluza-Klein scenario.

To achieve our aim, we need to make the extra dimension represented by a coordinate x_4 frozen, while the other spatial dimensions are expanding as in our universe. Such situation occurs for $p_1 = 3p_2$ in the power-law solutions. In that case, the spacetime metric is described by

$$ds^2 = -dt^2 + dx_4^2 + (M_{\mathrm{pl}5} t)^{8p_2}(dx^2 + dy^2 + dz^2). \tag{28}$$

The Hubble parameter in the four-dimensional spacetime is

$$H_4 = \frac{4p_2}{t}, \tag{29}$$

which should be positive to describe the expanding universe, i.e., $p_2 > 0$. Under the assumption, $p_1 = 3p_2$, the relations between the parameters (24)–(27) are reduced to

$$w = -1 + 12p_2, \qquad \gamma = \frac{p_2[-1+12p_2]}{3}|q|^{2(-1+12p_2)},$$
$$\lambda = \frac{p_2[1+12p_2]}{3|q|^2}, \qquad \xi = \frac{p_2(1-4p_2)}{|q|}. \tag{30}$$

Since the parameters γ and λ should be positive by definition, p_2 should satisfy[1]

$$p_2 > \frac{1}{12}. \tag{31}$$

It implies w is always positive.

If we divide this regime depending on the sign of ξ, they would be

$$\frac{1}{12} < p_2 < \frac{1}{4} \quad (\text{for } \xi > 0), \qquad \frac{1}{4} \leq p_2 \quad (\text{for } \xi \leq 0). \tag{32}$$

Since the second time derivative of the scale factor is proportional to $p_2^2(4p_2 - 1)$, we find that the existence of a cuscuton with $\xi < 0$ is essential to realize the accelerated expansion of the four-dimensional spacetime. Note that there is an interesting case $\xi = 0$ ($p_2 = 1/4$), i.e., $\mu^2 = 0$, where the kinetic term of the cuscuton vanishes. In this case, after integrating out ϕ in the action, the theory only includes a vector field with nonlinear terms of $F_{\mu\nu}F^{\mu\nu}$.

The solution (30) exists in the two-dimensional parameter space $\{p_2, |q|\}$ satisfying the condition (31). Here we mention that one can always characterize the solution by using $\{w, \lambda\}$ which appear in the original action (1) in contrast to $\{p_2, |q|\}$. Actually, in terms of $\{w, \lambda\}$, the solution is written by[2]

$$p_2(w) = \frac{1+w}{12}, \qquad \xi^2 = \frac{3p_2(w)(1-4p_2(w))^2}{(1+12p_2(w))}\lambda,$$
$$|q| = \frac{p_2(w)(1-4p_2(w))}{\xi(w,\lambda)}. \tag{33}$$

It is useful to evaluate the slow roll parameters, which characterize inflationary universe, and the ratio of the energy density of the vector field to that of the cuscuton. The slow roll parameters of the four-dimensional spacetime are calculated from (29) as

$$\epsilon_4 = -\frac{\dot{H}_4}{H_4^2} = \frac{1}{4p_2} = \frac{3}{1+w}, \quad \eta_4 = 2\epsilon_4 - \frac{\dot{\epsilon}_4}{2H_4\epsilon_4} = 2\epsilon_4, \tag{34}$$

where we have used the fact that $\dot{\epsilon}_4 = 0$ for power-law solutions. In particular, inflationary universe is realized if $4p_2 \gg 1$ is satisfied, which implies $\xi \ll 0 \Leftrightarrow w \gg 2$. Finally, the ratio of the energy density of the vector field to that of the cuscuton is found to be

$$\frac{\rho_A}{\rho_c} = \frac{\gamma}{\lambda|q|^{2(1+w)}} = \frac{-1+12p_2}{1+12p_2} = \frac{w}{2+w}. \tag{35}$$

It is almost the unity in the inflationary universe, $p_2 \gg 1$ ($w \gg 1$).

In the next section, we investigate the stability of the solution (30). We will see that the solution is an attractor in phase space as long as the inequality (31) is satisfied.

IV. STABILITY OF THE SOLUTION

In the previous section, we found exact power-law solutions describing expanding universe with a static extra dimension. Let us examine if the solutions are stable or not.

It is convenient to recast Eqs. (12)–(15) with following new dimensionless variables:

$$X = \frac{\dot{\sigma}}{\dot{\alpha}}, \qquad Y = -\mathrm{sgn}(\dot{\phi})\frac{\phi}{\dot{\alpha}M_{\mathrm{pl}5}^{1/2}},$$
$$\tilde{Y} = \mathrm{sgn}(\dot{\phi})\frac{\dot{\phi}}{\dot{\alpha}^2 M_{\mathrm{pl}5}^{1/2}}, \qquad Z = \frac{C^2}{12M_{\mathrm{pl}5}^5}\frac{M_{\mathrm{pl}5}^2}{\dot{\alpha}^2}f^{-1}e^{-6\alpha-6\sigma}. \tag{36}$$

Using these variables, Eqs. (12)–(15) can be rewritten as

$$\frac{dX}{d\alpha} = -4X + 3Z - X[-4\xi\tilde{Y} + 4\lambda Y^2 + Z - 4], \tag{37}$$

$$1 = X^2 + \lambda Y^2 + Z, \tag{38}$$

$$\frac{dZ}{d\alpha} = 2Z\left[1 + 4\xi\tilde{Y} - 4\lambda Y^2 - Z + w\frac{\tilde{Y}}{Y} - 3X\right], \tag{39}$$

$$0 = -4\xi + \lambda Y - \frac{wZ}{Y}, \tag{40}$$

---

[1]
$$p_2 < -\frac{1}{12}$$
is also another possibility, but this case realizes the contraction of three-dimensional space coordinate, and therefore, it is out of our interest here.

[2] For the set of model parameters (w, λ), we cannot freely choose the value of ξ to realize the Kaluza-Klein scenario. Even if the parameter ξ does not exactly satisfy $\xi^2 = \xi^2(w,\lambda)$, where $\xi^2(w,\lambda)$ is defined by (33), we can approximately realize the Kaluza-Klein scenario as far as $\xi^2 \sim \xi^2(w,\lambda)$. Then, the ratio of the expansion rate of the extra space dimension to that of our three-dimensional space, $(p_1 - 3p_2)/(p_1 + p_2)$, is small enough.

where we have used the e-folding number α as the time coordinate of the system. There are two constraint equations, one is the Friedmann equation (38) and the other is the equation of motion of the cuscuton (40). Note that the cuscuton is nondynamical scalar field and thus (40) is just a constraint equation. From (40), we can express Y in terms of Z as

$$Y = \frac{2\xi + \sqrt{w\lambda Z + 4\xi^2}}{\lambda}, \tag{41}$$

where we have used the fact $Y > 0$. Moreover, differentiating (41) with respect to t, one can reduce $\tilde{Y}$ as a function of X and Z,

$$\tilde{Y} = \frac{Y[(4\lambda Y^3 + YZ - 4Y)\sqrt{w\lambda Z + 4\xi^2} - 4\lambda w Y^2 Z - 3wXZ - wZ^2 + wZ]}{(4\xi Y^2 - Y)\sqrt{w\lambda Z + 4\xi^2} - 4\xi wYZ - w^2 Z}. \tag{42}$$

Finally, substituting (41) and (42), into (37)–(39), we get a closed autonomous system with a constraint equation relating X with Z.

On the other hand, the power-law solution (30) can be expressed in terms of X and Z as

$$X = \frac{1}{3}, \qquad Z = \frac{w}{27 p_2(w)}. \tag{43}$$

It is easy to check that the power-law solution (43) is indeed a fixed point in the phase space, which is defined by $\frac{dX}{d\alpha} = \frac{dZ}{d\alpha} = 0$.

In order to examine the stability of the fixed point, we first eliminate X from (39) by using (38), (41), and (42). Expanding Z around the power-law solution (43) as $Z = w/27p_2 + \delta Z$, we have

$$\frac{d\delta Z}{d\alpha} = -\frac{[-1 + 12p_2]}{3p_2}\delta Z = -\frac{w}{3p_2}\delta Z, \tag{44}$$

at the linear order. Equation (44) shows that the power-law solution is stable if the inequality $p_2 > 1/12$ is satisfied. The condition coincides with the existence condition of the power-law solution (31), so that the stability is guaranteed for the expanding power-law solutions (30) with (31). We have also numerically confirmed that the solution (43) is attractor in the phase space. It proved the stability at the nonlinear level.

Though we have not shown the stability for general Bianchi type I perturbations here, it is known that the unsourced anisotropy rapidly decays during the rapid cosmic expansion [35]. This fact can be understood as a consequence of the cosmic no-hair theorem [14]. Hence, the above analysis is sufficient for proving the stability of our solutions.

V. CONCLUSION

We studied the Kaluza-Klein scenario in cuscuton gravity. In our model of Kaluza-Klein scenario, a vector field coupled with a cuscuton has a vacuum expectation value along with the direction of the extra dimension and violates the rotational invariance of higher dimensional spaces. We found an exact power-law solution of the Einstein and the field equations. It was shown that the solution describes accelerating expansion of four-dimensional spacetime with a completely static extra dimension. To the best of our knowledge, this is the first concrete model of accelerating universes other than the de Sitter spacetime with a static extra dimension [28,29].

The stability of the solution was also investigated in Sec. IV. We performed dynamical system analysis (12)–(15) and revealed the condition for stability (44). It coincides with the condition for existence of the solution (31). Therefore, the solution is always stable if exists. All the discussion in the text focused on $(4+1)$-dimensional spacetime for simplicity; however, it is easy to extend it to general $(n+1)$-dimensional spacetime (see Appendix).

Although we used a one-form field to compactify one extra dimension in this paper, one can apply p one-form fields [35] or a p-form field [36] to compactify p extra dimensions simultaneously. It is interesting to explore cosmological perturbations in the Kaluza-Klein scenario. We leave these issues for future work.

ACKNOWLEDGMENTS

A. I. was supported by Grant-in-Aid for JSPS Research Fellow and JSPS KAKENHI Grant No. JP17J00216. Y. S. was supported by JSPS KAKENHI Grant No. JP17H02894. J. S. was in part supported by JSPS KAKENHI Grants No. JP17H02894, No. JP17K18778, No. JP15H05895, No. JP17H06359, and No. JP18H04589. J. S. is also supported by JSPS Bilateral Joint Research Projects (JSPS-NRF Collaboration) String Axion Cosmology.

APPENDIX: $(n+1)$-DIMENSIONAL SPACETIME

Though we have started with $(4+1)$-dimensional spacetime in the text, the discussion can be extended to the an arbitrary number of space dimensions. The argument is almost the same, and thus we shortly summarize the results.

The action in $(n+1)$-dimensional spacetime reads $(n > 2)$

$$\mathcal{S} = \int d^{n+1}x\sqrt{-g}\left[\frac{1}{2\kappa_n}R \pm \mu^2\sqrt{-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} - V(\phi) - \frac{1}{4}f(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{A1}$$

where $\kappa_n = 1/M_{\mathrm{pl}_{n+1}}^{n-1}$, $M_{\mathrm{pl}_{n+1}}$ is $(n+1)$-dimensional Planck scale, and the definition of field strength of the vector field A_μ is just the higher dimensional counterpart. We note that the mass dimensions of the field are $[\phi] = (n-1)/2$, $[A_\mu] = (n-1)/2$, and the parameter μ has the mass dimension $[\mu^2] = (n+1)/2$. The spacetime metric is defined with one anisotropic direction,

$$ds^2 = -dt^2 + e^{2\alpha(t)}(e^{-2(n-1)\sigma(t)}dx_n^2 + e^{2\sigma(t)}(dx_1^2 + \cdots + dx_{n-1}^2)), \tag{A2}$$

As the case of $(4+1)$-dimensional spacetime, we take the configuration of the vector field and the cuscuton as

$$A_\mu = (0, \ldots, 0, v(t)), \qquad \phi = \phi(t), \tag{A3}$$

Then, the background action is written as

$$\mathcal{S} = \int d^{n+1}x e^{n\alpha}\left[\frac{n(n-1)}{2\kappa_n}(-\dot{\alpha}^2 + \dot{\sigma}^2) \pm \mu^2 \mathrm{sgn}(\dot{\phi})\dot{\phi} - V(\phi) + \frac{1}{2}f(\phi)e^{-2\alpha+2(n-1)\sigma}\dot{v}^2\right], \tag{A4}$$

which leads to the equations of motion as follows:

$$\ddot{\alpha} = -n\dot{\alpha}^2 \mp \frac{\kappa_n\mu^2}{n-1}\mathrm{sgn}(\dot{\phi})\dot{\phi} + \frac{2\kappa_n}{n-1}V(\phi) + \frac{\kappa_n}{n(n-1)}f(\phi)e^{-2\alpha+2(n-1)\sigma}\dot{v}^2, \tag{A5}$$

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{2\kappa_n}{n(n-1)}\left[V(\phi) + \frac{1}{2}f(\phi)e^{-2\alpha+2(n-1)\sigma}\dot{v}^2\right], \tag{A6}$$

$$\ddot{\sigma} = -n\dot{\alpha}\dot{\sigma} + \frac{\kappa_n}{n}fe^{-2\alpha+2(n-1)\sigma}\dot{v}^2, \tag{A7}$$

$$\pm n\mu^2\dot{\alpha}\mathrm{sgn}(\dot{\phi}) + V_\phi - \frac{1}{2}f_\phi e^{-2\alpha+2(n-1)\sigma}\dot{v}^2 = 0, \tag{A8}$$

$$(e^{(n-2)\alpha+2(n-1)\sigma}f\dot{v})^\cdot = 0. \tag{A9}$$

We integrate (A9) and have

$$\dot{v} = f^{-1}e^{-(n-2)\alpha-2(n-1)\sigma}C, \tag{A10}$$

where C is a constant of integration with the mass dimension $[C] = (n+1)/2$. After substituting this solution into (A5)–(A8), we list the equations as follows:

$$\ddot{\alpha} = -n\dot{\alpha}^2 \mp \frac{\kappa_n\mu^2}{n-1}\mathrm{sgn}(\dot{\phi})\dot{\phi} + \frac{2\kappa_n}{n-1}V(\phi) + \frac{\kappa_n}{n(n-1)}f^{-1}e^{-2(n-1)\alpha-2(n-1)\sigma}C^2, \tag{A11}$$

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{2\kappa_n}{n(n-1)}\left[V(\phi) + \frac{1}{2}f^{-1}e^{-2(n-1)\alpha-2(n-1)\sigma}C^2\right], \tag{A12}$$

$$\ddot{\sigma} = -n\dot{\alpha}\dot{\sigma} + \frac{\kappa_n}{n}f^{-1}e^{-2(n-1)\alpha-2(n-1)\sigma}C^2, \tag{A13}$$

$$\pm n\mu^2\dot{\alpha}\mathrm{sgn}(\dot{\phi}) + V_\phi - \frac{1}{2}f_\phi f^{-2}e^{-2(n-1)\alpha-2(n-1)\sigma}C^2 = 0. \tag{A14}$$

The potential of the cuscuton and the function in front of the kinetic term of the vector field are defined by

$$V = \frac{1}{2}m^2\phi^2, \qquad f = f_0\left(\frac{\phi}{M_{\mathrm{pl}_{n+1}}^{\frac{n-1}{2}}}\right)^{2w}. \tag{A15}$$

1. Power-law solutions

We search for power-law solutions with

$$\alpha = p_1 \log M_{\mathrm{pl}_{n+1}}t, \tag{A16}$$

$$\sigma = p_2 \log M_{\mathrm{pl}_{n+1}}t, \tag{A17}$$

$$\phi = \frac{q}{t}M_{\mathrm{pl}_{n+1}}^{\frac{n-3}{2}}, \tag{A18}$$

and the potential of the cuscuton V and the function f are reduced to

$$V = \frac{1}{t^2}\frac{m^2q^2M_{\mathrm{pl}_{n+1}}^{n-3}}{2}, \quad f = f_0q^{2w}(M_{\mathrm{pl}_{n+1}}t)^{-2w}. \tag{A19}$$

To have power-law solutions, we should require

$$w - (n-1)p_1 - (n-1)p_2 + 1 = 0, \tag{A20}$$

and then, we have

$$-p_1 + np_1^2 = -n\xi|q| + n\lambda q^2 + \frac{\gamma}{q^{2w}}, \tag{A21}$$

$$p_1^2 - p_2^2 = \lambda q^2 + \frac{\gamma}{q^{2w}}, \tag{A22}$$

$$-p_2 + np_1p_2 = (n-1)\frac{\gamma}{q^{2w}}, \tag{A23}$$

$$n\xi|q|p_1 = \lambda q^2 - w\frac{\gamma}{q^{2w}}, \tag{A24}$$

where

$$\gamma = \frac{(C^2/M_{\mathrm{pl}_{n+1}}^{n+1})}{n(n-1)f_0}, \qquad \lambda = \frac{(m^2/M_{\mathrm{pl}_{n+1}}^{2})}{n(n-1)},$$
$$\xi = \pm\frac{(\mu^2/M_{\mathrm{pl}_{n+1}}^{\frac{n+1}{2}})}{n(n-1)}. \tag{A25}$$

We have seven parameters: $\{p_1, p_2, q, \gamma, w, \lambda, \xi\}$ and four-independent equations (A20), (A21), (A23), and (A24). We solve (A20) for w and (A23) for γ,

$$w = (n-1)(p_1+p_2) - 1, \tag{A26}$$

$$\gamma = \frac{1}{n-1}p_2(-1+np_1)q^{2w}, \tag{A27}$$

respectively. Using (A26) and (A27) in (A21) and (A24), we obtain

$$\lambda = \frac{(n-1)(p_1^2-p_2^2)+p_2(1-np_1)}{(n-1)|q|^2}, \tag{A28}$$

$$\xi = \frac{(p_1+p_2)(1-np_2)}{n|q|}. \tag{A29}$$

2. Kaluza-Klein solutions

We seek for the solution respecting $p_1 = (n-1)p_2$, which implies the spacetime metric is written as

$$ds^2 = -dt^2 + dx_n^2 + (M_{\mathrm{pl}_{n+1}}t)^{2np_2}(dx_1^2 + dx_2^2 + \cdots + dx_{n-1}^2). \tag{A30}$$

The effective Hubble parameter in the n-dimensional spacetime is

$$H_n = np_2. \tag{A31}$$

Then, the relations between the parameters become

$$w = -1 + n(n-1)p_2,$$
$$\gamma = \frac{p_2[-1+n(n-1)p_2]}{n-1}|q|^{2(-1+n(n-1)p_2)},$$
$$\lambda = \frac{p_2[1+n(n-1)(n-3)p_2]}{(n-1)|q|^2},$$
$$\xi = \frac{p_2(1-np_2)}{|q|}. \tag{A32}$$

From these relations, we can read off the condition for the existence of the solution from the positivity of γ and λ as

$$p_2 > \frac{1}{n(n-1)}, \tag{A33}$$

which implies w is always positive. If we divide this condition depending on the sign of ξ, they would be

$$\frac{1}{n(n-1)} < p_2 < \frac{1}{n} \quad (\text{for } \xi > 0), \qquad \frac{1}{n} \le p_2 \quad (\text{for } \xi \le 0). \tag{A34}$$

The slow roll parameter in n-dimensional spacetime is

$$\epsilon_n = \frac{1}{np_2} = \frac{n-1}{1+w}, \tag{A35}$$

which means that the slow roll condition is $np_2 \gg 1$. From (A32), we find that we need to satisfy $\xi < 0 \Leftrightarrow w > n-2$ for the realization of the accelerated expansion in n-dimensional spacetime.

3. Stability

Even in $(n+1)$-dimensional spacetime, the stability of the power-law solution (A32) does not change at all, since the perturbative equation is almost the same as (44),

$$\frac{d\delta Z}{d\alpha} = -\frac{[-1+n(n-1)p_2]}{(n-1)p_2}\delta Z = -\frac{w}{(n-1)p_2}\delta Z, \tag{A36}$$

where $\delta Z = Z - \bar{Z}$, which is the perturbation of

$$Z = \frac{C^2}{n(n-1)M_{\mathrm{pl}_{n+1}}^{n+1}}\frac{M_{\mathrm{pl}}{}^2}{\dot{\alpha}^2}f^{-1}e^{-2(n-1)\alpha-2(n-1)\sigma}, \tag{A37}$$

around $\bar{Z} = w/[(n-1)^3p_2(w)]$, where $p_2(w) = (1+w)/[n(n-1)]$, defined by the power-law solution (A32).

[1] T. Kaluza, Zum Unitatsproblem der Physik, The dawning of gauge theory, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921); On the unification problem in physics, Int. J. Mod. Phys. D **27**, 1870001 (2018).
[2] O. Klein, Quantum theory and five-dimensional theory of relativity. (In German and English), Gauge theories in the twentieth century, Z. Phys. **37**, 895 (1926).
[3] A. Chodos and S. L. Detweiler, Where has the fifth-dimension gone, Phys. Rev. D **21**, 2167 (1980).
[4] P. G. O. Freund and M. A. Rubin, Dynamics of dimensional reduction, Phys. Lett. B **97**, 233 (1980).
[5] K.-i. Maeda, Is cosmological dimensional reduction possible, Phys. Lett. **138B**, 269 (1984).
[6] K.-i. Maeda, Effect of particle creation on Kaluza-Klein cosmologies, Phys. Rev. D **30**, 2482 (1984).
[7] H. Ishihara, Kaluza-Klein Inflation, Prog. Theor. Phys. **72**, 376 (1984).
[8] T. Jacobson and D. Mattingly, Gravity with a dynamical preferred frame, Phys. Rev. D **64**, 024028 (2001).
[9] S. M. Carroll and H. Tam, Aether compactification, Phys. Rev. D **78**, 044047 (2008).
[10] S. M. Carroll, T. R. Dulaney, M. I. Gresham, and H. Tam, Instabilities in the Aether, Phys. Rev. D **79**, 065011 (2009).
[11] B. Himmetoglu, C. R. Contaldi, and M. Peloso, Instability of the Ackerman-Carroll-Wise model, and problems with massive vectors during inflation, Phys. Rev. D **79**, 063517 (2009).
[12] B. Himmetoglu, C. R. Contaldi, and M. Peloso, Instability of Anisotropic Cosmological Solutions Supported by Vector Fields, Phys. Rev. Lett. **102**, 111301 (2009).
[13] B. Himmetoglu, C. R. Contaldi, and M. Peloso, Ghost instabilities of cosmological models with vector fields nonminimally coupled to the curvature, Phys. Rev. D **80**, 123530 (2009).
[14] J. Soda, Statistical anisotropy from anisotropic inflation, Classical Quantum Gravity **29**, 083001 (2012).
[15] A. Maleknejad, M. M. Sheikh-Jabbari, and J. Soda, Gauge fields and inflation, Phys. Rep. **528**, 161 (2013).
[16] M.-a. Watanabe, S. Kanno, and J. Soda, Inflationary Universe with Anisotropic Hair, Phys. Rev. Lett. **102**, 191302 (2009).
[17] N. Afshordi, D. J. H. Chung, and G. Geshnizjani, Causal field theory with an infinite speed of sound, Phys. Rev. D **75**, 083513 (2007).
[18] N. Afshordi, Cuscuton and low energy limit of Horava-Lifshitz gravity, Phys. Rev. D **80**, 081502 (2009).
[19] J. Bhattacharyya, A. Coates, M. Colombo, A. E. Gumrukcuoglu, and T. P. Sotiriou, Revisiting the cuscuton as a Lorentz-violating gravity theory, Phys. Rev. D **97**, 064020 (2018).
[20] H. Gomes and D. C. Guariento, Hamiltonian analysis of the cuscuton, Phys. Rev. D **95**, 104049 (2017).
[21] S. Mukohyama and K. Noui, Minimally modified gravity: A Hamiltonian construction, J. Cosmol. Astropart. Phys. 07 (2019) 049.
[22] A. Iyonaga, K. Takahashi, and T. Kobayashi, Extended Cuscuton: Formulation, J. Cosmol. Astropart. Phys. 12 (2018) 002.
[23] E. Pajer and D. Stefanyszyn, Symmetric superfluids, J. High Energy Phys. 06 (2019) 008.
[24] N. Afshordi, D. J. H. Chung, M. Doran, and G. Geshnizjani, Cuscuton cosmology: Dark energy meets modified gravity, Phys. Rev. D **75**, 123509 (2007).
[25] S. S. Boruah, H. J. Kim, and G. Geshnizjani, Theory of cosmological perturbations with cuscuton, J. Cosmol. Astropart. Phys. 07 (2017) 022.
[26] S. S. Boruah, H. J. Kim, M. Rouben, and G. Geshnizjani, Cuscuton bounce, J. Cosmol. Astropart. Phys. 08 (2018) 031.
[27] A. Ito, A. Iyonaga, S. Kim, and J. Soda, Dressed power-law inflation with a cuscuton, Phys. Rev. D **99**, 083502 (2019).
[28] J. D. Barrow and J. Stein-Schabes, The stability of some Kaluza-Klein cosmological models, Phys. Lett. **167B**, 173 (1986).
[29] T. Bringmann, M. Eriksson, and M. Gustafsson, Cosmological evolution of homogeneous universal extra dimensions, Phys. Rev. D **68**, 063516 (2003).
[30] R. M. Wald, Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant, Phys. Rev. D **28**, 2118 (1983).
[31] Y. Sakakihara, J. Soda, and T. Takahashi, On cosmic no-hair in bimetric gravity and the Higuchi bound, Prog. Theor. Exp. Phys. **2013**, 033E02 (2013).
[32] S. Kanno, J. Soda, and M.-a. Watanabe, Anisotropic power-law inflation, J. Cosmol. Astropart. Phys. 12 (2010) 024.
[33] J. Ohashi, J. Soda, and S. Tsujikawa, Anisotropic power-law k-inflation, Phys. Rev. D **88**, 103517 (2013).
[34] A. Ito and J. Soda, Designing anisotropic inflation with form fields, Phys. Rev. D **92**, 123533 (2015).
[35] K. Yamamoto, M.-a. Watanabe, and J. Soda, Inflation with multi-vector-hair: The fate of anisotropy, Classical Quantum Gravity **29**, 145008 (2012).
[36] J. Ohashi, J. Soda, and S. Tsujikawa, Anisotropic non-gaussianity from a two-form field, Phys. Rev. D **87**, 083520 (2013).