Superpotential method for chiral cosmological models connected with modified gravity

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(Received 29 April 2019; published 19 September 2019)

We consider chiral cosmological models (CCMs) and modified gravity theories associated with them. Generalization of the superpotential method for a general CCM with several scalar fields is performed, and the method of construction CCMs admitting exact solutions is developed. New classes of exact solutions in the two-component CCM connected with an $f(R)$ gravity model with an additional scalar field have been constructed. We construct new cosmological solutions for a diagonal metric of the target space, including modified power-law solutions. In particular, we propose the reconstruction procedure based on the superpotential method and present examples of kinetic part reconstruction for periodic and hyperbolic Hubble parameters. We also focus on a cyclic type of Universe dubbed the quasi-steady-state (QSS) model, with the aim of constructing single- and double-field potentials for one and the same behavior of the Hubble parameter using the developed superpotential method for the CCM. The realization of this task includes a new set of solutions for a CCM with a scale factor characterized by the QSS theory. We also propose a method for reducing the two-field CCM to the single scalar field model.

DOI: 10.1103/PhysRevD.100.063522

I. INTRODUCTION

Scalar fields play an important role in model building for early Universe and late time cosmic evolution. Observations [1–3] show that the Universe evolution can be described by the spatially flat Friedmann-Lemaître–Robertson-Walker (FLRW) spacetime as background and cosmological perturbations. Models with scalar fields are well suited to describe such an evolution. As for the modified theories of gravity, in general, they can be thought of as the Einstein theory of general relativity plus extra degrees of freedom; for instance, $f(R)$ gravity models correspond to general relativity models with a single self-interacting scalar field.

Scalar fields are important in inflationary scenarios [4–9], including the Starobinsky R^2 model [10] and in Higgs-driven inflation [11]. Models with a single scalar field nonminimally coupled to gravity as well as $f(R)$ gravity models can always be transformed to models with a minimally coupled scalar field with a canonical kinetic term by the metric and scalar field transformations. On the other hand, models with a few fields nonminimally coupled with gravity, in general, do not admit such a transformation [12]. After the metric transformation, one obtains the chiral cosmological models (CCMs) in the Einstein frame [13]. Multifield inflationary models do not contradict the Planck data [3] and are being actively studied [13–16]. It was recently argued that, compared to single field models, a system with several scalar fields can be better reconciled with observation. For instance, it was shown in Ref. [13] that additional degrees of freedom can produce enough power in isocurvature perturbations, which could account for the anomaly in the Planck observation data. It is needless to mention that extra degrees of freedom are generic features of modified theories of gravity.

At the same time, it has been proven that at least one fundamental scalar field (the Higgs boson) exists. This gives good motivation to consider modified gravity models with an additional scalar field. For example, inflationary models obtained from $f(R)$ gravity models with additional scalar fields [17] are very popular now [18–22]. The

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implementation of the inflationary scenario within a well-defined model of particle physics consistent with collider phenomenology would be a fundamental step towards the unification of physics at all energy scales. The standard model of particle physics that includes only one fundamental scalar field could be an effective limit of some supersymmetry or grant unification theory that has the scalar sector with a few Higgs bosons, and there is no reason to assume that only one scalar field plays a role in the Universe's evolution. In the Einstein frame, all of the above-mentioned models are CCMs. Quantum motivated higher-order generalizations of general relativity under some conditions can be equivalent to adding several scalar fields to the Einstein-Hilbert action [23].

The CCM allows one to describe not only the inflationary epoch of the Universe evolution but also the present accelerated expansion of the Universe [24–26]. It has been shown in a recent paper [27] that the present value of the equation-of-state dark energy parameter has to be phantom-like, and for other redshifts, it either has to be a phantom or should have a phantom crossing. It has been shown [28] that such transitions are physically implausible in one-field models because they are either realized by a discrete set of trajectories in the phase space or are unstable with respect to the cosmological perturbations. This is a strong motivation to consider quintom models [29], which are a particular case of CCMs.

In the case of single scalar cosmology with a generic potential, the integration procedure is reduced to solving the Ivanov-Salopek-Bond equation [7,30]. There are many methods of obtaining physically relevant solutions, including that with the Higgs potential [30–33]. The reduction of a CCM to a single scalar field model was proposed in Ref. [34]. A new approach to studying a CCM when gravitational field equations are represented in a linear form under the point transformation was developed in Ref. [35]. For a specific geometry of the target space and a special form of the potential, the way to obtain the solutions to the gravitational field equation has been found.[1] In the case of several scalar fields with kinetic interaction, progress in obtaining exact solutions has been achieved for later Universe evolution [36], the emergent Universe [37], Einstein-Gauss-Bonnet cosmology [38], and the tensor-multiscalar model [39] as well.

The goal of this paper is to propose a way to get a particular solution of the CCM in the analytic form. We do not seek solutions for a given potential but construct the potential of the scalar field such that the resulting model has exact solutions with important physical properties. Such a method is similar to the Hamilton-Jacobi method (also known as the superpotential method or the first-order formalism) and is applied to cosmological models with minimally [7,30,33,40–49] and nonminimally [50] coupled scalar fields. This method has been used, in particular, to find exact solutions in single scalar field inflationary models[2] [8,59–62]. Note that a similar method is used for the reconstruction procedure in brane [44,63–66] and holographic models [67,68].

The key point of the superpotential method is that the Hubble parameter is considered a function of the scalar fields $\phi^A(t)$. Note that there is an important difference between one-field and multifield models. In the case of one-field models, the above-mentioned procedure is straightforward because only one superpotential (up to a constant) corresponds to the given scalar field $\phi(t)$. In the case of two or more fields, the knowledge of a particular solution $\phi^A(t)$ does not fix the potential. An explicit example of essentially different potentials of two-field models with the same particular solution $\phi^A(t)$ is given in Ref. [40]. On the other hand, presenting the Hubble parameter as a function of K scalar fields, which satisfy the first-order equations, one can get a K-parametric set of the exact solutions. One parameter corresponds to the shift of time, whereas other parameters correspond to different evolutions of the Universe depending on the initial conditions. An explicit example of a quintom model with a two-parametric set of exact solutions is given in Ref. [41]. In this paper, we generalize the superpotential method on the CCM by constructing new models with exact solutions. In Secs. IV–VI, two-field CCMs with two-parametric sets of exact solutions are constructed.

The term *multifield model* is similar to *chiral cosmological model*, which is defined as the self-gravitating nonlinear sigma model with the potential of (self-)interactions employed in cosmology. Let us mention that the term *multiscalar field cosmology* was first introduced as the collection of scalar fields with the sum of kinetic (canonical) parts and with the potential depending on all fields. The model with kinetic interaction between the scalar fields is represented in some articles (for example, in the recent work [35]), whereas the term *multifield* was first introduced in the work by de Alfaro *et al.* [69], with the aim of obtaining instanton and meron solutions in a 4D model. They introduced geometrical restriction: all fields take values in the n-dimensional sphere. The potential term was not presented in the model, which was called the "four-dimensional sigma model coupled to the metric tensor field." Perelomov in 1981 [70] introduced terminology by exchanging the term "group invariant sigma model" for "chiral model," and he also introduced the metric of a chiral model and extended the model from 2D for N-dimensional models, the so-called chiral models of general type. In Ref. [70], there was no connection with gravity. Ivanov [71],

[1]It is difficult to consider such solutions as exact ones without involving the dynamic equations of the chiral fields.

[2]Other methods allowing one to find exact solutions in inflationary models were presented in Refs. [26,31,32,34,51–58] (for a recent review, see Ref. [33]).

independently of Ref. [69], came to the "nonlinear sigma model coupled to gravity" by considering the Lorentz signature metric of spacetime, and scalar (chiral) fields as the source of gravity, besides the kinetic interaction, have been introduced as the metric of "chiral" space.

The potential of the interaction of chiral fields was introduced by Chervon in 1994 [72]. Such a model in Ref. [72] was called the "self-gravitating nonlinear sigma model with the potential." Then in further publications, using terminology introduced by Perelomov, the model was referred to as the "chiral inflationary model" and then the "chiral cosmological model." Thus, the term "chiral cosmological model" reflects the geometrical interactions of fields via the metric of the target (chiral) space which includes the kinetic interactions.

Let us stress the difference between the pure multifield model (without kinetic coupling and cross interaction between fields) and the CCM. Generally speaking, it is impossible to reduce a CCM with a functional component of the target space metric to a conformal Euclidean (or Lorentzian) diagonal metric. For example, in the two-dimensional case, when a surface is embedded in 3D Euclidean space for a C^2-smooth 2D metric component in some neighborhood of the point, it is possible to define the coordinates in which the metric takes the form of a conformal Euclidean diagonal metric. But to calculate the form of new coordinates, one needs to solve a rather complicated Beltrami equation [73].

In this paper, we generalize this reconstruction procedure on models with an arbitrary finite number of scalar fields minimally coupled to gravity. The structure of the paper is as follows. In Sec. II, we connect the CCM with modified gravity. In Sec. III, the superpotential method develops on CCMs with an arbitrary number of scalar fields. In Secs. IV–VII, we consider two-component CCMs. In Sec. IV, the considered CCMs correspond to $f(R)$ gravity models with an additional scalar field. In Secs. V and VI, we find models with Ruzmaikin solutions, solutions that correspond to the intermediate inflation and modified power-law solutions for models with the given kinetic terms of the actions due to the choice of the potential. A procedure for construction of CCMs with trigonometric and hyperbolic Hubble functions due to a suitable choice of the kinetic term is proposed in Sec. VII. In Sec. VIII, we construct models with the Hubble parameter that describe a cyclic type of Universe dubbed the quasi-steady-state. A method for reducing two-field CCMs to single scalar field models is proposed in Sec. IX. Our results are summarized in Sec. X.

II. THE CONNECTION BETWEEN CHIRAL COSMOLOGICAL MODELS AND MODIFIED GRAVITY

Chiral cosmological models with K scalar fields $\phi^A(\bar{\phi} = \phi^1, \phi^2 \dots, \phi^K)$ are described by the following action,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}h_{AB}(\bar{\phi})\partial_\mu\phi^A\partial_\nu\phi^B g^{\mu\nu} - V(\bar{\phi})\right], \tag{1}$$

where the functions $h_{AB}(\bar{\phi})$ and the potential $V(\bar{\phi})$ are differentiable functions; $M_{\rm Pl}$ denotes the reduced Planck mass: $M_{\rm Pl} \equiv 1/\sqrt{8\pi G}$. We assume that $h_{AB} = h_{BA}$ and that the determinant of this matrix is not equal to zero, so this matrix can be considered the field-space metric.

Varying action (2), we get the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{\rm Pl}^2}T_{\mu\nu}, \tag{2}$$

where the energy-momentum tensor is

$$T_{\mu\nu} = h_{AB}(\bar{\phi})\partial_\mu\phi^A\partial_\nu\phi^B - g_{\mu\nu}\left[\frac{1}{2}h_{AB}(\bar{\phi})\partial_\rho\phi^A\partial_\beta\phi^B g^{\rho\beta} + V(\bar{\phi})\right]. \tag{3}$$

Variation action (1) on the chiral field ϕ^C leads to the field equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}h_{CB}\phi^B_{,\nu}) - \frac{1}{2}g^{\mu\nu}h_{AB,C}\phi^A_{,\mu}\phi^B_{,\nu} - V_{,C}(\bar{\phi}) = 0, \tag{4}$$

where $h_{AB,C} \equiv \frac{\partial h_{AB}}{\partial\phi^C}$.

In the spatially flat FLRW metric with the interval

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2), \tag{5}$$

Einstein equations (2) have the following form:

$$3H^2 = \frac{\varrho}{M_{\rm Pl}^2}, \tag{6}$$

$$2\dot{H} + 3H^2 = -\frac{p}{M_{\rm Pl}^2}, \tag{7}$$

where

$$\varrho = -T_0^0 = \frac{1}{2}h_{AB}(\bar{\phi})\dot{\phi}^A\dot{\phi}^B + V(\bar{\phi}),$$
$$p = T_1^1 = \frac{1}{2}h_{AB}(\bar{\phi})\dot{\phi}^A\dot{\phi}^B - V(\bar{\phi}); \tag{8}$$

the dots denote the time derivative, and the Hubble parameter $H(t)$ is the logarithmic derivative of the scale factor, $H = \dot{a}/a$.

In the FLRW metric (5), Eq. (4) is transformed into

$$-h_{CB}(\ddot{\phi}^B + 3H\dot{\phi}^B) - h_{CB,D}\dot{\phi}^D\dot{\phi}^B + \frac{1}{2}h_{DB,C}\dot{\phi}^D\dot{\phi}^B - V_{,C} = 0. \quad (9)$$

Contracting this equation with h^{AC}, we obtain[3]

$$\ddot{\phi}^A + 3H\dot{\phi}^A + \Gamma^A_{DB}\dot{\phi}^D\dot{\phi}^B + h^{AC}V_{,C} = 0, \quad (10)$$

where Γ^A_{DB} are the Christoffel symbols for the field-space manifold defined by the metric h_{AB}.

Let us introduce a new variable,

$$X = h_{AB}\dot{\phi}^A\dot{\phi}^B. \quad (11)$$

From Eqs. (6) and (7), we get

$$\dot{H} = -\frac{X}{2M_{\rm Pl}^2}. \quad (12)$$

Multiplying Eq. (9) by $\dot{\phi}^C$, summing it, and using

$$\dot{X} = 2h_{AB}\ddot{\phi}^A\dot{\phi}^B + h_{AB,C}\dot{\phi}^A\dot{\phi}^B\dot{\phi}^C,$$

we get the following equation:

$$\frac{1}{2}\dot{X} + 3HX + \dot{V} = 0. \quad (13)$$

Note that Eq. (13) is a consequence of Eqs. (6) and (12).

Many modified gravity models are connected with chiral cosmological models. In particular, let us consider models with nonminimally coupled scalar fields that are described by the following action:

$$S_J = \int d^4x\sqrt{-\tilde{g}}\left[f(\bar{\phi})\tilde{R} - \frac{1}{2}\tilde{G}_{AB}\tilde{g}^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B - \tilde{V}(\bar{\phi})\right]. \quad (14)$$

By the conformal transformation of the metric

$$g_{\mu\nu} = \frac{2}{M_{\rm Pl}^2}f(\bar{\phi})\tilde{g}_{\mu\nu}, \quad (15)$$

one gets the following action in the Einstein frame [6]:

$$S_E = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}h_{AB}(\bar{\phi})g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B - V_E\right], \quad (16)$$

where

$$h_{AB}(\bar{\phi}) = \frac{M_{\rm Pl}^2}{2f(\bar{\phi})}\left[\tilde{G}_{AB} + \frac{3f_{,A}f_{,B}}{f(\bar{\phi})}\right], \qquad V_E = M_{\rm Pl}^4\frac{\tilde{V}}{4f^2},$$

and $f_{,A} = \partial f/\partial\phi^A$.

For the class of $f(R)$ gravity models with scalar fields described by

$$S_R = \int d^4\tilde{x}\sqrt{-\tilde{g}}\left[f(\bar{\phi},\tilde{R}) - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{G}_{AB}\partial_\mu\phi^A\partial_\nu\phi^B - \tilde{V}(\bar{\phi})\right], \quad (17)$$

one can introduce an additional scalar field without the kinetic term ϕ^{K+1} and rewrite S_R as follows [74]:

$$\tilde{S}_J = \int d^4\tilde{x}\sqrt{-\tilde{g}}\left[\frac{df(\bar{\phi},\phi^{K+1})}{d\phi^{K+1}}(\tilde{R} - \phi^{K+1}) + f(\bar{\phi},\phi^{K+1}) - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{G}_{AB}\partial_\mu\phi^A\partial_\nu\phi^B - \tilde{V}(\bar{\phi})\right]. \quad (18)$$

Therefore, we get the model with $K+1$ scalar fields described by action (14) and can transform it to the chiral cosmological model with action (16).

III. THE SUPERPOTENTIAL METHOD FOR THE CCM

Generalizing the superpotential method for multifield models with the standard kinetic term [7] and for two-field models with a constant kinetic term [40–42], we describe the superpotential for the general CCM.

We assume that functions ϕ^A are solutions of the following system of K ordinary differential equations:

$$\frac{d\phi^A}{dt} \equiv \dot{\phi}^A = F^A(\bar{\phi}). \quad (19)$$

In this case, the Hubble parameter $H(t)$ is a function of all scalar fields,

$$H(t) = W(\bar{\phi}) + C_W,$$

where superpotential W is a differentiable function, and C_W is a constant part of the Hubble function that plays a special role (see, for example, [33]). It is convenient to write C_W separately. Thus, Eqs. (6) and (7) take the following form:

$$3M_{\rm Pl}^2[W(\bar{\phi}) + C_W]^2 = \frac{1}{2}h_{AB}F^AF^B + V(\bar{\phi}), \quad (20)$$

$$M_{\rm Pl}^2(2W_{,A}F^A + 3[W(\bar{\phi}) + C_W]^2) = -\frac{1}{2}h_{AB}F^AF^B + V(\bar{\phi}). \quad (21)$$

[3]Note that the upper index A could not be moved down with the chiral metric h_{AB}.

Subtracting Eq. (20) from Eq. (21), we get

$$\left(W_{,A}+\frac{1}{2M_{\rm Pl}^2}h_{AB}F^B\right)F^A=0. \tag{22}$$

One can see that a sufficient condition to satisfy Eq. (22) is the following relation:

$$W_{,A}=-\frac{h_{AB}F^B}{2M_{\rm Pl}^2} \tag{23}$$

for all A. This is a rather tight restriction which is equivalent to the decomposition method used in Refs. [33,36,39].

Using $\ddot{\phi}^B=\dot{F}^B=F^B_{,D}F^D$, we rewrite field equation (9) as follows:

$$h_{CB}F^B_{,D}F^D+h_{CB,D}F^DF^B-\frac{1}{2}h_{DB,C}F^DF^B$$
$$+3[W(\bar{\phi})+C_W]h_{CB}F^B+V_{,C}=0. \tag{24}$$

From Eq. (20), it follows that

$$V_{,C}=6M_{\rm Pl}^2[W(\bar{\phi})+C_W]W_{,C}-\frac{1}{2}h_{DB,C}F^DF^B-h_{DB}F^D_{,C}F^B. \tag{25}$$

Substituting this expression of $V_{,C}$ into Eq. (24), we get

$$3[W(\bar{\phi})+C_W]\{2M_{\rm Pl}^2W_{,C}+h_{DC}F^D\}$$
$$=(h_{DB}F^D_{,C}-h_{DC}F^D_{,B})F^B+(h_{DB,C}-h_{CB,D})F^DF^B. \tag{26}$$

If condition (23) is satisfied, then

$$[h_{DB}F^D_{,C}-h_{DC}F^D_{,B}]F^B+(h_{DB,C}-h_{CB,D})F^DF^B=0. \tag{27}$$

Also, from the obvious equality $W_{,CB}=W_{,BC}$, we get

$$(h_{BD,C}-h_{CD,B})F^D=h_{CD}F^D_{,B}-h_{BD}F^D_{,C}. \tag{28}$$

Condition (27) is a consequence of Eq. (28). Matrix h_{AB} is symmetric, and the conditions in Eq. (28) are trivial at $B=C$, so it is enough to check Eq. (28) for all $B<C$ only.

Thus, the task of solving the dynamic equations of the model is reduced to Eqs. (20), (23), and (28). The last equation guarantees that the solution of chiral field equation (24) is true.

Further we will use the expression for the potential $V(\bar{\phi})$ in terms of the superpotential $W(\bar{\phi})$, which follows from Eqs. (20) and (23):

$$V(\bar{\phi})=3M_{\rm Pl}^2[W(\bar{\phi})+C_W]^2+M_{\rm Pl}^2W_{,A}F^A. \tag{29}$$

Applying Eq. (23) once more, the physical potential can be presented in the form

$$V(\bar{\phi})=3M_{\rm Pl}^2[W(\bar{\phi})+C_W]^2-2M_{\rm Pl}^4h^{AB}W_{,A}W_{,B}. \tag{30}$$

From Eq. (10), we obtain

$$\dot{F}^E+\Gamma^E_{DB}F^DF^B+3WF^E+h^{CE}V_{,C}=0. \tag{31}$$

To demonstrate how one can get exact solutions due to the superpotential method, we consider a diagonal matrix h_{AB} such that each h_{BB} depends only on ϕ^A, with $A\le B$, and has the following form:

$$h_{11}=s_1(\phi^1),\qquad h_{BB}=u_B(\phi^1,\ldots,\phi^{B-1})s_B(\phi^B) \tag{32}$$

for all $B=2,\ldots,K$.

Let us prove, for any h_{BB} given by Eq. (32), that we can construct the CCM with exact solutions obtained either in analytic form or in quadratures. We choose

$$H(t)=\sum_{A=1}^{K}W_A(\phi^A)+C_W, \tag{33}$$

where $W_A(\phi^A)$ are differentiable functions.

The function W fixes the potential V by Eq. (29). For any A, the function F^A is defined as follows:

$$F^A=-2M_{\rm Pl}^2\frac{W_{A,A}}{h_{AA}}. \tag{34}$$

Let us check conditions (28). For the matrix h_{AB}, defined by Eq. (32) and $B<C$, we get

$$\frac{d}{d\phi^B}(h_{CC}F^C)=0 \tag{35}$$

without summing on C. It is easy to see that the functions F^A, defined (for each index A) by Eq. (34) satisfy these conditions. Therefore, we obtain all functions F^A, and system (19) takes the following form:

$$\dot{\phi}^1=-2M_{\rm Pl}^2\frac{W_{1,1}(\phi^1)}{h_{11}(\phi^1)},$$
$$\dot{\phi}^2=-2M_{\rm Pl}^2\frac{W_{2,2}(\phi^2)}{h_{22}(\phi^1,\phi^2)},$$
$$\ldots$$
$$\dot{\phi}^K=-2M_{\rm Pl}^2\frac{W_{K,K}(\phi^K)}{h_{KK}(\phi^1,\phi^2,\ldots,\phi^K)}. \tag{36}$$

This system can be solved, at least in quadrature. Indeed, the first equation of this system, as the first-order autonomous differential equation, can be solved in quadrature. Let us assume that all ϕ^A are known for $A<B$, and consider the equation for ϕ^B. Using $h_{BB}=u_B(\phi^1,\ldots,\phi^{B-1})s_B(\phi^B)$, we get

$$\frac{s_B(\phi^B)}{W_{B,B}(\phi^B)}d\phi^B = -\frac{2M_{\rm Pl}^2}{u_B(\phi^1(t),\ldots,\phi^{B-1}(t))}dt. \qquad (37)$$

This equation is integrable for any u_B given as a function of t. So, the solution is found, and the statement is proven by induction.

Note that the superpotential method allows us to construct such a one-parametric set of the CCM that the corresponding Hubble parameters differ on an arbitrary constant C_W. We have shown that any of these CCMs has a K-parametric set of exact solutions that can be found explicitly or in quadrature.

In the next sections, we restrict ourselves to two-dimensional h_{AB}. In particular, we show that two-dimensional h_{AB} in the form of Eq. (32) naturally arise from $f(R)$ gravity models with one scalar field.

IV. EXACT SOLUTIONS FOR AN $f(R)$ GRAVITY MODEL WITH AN ADDITIONAL SCALAR FIELD

In this and the following sections, we consider two-component CCMs and denote

$$\phi^1 = \psi, \qquad \phi^2 = \chi,$$
$$\dot\phi^1 = F^1 = U(\psi,\chi), \qquad \dot\phi^2 = F^2 = S(\psi,\chi).$$

It was shown in Refs. [21,22] that under the metric transformation

$$g_{\mu\nu} = e^{\frac{\sqrt{2}\psi}{\sqrt{3}M_{\rm Pl}}}\tilde g_{\mu\nu},$$

the $f(R)$ gravity model with a scalar field χ, described by the action

$$S_{\rm J} = \int d^4\tilde x\sqrt{-\tilde g}\left[f(\chi,\tilde R) - \frac{1}{2}\tilde g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\right], \qquad (38)$$

transforms to the chiral cosmological model, described by action (1) with

$$h_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & K(\psi)\end{pmatrix}, \qquad (39)$$

where

$$\psi = \sqrt{\frac{3M_{\rm Pl}^2}{2}}\ln\left(\frac{2}{M_{\rm Pl}^2}\left|\frac{\partial f}{\partial\tilde R}\right|\right), \qquad K(\psi) = e^{C\psi}, \qquad (40)$$

and there is a constant $C = -\sqrt{\frac{2}{3M_{\rm Pl}^2}}$.

Let us consider this CCM to obtain exact solutions due to the superpotential method described in the previous section.

From Eq. (23), we get

$$W_{,1} \equiv W_{,\psi} = -\frac{1}{2M_{\rm Pl}^2}U(\psi,\chi),$$
$$W_{,2} \equiv W_{,\chi} = -\frac{K(\psi)}{2M_{\rm Pl}^2}S(\psi,\chi). \qquad (41)$$

To get W in the form of Eq. (33), we choose

$$U(\psi,\chi) = U(\psi), \qquad S(\psi,\chi) = \frac{Q(\chi)}{K(\psi)}, \qquad (42)$$

where $Q(\chi)$ is an arbitrary function.

The superpotential W is defined as follows:

$$W = -\frac{1}{2M_{\rm Pl}^2}\left(\int U d\psi + \int Q d\chi\right). \qquad (43)$$

To get an exact particular solution, we assume the explicit form of the functions $U(\psi)$ and $Q(\chi)$ and solve the corresponding system for the first-order differential equations:

$$\dot\psi = U(\psi), \qquad \dot\chi = \frac{Q(\chi)}{K(\psi)}. \qquad (44)$$

We choose

$$U(\psi) = F_0 e^{-\Lambda_1\psi}, \qquad (45)$$

where F_0 and Λ_1 are constants. Therefore,

$$\psi(t) = \frac{1}{\Lambda_1}\ln(\Lambda_1 F_0(t-t_0)) \qquad (46)$$

and

$$K(\psi) = e^{C\psi(t)} = (\Lambda_1 F_0(t-t_0))^{C/\Lambda_1}. \qquad (47)$$

Substituting the obtained $\psi(t)$, we get $\chi(t)$. For example, for

$$Q(\chi) = F_0 e^{-\Lambda_2\chi}, \qquad \Lambda_2 = \text{const}, \qquad (48)$$

we obtain

$$\chi(t) = -\frac{1}{\Lambda_2}\ln\left(\frac{C-\Lambda_1}{\Lambda_2(C_2 - [\Lambda_1 F_0(t-t_0)]^{1-C/\Lambda_1})}\right), \qquad (49)$$

where C_2 is an integration constant.

The Hubble parameter

$$H(t) = W_1(\psi) + W_2(\chi) + C_W, \qquad (50)$$

where

$$W_1(\psi) = \frac{F_0}{2\Lambda_1 M_{\rm Pl}^2} e^{-\Lambda_1 \psi}, \qquad W_2(\chi) = \frac{F_0}{2\Lambda_2 M_{\rm Pl}^2} e^{-\Lambda_2 \chi}. \tag{51}$$

So, the considered model with the potential

$$V = \frac{3}{4M_{\rm Pl}^2 \Lambda_1^2 \Lambda_2^2}(\Lambda_2 F_0 e^{-\Lambda_1 \psi} + \Lambda_1 F_0 e^{-\Lambda_2 \chi} + 2M_{\rm Pl}^2 \Lambda_1 \Lambda_2 C_W)^2 - \frac{F_0^2}{2}(e^{-C\psi - 2\Lambda_2 \chi} + e^{-2\Lambda_1 \psi}) \tag{52}$$

has exact solutions with the following Hubble parameter:

$$H = C_W + \frac{1}{2\Lambda_1^2 M_{\rm Pl}^2 (t - t_0)} + \frac{F_0 (C - \Lambda_1)}{2(\Lambda_2^2 M_{\rm Pl}^2 [C_2 - (\Lambda_1 F_0 (t - t_0))^{1 - C\Lambda_1}])}. \tag{53}$$

Note that the Hubble parameter depends on two integration constants, t_0 and C_2.

We get a one-parametric set of models with exact solutions. Let us check that for some values of model parameters we get a slow-roll regime that maybe suitable for inflation.

Let us assume that $\Lambda_1 > 0$ and that $\Lambda_2 > 0$. In this case, the potential has a finite non-negative limit

$$V \to 3M_{\rm Pl}^2 C_W^2$$

at the scalar fields, which tend to plus infinity such that $2\Lambda_2 \chi > -C\psi$.

We consider large initial values of the scalar fields and assume that the scalar fields monotonically decrease during inflation, choosing $F_0 < 0$. For large positive values of the scalar fields we have quasi–de Sitter solutions with $H \simeq C_W$. So, we get a model that looks suitable for describing inflation. Full analysis of the possible inflationary scenarios with calculations of the inflationary parameters will be a subject of further investigation.

V. DIAGONAL CONSTANT METRIC OF THE TARGET SPACE

Studying a canonical scalar field equation, we set the metric coefficient h_{11} equal to unity: $h_{11} = 1$. Therefore, it will be of interest to study the diagonal metric with a constant chiral metric component, $h_{22} = 1$.

Equation (22) takes the form

$$\frac{\partial W}{\partial \psi} U(\psi, \chi) + \frac{\partial W}{\partial \chi} S(\psi, \chi) = -\frac{1}{2} U^2 - \frac{1}{2} S^2. \tag{54}$$

Let us insert the metric components of the target space into the field equation (26). After simple algebra, we obtain

$$3W(2W_{,\psi} + U(\psi, \chi)) = S_{,\psi} S - U_{,\chi} S, \tag{55}$$

$$3W(2W_{,\chi} + S(\psi, \chi)) = U_{,\chi} U - S_{,\psi} U. \tag{56}$$

If we suggest that $U = U(\psi)$ and $S = S(\chi)$, then Eqs. (55) and (56) are reduced to

$$2W_{,\psi} + U(\psi) = 0, \tag{57}$$

$$2W_{,\chi} + S(\chi) = 0. \tag{58}$$

Let us note that the consistency relation $W_{,\psi\chi} = W_{,\chi\psi}$ is satisfied.

Such a representation gives us the possibility to perform integration and find the superpotential $W(\psi, \chi)$:

$$W = -\frac{1}{2M_{\rm Pl}^2} \left(\int U(\psi) d\psi + \int S(\chi) d\chi \right). \tag{59}$$

Let us choose the linear dependence of the chiral field derivatives

$$U(\psi) = \mu_1 \psi + c_1, \qquad S(\chi) = \mu_2 \chi + c_2. \tag{60}$$

From here, one can find the chiral field evolution

$$\psi = \frac{1}{\mu_1} e^{\mu_1 t} - \frac{c_1}{\mu_1}, \tag{61}$$

$$\chi = \frac{1}{\mu_2} e^{\mu_2 t} - \frac{c_2}{\mu_2}. \tag{62}$$

Then the superpotential can be obtained by the integration of Eq. (23) and reads

$$W(\psi, \chi) = -\frac{1}{2} \left(\frac{\mu_1}{2} \psi^2 + c_1 \psi + \frac{\mu_2}{2} \chi^2 + c_2 \chi \right). \tag{63}$$

Further inserting the chiral fields into Eq. (63), we can obtain the Hubble function $H = H(t)$ in two-parametric form:

$$H(t) = -\frac{1}{4} \left(\frac{e^{2\mu_1 t}}{\mu_1} - \frac{c_1^2}{\mu_1} + \frac{e^{2\mu_2 t}}{\mu_2} - \frac{c_2^2}{\mu_2} \right). \tag{64}$$

Thus, we obtain a double exponent solution for the scalar factor.

Let us choose the following dependence:

$$U(\psi) = \mu_1 \psi^{k_1}, \qquad S(\chi) = \mu_2 \chi^{k_2}. \tag{65}$$

The exceptional situation $k_1 = k_2 = 0$ leads to a very interesting solution,

$$W(\psi,\chi) = -\frac{1}{2}(\mu_1^2+\mu_2^2)t. \tag{66}$$

Remembering that $H(t) = W(\psi,\chi) + C_W$, we find that

$$a(t) = a_* \exp\left[-\frac{1}{4}(\mu_1^2+\mu_2^2)t^2 + C_W t\right]. \tag{67}$$

This scale factor corresponds to Ruzmaikin's solutions [75,76].

When k_1, $k_2 \neq 0, 1$, we have solutions for the fields

$$\psi(t) = [(1-k_1)(\mu_1 t + c_1)]^{\frac{1}{1-k_1}}, \tag{68}$$

$$\chi(t) = [(1-k_2)(\mu_2 t + c_2)]^{\frac{1}{1-k_2}}. \tag{69}$$

The Hubble function, once again in two-parametric form, is

$$H(t) = -\frac{1}{2}\left[\frac{\mu_1}{k_1+1}[(1-k_1)(\mu_1 t + c_1)]^{\frac{k_1+1}{1-k_1}} + \frac{\mu_2}{k_2+1}[(1-k_2)(\mu_2 t + c_2)]^{\frac{k_2+1}{1-k_2}} + C_W\right]. \tag{70}$$

The scale factor is

$$a(t) = a_* \exp\left[\frac{\mu_1}{k_1+1}(1-k_1)^{\frac{k_1+1}{1-k_1}}[(\mu_1 t + c_1)]^{\frac{2}{1-k_1}} + \frac{\mu_2}{k_2+1}(1-k_2)^{\frac{k_2+1}{1-k_2}}[(\mu_2 t + c_2)]^{\frac{2}{1-k_2}} + C_W t\right]. \tag{71}$$

Thus, the obtained solution corresponds to the intermediate inflation.

VI. MODIFIED POWER-LAW SOLUTIONS

As known, solutions with the power-law Hubble parameter correspond to the radiation and matter dominated epochs. It is interesting to get exact solutions with the Hubble parameter $H = C_0 + C_1/t$, where constants C_i can be chosen in such a way that the solution has both dark matter and dark energy parts.

We consider the CCM with the metric of the target space

$$h_{AB} = \begin{pmatrix} \frac{C_1 M_{\rm Pl}^2}{\psi^2} & 0 \\ 0 & \frac{C_2 M_{\rm Pl}^2}{\chi^2} \end{pmatrix}, \tag{72}$$

and we assume the following form of the superpotential,

$$W = -Y_1\psi^{m_1} - Y_2\chi^{m_2}, \tag{73}$$

where Y_i and m_i are constants.

Using expression (29), we get the potential

$$V = 3M_{\rm Pl}^2 C_W^2 - 6M_{\rm Pl}^2 C_W Y_1 \psi^{m_1} - 6M_{\rm Pl}^2 C_W Y_2 \chi^{m_2} + 6M_{\rm Pl}^2 Y_1\psi^{m_1} Y_2 \chi^{m_2} + \frac{M_{\rm Pl}^2 Y_1^2(3C_1 - 2m_1^2)}{C_1}\psi^{2m_1} + \frac{M_{\rm Pl}^2 Y_2^2(3C_2 - 2m_2^2)}{C_2}\chi^{2m_2}. \tag{74}$$

Equation (23) is

$$\dot\psi = 2\frac{Y_1 m_1}{C_1}\psi^{m_1+1}, \qquad \dot\chi = 2\frac{Y_2 m_2}{C_2}\chi^{m_2+1}. \tag{75}$$

This system has the following solution:

$$\psi = \left(\frac{-C_1}{2Y_1 m_1^2(t-t_0)}\right)^{1/m_1}, \qquad \chi = \left(\frac{-C_2}{2Y_2 m_2^2(t-\tilde t_0)}\right)^{1/m_2}, \tag{76}$$

where t_0 and $\tilde t_0$ are integration constants. Substituting the obtained solution for the superpotential, we get the Hubble parameter

$$H(t) = C_W + \frac{C_1}{2m_1^2(t-t_0)} + \frac{C_2}{2m_2^2(t-\tilde t_0)}. \tag{77}$$

Choosing $t_0 = 0$ and $\tilde t_0 = 0$, we get

$$H = C_W + \frac{1}{2t}\left(\frac{C_1}{m_1^2} + \frac{C_2}{m_2^2}\right). \tag{78}$$

So, the Hubble parameter is a sum of a constant that corresponds to the dark energy dominant epoch and the power-law function that corresponds to the radiation dominant epoch at $\frac{C_1}{m_1^2} + \frac{C_2}{m_2^2} = 1$ or to the matter dominant epoch at $\frac{C_1}{m_1^2} + \frac{C_2}{m_2^2} = \frac{4}{3}$. Choosing $\tilde t_0 \neq t_0$, we get more complicated solutions. So, starting from an exact solution with $H = C_0 + C_1/t$ and using the superpotential method, we not only reconstruct the corresponding potential but also find that the model obtained has a two-parametric set of the exact solution, described by formulas (76) and (77).

VII. TARGET SPACE RECONSTRUCTION FROM THE SUPERPOTENTIAL

A. The search of the CCM with the given $H(t)$

From the given superpotential (Hubble function) it is possible to reconstruct the kinetic part to get the exact solution of the model.

Considering single scalar field cosmology, we may find a different formulation of the problem because we have two independent differential equations with three functions. Therefore, we must choose which function we have to consider as the given one. To fix the potential energy (or, simply, the potential) is preferable because it may be collected from high energy physics. If we study a canonical scalar field, we suggest the unit multiplier in the kinetic energy term. Extending such an approach to CCMs, we assume that the metric of a chiral space should be fixed and the analogue to the canonical field will be the unit diagonal metric. In the models with symmetry (for example, $SO(3)$—invariant CCM), the chiral metric is fixed also.

Another approach can be proposed when we use the superpotential method. If we are looking for the CCM which obeys the given superpotential (or, equivalently, the Hubble function) we can use the so-called deformation of a chiral space method. That is, we define such metric components which match the given data. Such an approach is similar, in some sense, to the "fine tuning of the potential" method for a single scalar field, where a given Hubble function allows one to define the potential and kinetic energies. Analyzing Eq. (23), which takes for the first field ψ the following form,

$$2M_{\rm Pl}^2 W_{,\psi} = -h_{11}(\psi)\dot{\psi}, \tag{79}$$

one can come to the conclusion that the functional part of the lhs may be included in the chiral metric component $h_{11}(\psi)$, and one can set $\dot{\psi}$ equal to unity: $\dot{\psi} = 1$. This gives us the possibility (performing the same procedure for the second field χ) of choosing the linear dependence of the fields on time:

$$\psi = t + \psi_*, \qquad \chi = t + \chi_*. \tag{80}$$

B. Examples of periodic Hubble functions

To demonstrate such an approach, let us study an example of the following periodic Hubble function:

$$H(t) = H_0 \sin(\lambda t) + C_W. \tag{81}$$

We may represent the superpotential in the following form,

$$W(\psi, \chi) = H_0[(1-\lambda_0)\sin(\lambda\psi) + \lambda_0 \sin(\lambda\chi)], \tag{82}$$

where H_0, λ, and λ_0 are constants.

The field equations (23) are nothing but the definition of the chiral metric component

$$h_{11}(\psi) = -2H_0 M_{\rm Pl}^2 (1-\lambda_0)\lambda\cos(\lambda\psi),$$
$$h_{22}(\chi) = -2H_0 M_{\rm Pl}^2 \lambda_0 \lambda \cos(\lambda\chi). \tag{83}$$

The potential can be defined by Eq. (30):

$$V(\psi,\chi) = 3H_0^2 M_{\rm Pl}^2 \left[(1-\lambda_0)\sin(\lambda\psi) + \lambda_0 \sin(\lambda\chi) + \frac{C_W}{H_0}\right] + H_0 M_{\rm Pl}^2 \lambda((1-\lambda_0)\cos(\lambda\psi) + \lambda_0\cos(\lambda\chi)). \tag{84}$$

It is interesting to note that solution (83) belongs to the two-field case in Eq. (36) with $u = 1$. Putting $\psi_* = \chi_* = 0$, we obtain the Hubble parameter (81). If $\psi_* = \chi_* = -\pi/(2\lambda)$, then we get

$$H(t) = H_0 \cos(\lambda t) + C_W. \tag{85}$$

The model constructed has a two-parametric set of exact solutions with

$$H(t) = H_0[(1-\lambda_0)\sin(\lambda(t+\psi_*)) + \lambda_0 \sin(\lambda(t+\chi_*))] + C_W. \tag{86}$$

Note that the case

$$H(t) = H_0 \sin^2(\lambda t) + C_W \tag{87}$$

is reduced to the previous one due to the substitution $\sin^2(\lambda t) = 1–2\cos(2\lambda t)$.

For

$$H(t) = H_0 \exp(-\alpha t \sin t), \tag{88}$$

the same approach gives us the solution

$$H_0^{-1} W(\psi,\chi) = (1-\lambda_0)\exp(-\alpha\psi\sin\psi) + \lambda_0 \exp(-\alpha\chi\sin\chi), \tag{89}$$

$$h_{11}(\psi) = 2H_0 M_{\rm Pl}^2 \alpha(1-\lambda_0)[\sin(\psi) + \psi\cos(\psi)] \times \exp(-\alpha\psi\sin(\psi)), \tag{90}$$

$$h_{22}(\chi) = 2H_0 M_{\rm Pl}^2 \alpha\lambda_0 [\sin\chi + \chi\cos\chi]\exp(-\alpha\chi\sin(\chi)), \tag{91}$$

where $\psi = t + \psi_*$, $\chi = t + \chi_*$.

The potential $V(\psi,\chi)$ is

$$V(\psi,\chi) = 3H_0^2 M_{\rm Pl}^2 [(1-\lambda_0)\exp(-\alpha\psi\sin\psi) + \lambda_0\exp(-\alpha\chi\sin\chi)]^2 - H_0 M_{\rm Pl}^2 \alpha[(1-\lambda_0)(\sin\psi + \psi\cos\psi) \times \exp(-\alpha\psi\sin\psi) + \lambda_0(\sin\chi + \chi\cos\chi)\exp(-\alpha\chi\sin\chi)]. \tag{92}$$

C. Example of a hyperbolic Hubble function

Another example is connected with the scale factor

$$a(t) = a_0[k_* \sinh(\lambda t)]^{2/3},$$

which corresponds to the Λ cold dark matter model. The Hubble function is

$$H = H_0 \coth(\lambda t), \qquad H_0 = \frac{2}{3}\lambda.$$

The corresponding superpotential is

$$W(\psi, \chi) = H_0[(1-\lambda_0)\coth(\lambda\psi) + \lambda_0 \coth(\lambda\chi)].$$

The solution is

$$\psi = t + \psi_*, \tag{93}$$

$$\chi = t + \chi_*, \tag{94}$$

$$h_{11}(\psi) = 2M_{\rm Pl}^2 H_0(1-\lambda_0)\lambda \sinh^{-2}(\lambda\psi), \tag{95}$$

$$h_{22}(\chi) = 2M_{\rm Pl}^2 H_0 \lambda_0 \lambda \sinh^{-2}(\lambda\chi). \tag{96}$$

Here and further we assume that $\psi \to (\psi - \psi_*)$ and $\chi \to (\chi - \chi_*)$. The physical potential is

$$V(\psi, \chi) = 3M_{\rm Pl}^2 H_0^2[(1-\lambda_0)\coth(\lambda\psi) + \lambda_0\coth(\lambda\chi)]^2 - M_{\rm Pl}^2 H_0 \lambda[(1-\lambda_0)\sinh^{-2}(\lambda\psi) + \lambda_0 \sinh^{-2}(\lambda\chi)]. \tag{97}$$

Using the superpotential method, it is possible to get different models for the given time dependence of the Hubble parameter, which we demonstrate in the next section.

VIII. THE CYCLIC UNIVERSE

In this section, we focus on a cyclic type of Universe dubbed quasi-steady-state (QSS) introduced to address the outstanding problems of the hot big bang, for instance, the singularity problem—in particular, see Refs. [77–79] and the references therein. In this case, the Hubble parameter might increase at late times to account for the well-known tension between Planck and local observations. The proposed early Universe modifications—namely, the interaction between known matter components or their interaction with dark energy—do not seem to account for the discrepancy [80]. One might attribute the latter to late time physics, for instance, to the emergence of phantom behavior at late times. The quasi-steady-state model includes such a feature. In what follows, we construct single-(double-) field potentials corresponding to the QSS.

A. One-field models

Let us construct a CCM with the following form of scale factor that characterizes the quasi-steady-state theory,

$$a(t) = a_0 e^{C_W t}(1 + \alpha\cos(\mu t)), \tag{98}$$

where a_0, α, and μ are constants. This type of dynamics corresponds to the quasi-steady-state model [77–79]. We assume that $a(t) > 0$ for any values of t, so $|\alpha| < 1$. The shift of time $t \to t + \pi/\mu$ is equivalent to the change of the sign of α, so we can assume that $0 \le \alpha < 1$ without loss of generality.[4] For the same reason, we can put $\mu > 0$. The corresponding Hubble parameter is

$$H = C_W - \frac{\mu\alpha\sin(\mu t)}{1 + \alpha\cos(\mu t)}, \tag{99}$$

and it has the following time derivative:

$$\dot{H} = -\mu^2\alpha \frac{\alpha + \cos(\mu t)}{[1 + \alpha\cos(\mu t)]^2}. \tag{100}$$

Such behavior of the Hubble parameter can be reproduced in one-field models. Reconstructing the chiral metric component as in the previous section, we choose $\psi = \mu t$ and get

$$h_{11} = \frac{2\alpha(\cos\phi + \alpha)}{(1 + \alpha\cos\phi)^2} \tag{101}$$

and

$$V(\phi) = \frac{C_W^2 + \alpha^2(C_W^2 - \mu^2)\cos^2\phi + \alpha\cos\phi(2C_W^2 - \mu^2) - 2\alpha C_W \mu \sin\phi(1 - \alpha\cos\phi)}{(1 + \alpha\cos\phi)^2}. \tag{102}$$

The alternative is to consider a $\psi(t)$ that tends to finite limits at $t \to \pm\infty$:

$$\psi(t) = \frac{2}{\mu\sqrt{1-\alpha^2}} \arctan\left(\frac{\sqrt{1-\alpha^2}}{1+\alpha}\tan\left(\frac{\mu t}{2}\right)\right). \tag{103}$$

[4] The case $\alpha = 0$ corresponds to a de Sitter solution.

It is easy to check that

$$\cos^2\left(\frac{\mu t}{2}\right) = \frac{(1-\alpha)\cos^2\left(\frac{\mu\sqrt{1-\alpha^2}}{2}\psi\right)}{1+\alpha-2\alpha\cos^2\left(\frac{\mu\sqrt{1-\alpha^2}}{2}\psi\right)}$$

$$\Rightarrow \cos(\mu t) = \frac{\alpha-\cos\left(\mu\sqrt{1-\alpha^2}\psi\right)}{\alpha\cos\left(\mu\sqrt{1-\alpha^2}\psi\right)-1}. \quad (104)$$

Thus, the function $\psi(t)$ is a solution of the following equation:

$$\dot{\psi} = \frac{1}{1+\alpha\cos(\mu t)} = U(\psi) = \frac{1-\alpha\cos\left(\mu\sqrt{1-\alpha^2}\psi\right)}{1-\alpha^2}. \quad (105)$$

In the case of a one scalar field model,

$$\dot{H} = -\frac{1}{2M_{\rm Pl}^2}h_{11}(\psi)\dot{\psi}^2, \quad (106)$$

therefore,

$$h_{11} = 2M_{\rm Pl}^2\mu^2[\alpha^2+\alpha\cos(\mu t)] = 2M_{\rm Pl}^2\mu^2\alpha\left[\frac{(1-\alpha^2)\cos(\mu\sqrt{1-\alpha^2}\psi)}{1-\alpha\cos(\mu\sqrt{1-\alpha^2}\psi)}\right].$$

Using Eq. (23), we obtain

$$W'_{,\psi} = \mu^2\alpha\cos\left(\mu\sqrt{1-\alpha^2}\psi\right) \Rightarrow W = \frac{\mu\alpha}{\sqrt{1-\alpha^2}}\sin\left(\mu\sqrt{1-\alpha^2}\psi\right). \quad (107)$$

The potential of the model constructed is defined by Eq. (20) and has the following form:

$$V = \frac{M_{\rm Pl}^2\mu^2\alpha}{1-\alpha^2}\left[3\alpha-\cos\left(\mu\sqrt{1-\alpha^2}\psi\right)-2\alpha\cos^2\left(\mu\sqrt{1-\alpha^2}\psi\right)\right] - \frac{6\alpha\mu C_W M_{\rm Pl}^2}{\sqrt{1-\alpha^2}}\sin\left(\mu\sqrt{1-\alpha^2}\psi\right)+3M_{\rm Pl}^2C_W^2. \quad (108)$$

Note that h_{11} changes its sign during the scalar field evolution, so ψ is neither an ordinary scalar field nor a phantom scalar field [81,82]. Assuming that the considering one-field models describe the dark energy, we get the result that the obtained exact solutions have the state parameters crossing the cosmological constant barrier. It has been shown in Ref. [28] that such transitions are physically implausible in one-field models because they are either realized by a discrete set of trajectories in the phase space or are unstable with respect to the cosmological perturbations. To describe such a type of dark energy, one can use quintom models [29,40,41,83–86].

B. The CCM quasi-steady-state models

To construct a two-field CCM with the Hubble parameter given by Eq. (99), we use relations $W_{,\psi} = -\frac{1}{2}h_{11}(\psi)\dot{\psi}$, $W_{,\chi} = -\frac{1}{2}h_{22}(\chi)\dot{\chi}$, and

$$\frac{dW}{dt} = -\frac{1}{2}h_{11}(\psi)\dot{\psi}^2 - \frac{1}{2}h_{22}(\chi)\dot{\chi}^2. \quad (109)$$

First of all we can easily make a reconstruction of the chiral metric component as in previous section.

We choose the superpotential in the form

$$W(\psi,\chi) = -(1-\lambda_0)\frac{\alpha\mu\cos\mu t(\psi)}{(1+\alpha\cos\mu t(\psi))^2} - \lambda_0\frac{\alpha\mu\cos\mu t(\chi)}{(1+\alpha\cos\mu t(\chi))^2}. \quad (110)$$

Then we choose the dependence on t as

$$\psi = t+\psi_*, \qquad \chi = t+\chi_*. \quad (111)$$

The chiral metric components will be

$$h_{11}(\psi) = \frac{2\alpha\mu^2\cos\mu\psi}{(1+\alpha\cos\mu\psi)^2}, \quad (112)$$

$$h_{22}(\chi) = \frac{2\alpha\mu^2}{(1+\alpha\cos\mu\chi)^2}. \quad (113)$$

The physical potential can be easily derived by Eq. (29). (It can be written here, but it is rather large.)

The superpotential is evidently defined by Eq. (110) with the substitution of Eq. (111).

There are other possible solutions connecting with the choice of chiral metric components. Let us study a few of them.

From Eq. (110), we find that

$$\frac{dW}{dt} = -\frac{\alpha\mu^2}{(1+\alpha\cos(\mu t))^2}(\cos(\mu t)+\alpha). \quad (114)$$

Further we can decompose the expression above into the following two:

$$\frac{\partial W(\psi,\chi)}{\partial\psi} = -\frac{\alpha\mu^2\cos\mu t}{(1+\alpha\cos\mu t)^2} = -\frac{1}{2}h_{11}(\psi)\dot{\psi}^2, \quad (115)$$

$$\frac{\partial W(\psi,\chi)}{\partial\chi} = -\frac{\alpha^2\mu^2}{(1+\alpha\cos\mu t)^2} = -\frac{1}{2}h_{22}(\chi)\dot{\chi}^2. \quad (116)$$

Using relations (111), we can perform integration of the Eqs. (115) and (116) and obtain two parts of the superpotential $W_1(\psi)$ and $W_2(\chi)$ in terms of elementary functions, but in rather complicated form. Therefore, our task is to make a combination which allows to integrate the expressions for the fields and for the superpotential with a more suitable result.

To this end, we can choose

$$\dot{\psi}^2 = \frac{\mu^2}{(1+\alpha\cos\mu t)^2}, \qquad h_{11} = 2\alpha\cos\mu t, \tag{117}$$

$$\dot{\chi}^2 = \frac{\mu^2}{(1+\alpha\cos\mu t)^2}, \qquad h_{22} = 2\alpha^2. \tag{118}$$

As the result, we obtain the following solution for the chiral fields:

$$\psi(t) = \frac{2}{\sqrt{1-\alpha^2}}\arctan\left(\frac{1-\alpha}{\sqrt{1-\alpha^2}}\tan\left(\frac{\mu t}{2}\right)\right), \tag{119}$$

$$\chi(t) = \frac{2}{\sqrt{1-\alpha^2}}\arctan\left(\frac{1-\alpha}{\sqrt{1-\alpha^2}}\tan\left(\frac{\mu t}{2}\right)\right). \tag{120}$$

From solution (119), one can obtain the chiral metric component $h_{11}(\psi)$ in the form

$$h_{11}(\psi) = 2\alpha\frac{(1-\alpha)-(1+\alpha)\tan^2\left(\frac{\sqrt{1-\alpha^2}}{2}\psi\right)}{(1-\alpha)+(1+\alpha)\tan^2\left(\frac{\sqrt{1-\alpha^2}}{2}\psi\right)}. \tag{121}$$

Once again such a presentation does not give for us a suitable form for the superpotential and for physical potential.

For the same representation (117), one can choose another appearance for χ and h_{22}. For example,

$$\dot{\chi}^2 = 2\alpha^2, \qquad h_{22} = \frac{\mu^2}{(1+\alpha\cos\mu t)^2}. \tag{122}$$

Then the solution for χ is

$$\chi = \sqrt{2}\alpha t + \chi_*,$$

and for $h_{22}(\chi)$,

$$h_{22}(\chi) = \frac{\mu^2}{\left(1+\alpha\cos\left(\frac{\mu}{\sqrt{2}\alpha}\chi\right)\right)^2}.$$

Thus, we can state that the chiral metric, the fields, and the physical potential may be different with respect to the given Hubble function. To stress this fact and to find the suitable form of the superpotential, we generalize the procedure of solutions generated by introducing two arbitrary functions, $f(\mu t)$ and $y(\mu t)$, in the following way:

$$\alpha\mu^2\frac{f(\mu t)^2\cos\mu t}{f(\mu t)^2(1+\alpha\cos\mu t)^2} = \frac{1}{2}h_{11}(\psi)\dot{\psi}^2, \tag{123}$$

$$\alpha^2\mu^2\frac{y(\mu t)^2}{y(\mu t)^2(1+\alpha\cos\mu t)^2} = \frac{1}{2}h_{22}(\chi)\dot{\chi}^2. \tag{124}$$

The functions $f(\mu t)$ and $y(\mu t)$ can be selected in such a way that integration is performed while finding fields. For example, if we are finding the field ψ, we have to perform the integral

$$\psi = \int\frac{d(\mu t)}{f(\mu t)(1+\alpha\cos\mu t)}. \tag{125}$$

To make the integral with an elementary function a solution, one can choose as an example $f(\mu t) = \frac{1}{\sin\mu t}$. Then the solution is

$$\psi = -\frac{1}{\alpha}\ln(1+\alpha\cos\mu t), \quad \text{where } \alpha < 1. \tag{126}$$

Under suggestion (125), one can find the chiral metric component h_{11} in the following way:

$$h_{11}(\psi) = 2\alpha f(\mu t)^2\cos\mu t. \tag{127}$$

In our case, $h_{11} = 2\alpha\frac{\cos\mu t}{\sin^2\mu t}$. Finding dependence t on ψ from (126), one can obtain dependence h_{11} on ψ as follows:

$$h_{11}(\psi) = 2\alpha^2\frac{e^{-\alpha\psi}-1}{\alpha^2-(e^{-\alpha\psi}-1)^2}. \tag{128}$$

Further we consider which types of superpotential and physical potential will correspond to the solution for field ψ (126) and chiral metric component $h_{11}(\psi)$ (128) using the freedom of the possible choice of function $y(\chi)$.

Let us start with the solution for the ψ part in the form ($|\alpha| < 1$)

$$\psi = -\frac{1}{\alpha}\ln(1+\alpha\cos\mu t), \qquad f(\mu t) = \frac{1}{\sin\mu t}, \tag{129}$$

$$h_{11} = 2\alpha^2\frac{e^{-\alpha\psi}-1}{\alpha^2-(e^{-\alpha\psi}-1)^2}. \tag{130}$$

Note that

$$\dot{\psi}(\psi) = \frac{\mu}{\alpha}e^{\alpha\psi}\sqrt{\alpha^2-(e^{-\alpha\psi}-1)^2}.$$

To find the dependence W on ψ, we have to integrate the relation

$$W_{,\psi} = -\frac{1}{2}h_{11}(\psi)\dot{\psi},$$

which can be transformed to the following:

$$\frac{\partial W}{\partial \psi} = -\mu\alpha e^{-\alpha\psi}\frac{e^{-\alpha\psi}-1}{\sqrt{\alpha^2-(e^{-\alpha\psi}-1)^2}}. \tag{131}$$

The solution is

$$W_1(\psi) = -\frac{\mu}{a^2}F(z) + \frac{\mu\alpha^2}{a^3}\arctan\left(\frac{a^2 z-1}{aF(z)}\right), \tag{132}$$

where

$$z = \exp(\alpha\psi), \quad F(z) = \sqrt{-a^2z^2+2z-1}, \quad a^2 = 1-\alpha^2 > 0.$$

Further we have various possibilities to define χ and $h_{22}(\chi)$. Let us select the nontrivial choice $y(\mu t) = 1-\alpha\cos\mu t$. Then we have the solution for the field χ,

$$\chi = \frac{1}{\sqrt{1-\alpha^2}}\arctan\left[\frac{\tan\mu t}{\sqrt{1-\alpha^2}}\right], \qquad \alpha^2 < 1.$$

The chiral metric component h_{22} is

$$h_{22}(\chi) = 2\alpha^2\left(1-\alpha\left(1+(1-\alpha^2)\tan^2\left(\sqrt{1-\alpha^2}\chi\right)\right)^{-1/2}\right)^2.$$

Thus, we get

$$\frac{dW(\chi)}{d\chi} = -\frac{\alpha^2\mu}{(1-\alpha^2)^{3/2}}\left(\sqrt{1+(1-\alpha^2)v^2}-\alpha\right)^2(1+v^2)^{-2},$$
$$v = \tan(\sqrt{1-\alpha^2}\chi). \tag{133}$$

The solution for the χ part of the superpotential is

$$W_2(\chi) = -\frac{\alpha\mu}{(1-\alpha^2)^{2/3}}\left\{2(\alpha^2-1)\arctan(v/P(v)) - (2\alpha^2-1)\arctan(\alpha v/P(v)) + \frac{\alpha v(\alpha+1)P(v)}{1+v^2} - \frac{\sqrt{1-\alpha^2}}{2}\chi - \frac{\alpha^2 v}{v^2+1}\right\},$$
$$P(v) = \sqrt{v^2(1-\alpha^2)+1}. \tag{134}$$

The superpotential is equal to

$$W(\psi,\chi) = W_1(\psi) + W_2(\chi).$$

The next step is to calculate the potential $V(\psi,\chi)$ by Eq. (29). This is possible, but the answer will be too long.

It is possible to get the same time evolution of the scalar factor using a quintom model. The time derivative of the Hubble parameter (100) can be presented in the following form:

$$\dot{H} = -\frac{\mu^2\alpha(\alpha+\cos(\mu t))}{[1+\alpha\cos(\mu t)]^2} = \frac{-(\alpha^2+\alpha)\mu^2}{[1+\alpha\cos(\mu t)]^2} + \frac{2\mu^2\alpha\sin^2(\frac{\mu t}{2})}{[1-\alpha+2\alpha\cos^2(\frac{\mu t}{2})]^2}. \tag{135}$$

Let us introduce two scalar fields with the same time behavior:

$$\psi(t) = \frac{2}{\mu\sqrt{1-\alpha^2}}\arctan\left(\frac{\sqrt{1-\alpha^2}}{1+\alpha}\tan\left(\frac{\mu t}{2}\right)\right),$$
$$\chi(t) = \frac{2}{\mu\sqrt{1-\alpha^2}}\arctan\left(\frac{\sqrt{1-\alpha^2}}{1+\alpha}\tan\left(\frac{\mu t}{2}\right)\right). \tag{136}$$

Using Eq. (104), the time derivative of H can be rewritten as follows:

$$\dot{H} = -(1+\alpha)\alpha\mu^2\dot{\psi}^2 + 2\alpha\mu^2\left[1-\frac{(1-\alpha)\cos^2\left(\frac{\mu\sqrt{1-\alpha^2}}{2}\chi\right)}{1+\alpha-2\alpha\cos^2\left(\frac{\mu\sqrt{1-\alpha^2}}{2}\chi\right)}\right]\dot{\chi}^2. \tag{137}$$

So, we get the matrix h_{AB} as diagonal and

$$h_{11} = 2(1+\alpha)\alpha M_{\mathrm{Pl}}^2\mu^2,$$
$$h_{22} = -\frac{2\alpha(1+\alpha)\mu^2 M_{\mathrm{Pl}}^2\left[1-\cos\left(\mu\sqrt{1-\alpha^2}\chi\right)\right]}{1-\alpha\cos\left(\mu\sqrt{1-\alpha^2}\chi\right)}. \tag{138}$$

The field $\psi(t)$ is an ordinary scalar field, whereas $\chi(t)$ is a phantom scalar field.

We assume that the superpotential has the form given by Eq. (33):

$$W = W_1(\psi) + W_2(\chi). \tag{139}$$

Using Eq. (23), we get

$$W_1 = \frac{\mu^2\alpha}{\alpha-1}\psi - \frac{\mu\alpha^2}{\sqrt{1-\alpha^2}(\alpha-1)}\sin\left(\mu\sqrt{1-\alpha^2}\psi\right), \tag{140}$$

$$W_2 = \frac{\mu^2\alpha}{\alpha-1}\chi - \frac{\mu\alpha}{\sqrt{1-\alpha^2}(\alpha-1)}\sin\left(\mu\sqrt{1-\alpha^2}\chi\right). \tag{141}$$

Using Eq. (29), we get the potential of the obtained two-field model in the following form:

$$V = \frac{3M_{\rm Pl}^2}{(\alpha-1)^3(\alpha+1)}\Big[\alpha^2\mu^4(\alpha^2-1)(\psi^2+\chi^2)+2\mu^4\alpha^2(\alpha^2-1)\psi\chi$$
$$+2\mu^2\alpha C_W(\alpha+1)(\alpha-1)^2\psi+2\mu^2\alpha C_W(\alpha+1)(\alpha-1)^2\chi+(\alpha+1)(\alpha-1)^3C_W^2$$
$$-\alpha^2(\alpha^2+1)\mu^2-2\alpha^3\mu^2\sin\left(\mu\sqrt{1-\alpha^2}\psi\right)\sin\left(\mu\sqrt{1-\alpha^2}\chi\right)$$
$$+2\alpha\mu\sqrt{1-\alpha^2}[\alpha\mu^2(\chi+\psi)+(\alpha-1)C_W]\sin\left(\mu\sqrt{1-\alpha^2}\chi\right)$$
$$+2\alpha^2\mu\sqrt{1-\alpha^2}[\alpha\mu^2(\chi+\psi)+(\alpha-1)C_W]\sin\left(\mu\sqrt{1-\alpha^2}\psi\right)$$
$$+\frac{1}{3}\mu^2\alpha^2[(\alpha+2)\cos\left(\mu\sqrt{1-\alpha^2}\chi\right)^2+(2\alpha^2+\alpha)\cos\left(\mu\sqrt{1-\alpha^2}\psi\right)^2]$$
$$+\frac{1}{3}\mu^2\left(\alpha(1-\alpha^2)\cos\left(\mu\sqrt{1-\alpha^2}\chi\right)+2\alpha^2(\alpha-1)\cos\left(\mu\sqrt{1-\alpha^2}\psi\right)\right)\Big]. \tag{142}$$

IX. REDUCING TWO-FIELD DYNAMIC EQUATIONS TO THE SINGLE-FIELD ONES

For simplification of the generation of the exact solutions for CCMs, we consider the possibility of reducing the dynamic equations in such a type of model to a single-field case.

For this aim, we write the dynamic equations in terms of the effective field φ, which is connected with CCM fields ϕ^A by the following relation [34]:

$$\dot\varphi^2 = h_{AB}\dot\phi^A\dot\phi^B, \tag{143}$$

where ϕ^A are the fields of the CCM, and h_{AB} is the metric tensor of a target space.

The double kinetic energy $X=\dot\varphi^2$ of the effective field φ was considered earlier the X field (11). For positive $X=\dot\varphi^2>0$, one has the canonical effective scalar field φ, and $X<0$ corresponds to the phantom one. The dynamic equations in CCMs (6)–(10) can be noted as

$$3H^2M_{\rm Pl}^2 = \frac{1}{2}\dot\varphi^2+V(\varphi), \tag{144}$$

$$\dot\varphi^2 = -2\dot H M_{\rm Pl}^2, \tag{145}$$

$$D_t\dot\phi^A+3H\dot\phi^A+h^{AB}V_{,B}=0, \tag{146}$$

where

$$D_t\dot\phi^A = \frac{d\dot\phi^A}{dt}+\Gamma^A_{BC}\dot\phi^B\dot\phi^C \tag{147}$$

is a covariant derivative in a target space.

Also, one can rewrite the first dynamic equation (144) on the basis of Eq. (145) in the following form:

$$V(\varphi) = M_{\rm Pl}^2(3H^2+\dot H). \tag{148}$$

Therefore, in the general case, the connection between particular solutions of CCMs and one-field models is defined on the basis of Eq. (146) as follows:

$$\ddot\phi^A+\Gamma^A_{BC}\dot\phi^B\dot\phi^C+3H\dot\phi^A+h^{AB}V_{,B}=0$$
$$\Leftrightarrow \ddot\varphi+3H\dot\varphi+V_{,\varphi}=0 \tag{149}$$

for all A.

Further we consider the fulfillment of this condition for the case of a CCM with two identical scalar fields $\chi=\psi$ by the specific connections between components of the tensor h_{AB}.

A. The CCM with two scalar fields

Now, we consider the partial case of a CCM with two identical scalar fields $\chi=\psi$ and the specific connections between components of the tensor h_{AB}.

Firstly, we write the dynamic equations (144)–(146) for a CCM with two fields:

$$3H^2M_{\rm Pl}^2 = \frac{1}{2}h_{11}\dot\chi^2+h_{12}\dot\chi\dot\psi+\frac{1}{2}h_{22}\dot\psi^2+V(\psi,\chi), \tag{150}$$

$$-\dot HM_{\rm Pl}^2 = \frac{1}{2}h_{11}\dot\chi^2+h_{12}\dot\chi\dot\psi+\frac{1}{2}h_{22}\dot\psi^2, \tag{151}$$

$$3H(h_{11}\dot\chi+h_{12}\dot\psi)+\frac{\partial}{\partial t}(h_{11}\dot\chi+h_{12}\dot\psi)-\frac{1}{2}\frac{\partial h_{11}}{\partial\chi}\dot\chi^2$$
$$-\frac{\partial h_{12}}{\partial\chi}\dot\chi\dot\psi-\frac{1}{2}\frac{\partial h_{22}}{\partial\chi}\dot\psi^2+\frac{\partial V}{\partial\chi}=0, \tag{152}$$

$$3H(h_{12}\dot\chi+h_{22}\dot\psi)+\frac{\partial}{\partial t}(h_{12}\dot\chi+h_{22}\dot\psi)-\frac{1}{2}\frac{\partial h_{11}}{\partial\psi}\dot\chi^2$$
$$-\frac{\partial h_{12}}{\partial\psi}\dot\chi\dot\psi-\frac{1}{2}\frac{\partial h_{22}}{\partial\psi}\dot\psi^2+\frac{\partial V}{\partial\psi}=0. \tag{153}$$

Secondly, for models with the following metric tensor of the target space,

$$h_{AB} = \begin{pmatrix} \frac{n}{2} - h_{12}(\psi,\chi) & h_{12}(\psi,\chi) \\ h_{12}(\psi,\chi) & \frac{n}{2} - h_{12}(\psi,\chi) \end{pmatrix}, \tag{154}$$

where $h_{11} = h_{22} = \frac{n}{2} - h_{12}$, $h_{21} = h_{12}$ and n is an arbitrary constant, under condition $\chi = \psi$, we have

$$h_{11}\dot{\chi} + h_{12}\dot{\psi} = h_{12}\dot{\chi} + h_{22}\dot{\psi} = \frac{n}{2}\dot{\chi} = \frac{n}{2}\dot{\psi}, \tag{155}$$

$$\frac{1}{2}\frac{\partial h_{11}}{\partial\chi}\dot{\chi}^2 + \frac{\partial h_{12}}{\partial\chi}\dot{\chi}\dot{\psi} + \frac{1}{2}\frac{\partial h_{22}}{\partial\chi}\dot{\psi}^2 = \frac{1}{2}\frac{\partial h_{11}}{\partial\psi}\dot{\chi}^2 + \frac{\partial h_{12}}{\partial\psi}\dot{\chi}\dot{\psi} + \frac{1}{2}\frac{\partial h_{22}}{\partial\psi}\dot{\psi}^2 = 0, \tag{156}$$

$$\frac{1}{2}h_{11}\dot{\chi}^2 + h_{12}\dot{\chi}\dot{\psi} + \frac{1}{2}h_{22}\dot{\psi}^2 = \frac{n}{2}\dot{\chi}^2 = \frac{n}{2}\dot{\psi}^2. \tag{157}$$

Thus, from Eqs. (150)–(153), we obtain

$$V(\psi,\chi) = M_{\rm Pl}^2(3H^2 + \dot{H}) = V(\varphi), \tag{158}$$

$$-\dot{H}M_{\rm Pl}^2 = \frac{n}{2}\dot{\chi}^2 = \frac{n}{2}\dot{\psi}^2 = \frac{1}{2}\dot{\varphi}^2, \tag{159}$$

$$\ddot{\chi} + 3H\dot{\chi} + \frac{2}{n}\frac{\partial V}{\partial\chi} = \ddot{\psi} + 3H\dot{\psi} + \frac{2}{n}\frac{\partial V}{\partial\psi} = 0. \tag{160}$$

From Eq. (160), taking into account the equality of the scalar fields $\chi = \psi$, we have the condition of symmetry of the potential $\frac{\partial V}{\partial\chi} = \frac{\partial V}{\partial\psi}$ with respect to these fields.

Therefore, one can write

$$\frac{dV}{d\varphi} = \frac{\partial V}{\partial\chi}\frac{d\chi}{d\varphi} + \frac{\partial V}{\partial\psi}\frac{d\psi}{d\varphi} = 2\frac{\partial V}{\partial\chi}\frac{d\chi}{d\varphi} = 2\frac{\partial V}{\partial\psi}\frac{d\psi}{d\varphi}. \tag{161}$$

Finally, we note that from Eqs. (158)–(159), one can obtain Eq. (160) and the equation

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0 \tag{162}$$

is a differential consequence of Eqs. (158) and (159) as well, which can easily be obtained by substituting the effective field

$$\varphi = \pm\frac{\sqrt{n}}{2}(\chi + \psi) \tag{163}$$

in the form $\varphi = \pm\sqrt{n}\chi$ or $\varphi = \pm\sqrt{n}\psi$ into Eq. (162) using the relations (161).

Therefore, in this case, we have the fulfillment of condition (149), which allows one to reduce the initial CCM with two scalar fields to the one-field model.

Further we consider the following superpotential of the effective field

$$W(\varphi) \equiv H(t) \tag{164}$$

and write the dynamic equations (158)–(160) as

$$V(\varphi) = M_{\rm Pl}^2\left[3W^2(\varphi) - 2M_{\rm Pl}^2\left(\frac{dW(\varphi)}{d\varphi}\right)^2\right], \tag{165}$$

$$\dot{\varphi} = -2M_{\rm Pl}^2\left(\frac{dW(\varphi)}{d\varphi}\right) \tag{166}$$

for the effective field (163), where the constant parameter n defines the character of an effective field φ—namely, this field can be canonical or phantom for the different signs of n.

Thus, we have a connection between chiral cosmological models with two canonical scalar fields χ and ψ, and a single-field model with a effective field φ. The sign on the parameter n depends on the choice of field φ: for a canonical effective field one has $n > 0$, for a phantom one the parameter $n < 0$.

As an example of the proposed approach, we will obtain the exact solutions for two identical scalar fields χ and ψ with linear dependence from cosmic time for an arbitrary function $h_{12}(\chi,\psi)$ in the metric tensor of the target space (154) by choosing the special form of the superpotential $W(\varphi)$. Thus, these solutions will differ from ones considered earlier in Sec. VII and Sec. VIII with a specific expressions of the components of the tensor h_{AB} for a given type of the evolution of scalar fields.

For the following superpotential,

$$W(\varphi) = -\frac{\alpha}{2M_{\rm Pl}^2}\varphi, \tag{167}$$

from Eq. (166), we have

$$\varphi(t) = \alpha t - \beta, \tag{168}$$

where β is the constant of integration.

The Hubble parameter and the scale factor are

$$H(t) = -\frac{\alpha}{2M_{\rm Pl}^2}(\alpha t - \beta), \tag{169}$$

$$a(t) = a_0 \exp\left[\frac{\alpha}{4M_{\rm Pl}^2}t(2\beta - \alpha t)\right], \tag{170}$$

corresponding to the Ruzmaikin solutions [75,76].

From Eq. (165), we obtain the following potential of the effective field:

$$V(\varphi) = \left(\frac{\alpha}{2M_{\rm Pl}}\right)^2[3\varphi^2 - 2M_{\rm Pl}^2]. \tag{171}$$

After substituting Eq. (163) into the solutions for φ, we obtain

$$\chi(t) = \psi(t) = \pm\frac{1}{\sqrt{n}}(\alpha t - \beta), \tag{172}$$

$$V(\psi,\chi) = \left(\frac{\alpha}{2M_{\rm Pl}}\right)^2\left[\frac{3n}{4}(\chi+\psi)^2 - 2M_{\rm Pl}^2\right], \tag{173}$$

the potential and evolution of the CCM fields corresponding to the same dynamics (169) and (170) of the early Universe.

For the other example, we consider the exact solutions defined by the following superpotential,

$$W(\varphi) = \frac{A}{8M_{\rm Pl}^2}\varphi^2 + \lambda, \tag{174}$$

where A and λ are arbitrary constants.

From Eqs. (165) and (166), one has

$$H(t) = B\exp(-At) + \lambda, \tag{175}$$

$$a(t) = a_0\exp\left(\lambda t - \frac{B}{A}e^{-At}\right), \tag{176}$$

$$\varphi(t) = \sqrt{\frac{8B}{A}}\exp\left(-\frac{A}{2}t\right), \tag{177}$$

$$V(\varphi) = 3\left(\frac{A}{8M_{\rm Pl}}\right)^2\varphi^4 + \frac{A}{4}\left(3\lambda - \frac{A}{2}\right)\varphi^2 + 3\lambda^2, \tag{178}$$

which correspond to the Higgs potential.

As a special case for $\lambda = A/6$, one has the potential for chaotic inflation

$$V(\varphi) = 3\left(\frac{A}{8M_{\rm Pl}}\right)^2\varphi^4 + 3\lambda^2. \tag{179}$$

After replacing the effective field φ on CCM fields,

$$\chi(t) = \psi(t) = \pm\sqrt{\frac{8B}{An}}\exp\left(-\frac{A}{2}t\right), \tag{180}$$

we have

$$V(\psi,\chi) = \frac{3}{16}\left(\frac{An}{8M_{\rm Pl}}\right)^2(\chi+\psi)^4 + \frac{nA}{16}\left(3\lambda - \frac{A}{2}\right)(\chi+\psi)^2 + 3\lambda^2. \tag{181}$$

For $\lambda = A/6$, the potential of the CCM fields is

$$V(\psi,\chi) = \frac{3}{16}\left(\frac{An}{8M_{\rm Pl}}\right)^2(\chi+\psi)^4 + 3\lambda^2. \tag{182}$$

Under condition $h_{12}(\chi,\psi) = 0$, one has the same solutions for the trivial case of a constant diagonal tensor h_{AB}.

Similarly, one can generalize any exact solutions in single-field models (see, for example, Refs. [33,87]) on this special class of chiral cosmological models with two components.

B. The generalization of the exact solutions for a CCM with an arbitrary number of fields

Now, we generalize the proposed method on the case of a CCM with an arbitrary number of interacting similar scalar fields $\phi^1(t) = \phi^2(t) = \ldots = \phi^K(t)$. In this case, we determine the connection between the diagonal and nondiagonal components of the metric tensor of the target space h_{AB} as

$$\sum_{B=1}^{K} h_{CB} = \frac{n}{K} \tag{183}$$

for all C, with the following condition for nondiagonal components $h_{CB} = h_{BC}$.

Hence, the first diagonal component is determined as

$$h_{11} = \frac{n}{K} - h_{12} - h_{13} - \cdots - h_{1K}, \tag{184}$$

and the other components are defined similarly.

In this case, Eqs. (144)–(146) are reduced to

$$V(\bar{\phi}) = M_{\rm Pl}^2(3H^2 + \dot{H}) = V(\varphi), \tag{185}$$

$$-\dot{H}M_{\rm Pl}^2 = \frac{n}{2}\dot{\phi}^A\dot{\phi}^A = \frac{1}{2}\dot{\varphi}^2, \tag{186}$$

$$\ddot{\phi}^A + 3H\dot{\phi}^A + \frac{2}{n}\frac{\partial V(\phi^A)}{\partial\phi^A} = 0. \tag{187}$$

Therefore, we can generalize the exact solutions (164)–(166) for the effective field

$$\varphi = \pm\frac{\sqrt{n}}{2}\left(\sum_{A=1}^{K}\phi^A\right). \tag{188}$$

For a CCM with K components, any (similar) field ϕ^A can be obtained from the effective field as follows:

$$\phi^A(t) = \pm\frac{2}{K\sqrt{n}}\varphi(t). \tag{189}$$

For example, we can generalize solutions (167)–(170) for the case of K number of fields:

$$\phi^1(t) = \phi^2(t) = \ldots = \phi^K(t) = \pm\frac{2}{K\sqrt{n}}(\alpha t - \beta), \tag{190}$$

$$V(\bar{\phi}) = \left(\frac{\alpha}{2M_{\rm Pl}}\right)^2 \left[\frac{3n}{4}\left(\sum_{A=1}^{K}\phi^A\right)^2 - 2M_{\rm Pl}^2\right]. \qquad (191)$$

In the same way, it is possible to generalize exact solutions for any other models with one scalar field.

Thus, this method gives an integrable class of exact solutions of Eqs. (6)–(9) for a special case of identical scalar fields and relation (183) between the components of the metric of the target space.

X. CONCLUSIONS

In this paper, we develop the superpotential technique for chiral cosmological models. The key point in this method is that the Hubble parameter is considered a function of the scalar fields, and this allows one to reconstruct the scalar field potential. The CCM models are actively used in cosmology and can be connected with modified gravity models due to the conformal transformation of the metric. So, the proposed method allows one to construct modified gravity models with exact solutions. In particular, CCMs that correspond to $f(R)$ gravity models with one scalar field are considered in Sec. IV. Corresponding two-field CCMs with asymptotic de Sitter solutions are constructed. In the future, we shall explore the possibility of applying our results to inflation.

The superpotential method is an effective procedure to construct models with exact particular solutions. In the case of a model with K scalar fields, the superpotential method gives the possibility of getting a K-parametric set of solutions, as we show in Sec. III. In Secs. IV–VI, we find two-parametric sets of exact solutions for two-field CCMs.

To demonstrate that the proposed reconstruction procedure is powerful, we construct the CCM with different behaviors of the Hubble parameter that are actively used in cosmology. In particular, in Secs. V and VI, we find models with Ruzmaikin solutions that correspond to the intermediate inflation and modified power-law solutions, for which the Hubble parameter is the sum of a constant and a function inverse proportional to the cosmic time. These solutions are found for models with the given kinetic terms of the actions. In our case, it is possible to choose both the potential and the function that defines the kinetic term. The construction of trigonometric and hyperbolic Hubble functions due to a suitable choice of kinetic term is proposed in Sec. VII.

In Sec. VIII, we construct one- and two-field models that correspond to the Hubble parameter that describes a cyclic type of Universe dubbed the quasi-steady-state. We demonstrate that the superpotential method allows one to construct different models with one and the same Hubble parameter.

In Sec. IX, we show that exact solutions of the CCMs can be obtained by the single-field superpotential method. Comparing this method with the multifield superpotential method developed in Sec. III, one can see that the use of the single-field superpotential method allows one to obtain only a one-parametric set of solutions, whereas the multifield superpotential method gives rise to a K-parametric set of exact solutions if system (19) is integrable. Also, the multifield superpotential method is preferable to get exact soluble models with a nonmonotonic Hubble parameter, which corresponds to models with both ordinary and phantom scalar fields. At the same time, for a nonintegrable system (19), the method proposed in Sec. IX is a more simple way to obtain exact solutions.

The correspondence between one- and multifield models is actively used for multifield inflationary models in the method of cosmological attractors [15,16]. Let us note that, compared to the method of cosmological attractors, the proposed algorithm allows one to obtain the exact solutions. The superpotential method is suitable for construction of inflationary scenarios in one-field models [8,59–62]. In the future, we plan to generalize this method to the chiral cosmological inflationary models with many scalar fields.

ACKNOWLEDGMENTS

This work is partially supported by Indo-Russia Project; S. V. C., I. V. F., E. O. P., and S. Yu. V. are supported by RFBR Grant No. 18-52-45016, and M. S. is supported by Grant No. INT/RUS/RFBR/P-315 of the Department of science and technology of India government. S. V. C. is grateful for the support of the Program of Competitive Growth of Kazan Federal University.

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