Ultraviolet to infrared evolution and nonperturbative behavior of $\mathrm{SU}(N)\otimes\mathrm{SU}(N-4)\otimes\mathrm{U}(1)$ chiral gauge theories

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We analyze the ultraviolet to infrared evolution and nonperturbative properties of asymptotically free $\mathrm{SU}(N)\otimes\mathrm{SU}(N-4)\otimes\mathrm{U}(1)$ chiral gauge theories with N_f copies of chiral fermions transforming according to $([2]_N,1)_{N-4}+([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)}+(1,(2)_{N-4})_N$, where $[k]_N$ and $(k)_N$ denote the antisymmetric and symmetric rank-k tensor representations of $\mathrm{SU}(N)$ and the rightmost subscript is the U(1) charge. We give a detailed discussion for the lowest nondegenerate case, $N=6$. These theories can exhibit both self-breaking of a strongly coupled gauge symmetry and induced dynamical breaking of a weakly coupled gauge interaction symmetry due to fermion condensates produced by a strongly coupled gauge interaction. A connection with the dynamical breaking of $\mathrm{SU}(2)_L\otimes\mathrm{U}(1)_Y$ electroweak gauge symmetry by the quark condensates $\langle\bar{q}q\rangle$ due to color $\mathrm{SU}(3)_c$ interactions is discussed. We also remark on direct-product chiral gauge theories with fermions in higher-rank tensor representations.

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I. INTRODUCTION

A problem of basic field-theoretic interest concerns the behavior of strongly coupled chiral gauge theories. In general, there are two types of chiral gauge theories, namely those based on a single gauge group and those with a direct-product (dp) gauge group of the form

$$G_{dp}=\bigotimes_{i=1}^{N_G}G_i \tag{1.1}$$

with $N_G\geq 2$. Strongly coupled direct-product chiral gauge theories are of particular interest because they can exhibit a phenomenon that cannot occur in a chiral gauge theory with a single gauge group, namely the induced dynamical breaking of a weakly coupled gauge symmetry by a different, strongly coupled, gauge interaction. This phenomenon is important not only from the point of view of abstract quantum field theory, but also because it actually occurs in nature. In the Standard Model (SM), with the gauge group $G_{\rm SM}=\mathrm{SU}(3)_c\otimes G_{\rm EW}$, where the electroweak gauge group is $G_{\rm EW}=\mathrm{SU}(2)_L\otimes\mathrm{U}(1)_Y$, the bilinear quark condensates $\langle\bar{q}q\rangle$ produced by the strongly coupled $\mathrm{SU}(3)_c$ color gauge interaction dynamically break $G_{\rm EW}$ to the elctromagnetic gauge symmetry, $\mathrm{U}(1)_{\rm em}$. This breaking contributes terms of the form $g^2f_\pi^2/4$ and $(g^2+g'^2)f_\pi^2/4$ to the squared masses of the W and Z bosons, m_W^2 and m_Z^2, respectively, where g and g' are the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge couplings, and $f_\pi=93$ MeV is the pion decay constant. Thus, although textbook discussions usually mention only the vacuum expectation value (VEV)

$$\langle\phi\rangle_0=\begin{pmatrix}0\\ \frac{v}{\sqrt{2}}\end{pmatrix} \tag{1.2}$$

of the Higgs field

$$\phi=\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \tag{1.3}$$

as the source of electroweak symmetry breaking in the SM, this breaking really arises from two different sources, one of which is the Higgs VEV (1.2), yielding $m_W^2=g^2v^2/4$ and $m_Z^2=(g^2+g'^2)v^2/4$, where $v=246$ GeV, and the other of which is the above-mentioned dynamical contribution due to the formation of bilinear quark condensates in quantum chromodynamics (QCD).

Although this dynamical breaking of electroweak gauge symmetry by the $\mathrm{SU}(3)_c$ color gauge interaction is very small compared with the contribution due to the VEV of the Higgs field, it is important as a physical example of how, in a direct-product chiral gauge theory, one strongly coupled gauge interaction can induce the breaking of a weakly coupled one [1,2]. Indeed, a *gedanken* modification of the Standard Model in which the Higgs field is removed is a perfectly well-defined theory in which the W and Z masses are entirely due to the dynamical breaking of the electroweak gauge symmetry by the $\mathrm{SU}(3)_c$ interaction [2,3]. In the Standard Model, the $\mathrm{SU}(3)_c$ gauge interaction is vectorial, while G_{EW} is chiral, but this mechanism can also break a vectorial gauge symmetry; in Ref. [3] it was shown that in this *gedanken* modification of the SM without any Higgs field, if one reversed the order of the coupling strengths of the non-Abelian gauge interactions so that the $\mathrm{SU}(2)_L$ coupling were much stronger than the $\mathrm{SU}(3)_c$ coupling, then the $\mathrm{SU}(2)_L$ gauge interaction would produce bilinear fermion condensates of quarks and leptons that would break the vectorial $\mathrm{SU}(3)_c$, as well as $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_{\mathrm{em}}$, [3], while preserving $\mathrm{SU}(2)_L$.

Since dynamical symmetry breaking of a weakly coupled gauge symmetry occurs in nature, as shown by the breaking of electroweak gauge symmetry G_{EW} by the $\langle \bar{q}q \rangle$ quark condensates produced by $\mathrm{SU}(3)_c$ gauge interaction, there is a motivation to investigate chiral gauge theories that can exhibit this phenomenon of the dynamical breaking of a weakly coupled gauge symmetry by a different, strongly coupled gauge interaction. As noted above, this requires that one consider theories with direct-product chiral gauge symmetries. Some previous studies of strongly coupled chiral gauge theories with direct-product gauge groups (and without any fundamental scalar fields) include [3–13], [14,15]. See also [16].

In this paper we shall analyze chiral gauge theories with the direct-product gauge group

$$G = \mathrm{SU}(N) \otimes \mathrm{SU}(N-4) \otimes \mathrm{U}(1). \tag{1.4}$$

This group is of the form (1.1) with $N_G = 3$, $G_1 = \mathrm{SU}(N)$, $G_2 = \mathrm{SU}(N-4)$, and $G_3 = \mathrm{U}(1)$. The group (1.4) has order $o(G)$ and rank $rk(G)$ given by

$$o(G) = 2N^2 - 8N + 15, \qquad rk(G) = 2N - 5. \tag{1.5}$$

The fermion content of the theory consists of N_f copies ("flavors") of chiral fermions transforming as

$$([2]_N, 1)_{N-4} + ([\bar{1}]_N, [\bar{1}]_{N-4})_{-(N-2)} + (1, (2)_{N-4})_N, \tag{1.6}$$

where the meaning of the notation

$$(R_1, R_2)_q \tag{1.7}$$

is as follows: the first and second entries refer to the representation R_1 of $G_1 = \mathrm{SU}(N)$ and R_2 of $G_2 = \mathrm{SU}(N-4)$, and the subscript q is the U(1) charge of the given fermion. The symbols $[k]_N$ and $(k)_N$ denote the k-fold antisymmetric and symmetric tensor representations of $\mathrm{SU}(N)$, respectively, and $R_i = 1$ denotes a singlet of G_i, where $i = 1$ or $i = 2$. The fermion fields are denoted explicitly as

$$\begin{aligned} &([2]_6, 1)_2\colon\ \psi^{ij}_{p,L}, \\ &([\bar{1}]_6\colon\ [\bar{1}]_2)_{-4}\colon\ \chi_{i,\alpha,p,L}, \\ &(1, (2)_2)_6\colon\ \omega^{\alpha\beta}_{p,L}, \end{aligned} \tag{1.8}$$

where i, j are $\mathrm{SU}(N)$ group indices, α, β are $\mathrm{SU}(N-4)$ group indices, and p is a copy (flavor) index, running from 1 to N_f. We exclude the trivial value $N_f = 0$, because it does not produce a chiral gauge theory, but instead just a set of three decoupled pure gauge theories. There are no bare fermion masses in the theory, since they are forbidden by the chiral gauge symmetry. Without loss of generality, we write the fermions as left-handed. This theory is free of anomalies in gauged currents, as is necessary for renormalizability, and is also free of global anomalies and mixed gauge-gravitational anomalies [12,17].

We note two equivalent theories with the same gauge group, (1.4). The first of these has all of the representations of the left-handed chiral fermions in (1.6) conjugated. The second has the representations of $\mathrm{SU}(M)$ conjugated relative to those of $\mathrm{SU}(N)$, i.e., its fermion content consists of N_f copies of the set

$$([2]_N, 1)_{N-4} + ([\bar{1}]_N, [1]_{N-4})_{-(N-2)} + (1, (\bar{2})_{N-4})_N. \tag{1.9}$$

Since these theories are equivalent to (1.4) with (1.6), it will suffice to study only the latter.

This model is of particular interest for the following reason. A natural construction of a chiral gauge theory with a non-Abelian gauge group uses (left-handed chiral) fermions transforming according to an antisymmetric or symmetric rank-k tensor representation of the gauge group, together with the requisite number of fermions transforming according to the conjugate fundamental representation, so as to yield zero gauge anomaly. The simplest of these uses $k = 2$, so let us focus on these theories with $k = 2$. With a special unitary gauge group, there are two such constructions: (i) $G = \mathrm{SU}(N)$ and chiral fermion content consisting of N_f copies of the set

$$[2]_N + (N-4)[\bar{1}]_N \tag{1.10}$$

and (ii) $G = \mathrm{SU}(M)$ and chiral fermion content consisting of N_f copies of the set

$$(2)_M + (M+4)[\bar{1}]_M. \tag{1.11}$$

A basic question in the analysis of chiral gauge theories is whether one can combine these two separate single-gauge-group theories (i) and (ii) into a single chiral gauge theory with a direct-product gauge group that contains $\mathrm{SU}(N) \otimes \mathrm{SU}(M)$ such that it is again anomaly-free. The answer is yes, if we set $M = N - 4$, and the theory (1.4) with (1.6) provides an explicit realization of this combination. Indeed, not only does this theory successfully combine the two separate chiral gauge symmetries $\mathrm{SU}(N)$ and $\mathrm{SU}(M)$ in an anomaly-free manner; it also incorporates a third gauge symmetry, namely the U(1).

A general classification of chiral gauge theories with direct-product gauge groups was given in Ref. [13]. In this classification, a factor group G_i is labeled as G_c if it has complex representations and G_r if it has (only) real or pseudoreal representations. If a group G_c has no gauge anomaly from any of its representations, then it was denoted as G_{cs}, where the subscript s stands for "safe." In this classification, if $N \geq 7$, then the gauge group (1.4) is of the form (G_c, G_c, G_c). In contrast, if $N = 6$, then the second factor group is SU(2), which has (pseudo)real representations, so that the $N = 6$ special case of (1.4) is of the form (G_c, G_r, G_c) in this classification.

In accordance with the order of labeling of the G_i factor groups, we denote the corresponding running gauge couplings as $g_1(\mu)$ for $G_1 = \mathrm{SU}(N)$, $g_2(\mu)$ for $G_2 = \mathrm{SU}(N-4)$, and $g_3(\mu)$ for $G_3 = \mathrm{U}(1)$, where μ is the Eucidean energy/momentum reference scale where $g_i(\mu)$ is measured. We further define $\alpha_i(\mu) = g_i(\mu)^2/(4\pi)$ and $a_i(\mu) = g_i(\mu)^2/(16\pi^2)$, with $i = 1, 2, 3$. (The argument μ will sometimes be suppressed in the notation.) As usual with a U(1) gauge interaction, the U(1) charge assignments in (1.6) involve an implicit normalization convention; the physics is unchanged if one redefines $q_f \to \lambda q_f$ for each fermion f and $g_3 \to \lambda^{-1} g_3$, since only the product $q_f g_3$ appears in the U(1) covariant derivative.

Each of the two non-Abelian gauge interactions is required to be asymptotically free (AF), because this enables us to calculate the corresponding beta functions self-consistently at a high scale $\mu = \mu_{\mathrm{UV}}$ in the deep ultraviolet (UV) region, where they are weakly coupled. These beta functions then describe the running of the non-Abelian couplings toward the infrared (IR) at small μ, where these couplings become larger. Since we are interested in the nonperturbative behavior of the non-Abelian gauge interactions, we will assume the U(1) gauge interaction to be weakly coupled at the initial reference scale μ_{UV}; owing to the property that the beta function for this U(1) interaction is nonasymptotically free, the U(1) coupling $\alpha_3(\mu)$ becomes even weaker as μ decreases below μ_{UV} and hence can be treated perturbatively in the full range $\mu < \mu_{\mathrm{UV}}$ under consideration here.

In addition to the phenomenon of a strongly coupled gauge interaction inducing the dynamical breaking of a different gauge symmetry, a chiral gauge theory can also exhibit a different phenomenon in which a strongly coupled gauge interaction corresponding to a given gauge symmetry produces fermion condensates that break this gauge symmetry itself [1,18]. In particular, for a given gauge interaction corresponding to the non-Abelian gauge group G_i, as μ decreases from μ_{UV} and $\alpha_i(\mu)$ grows, it may become large enough at a certain scale, which we will denote as $\mu = \Lambda_1$, to produce a fermion condensate that breaks the gauge symmetry G_i to a subgroup $H_i \subset G_i$. The fermions involved in this condensate gain dynamical masses of order Λ_1 and are integrated out of the low-energy effective field theory (EFT) that describes the physics as μ decreases below Λ_1. The gauge bosons in the coset space G_i/H_i pick up dynamical masses of order $g_i(\Lambda_1)\Lambda_1$ and are also integrated out of the low-energy effective theory. This low-energy theory has a gauge coupling inherited from the UV theory, but since the fermion and gauge boson content is different, this gauge coupling runs according to a different beta function. Then this process of self-breaking of a gauge can repeat at one or more lower scales. The final low-energy effective field theory may be a vectorial theory that confines and produces fermion condensates with associated spontaneous chiral symmetry breaking (SχSB) but no further gauge self-breaking.

Besides being of abstract field-theoretic interest, this mechanism of gauge self-breaking has been used in constructions and studies of reasonably ultraviolet-complete models of dynamical electroweak symmetry breaking (EWSB) and fermion mass generation [4–8], [11]. In these constructions, one starts with an asymptotically free chiral gauge theory that undergoes either self-breaking or a combination of self-breaking and induced symmetry breaking in a sequence of three different scales, $\Lambda_1 > \Lambda_2 > \Lambda_3$, with an associated breaking of the UV chiral gauge symmetry $G_{\mathrm{UV}} \to H_1 \to H_2 \to H_3$, where the H_3 symmetry is vectorial. At a lower scale Λ_T of order 1 TeV, the H_3 gauge interaction confines and produces condensates that break G_{EW}. It also produces a spectrum of H_3-singlet bound states. Gauge bosons in the coset space G_{UV}/H_1 gain dynamical masses of order Λ_1, while gauge bosons in the coset spaces H_1/H_2 and H_2/H_3 gain dynamical masses of order Λ_2 and Λ_3, respectively. Exchanges of these three different types of massive vector bosons produce the three generations of quark and lepton masses. More complicated exchanges can also produce light neutrino masses via an appropriate seesaw mechanism [5]. This scenario has the potential to naturally explain the generational hierarchy in fermion masses, which reflects the hierarchy of self-breaking scales Λ_i, $i = 1, 2, 3$. This construction is also an ultraviolet completion of low-energy effective Lagrangians for dynamical EWSB that use four-fermion operators [19] and predicts the coefficients of these four-fermion operators.

Our theory does not include any fundamental scalar fields. Thus, the pattern of possible dynamical gauge symmetry breaking depends only on the gauge and fermion content, and the initial values of the gauge couplings at the reference scale μ_{UV}. This is in contrast with theories in which gauge symmetry breaking is produced by VEVs of Higgs fields, because in these latter theories, the nature of the symmetry breaking depends on various parameters in the Higgs potential, which can be chosen at will, subject to the constraint that this Higgs potential should be bounded from below [20,21].

An alternate application of strongly coupled chiral gauge theories was to efforts at modeling the quarks and leptons as composites of more fundamental fermions, commonly called preons. This involved a scenario in which it was envisioned that the strongly coupled gauge interaction would produce confinement of the preons in gauge-singlet composite fermions, but no spontaneous chiral symmetry breaking. The presumed absence of SχSB was necessary in order for the composite fermions to be very light compared to the inverse of the spatial compositeness scale $\Lambda_{\rm comp} = 1/r_{\rm comp}$ For this purpose, theories were constructed that satisfied certain matching conditions of chiral symmetries between preons and the composite fermions [22,23]. In the present paper we will focus on studying possible patterns of bilinear fermion condensate formation and resultant dynamical gauge symmetry breaking in the strongly coupled gauge theory (1.4) with (1.6) and (1.12)–(1.13) rather than on possible scenarios with light composite fermions.

In addition to our analysis of the general theory (1.4) with (1.6), we will study the $N = 6$ special case in detail. This $N = 6$ theory, with the gauge group

$$G_{N=6} = {\rm SU}(6) \otimes {\rm SU}(2) \otimes {\rm U}(1), \qquad (1.12)$$

is of particular interest because it is the lowest nondegenerate member of this family. [If $N = 5$, then the SU$(N-4)$ group is trivial.] It is also special in two related aspects, namely that (i) as mentioned above, the resultant second factor group is SU(2), with (pseudo)real representations, in contrast to the situation for $N \geq 7$, where the SU$(N-4)$ group has complex representations; and (ii) the symmetric rank-2 tensor representation $(2)_2$ of SU(2) is the adjoint representation. The fermion content for this $N = 6$ theory, comprised of N_f copies of

$$([2]_6, 1)_2 + ([\bar{1}]_6, [\bar{1}]_2)_{-4} + (1, (2)_2)_6, \qquad (1.13)$$

can also be conveniently expressed in terms of the dimensionalities of the representations as

$$(15, 1)_2 + (\bar{6}, 2)_{-4} + (1, 3)_6. \qquad (1.14)$$

Owing to property (ii) above, we will often use the equivalent isovector notation $\vec{\omega}_{p,L}$ for the $\omega^{\alpha\beta}_{p,L}$ fermion.

Thus, the theory (1.4) with (1.6) and, in particular, the $N = 6$ special case, provide valuable theoretical laboratories for the study of nonperturbative properties of chiral gauge theories, including self-breaking of a strongly coupled chiral gauge symmetry, induced breaking of a weakly coupled gauge symmetry by a strongly coupled gauge interaction, and the sequential construction of low-energy effective field theories. This paper is organized as follows. The general methods used in our analysis are described in Sec. II. In Sec. III we analyze the UV to IR evolution, possible fermion condensation channels, and corresponding gauge symmetry breaking patterns of the theory (1.4) with (1.6). In Secs. IV–VII we present a detailed analysis of the $N = 6$ theory. Some remarks on related constructions of direct-product chiral gauge theories with fermions in higher-rank tensor representations are given in Sec. VIII. Our conclusions are contained in Sec. IX.

II. RENORMALIZATION-GROUP EVOLUTION AND FERMION CONDENSATES

A. Beta functions

In this section we discuss the general methods that are used for our analysis. We first explain our application of the renormalization group (RG). Recall our labeling conventions given above for the gauge couplings, namely $g_1(\mu)$ for SU(N), $g_2(\mu)$ for SU$(N-4)$, and $g_3(\mu)$ for U(1). The evolution of the three gauge couplings $g_i(\mu)$, or equivalently, the corresponding $\alpha_i(\mu)$ with $i = 1, 2, 3$, is determined by the RG beta functions

$$\beta_{G_i} = \frac{d\alpha_i(\mu)}{d\ln\mu}. \qquad (2.1)$$

These have the series expansions

$$\beta_{G_i} = -8\pi a_i \Bigg[b^{(G_i)}_{1\ell,i} a_i + \sum_{j=1}^{3} b^{(G_i)}_{2\ell;ij} a_i a_j + \sum_{j,k=1}^{3} b^{(G_i)}_{3\ell;ijk} a_i a_j a_k + \cdots \Bigg], \qquad (2.2)$$

where an overall minus sign is extracted, the dots … indicate higher-loop terms, and there is no sum on repeated i indices in the square bracket. Here, $b^{(G_i)}_{1\ell,i}$ is the one-loop (1ℓ) coefficient, multiplying a_i inside the square bracket in (2.2); $b^{(G_i)}_{2\ell;ij}$ is the two-loop coefficient, multiplying $a_i a_j$ in the square bracket, and so forth for higher-loop terms. The one-loop coefficients $b^{(G_i)}_{1\ell,i}$ are scheme-independent.

We focus on the beta functions for the two non-Abelian gauge interactions, since these determine the upper bound on N_f and are relevant for the formation of various possible fermion condensates as $\alpha_1(\mu)$ or $\alpha_2(\mu)$ become large in the infrared. The one-loop coefficients in Eq. (2.2) are

$$b^{({\rm SU}(N))}_{1\ell,1} = \frac{1}{3}[11N - 2N_f(N-3)], \qquad (2.3)$$

$$b^{({\rm SU}(N-4))}_{1\ell,2} = \frac{1}{3}[11(N-4) - 2N_f(N-1)], \qquad (2.4)$$

and

$$b_{1\ell,3}^{(\mathrm{U}(1))} = -\frac{4}{3}N_f N(N-1)(N-3)(N-4). \quad (2.5)$$

As mentioned before, we assume that the U(1) gauge interaction is weakly coupled at the UV reference scale μ_{UV}; then its gauge coupling decreases as μ decreases from the UV to the IR, and hence can be treated perturbatively.

The requirements that the SU(N) and SU($N-4$) gauge interactions must be asymptotically free are that $b_{1\ell,1}^{(\mathrm{SU}(N))} > 0$ and $b_{1\ell,2}^{(\mathrm{SU}(N-4))} > 0$. These impose the respective upper limits $N_f < N_{f,b1z}$ and $N_f < N'_{f,b1z}$, where

$$N_{f,b1z} = \frac{11N}{2(N-3)} \quad (2.6)$$

and

$$N'_{f,b1z} = \frac{11(N-4)}{2(N-1)}, \quad (2.7)$$

where we use a prime to indicate the upper limit on N_f from the condition $b_{1\ell}^{(\mathrm{SU}(N-4))} > 0$. The upper bound (2.7), is more restrictive than the upper bound (2.6), as is clear, since the difference

$$N_{f,b1z} - N'_{f,b1z} = \frac{33(N-2)}{(N-1)(N-3)} \quad (2.8)$$

is positive for all of the relevant values of N under consideration here. Hence, we restrict

$$N_f < \frac{11(N-4)}{2(N-1)}. \quad (2.9)$$

The (nonzero) values of N_f that are allowed by the inequality (2.9) depend on N and are as follows:

(1) $1 \le N_f \le 2$ if $6 \le N \le 7$
(2) $1 \le N_f \le 3$ if $8 \le N \le 12$
(3) $1 \le N_f \le 4$ if $13 \le N \le 34$
(4) $1 \le N_f \le 5$ if $N_f \ge 35$.

As $N \to \infty$, the upper limit on N_f (formally generalized to a non-negative real number) approaches $11/2$, thus allowing physical integral values up to 5, inclusive, as indicated above.

In general, the set of equations (2.2) is comprised of three coupled nonlinear first-order ordinary differential equations for the quantities α_i, $i = 1, 2, 3$. The solutions for the three $\alpha_i(\mu)$ depend on N_f and the three initial values $\alpha_i(\mu_{\mathrm{UV}})$ at the UV reference scale μ_{UV}. Since we do not assume that the group (1.4) is embedded in a single gauge group higher in the UV, we may choose these initial values $\alpha_i(\mu_{\mathrm{UV}})$ arbitrarily, subject to the constraint that for $\mu = \mu_{\mathrm{UV}}$, the values are sufficiently small that the perturbative calculation of the beta functions β_{α_i} are self-consistent. To leading order, i.e., to one-loop order, the differential equations making up this set decouple from each other, and one has the simple solution for each $i = 1, 2, 3$:

$$\alpha_i(\mu_1)^{-1} = \alpha_i(\mu_2)^{-1} - \frac{b_{1\ell,i}^{(G_i)}}{2\pi}\ln\left(\frac{\mu_2}{\mu_1}\right), \quad (2.10)$$

where we take $\mu_1 < \mu_2$.

At the level of two loops and higher, due to the fact that each of the fermions has nonzero U(1) charge and one of the fermions, $\chi_{i,\alpha,p,L}$, is a nonsinglet under both of the non-Abelian gauge groups, there are mixed terms $a_i a_j$, $a_i a_j a_k$, etc., that involve different gauge interactions, in the three beta functions β_{α_i}, so that the three beta functions become coupled differential equations. In view of the mixing terms in (2.2) at the two-loop level, it is natural to focus first on two special cases, namely those in which one of the non-Abelian gauge interactions is much stronger than the other. This can be arranged by specifying appropriate initial values of $\alpha_1(\mu_{\mathrm{UV}})$ and $\alpha_2(\mu_{\mathrm{UV}})$ at the UV scale μ_{UV}. In these two cases, one can neglect the two-loop term that mixes these two non-Abelian gauge interactions in Eq. (2.2), so that, to two-loop level, these interactions decouple, and the corresponding beta functions have the form, to this level,

$$\beta_{\alpha_1} = \frac{d\alpha_1}{d\ln\mu} = -8\pi a_1[b_{1\ell,1}^{(\mathrm{SU}(N))}a_1 + b_{2\ell;11}^{(\mathrm{SU}(N))}a_1^2] \quad (2.11)$$

and

$$\beta_{\alpha_2} = \frac{d\alpha_2}{d\ln\mu} = -8\pi a_2[b_{1\ell,2}^{(\mathrm{SU}(N-4))}a_2 + b_{2\ell;22}^{(\mathrm{SU}(N-4))}a_2^2], \quad (2.12)$$

where the one-loop coefficients $b_{1\ell,1}^{(\mathrm{SU}(N))}$ and $b_{1\ell,2}^{(\mathrm{SU}(N-4))}$ were given above in Eqs. (2.3)–(2.5), and the two-loop coefficients are

$$b_{2\ell;11}^{(\mathrm{SU}(N))} = \frac{1}{6N}[68N^3 - N_f(N-3)(29N^2 - 3N - 12)] \quad (2.13)$$

and

$$b_{2\ell;22}^{(\mathrm{SU}(N-4))} = \frac{1}{6(N-4)}[68(N-4)^3 - N_f(N-1) \times (29N^2 - 229N + 440)]. \quad (2.14)$$

Both of these two-loop coefficients for the non-Abelian gauge couplings are positive for small N_f and decrease with increasing N_f, eventually passing through zero to negative values. We denote the values of N (formally generalized from positive integers $N \geq 6$ to positive real numbers) at which $b_{2\ell;11}^{(\mathrm{SU}(N))}$ and $b_{2\ell;22}^{(\mathrm{SU}(N-4))}$ pass through zero as $N_{f,b2z}^{(\mathrm{SU}(N))}$ and $N_{f,b2z}^{(\mathrm{SU}(N-4))}$. These are

$$N_{f,b2z}^{(\mathrm{SU}(N))} = \frac{68N^3}{(N-3)(29N^2-3N-12)} \qquad (2.15)$$

and

$$N_{f,b2z}^{(\mathrm{SU}(N-4))} = \frac{68(N-4)^3}{(N-1)(29N^2-229N+440)}. \qquad (2.16)$$

As $N \to \infty$, $N_{f,b2z}^{(\mathrm{SU}(N))}$ approaches $68/29 = 2.34483$ from above, while $N_{f,b2z}^{(\mathrm{SU}(N-4))}$ approaches the same value from below.

With these inputs, we can investigate the presence or absence of an IR zero in the respective two-loop beta functions for the SU(N) and SU($N-4$) theories. The two-loop beta function for SU(N) has no IR zero for $N_f = 1$ or $N_f = 2$; it does have an IR zero for higher values of N_f, as allowed by the asymptotic freedom requirement for a fixed N. With a given N, for the range of N_f such that $b_{1\ell,1}^{(\mathrm{SU}(N))} > 0$ and $b_{2\ell;11}^{(\mathrm{SU}(N))} < 0$, the IR zero of the two-loop SU(N) beta function occurs at

$$\alpha_{1,IR,2\ell} = \frac{8\pi N[11N - 2N_f(N-3)]}{N_f(N-3)(29N^2-3N-12) - 68N^3}. \qquad (2.17)$$

Similarly, given a value of N, for the range of N_f such that $b_{1\ell,2}^{(\mathrm{SU}(N-4))} > 0$, while $b_{2\ell;22}^{(\mathrm{SU}(N-4))} < 0$, the two-loop SU($N-4$) beta function has an IR zero at

$$\alpha_{2,IR,2\ell} = \frac{8\pi(N-4)[11(N-4) - 2N_f(N-1)]}{N_f(N-1)(29N^2-229N+440) - 68(N-4)^3}. \qquad (2.18)$$

As $N \to \infty$, the rescaled IRFP values of the SU(N) and SU($N-4$) gauge interactions have the same limit:

$$\lim_{N\to\infty} \alpha_{1,IR,2\ell} N = \lim_{N\to\infty} \alpha_{2,IR,2\ell} N = \frac{8\pi(11-2N_f)}{29N_f - 68}. \qquad (2.19)$$

We will analyze the UV to IR evolution using these beta functions below.

B. Global flavor symmetries

The theory (1.4) with (1.6) has the classical global flavor (cgb) symmetry

$$G_{cgb} = \mathrm{U}(N_f)_\psi \otimes \mathrm{U}(N_f)_\chi \otimes \mathrm{U}(N_f)_\omega, \qquad (2.20)$$

where, for each fermion $f = \psi^{ij}_{p,L}$, $\chi_{i,\alpha,p,L}$, and $\vec{\omega}_{p,L}$, the elements of the group $\mathrm{U}(N_f)_f$ act on the flavor indices p, leaving all gauge indices unchanged. Each $\mathrm{U}(N_f)_f$ factor group in (2.20) can equivalently be written as $\mathrm{SU}(N_f)_f \otimes \mathrm{U}(1)_f$. The instantons present in the SU(N) gauge sector break both of the global Abelian symmetries $\mathrm{U}(1)_\psi$ and $\mathrm{U}(1)_\chi$. Separately, the instantons in the SU($N-4$) gauge sector break both the $\mathrm{U}(1)_\chi$ and $\mathrm{U}(1)_\omega$ symmetries.

There are two special cases that will be of particular interest, namely the respective cases in which one non-Abelian gauge interaction is much stronger than the other. First, let us consider the case in which the SU(N) gauge interaction is much stronger than the SU($N-4$) gauge interaction, which, like the U(1) interaction, is weakly coupled. In this theory, the effects of instantons in the SU($N-4$) gauge sector are exponentially suppressed and can be neglected [24]. Although the SU(N) instantons break the global $\mathrm{U}(1)_\psi$ and $\mathrm{U}(1)_\chi$ flavor symmetries, one can construct a current which is a linear combination of the $\mathrm{U}(1)_\psi$ and $U(1)\chi$ currents and is conserved in the presence of the SU(N) instantons (see, e.g., Sec. V of [25]), which we denote as $\mathrm{U}(1)_{\psi\chi}$. The effective nonanomalous global flavor (gb) symmetry of this theory is thus $G_{gb} = \mathrm{SU}(N_f)_\psi \otimes \mathrm{SU}(N_f)_\chi \otimes \mathrm{U}(1)_{\psi\chi} \otimes \mathrm{U}(N_f)_\omega$. Similarly, in the other case, in which the SU(N) and U(1) gauge interactions are weak, and the SU($N-4$) gauge interaction is strong, the effects of SU(N) instantons are exponentially suppressed and are negligible. Although the SU($N-4$) instantons break the global $\mathrm{U}(1)_\omega$ and $\mathrm{U}(1)_\chi$ flavor symmetries, one can construct a current which is a linear combination of the $\mathrm{U}(1)_\omega$ and $\mathrm{U}(1)\chi$ currents and is conserved in the presence of the SU(N) instantons, which we denote as $\mathrm{U}(1)_{\omega\chi}$. The effective nonanomalous global flavor symmetry of this theory is thus $G_{gb} = \mathrm{SU}(N_f)_\omega \otimes \mathrm{SU}(N_f)_\chi \times \mathrm{U}(1)_{\omega\chi} \otimes \mathrm{U}(N_f)_\psi$.

C. UV to IR evolution and fermion condensates

We next discuss the UV to IR evolution of this theory and the general analysis of possible fermion condensate formation in various channels. We begin with the two respective cases in which one of the two non-Abelian gauge interactions is much stronger than the other and then remark on the case where both are present with comparable strength. Let us denote the dominant coupling as $\alpha_i(\mu)$.

As the reference scale μ decreases below μ_{UV}, the coupling $\alpha_i(\mu)$ for this interaction increases. There are two general possibilities for the associated beta function, β_{α_i}: (i) it does not have an IR zero or (ii) it has an IR zero. In the first case, (i), the coupling continues to increase with decreasing μ until it eventually exceeds the range where it can be calculated with the perturbative beta function. This can then lead to the formation of (bilinear) fermion condensates. In the second case, let us denote the value of α_i at this IR zero as $\alpha_{\rm IR}$, and consider a possible condensation channel,

$$R \times R' \to R_c, \tag{2.21}$$

where R and R' denote fermion representations under the strongly coupled gauge symmetry G_i, and R_c denotes the representation of the condensate under G_i. Assuming that this is an attractive channel, we denote the minimal critical coupling for condensation in this channel as α_{cr}. If the beta function does not have an IR zero, then α_i will certainly exceed α_{cr} as μ decreases to some scale. If the beta function β_{α_i} does have an IR zero, then there are two subcases: (iia) $\alpha_{\rm IR} \ge \alpha_{cr}$ and (iib) $\alpha_{\rm IR} < \alpha_{cr}$. In case (iia), the condensate can form, similarly to case (i), while in case (iib), this condensate will not form. For the possible condensation channel (2.21), an approximate measure of its attractiveness (motivated by iterated one-gluon exchange) is

$$\Delta C_2 = C(R) + C(R') - C(R_c), \tag{2.22}$$

where $C_2(R)$ is the quadratic Casimir invariant for the representation R [26]. Among several possible fermion condensation channels, the one with the largest (positive) value of ΔC_2 is commonly termed the most attractive channel (MAC) and is the one that is expected to occur.

Approximate solutions of Schwinger-Dyson equations for the fermion propagator in a vectorial theory have shown that if one starts with a massless fermion, it follows that if $\alpha > \alpha_{cr}$, where $3\alpha_{cr} C_2(R)/\pi = 1$, then the Schwinger-Dyson equation has a solution with a dynamically generated mass, indicating spontaneous chiral symmetry breaking and associated bilinear fermion condensate formation [27]. In a vectorial gauge theory such as quantum chromodynamics, the condensate is a gauge-singlet, so $\Delta C_2 = 2C_2(R)$. Hence, one can write the condition for the critical coupling in the form that can be taken over for a chiral gauge theory, namely $3\alpha_{cr}\Delta C_2/(2\pi) = 1$, so that

$$\alpha_{cr} = \frac{2\pi}{3\Delta C_2}. \tag{2.23}$$

Because this is based on a rough approximation (an iterated one-gluon exchange approximation to the Schwinger-Dyson equation), it is used only as a rough estimate.

Since without loss of generality we write all fermions as left-handed, the Lorentz-invariant bilinears involving two fermion fields f_L and f'_L are of the form $f_L^T C f'_L$, where C is the Dirac charge-conjugation matrix satisfying $C\gamma_\mu C^{-1} = -(\gamma_\mu)^T$. If f_L and f'_L transform according to the same representation R_1 of a symmetry group G_1 and R_2 of a symmetry group G_2, then we may write the bilinear fermion operator product abstractly as

$$f^T_{\mathcal{R},p,L} C f_{\mathcal{R},p',L}, \tag{2.24}$$

where gauge group indices are suppressed in the notation, $\mathcal{R}$ denotes the representations under the gauge groups, and, as before, p and p' are flavor indices. From the property $C^T = -C$ together with the anticommutativity of fermion fields, it follows that the bilinear fermion operator product (2.24) is symmetric under interchange of the order of fermion fields and therefore is symmetric in the overall product

$$\prod_i (R_i \times R_i)] R_{fl}, \tag{2.25}$$

where R_{fl} abstractly denotes the symmetry property under interchange of flavors [13]. For our theory, with its two non-Abelian groups, this means that the fermion bilinears are of the form

$$(s,s,s), \quad (s,a,a), \quad (a,s,a), \quad \text{or} \quad (a,a,s), \tag{2.26}$$

where here s and a indicate symmetric and antisymmetric, and the three entries refer to the representations R_1 of G_1, R_2 of G_2, and R_{fl}.

If, as μ decreases through a scale Λ_1 and the coupling $\alpha_i(\mu)$ of the strongly coupled gauge interaction corresponding to the factor group G_i increases beyond α_{cr} for the condensation channel (2.21) and the condensate forms, then the fermions involved in the condensate gain dynamical masses of order Λ_1 and are integrated out of the low-energy effective field theory that describes the physics as μ decreases below Λ_1. If this condensate either self-breaks the G_i symmetry or produces induced breaking of a weakly coupled gauge symmetry G_j to a respective subgroup $H_i \subset G_i$ or $H_j \subset G_j$, then the gauge bosons in the respective coset spaces G_i/H_i or G_j/H_j pick up dynamical masses of order $g_i(\Lambda_1)\Lambda_1$ or $g_j(\Lambda_1)\Lambda_1$, respectively. Hence, like the fermions with dynamically generated masses, these now massive vector bosons are integrated out of the low-energy effective field theory applicable as μ decreases below Λ.

III. THEORY WITH GENERAL N

In this section we analyze possible fermion condensation channels in the general-N theory (1.4) with fermion content (1.6).

A. SU(N) gauge interaction dominant

We begin by focusing on the case where the SU(N) gauge interaction is much stronger than the SU($N-4$) [and U(1)] gauge interactions. This theory is labeled SUND, standing for "SU(N) dominant." Although we keep α_2 and α_3 nonzero, we note parenthetically that if one were to set $\alpha_2 = \alpha_3 = 0$, then the resultant theory would be the $k = 2$ special case of a family of chiral gauge theories analyzed in Ref. [25] with a single gauge group $G = \mathrm{SU}(N)$ and an anomaly-free content of chiral fermions transforming as $[k]_N$ and $n_{\bar{F}}$ copies of $[\bar{1}]_N$, where $n_{\bar{F}} = (N-3)!(N-2k)/[(N-k-1)!(k-1)!]$ [plus SU(N)-singlet fermions]. Since the SU(N) gauge interaction is asymptotically free, $\alpha_1(\mu)$ increases as μ decreases from the initial reference scale μ_{UV} in the UV. We focus on the subset of values of N_f such that the beta function β_{α_1} either has no IR zero at the two-loop level or has an IRFP at a sufficiently large value that fermion condensation can take place.

There are three possible (bilinear) fermion condensation channels. We give shorthand names to these based on the fermions involved in each condensate. The first is the $\psi\chi$ channel,

$$\psi\chi\colon\ ([2]_N,1)_{N-4} \times ([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)} \to ([1]_N,[\bar{1}]_{N-4})_{-2}, \tag{3.1}$$

with associated condensate

$$\langle \psi^{ij\,T}_{p,L} C \chi_{j,\beta,p',L} \rangle, \tag{3.2}$$

where i, j are SU(N) group indices and β is an SU($N-4$) group index. The condensate (3.2) transforms as the fundamental ($[1]_N$) representation of SU(N) and the conjugate fundamental ($[\bar{1}]_{N-4}$) representation of SU($N-4$), so it self-breaks SU(N) to SU($N-1$) and produces an induced breaking of the weakly coupled SU($N-4$) to SU($N-5$). Since the condensate (3.2) has a nonzero U(1) charge, $q_{\psi\chi} = -2$, it also breaks U(1). Thus, here the residual gauge symmetry in the effective field theory that is applicable as μ decreases below Λ_1 is

$$\mathrm{SU}(N-1) \otimes \mathrm{SU}(N-5). \tag{3.3}$$

If $N = 6$, then the residual gauge symmetry is just SU(5). For this channel we calculate

$$(\Delta C_2)_{\psi\chi} = C_2([2]_N) = \frac{(N-2)(N+1)}{N}. \tag{3.4}$$

For this and other possible fermion condensation channels, we record these properties in Table I. This table refers to the possible initial condensation patterns at the highest condensation scale; subsequent evolution further into the infrared is discussed below.

The second possible channel is the $\psi\psi$ channel,

$$\psi\psi\colon\ ([2]_N,1)_{N-4} \times ([2]_N,1)_{N-4} \to ([4]_N,1)_{2(N-4)}. \tag{3.5}$$

Note that $[4]_N \approx \overline{[N-4]}_N$, where $R \approx R'$ means that the representations R and R' are equivalent. The associated condensate is

$$\epsilon_{\ldots k\ell mn} \langle \psi^{k\ell\,T}_{p,L} C \psi^{mn}_{p',L} \rangle, \tag{3.6}$$

where the antisymmetric tensor $\epsilon_{\ldots k\ell mn}$ has N indices, four of which are indicated explicitly, with the rest implicit. From the general group-theoretic analysis in [25,28], it follows that since the condensate (3.6) transforms as a $[4]_N$ of SU(N), it breaks SU(N) to SU($N-4$) ⊗ SU(4). Since the $\psi^{k\ell}_{p,L}$ are singlets under the original SU($N-4$) group in (1.4), this condensate is obviously invariant under this SU($N-4$). Furthermore, since this condensate has a nonzero U(1) charge [namely, $q_{\psi\psi} = 2(N-4)$], it breaks the U(1) gauge symmetry. Hence, the condensate (3.6) breaks G to

TABLE I. Properties of possible initial (highest-scale) bilinear fermion condensates in the UV theory (1.4) with (1.6) for $N \geq 7$. The shorthand name of the condensation channel is listed in the first column, and the corresponding condensate is displayed in the second column. The third and fourth columns list the values of ΔC_2 with respect to the SU(N) and SU($N-4$) gauge interactions. The entries in the fifth, sixth, and seventh columns indicate whether a given condensate is invariant (inv.) under the SU(N), SU($N-4$), and U(1) gauge symmetries, respectively, or breaks (bk.) one or more of these symmetries. The entry in the eighth column gives the representation $(R_1,R_2)_q$ of the condensate under the group (1.4), following the notation of Eq. (1.7). The ninth column lists the continuous gauge symmetry group under which a given condensate is invariant. The tensors $\epsilon_{\ldots k\ell mn}$ and $\epsilon^{\cdots mn}$ are antisymmetric SU(N) tensors, while $\epsilon^{\cdots\alpha\beta}$ is an antisymmetric SU($N-4$) tensor. These results are for the case $N_f = 1$ and for $N_f \geq 2$ with condensates symmetrized over the flavor indices, which are suppressed in the notation. The $\psi\chi$ channel is the MAC for the SU(N)-dominant case, while the $\chi\omega$ channel is the MAC for the SU($N-4$)-dominant case in this $N \geq 7$ range. See text for further discussion.

Name	Condensate	$(\Delta C_2)_{\mathrm{SU}(N)}$	$(\Delta C_2)_{\mathrm{SU}(N-4)}$	SU(N)	SU($N-4$)	U(1)	$(R_1,R_2)_q$	H_{inv}
$\psi\chi$	$\langle \psi^{ijT}_L C \chi_{j,\beta,L} \rangle$	$\frac{(N-2)(N+1)}{N}$	0	bk.	bk.	bk.	$([1]_N,[\bar{1}]_{N-4})_{-2}$	$\mathrm{SU}(N-1)\otimes\mathrm{SU}(N-5)$
$\chi\omega$	$\langle \chi^T_{i,\alpha,L} C \omega^{\alpha\beta}_L \rangle$	0	$\frac{(N-2)(N-5)}{N-4}$	bk.	bk.	bk.	$([\bar{1}]_N,[1]_{N-4})_2$	$\mathrm{SU}(N-1)\otimes\mathrm{SU}(N-5)$
$\psi\psi$	$\epsilon_{\ldots k\ell mn}\langle \psi^{k\ell T}_L C \psi^{mn}_L \rangle$	$\frac{4(N+1)}{N}$	0	bk.	inv.	bk.	$([4]_N,1)_{2(N-4)}$	$[\mathrm{SU}(N-4)\otimes\mathrm{SU}(4)]$ $\otimes\mathrm{SU}(N-4)$
$\chi\chi$	$\epsilon^{\cdots mn}\epsilon^{\cdots\alpha\beta}\langle \chi^T_{m,\alpha,L} C \chi_{n,\beta,L} \rangle$	$\frac{N+1}{N}$	$\frac{N-3}{N-4}$	bk.	bk.	bk.	$([\bar{2}]_N,[\bar{2}]_{N-4})_{-2(N-2)}$	$[\mathrm{SU}(N-2)\otimes\mathrm{SU}(2)']\otimes$ $\otimes[\mathrm{SU}(N-6)\otimes\mathrm{SU}(2)'']$

$$[\mathrm{SU}(N-4)\otimes\mathrm{SU}(4)]\otimes\mathrm{SU}(N-4), \tag{3.7}$$

where we have inserted brackets to distinguish the two different $\mathrm{SU}(N-4)$ groups. The measure of attractiveness for this condensation channel is

$$(\Delta C_2)_{\psi\psi} = 2C_2([2]_N) - C_2([4]_N) = \frac{4(N+1)}{N}. \tag{3.8}$$

The third possible channel is the $\chi\chi$ channel,

$$\chi\chi:\ ([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)} \times ([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)} \to ([\bar{2}]_N,[\bar{2}]_{N-4})_{-2(N-2)}, \tag{3.9}$$

with associated condensate

$$\epsilon^{\cdots mn}\epsilon^{\cdots\alpha\beta}\langle \chi^T_{m\alpha,p,L} C\chi_{n,\beta,p',L}\rangle, \tag{3.10}$$

where $\epsilon^{\cdots mn}$ and $\epsilon^{\cdots\alpha\beta}$ are antisymmetric tensors under $\mathrm{SU}(N)$ and $\mathrm{SU}(N-4)$, respectively, with two indices shown explicitly and the rest understood implicitly. From the general group-theoretic analysis [25,28], it follows that since the condensate (3.10) transforms as a $[\bar{2}]_N$ representation of $\mathrm{SU}(N)$, it breaks $\mathrm{SU}(N)$ to $\mathrm{SU}(N-2)\otimes\mathrm{SU}(2)'$, and similarly, since it transforms as a $[\bar{2}]_{N-4}$ representation of $\mathrm{SU}(N-4)$, it breaks $\mathrm{SU}(N-4)$ to $\mathrm{SU}(N-6)\otimes\mathrm{SU}(2)''$. Here we append a single prime to the first SU(2) and a double prime to the second SU(2) to distinguish them and also to distinguish them from the SU(2) group of the $N=6$ theory (1.12). Since the condensate (3.10) carries a nonzero U(1) charge [namely, $q_{\chi\chi} = -2(N-2)$], it breaks the U(1) gauge symmetry. Thus, this condensate (3.10) breaks G to the group

$$[\mathrm{SU}(N-2)\otimes\mathrm{SU}(2)']\otimes[\mathrm{SU}(N-6)\otimes\mathrm{SU}(2)''], \tag{3.11}$$

where the square brackets here are inserted to indicate the origin of the different factor groups from the original $\mathrm{SU}(N)$ and $\mathrm{SU}(N-4)$ factor groups in (1.4). We find

$$(\Delta C_2)_{\chi\chi} = 2C_2([\bar{1}]_N) - C_2([\bar{2}]_N) = \frac{N+1}{N}. \tag{3.12}$$

From these results we calculate the relative attractiveness of these three possible fermion condensation channels in this $\mathrm{SU}(N)$-dominant case. We compute the differences

$$(\Delta C_2)_{\psi\chi} - (\Delta C_2)_{\psi\psi} = \frac{(N-6)(N+1)}{N} \tag{3.13}$$

and

$$(\Delta C_2)_{\psi\psi} - (\Delta C_2)_{\chi\chi} = \frac{3(N+1)}{N}, \tag{3.14}$$

whence

$$(\Delta C_2)_{\psi\chi} - (\Delta C_2)_{\chi\chi} = \frac{(N-3)(N+1)}{N}, \tag{3.15}$$

and the ratios

$$\frac{(\Delta C_2)_{\psi\chi}}{(\Delta C_2)_{\psi\psi}} = \frac{N-2}{4} \tag{3.16}$$

and

$$\frac{(\Delta C_2)_{\psi\psi}}{(\Delta C_2)_{\chi\chi}} = 4, \tag{3.17}$$

whence

$$\frac{(\Delta C_2)_{\psi\chi}}{(\Delta C_2)_{\chi\chi}} = N-2. \tag{3.18}$$

Therefore, in this $\mathrm{SU}(N)$-dominant case with $N\geq 7$, the $\psi\chi$ channel is the MAC, with greater attractiveness than the $\psi\psi$ channel, which, in turn, is more attractive than the $\chi\chi$ channel. Summarizing,

$$\mathrm{SU}(N) - \text{dominant with } N\geq 7 \Rightarrow \psi\chi \text{ channel is the MAC.} \tag{3.19}$$

One interesting feature of these comparisons is that the ratio $(\Delta C_2)_{\psi\psi}/(\Delta C_2)_{\chi\chi}$ is independent of N. As is evident from these results, in the lowest nondegenerate case, namely $N=6$, the $\psi\chi$ and $\psi\psi$ channels are equally attractive, and are a factor 4 more attractive than the $\chi\chi$ channel. Thus,

$$\mathrm{SU}(N) - \text{dominant with } N=6 \Rightarrow \psi\chi \text{ and } \psi\psi \text{ channels are the MACs.} \tag{3.20}$$

We focus here on the range $N\geq 7$; a detailed analysis of the $N=6$ case will be given below. Since the $\psi\chi$ channel is the MAC, it is expected that as the Euclidean reference scale μ decreases below a value that we denote as Λ_1, the coupling $\alpha_1(\mu)$ increases sufficiently to cause condensation in this channel. This condensation self-breaks $\mathrm{SU}(N)$ to $\mathrm{SU}(N-1)$ and breaks the weakly coupled gauge symmetry $\mathrm{SU}(N-4)$ to $\mathrm{SU}(N-5)$ and also the Abelian symmetry U(1). Without loss of generality, we may choose the $\mathrm{SU}(N)$ group index i in the condensate (3.2) to be $i=N$ and the $\mathrm{SU}(N-4)$ group index to be $\alpha = N-4$. The condensate (3.2) also spontaneously breaks the global flavor group G_{gb} for this theory, producing a set of Nambu-Goldstone bosons (NGBs). Earlier works in related chiral gauge theories have studied the resultant change in counts of the UV versus IR degrees of freedom [25], [29–32]. Here we focus on the dynamical self-breaking and induced breaking of gauge symmetries, together with the construction of resultant low-energy effective field theories. The fermions involved in the condensate (3.2), namely $\psi^{Nj}_{p,L} = -\psi^{jN}_{p,L}$ with $1 \le j \le N-1$ and $\chi_{j,N-4,p',L}$ with $1 \le j \le N-1$, gain dynamical masses of order Λ_1. The $2N-1$ SU(N) gauge bosons in the coset space SU(N)/SU($N-1$) gain dynamical masses of order $g_1(\Lambda_1)\Lambda_1$, while the $(2N-9)$ SU($N-4$) gauge bosons in the coset space SU($N-4$)/SU($N-5$) gain dynamical masses of order $g_2(\Lambda_1)\Lambda_1$. Finally, the U(1) gauge boson picks up a dynamical mass of order $g_3(\Lambda_1)\Lambda_1$. These massive fields are integrated out of the low-energy effective field theory that describes the physics as the reference scale μ decreases below Λ_1.

This low-energy effective field theory that is applicable as μ decreases below Λ_1 is invariant under the gauge symmetry (3.3). The massless gauge-nonsinglet fermion content of this EFT consists of

(1) $\psi^{ij}_{p,L}$ with $1 \le i,j \le N-1$, $1 \le p \le N_f$, forming N_f copies of a $([2]_{N-1}, 1)$ representation under the group (3.3),

(ii) $\chi_{j,\beta,p',L}$ with $1 \le j \le N-1$, $1 \le \beta \le N-5$, and $1 \le p' \le N_f$, comprising N_f copies of the $([\bar{1}]_{N-1}, [\bar{1}]_{N-5})$ representation of (3.3), and

(iii) $\omega^{\alpha\beta}_{p,L}$ with $1 \le \alpha,\beta \le N-5$ and $1 \le p \le N_f$, comprising N_f copies of $(1, (2)_{N-5})$.

[We do not list the U(1) charges, since there is no U(1) gauge symmetry in this low-energy effective theory.] The condensation process then repeats, with the $\psi\chi$ condensation channel again being the MAC in this SU($N-1$) $\otimes$ SU($N-5$) theory. One can treat the successive self-breakings and induced dynamical breakings iteratively at the various steps.

B. SU($N-4$) gauge interaction dominant, $N \ge 7$

Here we analyze the case in which the SU($N-4$) gauge interaction is strongly coupled and dominates over the SU(N) gauge interaction [as well as the weakly coupled U(1) gauge interaction]. We restrict our analysis to the range $N \ge 7$ here and will consider $N = 6$ in detail below. It will sometimes be convenient to use the quantity $M = N-4$ as defined before. We will denote this theory as SUMD, standing for "SU(M) dominant." If we were to completely turn off the SU(N) and U(1) gauge interactions, then this theory would be equivalent to a chiral gauge theory with an SU(M) gauge group, and N_f flavors of chiral fermions transforming according to the anomaly-free set $(2)_M + M + 4$ copies of $[\bar{1}]_M$, which has been studied in [29–32]. However, here we do not completely turn off the SU(N) or U(1) gauge interactions.

There are two possible (bilinear) fermion condensation channels. The first is

$$\chi\omega\colon\ ([\bar{1}]_N, [\bar{1}]_{N-4})_{-(N-2)} \times (1, (2)_{N-4})_N \to \to ([\bar{1}]_N, [1]_{N-4})_2, \tag{3.21}$$

with associated condensate

$$\langle \chi^T_{i,\alpha,p,L} C \omega^{\alpha\beta}_{p',L} \rangle, \tag{3.22}$$

where i is an SU(N) group index and α, β are SU($N-4$) group indices. The value of ΔC_2 for this condensation, as produced by the SU($N-4$) gauge interaction, is

$$(\Delta C_2)_{\chi\omega,\mathrm{SUMD}} = C_2((2)_{N-4}) = \frac{(N-2)(N-5)}{N-4} \quad \text{in SU}(N-4). \tag{3.23}$$

This condensate transforms as $([\bar{1}]_N, [1]_{N-4})_2$ and hence self-breaks SU($N-4$) to SU($N-5$) and produces induced breaking of the weakly coupled symmetries SU(N) to SU($N-1$) and of U(1). It leaves invariant the same residual gauge symmetry, (3.3), as the $\psi\chi$ condensate (3.2), which is the MAC for the SU(N)-dominant case (3.3). By convention, one may choose the SU($N-4$) index β in the condensate (3.22) to be $\beta = N-4$ and the SU(N) index i to be $i = N$. Then the fermions $\chi_{N,\alpha,p,L}$ and $\omega^{\alpha,N-4}_{p',L}$ with $1 \le \alpha \le N-4$, $1 \le p, p' \le N_f$ involved in the condensate pick up dynamical masses of order Λ_1. The dynamical mass generation for the SU(N) and SU($N-4$) gauge bosons in the respective coset spaces SU(N)/SU($N-1$) and SU($N-4$)/SU($N-5$) is the same as described above in the SU(N)-dominant scenario, as is the dynamical mass generation for the U(1) gauge boson.

A second possible condensation channel is the $\chi\chi$ channel (3.9), with associated condensate (3.10). This condensate breaks G to the group given above in Eq. (3.11). The measure of attractiveness of this condensation channel, as produced by the SU(M) = SU($N-4$) gauge interaction, is

$$(\Delta C_2)_{\chi\chi} = 2C_2([\bar{1}]_M) - C_2([\bar{2}]_M) = \frac{M+1}{M} = \frac{N-3}{N-4}. \tag{3.24}$$

Comparing the attractiveness measure of the channels (3.21) and (3.9), we calculate the difference

$$(\Delta C_2)_{\chi\omega} - (\Delta C_2)_{\chi\chi} = \frac{N^2 - 8N + 13}{N-4} \tag{3.25}$$

and the ratio

$$\frac{(\Delta C_2)_{\chi\omega}}{(\Delta C_2)_{\chi\chi}} = \frac{(N-2)(N-5)}{N-3}. \tag{3.26}$$

For the range $N \ge 6$, the difference $(\Delta C_2)_{\chi\omega} - (\Delta C_2)_{\chi\chi}$ is positive and, equivalently, the ratio $(\Delta C_2)_{\chi\omega}/(\Delta C_2)_{\chi\chi} > 1$.

Hence, the $\chi\omega$ channel is always more attractive than the $\chi\chi$ channel in this $\mathrm{SU}(N-4)$-dominant case. Thus,

$$\begin{aligned}&\mathrm{SU(N-4)-dominant\ with}\ N\geq 7:\\ &\quad\Rightarrow \chi\omega\ \text{channel is the MAC.}\end{aligned} \tag{3.27}$$

In addition to breaking gauge symmetries, the MAC condensate (3.22) spontaneously breaks the global symmetry G_{gb} for this theory, yielding a set of NGBs. Here we focus on the gauge symmetry breaking. We have restricted our analysis to the range $N \geq 7$; as will be discussed below, the MAC is different in the special case $N = 6$, where it is the $\omega\omega$ channel.

Although the $\chi\chi$ channel is not the MAC, we comment on its symmetry properties. It breaks $\mathrm{SU}(N-4)$ to $\mathrm{SU}(N-6)\otimes \mathrm{SU}(2)$ and also breaks U(1), since the condensate has nonzero U(1) charge $q_{\chi\chi} = -2(N-2)$. In terms of $\mathrm{SU}(N-4)$, the associated condensate has the form

$$\epsilon^{\cdots\alpha\beta}\langle\chi^T_{i,\alpha,p,L}C\chi_{j,\beta,p',L}\rangle, \tag{3.28}$$

where $\epsilon^{\cdots\alpha\beta}$ is an antisymmetric $\mathrm{SU}(N-4)$ tensor and we have indicated $N-6$ of the indices implicitly with dots. For this $\chi\chi$ channel, as regards the $\mathrm{SU}(N)$ and flavor symmetry, there are two channels and corresponding condensates. The (3.9) channel that involves an antisymmetric structure for $\mathrm{SU}(N)$ group indices is

$$\begin{aligned}&([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)}\times([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)}\to\\ &\quad\to([\bar{2}]_N,[\bar{2}]_{N-4})_{-2(N-2)},\end{aligned} \tag{3.29}$$

with corresponding condensate

$$\epsilon^{\cdots mn}\epsilon^{\cdots\alpha\beta}\langle\chi^T_{m,\alpha,p,L}C\chi_{n,\beta,p',L}\rangle. \tag{3.30}$$

Here $\epsilon^{\cdots mn}$ is an antisymmetric tensor under $\mathrm{SU}(N)$, $\epsilon^{\cdots\alpha\beta}$ was defined, and we indicate the rest of the indices in each tensor implicitly with dots. This condensate is automatically symmetrized in the flavor indices p and p' and is of the form (a,a,s) in the classification of Ref. [13]. The (3.9) channel that involves a symmetric structure for $\mathrm{SU}(N)$ group indices is

$$\begin{aligned}&([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)}\times([\bar{1}]_N,[\bar{1}]_{N-4})_{-(N-2)}\to\\ &\quad\to((\bar{2})_N,[\bar{2}]_{N-4})_{-2(N-2)},\end{aligned} \tag{3.31}$$

with corresponding condensate

$$\epsilon^{\cdots\alpha\beta}\langle\chi^T_{i,\alpha,p,L}C\chi_{j,\beta,p',L}\rangle-(p\leftrightarrow p'). \tag{3.32}$$

Because this condensate is antisymmetrized in flavor indices, it is automatically symmetric in $\mathrm{SU}(N)$ group indices and is thus of the form (s,a,a) in the classification of Ref. [13].

C. SU(N) and SU($N-4$) gauge interactions of comparable strength

Finally, we analyze the situation in which the $\mathrm{SU}(N)$ and $\mathrm{SU}(N-4)$ gauge interactions are of comparable strength at the scale relevant for the initial condensation. We restrict to $N\geq 7$ here and will discuss the $N=6$ theory below. The value of ΔC_2 for the most attractive channel, $\psi\chi$, in the $\mathrm{SU}(N)$-dominant case was given in Eq. (3.4), and the value of ΔC_2 for the MAC $\chi\omega$ in the $\mathrm{SU}(N-4)$-dominant case was given in Eq. (3.23) above. The difference is

$$(\Delta C_2)_{\psi\chi,\mathrm{SUND}}-(\Delta C_2)_{\chi\omega,\mathrm{SUMD}}=\frac{4(N-2)}{N(N-4)}. \tag{3.33}$$

Since this is positive for the relevant range of N considered here, it follows that, as the reference scale decreases and the $\mathrm{SU}(N)$ and $\mathrm{SU}(N-4)$ couplings increase, the minimal value of α for condensation is reached first for the $\mathrm{SU}(N)$-induced $\psi\chi$ condensate, at a scale μ that we may again denote Λ_1, where $\alpha_1(\Lambda_1)$ exceeds α_{cr} for the $\psi\chi$ condensation. At a slightly lower scale, $\Lambda_1'\lesssim\Lambda_1$, the $\mathrm{SU}(N-4)$ gauge interaction, of comparable strength, increases through the slightly larger critical value for condensation in the $\chi\omega$ channel. These condensates both break the gauge symmetry in the same way, to the residual subgroup $\mathrm{SU}(N-1)\otimes\mathrm{SU}(N-5)$, as given in Eq. (3.3). We have described the fermions and gauge bosons that gain dynamical masses from the $\psi\chi$ and $\chi\omega$ condensations above, and we combine these results here. By convention, one may choose the $\mathrm{SU}(N)$ index i and the $\mathrm{SU}(N-4)$ index α in the $\psi\chi$ condensate $\langle\psi^{ij\,T}_{p,L}C\chi_{j,\alpha,p',L}\rangle$ in Eq. (3.2) to be $i=N$ and $\alpha=N-4$, respectively. The fermions involved in this condensate are then $\psi^{Nj}_{p,L}$ and $\chi_{j,N-4,p',L}$ with $1\leq j\leq N-1$. These gain dynamical masses of order Λ_1. The $2N-1$ $\mathrm{SU}(N)$ gauge bosons in the coset $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ and the $2M-1=2N-9$ $\mathrm{SU}(M)$ gauge bosons in the coset $\mathrm{SU}(M)/\mathrm{SU}(M-1)=\mathrm{SU}(N-4)/\mathrm{SU}(N-5)$ gain dynamical masses of order $\simeq g_1(\Lambda_1)\Lambda_1$ and $\simeq g_2(\Lambda_1)\Lambda_1$, while the U(1) gauge boson gains a dynamical mass $\simeq g_3(\Lambda_1)\Lambda_1$. A vacuum alignment argument [1,33] suggests that the condensation process would be such as to preserve the maximal residual gauge symmetry, with gauge group of the largest order, thereby minimizing the number of gauge bosons that pick up masses. In the present case, one can use this argument to infer that in the condensate $\langle\chi^T_{i,\alpha,p,L}C\omega^{\alpha\beta}_{p',L}\rangle$ in Eq. (3.22), the $\mathrm{SU}(N)$ index is the same as the unmatched index in the $\langle\psi^{ij\,T}_{p,L}C\chi_{j,\alpha,p',L}\rangle$ condensate, namely $i=N$, and the β index is the same as the unmatched $\mathrm{SU}(N-4)$ index α in the $\psi\chi$ condensate, namely $N-4$, so that these two condensates break the initial UV gauge group G in the same way, to the subgroup $\mathrm{SU}(N-1) \otimes \mathrm{SU}(N-5)$ in Eq. (3.3). Then the fermions involved in the $\chi\omega$ condensate, $\chi_{N,\alpha,p,L}$ and $\omega^{\alpha\beta}_{p',L}$ with $1 \le \alpha \le N-4$ and $\beta = N-4$, gain dynamical masses of order Λ_1 and Λ'_1.

The resultant low-energy effective field theory that describes the physics as the reference scale μ decreases below Λ'_1 contains the following massless fermions that are nonsinglets under the residual gauge group $\mathrm{SU}(N-1) \otimes \mathrm{SU}(N-5)$:

(1) $\psi^{ij}_{p,L}$ with $1 \le i,j \le N-1$, $1 \le p \le N_f$, forming N_f copies of a $([2]_{N-1}, 1)$ representation under the group (3.3),

(2) $\chi_{j,\alpha,p',L}$ with $1 \le j \le N-1$, $1 \le \beta \le N-5$, and $1 \le p' \le N_f$, forming N_f copies of the $([\bar{1}]_{N-1}, [\bar{1}]_{N-5})$ representation of (3.3), and

(3) $\omega^{\alpha\beta}_{p',L}$ with $1 \le \alpha, \beta \le N-5$, forming N_f copies of the $(1, (2)_{N-5})$ representation of the group (3.3).

This theory also includes certain massless fermions that are singlets under the gauge group (3.3), e.g., $\chi_{N,N-4,p,L}$.

IV. $N=6$ THEORY

A. Beta function and constraints on N_f

In this section we study the lowest nondegenerate case of the chiral gauge theory (1.4) with the fermions (1.6), namely the $N = 6$ theory, for which the fermion content was given in Eq. (1.13). From the general formulas (2.3) and (2.4), it follows that the one-loop coefficients for the SU(6) and SU(2) gauge interactions in this theory are

$$b^{(\mathrm{SU}(6))}_{1\ell,1} = 2(11 - N_f) \tag{4.1}$$

and

$$b^{(\mathrm{SU}(2))}_{1\ell,2} = \frac{2}{3}(11 - 5N_f). \tag{4.2}$$

Substituting $N = 6$ into the upper bound on N_f in Eq. (2.9), we find that $N_f < 11/5$, i.e., for physical integral values,

$$N = 6 \Rightarrow N_f = 1, 2, \tag{4.3}$$

in accord with the general result given in Sec. II. When discussing the $N_f = 1$ case, we will suppress the flavor indices in the notation, since they are all the same.

For the study of the UV to IR evolution of this theory, we substitute $N = 6$ into the general formulas (2.13) and (2.14) to obtain the two-loop coefficients in the SU(6) and SU(2) beta functions, which are

$$b^{(\mathrm{SU}(6))}_{2\ell;11} = \frac{1}{2}(816 - 169N_f) \tag{4.4}$$

and

$$b^{(\mathrm{SU}(2))}_{2\ell;22} = \frac{1}{6}(272 - 275N_f). \tag{4.5}$$

V. $N=6$ THEORY WITH SU(2) GAUGE INTERACTION DOMINANT

A. RG evolution from UV

As before, it is natural to begin by analyzing the UV to IR evolution in the case where one non-Abelian gauge interaction is much stronger than the other. We start with the situation in which the SU(2) gauge interaction is much stronger than the SU(6) interaction, so that, to first approximation, we may treat the SU(6) [as well as U(1)] gauge interaction perturbatively. By analogy with our notation above, this will be denoted as the SU2D case, where again, D stands for "dominant." Then, since $b^{(\mathrm{SU}(2))}_{1\ell,2} > 0$ while $b^{(\mathrm{SU}(2))}_{2\ell,22} < 0$, the two-loop beta function β_{α_2} for the SU(2) gauge interaction has an IR zero at

$$\alpha_{2,IR,2\ell} = -\frac{4\pi b^{(\mathrm{SU}(2))}_{1\ell,2}}{b^{(\mathrm{SU}(2))}_{2\ell;22}} = \frac{16\pi(11 - 5N_f)}{275N_f - 272}. \tag{5.1}$$

For $N_f = 1$, $\alpha_{2,IR,2\ell} = 32\pi = 100.5$, while for $N_f = 2$, $\alpha_{2,IR,2\ell} = 8\pi/139 = 0.181$. The IRFP value for $N_f = 1$ is too large for the two-loop calculation to be considered to be quantitatively accurate, but it does indicate that the theory becomes strongly coupled in the IR. The IRFP value for $N_f = 2$ is considerably smaller than the estimates of the critical values α_{cr} for any of the three attractive condensation channels (which will be given below). Hence, this theory with $N_f = 2$ is expected to evolve in the IR limit to an exact IR fixed point (IRFP) in a scale-invariant and conformally invariant non-Abelian Coulomb phase (NACP), without any spontaneous chiral symmetry breaking or associated fermion condensate formation. We therefore focus on the $N_f = 1$ case. Since the flavor subscripts p, p' are always equal to 1, they are suppressed in the notation.

B. Condensation at scale Λ_1

We proceed to determine the most attractive channel for the formation of bilinear condensates of SU(2)-nonsinglet fermions in this $N_f = 1$ case. There are, *a priori*, several possible channels. The first is

$$\omega\omega\colon\ (1, Adj)_6 \times (1, Adj)_6 \to (1, 1)_{12}, \tag{5.2}$$

where Adj is the adjoint (triplet) representation of SU(2) and the notation follows Eq. (1.7). The shorthand name for this channel, $\omega\omega$, follows from the condensate, which is

$$\langle \vec{\omega}^T_L C \cdot \vec{\omega}_L \rangle. \tag{5.3}$$

In terms of dimensions of the SU(2) representations, this channel has the form $3 \times 3 \to 1$. The measure of attractiveness of this channel due to the strongly coupled SU(2) gauge interaction is

$$\Delta C_2 = 2C_2(Adj) = 4 \quad \text{for } 3\times 3 \to 1 \text{ in SU}(2). \tag{5.4}$$

This is the most attractive channel:

$$\text{SU}(2) - \text{dominant} \Rightarrow \omega\omega \text{ channel is the MAC.} \tag{5.5}$$

The rough estimate of the minimal (critical) coupling $\alpha_2(\mu) = \alpha_{cr}$ for this channel is given by Eq. (2.23) as $\alpha_{cr} \simeq \pi/6 = 0.5$. Since the condensate involves the SU(6)-singlet fermion $\vec{\omega}_L$, it obviously preserves the SU(6) gauge symmetry. As a scalar product of the isovector $\vec{\omega}_L$ with itself, this condensate is also invariant under the strongly coupled SU(2) gauge symmetry. Because the condensate has a nonzero U(1) charge (namely, $q = 12$), it breaks the U(1) gauge symmetry. The continuous gauge symmetry under which the condensate (5.3) is invariant is therefore

$$\text{SU}(4) \otimes \text{SU}(2). \tag{5.6}$$

This residual symmetry group has order 38 and rank 6. For this and other possible fermion condensation channels, we record these properties in Table II. This table refers to the possible initial condensation patterns at the highest condensation scale; subsequent evolution further into the infrared is discussed below

A second possible condensation channel is

$$\chi\chi\colon\ ([\bar{1}]_6, [\bar{1}]_2)_{-4} \times ([\bar{1}]_6, [\bar{1}]_2)_{-4} \to ([\bar{2}]_6, 1)_{-8}, \tag{5.7}$$

where the shorthand name $\chi\chi$ reflects the associated condensate, $\epsilon^{\alpha\beta}\langle\chi^T_{i,\alpha,L} C \chi_{j,\beta,L}\rangle$. Since SU(2) has only pseudoreal representations, this channel has the form $2\times 2 \to 1$ with respect to SU(2). The measure of attractiveness of this channel due to the strongly coupled SU(2) gauge interaction is

$$\Delta C_2 = 2C_2([\bar{1}]_2) = \frac{3}{2} \quad \text{for } 2\times 2 \to 1 \text{ in SU}(2). \tag{5.8}$$

From (2.23), we find that the minimal critical coupling for condensation in this channel is $\alpha_{cr} \simeq 4\pi/9 = 1.4$. From the general structural analysis of fermion condensates given above, it follows that, since the SU(2) tensor $\epsilon^{\alpha\beta}$ is antisymmetric, the condensate must be of the form (a, a, s). [It cannot be of the form (s, a, a) because with $N_f = 1$, this would vanish identically.] Hence, under SU(6), it transforms as $[4]_6$, or equivalently, as $[\bar{2}]_6$, as indicated in Eq. (5.7). Consequently, it is proportional to

$$\epsilon^{ijk\ell mn}\epsilon^{\alpha\beta}\langle\chi^T_{m,\alpha,L} C \chi_{n,\beta,L}\rangle, \tag{5.9}$$

where i, j, k, ℓ, m, n are SU(6) indices and α, β are SU(2) indices. This condensation channel thus preserves SU(2) while breaking U(1). As regards SU(6), from a general group-theoretic analysis [25,28], one infers that the condensate (5.9) breaks this SU(6) gauge symmetry to the subgroup $\text{SU}(4) \otimes \text{SU}(2)'$, where we mark the $\text{SU}(2)'$ with a prime to distinguish it from the SU(2) in the original gauge group (1.12). Hence, the full continuous gauge symmetry under which the condensate (5.9) is invariant is

$$[\text{SU}(4) \otimes \text{SU}(2)'] \otimes \text{SU}(2), \tag{5.10}$$

where we insert the brackets to indicate the origin of the $[\text{SU}(4) \otimes \text{SU}(2)']$ group from the breaking of the original SU(6) in (1.12). This residual symmetry group has order 21 and rank 5.

A third type of condensation channel is

$$\chi\omega\colon\ ([\bar{1}]_6, [\bar{1}]_2)_{-4} \times (1, Adj)_6 \to ([\bar{1}]_6, [1]_2)_2, \tag{5.11}$$

where the shorthand name $\chi\omega$ reflects the condensate

TABLE II. Properties of possible initial bilinear fermion condensates in the UV theory (1.12) with (1.13). The shorthand name of the condensation channel and the condensate in this channel are displayed in the first and second columns. The third and fourth columns list the values of ΔC_2 with respect to the SU(6) and SU(2) gauge interactions. The entries in the fifth, sixth, and seventh columns indicate whether a given condensate is invariant (inv.) under the SU(6), SU(2), and U(1) gauge symmetries, respectively, or breaks (bk.) one or more of these symmetries. The entry in the eighth column gives the representation $(R_1, R_2)_q$ of the condensate under the group (1.12), following the notation of Eq. (1.7). The ninth column lists the continuous gauge symmetry group under which a given condensate is invariant. These results are for the case $N_f = 1$ and for $N_f = 2$ with condensates symmetrized over the flavor indices, which are suppressed in the notation. The $\psi\psi$ and $\psi\chi$ channels are the MACs for the SU(6)-dominant case, while the $\omega\omega$ is the MAC for the SU(2)-dominant case. See text for further discussion.

Name	Condensate	$(\Delta C_2)_{\text{SU}(6)}$	$(\Delta C_2)_{\text{SU}(2)}$	SU(6)	SU(2)	U(1)	$(R_1, R_2)_q$	H_{inv}
$\omega\omega$	$\langle\vec{\omega}^T_L C \cdot \vec{\omega}_L\rangle$	0	4	inv.	inv.	bk.	$(1, 1)_{12}$	$\text{SU}(4) \otimes \text{SU}(2)$
$\psi\chi$	$\langle\psi^{ij\,T}_L C \chi_{j,\beta,L}\rangle$	$\frac{14}{3}$	0	bk.	bk.	bk.	$([1]_6, [\bar{1}]_2)_{-2}$	SU(5)
$\chi\omega$	$\langle\chi^T_{i,\alpha,L} C \omega^{\alpha\beta}_L\rangle$	0	2	bk.	bk.	bk.	$([\bar{1}]_6, [1]_2)_2$	SU(5)
$\psi\psi$	$\epsilon_{ijk\ell mn}\langle\psi^{k\ell\,T}_L C \psi^{mn}_L\rangle$	$\frac{14}{3}$	0	bk.	inv.	bk.	$([\bar{2}]_6, 1)_4$	$[\text{SU}(4) \otimes \text{SU}(2)'] \otimes \text{SU}(2)$
$\chi\chi$	$\epsilon^{ijk\ell mn}\epsilon^{\alpha\beta}\langle\chi^T_{m,\alpha,L} C \chi_{n,\beta,L}\rangle$	$\frac{7}{6}$	$\frac{3}{2}$	bk.	inv.	bk.	$([\bar{2}]_6, 1)_{-8}$	$[\text{SU}(4) \otimes \text{SU}(2)'] \otimes \text{SU}(2)$

$$\langle \chi^T_{i,\alpha,L} C \omega^{\alpha\beta}_L \rangle. \tag{5.12}$$

With respect to SU(2), this channel is $2 \times 3 \to 2$. The measure of attractiveness for this channel due to SU(2) gauge interactions is

$$\Delta C_2 = C_2(Adj) = 2 \quad \text{for } 2 \times 3 \to 2 \, \text{in} \, \text{SU}(2). \tag{5.13}$$

The corresponding estimate of the critical coupling from Eq. (2.23) is $\alpha_{cr} = \pi/3$. Evidently, this channel is more attractive than the $\chi\chi$ channel (5.7), but less attractive than the $\omega\omega$ channel (5.2).

All of these three types of fermion condensation exhibit the phenomenon of a strongly coupled gauge interaction producing condensate(s) that dynamically break a more weakly coupled gauge interaction, namely U(1). Furthermore, the condensate in the $\chi\chi$ channel (5.7) dynamically breaks not only the U(1) gauge symmetry, but also the more weakly coupled SU(6) gauge symmetry. If a condensate were to form in the $\chi\omega$ channel (5.11), it would self-break the strongly coupled SU(2) symmetry, as well as breaking the weakly coupled SU(6) symmetry. However, as will be shown below, a condensate is not likely to form in the $\chi\omega$ channel.

Since the $\omega\omega$ channel (5.2) is the MAC, one expects that, as this theory evolves from the UV to the IR, at a scale that we denote $\mu = \Lambda_1$ where the running coupling $\alpha_2(\mu)$ increases above the critical value for condensation in this $\omega\omega$ channel, the condensate (5.3) forms, breaking the U(1) gauge symmetry, but leaving the SU(2) and SU(6) symmetries intact. As the condensate $\langle \vec{\omega}^T_L C \cdot \vec{\omega}_L \rangle$ in Eq. (5.3) maintains the SU(2) symmetry, all of the three components of the fermion $\vec{\omega}_L$ involved in this condensate gain equal dynamical masses $\sim\Lambda_1$ and are integrated out of the low-energy effective field theory that describes the physics as the reference scale μ decreases below Λ_1. The U(1) gauge field gains a mass $\sim g_3(\Lambda_1)\Lambda_1$. With these fermion and vector boson fields integrated out, the one-loop and two-loop coefficients in the SU(2) beta function in the low-energy effective theory have the same sign, so as the reference momentum scale μ decreases below Λ_1, the coupling $\alpha_2(\mu)$ continues to increase. Because the $\vec{\omega}_L$ fermions have been integrated out at the scale Λ_1, they are no longer available to form a condensate in the $\chi\omega$ channel (5.11) in the low-energy effective theory below Λ_1.

C. EFT below Λ_1 and condensation at scale Λ_2

In Ref. [32] it was proved that if one starts with a chiral gauge theory with gauge group G that is free of gauge and global anomalies, and it is broken dynamically to a theory with gauge group $H \subseteq G$, with some set of fermions gaining dynamical masses and being integrated out, then the low-energy theory with the gauge group H is also free of gauge and global anomalies. As a special case of this theorem, the low-energy theory that is operative here as μ decreases below Λ_1 is also an anomaly-free theory. One easily checks that it is chiral.

As μ decreases below a lower scale that we denote as Λ_2, $\alpha_2(\mu)$ increases past the critical value for the attractive $\chi\chi$ condensation channel (5.7), which is the MAC in this low-energy effective theory, and the condensate (5.9) is expected to form. As noted above, this leaves SU(2) invariant and breaks SU(6) to SU(4) $\otimes$ SU(2)$'$. By convention, one may label the SU(6) indices i, j of the fermions in the condensate (5.9) as $m = 5$ and $n = 6$. Then the fermions $\chi_{5\alpha,L}$ and $\chi_{6\beta,L}$ that are involved in this condensate gain dynamical masses of order Λ_2 and are integrated out of the low-energy effective theory applicable for $\mu < \Lambda_2$. Furthermore, the gauge fields in the coset space SU(6)/[SU(4) $\otimes$ SU(2)] gain dynamical masses of order $g_1(\Lambda)\Lambda_2$.

D. EFT below Λ_2 and further condensation

By the same theorem as before, this low-energy theory is anomaly-free and one can again check that it is chiral. The low-energy effective theory below Λ_2 thus has a gauge symmetry [SU(4) $\otimes$ SU(2)$'$] $\otimes$ SU(2), where the SU(2)$'$ arises from the breaking of the SU(6) and the second SU(2) was present in the original theory. The fermions that have gained masses and have been integrated out are no longer dynamical. The elements of the residual SU(4) subgroup of SU(6) operate on the indices $1 \le i \le 4$, while the elements of SU(2)$'$ operate on the indices $i = 5, 6$. Thus, the massless fermions in this effective field theory below Λ_2 are as follows, where we categorize them with a three-component vector, indicating the representations with respect to the group (5.10) in the indicated order:

(1) ψ^{ij}_L with $1 \le i, j \le 4$, forming a (self-conjugate) $([2]_4, 1, 1)$ representation of the group SU(4) $\otimes$ SU(2)$'$ $\otimes$ SU(2) in (5.10),
(2) ψ^{i5}_L and ψ^{i6}_L, forming a $([1]_4, [1]_{2'} 1, 1)$ representation of (5.10),
(3) $\chi_{i,\alpha,L}$ with $1 \le i \le 4$, forming a $([\bar{1}]_4, 1, [\bar{1}]_2)$ representation of (5.10).

In this low-energy EFT below Λ_2, the MAC for SU(4)-induced condensate formations is $[2]_4 \times [2]_4 \to 1$ with the self-conjugate ψ^{ij}_L transforming as $[2]_4$ of SU4), producing the condensate

$$\sum_{i,j,k,\ell=1}^{4} \epsilon_{ijk\ell} \langle \psi^{ij\,T}_L C \psi^{k\ell}_L \rangle. \tag{5.14}$$

This is a singlet under the SU(4) gauge symmetry and is obviously invariant under the two other gauge symmetries, SU(2)$'$ $\otimes$ SU(2), since the fermions in (5.14) are singlets under these groups. Let us denote the scale at which this condensate forms as Λ_3. The SU(4)-induced condensation producing this condensate (5.14) has $\Delta C_2 = 5$. The ψ^{ij}_L with $1 \le i, j \le 4$ involved in this condensate pick up dynamical masses of order Λ_3 and are integrated out of the low-energy EFT that is operative below Λ_3. The SU(4) gauge interaction can also produce the condensate

$$\sum_{i=1}^{4} \langle \psi_L^{ir\,T} C \chi_{i,\alpha} \rangle, \quad (5.15)$$

where $r = 5,\ 6$. This condensate is invariant under SU(4) and breaks $\mathrm{SU}(2)' \otimes \mathrm{SU}(2)$ [since it involves the uncontracted $\mathrm{SU}(2)'$ index r and the uncontracted SU(2) index α]. For this condensation, $\Delta C_2 = 15/4$. With these condensates, only the SU(4) gauge symmetry remains, and all SU(4)-nonsinglet fermions have picked up dynamical masses. This vectorial SU(4) theory confines and produces a spectrum of SU(4)-singlet bound state hadrons.

VI. $N = 6$ THEORY WITH SU(6) GAUGE INTERACTION DOMINANT

A. RG evolution from UV

In this section we analyze the $N = 6$ theory for the case in which the SU(6) gauge interaction becomes strongly coupled and is dominant over the weakly coupled SU(2) [and U(1)] gauge interactions. We denote this as the SU6D case. The one-loop and two-loop terms in the beta function were given above in Eqs. (4.1) and (4.4). For both of the cases allowed by the requirement of asymptotic freedom for the SU(6) and SU(2) gauge interactions, namely $N_f = 1$ and $N_f = 2$, these coefficients have the same sign, so that the two-loop beta function of this SU(6) theory has no IR zero. Hence, as the scale μ decreases from μ_{UV} to the IR, $\alpha_1(\mu)$ increases until it eventually exceeds the range of values where it can be calculated perturbatively.

B. Highest-scale condensation channels

We examine the various possible fermion condensation channels produced by the strongly coupled SU(6) gauge interaction. The first is the $\psi\psi$ channel

$$\psi\psi\colon\ ([2]_6, 1)_2 \times ([2]_6, 1)_2 \to ([4]_6, 1)_4 \approx ([\bar{2}]_6, 1)_4, \quad (6.1)$$

with associated condensate

$$\epsilon_{ijk\ell mn} \langle \psi_{p,L}^{k\ell\,T} C \psi_{p',L}^{mn} \rangle. \quad (6.2)$$

This condensate is automatically symmetrized in the flavor indices. Since it transforms as a $[\bar{2}]_6$ representation of SU(6), it breaks SU(6) to $\mathrm{SU}(4) \otimes \mathrm{SU}(2)'$. Because the constituent fermion fields in (6.2) are singlets under SU(2), this condensate is obviously SU(2)-invariant. Finally, owing to the property that the condensate (6.2) has nonzero U(1) charge, it also breaks U(1). The residual subgroup of the original group (1.12) that is left invariant by the condensate (6.2) is thus $[\mathrm{SU}(4) \otimes \mathrm{SU}(2)'] \otimes \mathrm{SU}(2)$ [see Eq. (5.10)], as in the condensation process (5.7). The condensation (6.1) thus provides another example of an induced, dynamical breaking of one gauge symmetry, namely U(1), by a different, strongly coupled, gauge interaction in a direct-product chiral gauge theory. The measure of attractiveness of this condensation channel involving the SU(6) gauge interaction is

$$\Delta C_2 = C_2([\bar{2}]_6) = \frac{14}{3} \text{ for } [2]_6 \times [2]_6 \to [\bar{2}]_6 \text{ in } SU(6). \quad (6.3)$$

From the rough estimate for the minimal critical coupling strength to produce this condensate, (2.23), one has $\alpha_{cr} \simeq \pi/7 = 0.45$.

A second possible condensation channel is

$$\psi\chi\colon\ ([2]_6, 1)_2 \times ([\bar{1}]_6, [\bar{1}]_2)_{-4} \to ([1]_6, [\bar{1}]_2)_{-2} \quad (6.4)$$

with associated condensate

$$\langle \psi_{p,L}^{ij\,T} C \chi_{j,\beta,p',L} \rangle. \quad (6.5)$$

This condensation breaks SU(6) to SU(5) and also breaks SU(2) and U(1), so that the residual invariance group is SU(5), with order 24 and rank 4. The total number of broken generators is thus 15 and the reduction in rank is by 3. Again, this illustrates the dynamical breaking of more weakly coupled gauge symmetries by a strongly coupled gauge interaction in a direct-product gauge theory. The measure of attractiveness of this channel (6.4) is

$$\Delta C_2 = C_2([2]_6) = \frac{14}{3} \quad \text{for } [2]_6 \times [\bar{1}]_6 \to [1]_6 \text{ in } SU(6). \quad (6.6)$$

Evidently, this is the same as the attractiveness for the channel (6.1), so the critical coupling α_{cr} is also the same as for that channel. This $\Delta C_2 = 14/3$ is also larger than the ΔC_2 for the third channel (to be discussed below), so that, as was stated above in (3.20), for this $N = 6$ theory, with SU(6) being the dominant gauge interaction, the $\psi\psi$ and $\psi\chi$ channels are the MACs.

A third condensation channel produced by the dominant SU(6) gauge interaction is

$$\chi\chi\colon\ [\bar{1}]_6 \times [\bar{1}]_6 \to [\bar{2}]_6 \approx [4]_6 \text{ in SU}(6) \quad (6.7)$$

with condensate

$$\epsilon^{ijk\ell mn} \langle \chi_{m,\alpha,p,L}^{T} C \chi_{n,\beta,p',L} \rangle. \quad (6.8)$$

Although we use the same shorthand name, $\chi\chi$, for this channel as in Eq. (5.7), it is understood that here it is the SU(6) gauge interaction that is responsible for the formation of this condensate, rather than the SU(2) gauge interaction in (5.7). The measure of attractiveness for this condensation, as produced by the SU(6) gauge interaction, is

$$\Delta C_2 = 2C_2([\bar{1}]_6) - C_2([2]_6) = \frac{7}{6} \quad \text{for } [\bar{1}]_6 \times [\bar{1}]_6 \to [\bar{2}]_6 \, \text{in SU}(6). \tag{6.9}$$

This ΔC_2 is a factor of 4 smaller than the common value $\Delta C_2 = 14/3$ for the condensation channels (6.1) and (6.4) and hence is predicted not to occur in this SU(6)-dominant case. We proceed to discuss in greater detail the two different patterns of UV to IR evolution for the most attractive condensation channels in this SU(6)-dominant case.

C. $\psi\psi$ condensation channel

Here we consider the $\psi\psi$ condensation channel (6.1), i.e., $([2]_6, 1)_2 \times ([2]_6, 1)_2 \to ([\bar{2}]_6, 1)_4$. We denote the scale at which the condensate (6.2) forms as Λ_1. [To avoid cumbersome notation, we use the same symbol for this highest-level condensation as we did in the subsection dealing with the case where the SU(2) gauge interaction is dominant, but it is understood implicitly that this scale has generically different values for these different cases.] Without loss of generality, one may choose the SU(6) group indices of the ψ fermions involved in the condensate (6.2) to be $k, \ell, m, n \in \{1,2,3,4\}$ and the uncontracted indices in (6.2) to be $i, j \in \{5,6\}$. The ψ fermions involved in the condensate (6.2) gain dynamical masses of order Λ_1. The gauge bosons in the coset $\text{SU}(6)/[\text{SU}(4) \otimes \text{SU}(2)']$ pick up dynamical masses of order $g_1(\Lambda_1)\Lambda_1$, and the U(1) gauge boson picks up a dynamical mass $\simeq g_3(\Lambda_1)\Lambda_1$. These massive fermion and vector boson fields are integrated out of the low-energy effective field theory that describes the physics as the reference scale μ decreases below Λ_1. The resultant low-energy effective theory contains the following massless fermions: (1) $\text{SU}(4) \otimes \text{SU}(2)'$-nonsinglets $\psi^{ia}_{p,L}$ with $1 \le i \le 4$, $a \in \{5,6\}$, and $1 \le p \le N_f$, which are singlets under SU(2); (2) $\text{SU}(4) \otimes \text{SU}(2)$-nonsinglets $\chi_{i\alpha,p,L}$ with $1 \le i \le 4$, $\alpha = 1, 2$, and $1 \le p \le N_f$, which are singlets under $\text{SU}(2)'$; and (3) $\text{SU}(2)' \otimes \text{SU}(2)$-nonsinglets $\chi_{i,\alpha,p,L}$ with $i = 5, 6$, which are singlets under SU(4). There are also the massless fermions $\psi^{56}_{p,L}$ with $1 \le p \le N_f$, which are singlets under all three factor groups in (5.10). The fermions (1) transform as $2N_f$ fundamental representations $F = [1]_4$ of SU(4), while the fermions (2) transform as $2N_f$ conjugate fundamental representations $\bar{F} = [\bar{1}]_4$ of SU(4), so that the SU(4) gauge symmetry is vectorial. Combining this property with the fact that the $\text{SU}(2)'$ and SU(2) groups have only real representations, it follows that this low-energy theory is vectorial. The action of an element $U \in \text{SU}(4)$ is

$$\psi^{ia}_{p,L} = U^i_j \psi^{ja}_{p,L}$$
$$\chi_{ia,p,L} = (U^\dagger)^j_i \chi_{ja,p,L}, \tag{6.10}$$

with fixed $a = 5$, 6 and $1 \le p \le N_f$. The elements of $\text{SU}(2)'$ operate on the indices $a = 5$, 6 of the fermions (1) and (3). [The operation of the elements of SU(2) on the α, β indices has already been discussed.] The couplings of the SU(4) and $\text{SU}(2)'$ gauge interactions start out equal at $\mu = \Lambda_1$, as descendents of the gauge coupling α_1 of the UV gauge coupling for the SU(6) gauge interaction.

As the theory evolves further into the IR, several possible patterns of gauge symmetry breaking are possible. The SU(4) gauge interaction can produce a condensate in the $[1]_4 \times [\bar{1}]_4 \to 1$, i.e., $F \times \bar{F} \to 1$ channel:

$$\left\langle \sum_{i=1}^{4} \psi^{ia\,T}_{p,L} C \chi_{i,\alpha,p',L} \right\rangle, \tag{6.11}$$

where, as indicated, the sum on i is over the active SU(4) gauge indices, while the other indices take on the values $a = 5$, 6, $\alpha = 1$, 2, and $1 \le p, p' \le N_f$. The measure of attractive of this condensation, as produced by the SU(4) gauge interaction, is $\Delta C_2 = 2C_2([1]_4) = 15/4 = 3.75$. This condensate preserves the SU(4) gauge symmetry and breaks the $\text{SU}(2)'$ gauge symmetry operating on the indices $a = 5$, 6 and the SU(2) gauge symmetry operating on the indices $\alpha = 1$, 2.

In contrast, the $\text{SU}(2)'$ gauge interaction could produce the condensate

$$\sum_{a,b=5}^{6} \epsilon_{ab} \langle \psi^{ia\,T}_{p,L} C \psi^{jb}_{p',L} \rangle. \tag{6.12}$$

The measure of attractiveness for this condensation, as produced by the $\text{SU}(2)'$ interaction, is $\Delta C_2 = 3/2$. Since the fermions involved in this condensate are SU(2)-singlets, it obviously preserves SU(2). With the contraction on the $\text{SU}(2)'$ indices $a, b \in \{5,6\}$, it also preserves $\text{SU}(2)'$. If $N_f = 1$, then the condensate is automatically symmetric in the single flavor index, so it has the form (a, a, s) in the notation of Eq. (2.26) and hence transforms like the $[2]_4$ representation of SU(4). This breaks SU(4) to $SU(2)'' \otimes SU(2)'''$, where we use repeated primes to indicate that these SU(2) subgroups of SU(4) are distinct from both the original UV SU(2) symmetry and the $\text{SU}(2)'$ symmetry. If $N_f = 2$, then there are two possibilities; (a, a, s) if one constructs a linear combination that is symmetrized in flavor indices, and (a, s, a), if one antisymmetrizes over flavor indices. For each of these possibilities, one can track the evolution further into the IR using the same methods as above.

D. $\psi\chi$ condensation channel

Here we consider the $\psi\chi$ condensation channel (6.4), i.e., $([2]_6, 1)_2 \times ([\bar{1}]_6, [\bar{1}]_2)_{-4} \to ([1]_6, [\bar{1}]_2)_{-2}$, with the associated condensate $\langle \psi^{ij\,T}_{p,L} C \chi_{j,\beta,p',L} \rangle$ in Eq. (6.5). By convention, we may choose the SU(6) index $i = 6$ and the SU(2) index $\beta = 2$ in this condensate. Then the fermions involved in the condensate, namely $\psi^{6j}_{p,L}$ and $\chi_{j,2,p',L}$ with $1 \le j \le 5$ gain dynamical masses of order Λ_1 and are integrated out of the low-energy effective theory applicable for $\mu < \Lambda_1$. The 11 SU(6) gauge bosons in the coset SU(6)/SU(5) gain dynamical masses of order $g_1(\Lambda_1)\Lambda_1$, while the SU(2) and U(1) gauge bosons gain masses of order $g_i(\Lambda_1)\Lambda_1$ with $i = 2, 3$, respectively. These fields are integrated out of the low-energy effective theory applicable for $\mu < \Lambda_1$.

For this channel, the low-energy effective theory that describes the physics as μ decreases below Λ_1 has an SU(5) gauge symmetry with (massless) SU(5)-nonsinglet fermions $\psi^{ij}_{p,L}$ and $\chi_{i,1,p',L}$, where $1 \le i, j \le 5$ and $1 \le p$, $p' \le N_f$. In addition, there are massless SU(5)-singlet fermions $\chi_{6,\beta,p',L}$ and $\omega^{\alpha\beta}_{p,L}$ with $1 \le \alpha, \beta \le 2$ and $1 \le p, p' \le N_f$ remaining from the UV theory.

In this low-energy theory, the SU(5) gauge coupling inherited from the SU(6) UV theory continues to increase as μ decreases below Λ_1, and is expected to trigger a further fermion condensation

$$[2]_5 \times [2]_5 \to [\bar{1}]_5 \tag{6.13}$$

with $\Delta C_2 = 24/5$ and associated condensate

$$\sum_{i,j,k,\ell,m=1}^{5} \epsilon_{ijk\ell m} \langle \psi^{jk\,T}_{p,L} C \psi^{\ell m}_{p',L} \rangle, \tag{6.14}$$

where the indices i, j, k, ℓ, m are SU(5) group indices. By convention, we may choose the uncontracted SU(5) group index in (6.14) to be $i = 5$. This condensate breaks SU(5) to SU(4). The fermions $\psi^{jk}_{p,L}$ with $j, k \in \{1, 2, 3, 4\}$ and $1 \le p \le N_f$ gain dynamical masses of order Λ_2. The 9 gauge bosons in the coset SU(5)/SU(4) gain dynamical masses of order $g_1(\Lambda_2)\Lambda_2$. All of these fields are integrated out of the low-energy effective theory that describes the physics at scales $\mu < \Lambda_2$.

The low-energy theory that is operative for $\mu < \Lambda_2$ has a gauge group SU(4) and (massless) SU(4)-nonsinglet fermion content consisting of $\psi^{ij}_{p,L}$ with $1 \le i, j \le 4$ and $1 \le p \le N_f$. However, this representation, $[2]_4$, in SU(4) is self-conjugate, i.e., $[2]_4 \approx [\bar{2}]_4$, so this theory is vectorial. The two-loop beta function for this theory has no IR zero and as μ continues to decrease, the SU(4) coupling inherited from the SU(5) theory continues to increase. Because of the vectorial nature of this descendent SU(4) theory, the condensate that forms is in the channel $[2]_4 \times [2]_4 \to 1$, with condensate

$$\sum_{i,j,k,\ell=1}^{4} \epsilon_{ijk\ell} \langle \psi^{ij\,T}_{p,L} C \psi^{k\ell}_{p',L} \rangle, \tag{6.15}$$

with $\Delta C_2 = 5$, where here, i, j, k, ℓ are SU(4) group indices. This condensate preserves the SU(4) gauge symmetry, while breaking global chiral symmetries spontaneously. The fermions involved in this condensate pick up dynamical masses of order the condensation scale. This theory confines and produces a spectrum of SU(4)-singlet bound state hadrons.

VII. $N = 6$ THEORY WITH SU(6) AND SU(2) GAUGE INTERACTIONS COMPARABLE IN STRENGTH

A. General discussion

In this section we consider the situation in which both the SU(6) and SU(2) gauge interactions are of comparable strength and hence must be treated together [with the U(1) gauge interaction still being weak]. In this case, one cannot neglect the mixing terms at the two-loop and higher-loop level in the beta functions β_{α_i}, Eq. (2.2), so the calculation the evolution of the gauge couplings down from the initial reference point $\mu = \mu_{\rm UV}$ in the UV is more complicated. For our present purposes, it will suffice to consider a case in which $\alpha_1(\mu) \simeq \alpha_2(\mu) \simeq O(1)$ at a lower scale μ. Since the SU(2) interaction by itself would evolve to a relatively weakly coupled IRFP if $N_f = 2$, expected to be in the non-Abelian Coulomb phase, we will assume $N_f = 1$ here, to guarantee that not just the SU(6) interaction, but also the SU(2) interaction become strongly coupled in the infrared.

B. Analysis of possible condensation channels

1. Condensation(s) involving SU(6)-nonsinglet fermions

We have shown above that the most attractive condensation channels are different in the simple situations where either the SU(6) or the SU(2) gauge interactions are dominant. Specifically, in the SU(2)-dominant case, the MAC is the $\omega\omega$ channel, with $\Delta C_2 = 4$, while in the SU(6)-dominant case, the MACs are the $\psi\psi$ and $\psi\chi$ channels, with the same measure of attractiveness, $\Delta C_2 = 14/3 = 4.7$. One would thus expect that as the reference scale decreases, the first condensate(s) to form would be in the $\psi\psi$ and/or $\psi\chi$ channels, as produced by the SU(6) gauge interaction. Since the $\psi\psi$ channels involves SU(2)-singlet fermions, it would not be affected by the fact that the SU(2) gauge interaction is also strongly coupled. The other SU(6) MAC, namely the $\psi\chi$ channel involves the SU(2)-singlet fermion ψ and the SU(2)-nonsinglet fermion χ, so the binding is only caused by the SU(6) interaction. Since the $\psi\chi$ condensation leaves the residual gauge symmetry group SU(5), of order 24, while the $\chi\chi$ condensation would leave the residual gauge symmetry (5.10), of order 21, a vacuum alignment argument suggests that the $\psi\chi$ condensation channel is preferred over the $\chi\chi$ channel. Thus, the $\psi\chi$ condensate (6.5) is expected to form at a scale that we will denote as Λ_1, self-breaking SU(6) to SU(5) and also producing induced dynamical breaking of U(1). The 11 gauge bosons in the coset SU(6)/SU(5) gain dynamical masses of order $g_1(\Lambda_1)\Lambda_1$, and the U(1) gauge boson gains a mass of order $g_3(\Lambda_1)\Lambda_1$.

2. EFT below Λ_1

Next, one would expect that condensation would occur in the $\omega\omega$ channel, as produced by the strong SU(2) gauge interaction. Owing to the fact that the value of ΔC_2 for this condensation is equal to 4, slightly less than the value of 4.7 for the $\psi\psi$ condensation, one expects that this occurs at a slightly lower scale. Because this second condensation would give dynamical masses to the ω fermions, which would thus be integrated out of the low-energy theory applicable below this condensation scale, it would preclude the formation of an SU(2)-induced condensate in the $\chi\omega$ channel.

There remains the $\chi\chi$ condensation channel. Although the value of ΔC_2 for this condensation, as produced by the SU(6) interaction, is $7/6$, which is a factor of 4 smaller than the value of $14/3$ for the MACs, and although the value of ΔC_2 for this condensation, as produced by the SU(2) interaction, is $3/2$, considerably smaller than the value $\Delta C_2 = 4$ for the SU(2)-induced MAC channel, $\omega\omega$, the $\chi\chi$ channel has the special property that it involves both the SU(2) and SU(6) gauge interactions, in contrast to all of the other possible condensation channels ($\psi\psi$, $\psi\chi$, $\omega\omega$, and $\chi\omega$), each of which only involves one of these two non-Abelian gauge interactions. If $\alpha_1(\mu) = \alpha_2(\mu)$ and one were simply to add the two terms $(7/6)\alpha_1(\mu) + (3/2)\alpha_2(\mu) = (8/3)\alpha_1(\mu)$, the effective ΔC_2 would be $8/3 = 3.7$, which is still less than values for the MACs for both the SU(6)-induced condensates and the SU(2)-induced condensates.

VIII. RELATED CONSTRUCTIONS

At the beginning of this paper we remarked on how the theory (1.4) with (1.6) successfully combines two different (anomaly-free) chiral gauge theories, SU(N) with N_f copies of (1.10), and SU(M) with N_f copies of (1.11), where $M = N - 4$. A natural question concerns related constructions of direct-product chiral gauge theories with fermions in higher-rank tensor representations of the factor groups. The next step up in complexity involves rank-3 antisymmetric and symmetric tensor representations for the fermions. Two theories with these rank-3 representations use a gauge group of the form

$$\mathrm{SU}(N) \otimes \mathrm{SU}(M) \otimes \mathrm{U}(1), \tag{8.1}$$

where now M can take on two different values as a function of N, namely $M = N - 3$ or $N = N - 6$. In both cases, the fermion content consists of N_f copies of the set [12,17]

$$([3]_N, 1)_{q_{30}} + ([\bar{2}]_N, (\bar{1})_M)_{q_{21}} + ([1]_N, (2)_M)_{q_{12}} + (1, (\bar{3})_M)_{q_{03}} \tag{8.2}$$

with

$$M = N - 3 \Rightarrow (q_{30}, q_{21}, q_{12}, q_{03}) = \\ = (-(N-3), (N-2), -(N-1), N) \tag{8.3}$$

and

$$M = N - 6 \Rightarrow (q_{30}, q_{21}, q_{12}, q_{03}) = \\ = (-(N-6), (N-4), -(N-2), N). \tag{8.4}$$

Owing to the presence of the factor group SU(M) in (8.1), the lowest nondegenerate cases are $N = 5$ if $M = N - 3$ and $N = 8$ if $M = N - 6$.

As before, there are equivalent theories. One has all of the representations of the (left-handed chiral) fermions conjugated. The second has the SU(M) representations conjugated relative to the SU(N) representations, i.e., it has a fermion content comprised of N_f copies of the set

$$([3]_N, 1)_{q_{30}} + ([\bar{2}]_N, (1)_M)_{q_{21}} + ([1]_N, (\bar{2})_M)_{q_{12}} + (1, (3)_M)_{q_{03}}. \tag{8.5}$$

Since these are equivalent to the theory with gauge group (8.1) and fermions (8.2), with the indicated U(1) charges for $M = N - 3$ and $M = N - 6$, it suffices to discuss only the latter theories.

However, none of these theories satisfies the requisite condition for our analysis, that both the SU(N) and SU(M) gauge interactions are asymptotically free (AF). The reason for this is as follows. In the theory (1.4) with (1.6), the one-loop term in the SU(N) and SU($N-4$) beta functions involves the trace invariants for the fundamental and symmetric or antisymmetric rank-2 representations. While $T([2]_N) = (N-2)/2$ and $T((2)_N)$ are linear functions of N and hence enter the one-loop coefficients in the beta functions with the same polynomial degree as the pure gauge contribution, $T([3]_N)$ and $T((3)_M)$ are quadratic functions of N and M, respectively, namely $T([3]_N = (N-3)(N-4)/4)$ and $T((3)_M) = (M+3)(M+4)/4$. The most stringent restriction arises from the constraint that the SU(M) beta function be negative. The one-loop coefficient in this beta function is

$$b_{1\ell;11}^{(\mathrm{SU}(M))} = \frac{1}{3}\left[11M - N_f\left\{\frac{N(N-1)}{2} + N(M+2) + \frac{(M+2)(M+3)}{2}\right\}\right]. \tag{8.6}$$

For the theory with $M = N - 3$, this is

$$b_{1\ell;11}^{(\mathrm{SU}(N-3))} = \frac{1}{3}[11(N-3) - 2N_f N(N-1)]. \tag{8.7}$$

This one-loop coefficient is negative if $N_f > N_{f,b1z,k3a}$, where

$$N_{f,b1z,k3a} = \frac{11(N-3)}{2N(N-1)}. \tag{8.8}$$

We find that $N_{f,b1z,k3a} < 1$ for all N in the relevant range $N \geq 5$. Hence, the AF constraint does not allow any nonzero value of N_f. Similarly, for the theory with $M = N - 6$,

$$b_{1\ell;11}^{(\mathrm{SU}(N-6))} = \frac{1}{3}[11(N-6) - 2N_f(N^2 - 4N + 3)]. \tag{8.9}$$

This one-loop coefficient is negative if $N_f > N_{f,b1z,k3b}$, where

$$N_{f,b1z,k3b} = \frac{11(N-6)}{2N(N^2 - 4N + 3)}. \tag{8.10}$$

The value of $N_{f,b1z,k3b}$ is less than 1 for all N in the relevant range, $N \geq 8$. Therefore, the AF constraint does not allow any nonzero value of N_f. We recall that N_f must be nonzero in order for the theory to be a chiral gauge theory, since if $N_f = 0$, then the theory degenerates into decoupled purely gluonic sectors. Thus, in neither of these theories with rank-3 fermion representations and $M = N - 3$ or $M = N - 6$ is the SU(M) gauge interaction asymptotically free. Similar comments apply to $\mathrm{SU}(N) \otimes \mathrm{SU}(M) \otimes \mathrm{U}(1)$ theories with fermions in sets of representations containing antisymmetric and symmetric rank-k tensor representations of the non-Abelian gauge groups with $k \geq 4$. As was discussed above, the requirement of asymptotic freedom of both of the non-Abelian gauge interactions was imposed because of (i) the purpose of studying the strong-coupling behavior of one or both of these interactions as the theory evolves from the UV to the IR and (ii) the necessity to be able to carry out a self-consistent perturbative calculation of the beta functions for these interactions at a reference scale, μ_{UV}.

IX. CONCLUSIONS

In nature, the $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ electroweak symmetry is broken not only by the vacuum expectation value of the Higgs field, but also dynamically, by the $\langle \bar{q}q \rangle$ quark condensates produced by the color $\mathrm{SU}(3)_c$ gauge interaction. Moreover, sequential self-breakings of strongly coupled chiral gauge symmetries have also been used in models of dynamical generation of fermion masses. In this paper we have investigated a chiral gauge theory that serves as a theoretical laboratory that exhibits both induced breaking of a weakly coupled gauge symmetry via condensates formed by a different, strongly coupled gauge interaction, and also self-breaking of strongly coupled chiral gauge symmetries. We have studied an asymptotically free chiral gauge theory with the direct-product gauge group $\mathrm{SU}(N) \otimes \mathrm{SU}(N-4) \otimes \mathrm{U}(1)$ and chiral fermion content consisting of N_f flavors of fermions transforming according to the representations $([2]_N, 1)_{N-4} + ([\bar{1}]_N, [\bar{1}]_{N-4})_{-(N-2)} + (1, (2)_{N-4})_N$. One of the reasons for interest in this theory is that it may be viewed as a combination of two separate (anomaly-free) chiral gauge theories, namely (i) an SU(N) theory with fermion content consisting of N_f flavors of fermions in the $[2]_N$ and $N - 4$ copies of $[\bar{1}]_N$, and (ii) an SU(M) theory with fermions consisting of N_f flavors of fermions in the $(2)_M$ and $M + 4$ copies of $[\bar{1}]_M$, with $M = N - 4$, which also incorporates a U(1) gauge symmetry. We have analyzed the UV to IR evolution of this theory and have investigated patterns of possible bilinear condensate formation. A detailed discussion of the lowest nondegenerate case, $N = 6$ was given. This analysis involved a sequential construction and analysis of low-energy effective field theories that describe the physics as the theory evolves through various condensation scales and certain fermions and gauge bosons pick up dynamically generated masses. Our findings provide new insights into the phenomenon of induced breaking of a weakly coupled gauge symmetry by a different, strongly coupled gauge interaction, and self-breaking of a strongly coupled chiral gauge symmetry.

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[33] In passing, we recall that a vacuum alignment argument explains a basic property of the dynamical breaking of G_{EW} to $\mathrm{U}(1)_{\mathrm{em}}$ by quark condensates. At the level of $\mathrm{SU}(3)_c$ by itself, the quark condensates $\langle \bar{u} d \rangle$ and $\langle \bar{u} s \rangle$ are of the same $\bar{3} \times 3 \to 1$ form as the $\langle \bar{u} u \rangle$, $\langle \bar{d} d \rangle$, and $\langle \bar{s} s \rangle$ condensates and have the same attractiveness measure, $\Delta C_2 = 8/3$. However, if $\langle \bar{u} d \rangle$ or $\langle \bar{u} s \rangle$ were nonzero, this would violate $\mathrm{U}(1)_{\mathrm{em}}$. The vacuum alignment precludes the $\langle \bar{u} d \rangle$ and $\langle \bar{u} s \rangle$ condensates and thus preserves $\mathrm{U}(1)_{\mathrm{em}}$ as the maximal residual gauge symmetry.