Radial and orbital Regge trajectories in heavy quarkonia

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(Received 11 March 2019; published 26 September 2019)

The spectra of heavy quarkonia are studied in two approaches: with the use of the Afonin-Pusenkov representation of the Regge trajectories for the squared excitation energy $E^2(nl)$ (ERT) and using the relativistic Hamiltonian with the universal interaction. The parameters of the ERTs are extracted from the experimental masses and in both cases, bottomonium and charmonium, the value of the radial slope $b_n(Q\bar{Q})$ appears to be ~40% larger than the orbital slope $b_l(Q\bar{Q})$: in particular in bottomonium the orbital slope $b_l(b\bar{b})=0.50(1)\,\mathrm{GeV}^2$, the radial slope $b_n(b\bar{b}) = 0.71(1)$ GeV^2 and in charmonium $b_l(c\bar{c})=0.76(2)\,\mathrm{GeV}^2$, $b_n(c\bar{c}) = 1.02(8)$ GeV^2; in both cases the slopes do not depend on the total momentum J, while the intercepts are different for the states with different J. For the resonances above the $D\bar{D}$ threshold, we predict $M(\chi_{c0}(nP))$ (in MeV) as 3415, 3862(8), 4196(12), 4475(15), and 4720(17); $M(\chi_{c1}(nP))$ (in MeV) as 3511, 3943(7), 4274(11), and 4553(15); and $M(\chi_{c2})(nP))$ (in MeV) as 3556, 3.928(4), 4223(6), and 4474(8) for $n_r = 0, 1, 2, 3$, which are in good agreement with the experimental masses, and for the n^3D_3 states the masses $M(\psi_3(n^3D_3))$ (in MeV) as 3857(8), 4197(11), 4479(13), and 4700(13) are predicted. In bottomonium above the $B\bar{B}$ threshold, the resonances $\Upsilon(3^3D_1)$ with the mass 10698(8) MeV and $\chi_{b1}(4^3P_1)$ with the mass 10758(3) MeV are obtained.

DOI: 10.1103/PhysRevD.100.054036

I. INTRODUCTION

In recent years, a large number of new resonances was observed in heavy quarkonia (HQ) [1–8], and among them, the resonances $X(4500)$ and $X(4700)$ with $J^{PC} = 0^{++}$ [6], being the highest excitations in the meson sector, are particularly interesting. The discovery of these resonances has stimulated new theoretical studies [8–16], and different conceptions about their nature have been presented, including diquark-antidiquark $cs\bar{c}\bar{s}$ types of tetraquarks [10,12,13,15,16]. However, even within the tetraquark $cs\bar{c}\bar{s}$ picture, different interpretations have been suggested. These resonances were also studied as conventional $c\bar{c}$ mesons [17–21], which implies that the $c\bar{c}$ component dominates in the wave function (w.f.) of a resonance but does not exclude that other components, like diquark-antidiquark or meson-meson, can also be present in the w.f. [17]. For decades, the spectra and other properties of HQ were studied in different potential models (PMs), both nonrelativistic and relativistic [22–31], which have allowed us to successfully describe low-lying HQ states. However, the masses of the high HQ excitations strongly depend on the $Q\bar{Q}$ interaction at large distances as well as on the heavy quark mass used, and the predicted values can differ by approximately (100–150) MeV (see the compilations in Refs. [11,27]). This happens because, using in PMs several fitting parameters, the first two or three excitations can be easily described with good accuracy, while the masses of the high excitations appear to be very sensitive to the behavior of the $Q\bar{Q}$ potential at large distances as well as to the quark masses and kinematics used. For example, in Ref. [21], in which the screened confining potential is taken, the resonance $X(4140)$ is considered as a candidate of $\chi_{c1}(3P)$, while with a similar potential in a nonrelativistic model, the mass $M_{c1}(3P)$, larger by approximately 140 MeV, is obtained [17], and this state is identified as $X(4274)$.

To study HQ spectra, it is useful to present the masses of high excitations via the radial Regge trajectories (RTs)—it allows us to distinguish between conventional $Q\bar{Q}$ mesons and exotic states. The parameters of RTs can be determined either in dynamical calculations [25–27,29–31] or from the

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analysis of the experimental masses [32,33], assuming that the HQ high excitations can be described by the linear radial RT, similar to that in light mesons. The linear radial RTs in light mesons were introduced in Ref. [34] and later developed in many studies [35–40],

$$M^2(nl) = M^2(l) + \mu^2 n, \qquad (n = n_r). \tag{1}$$

From the experimental data on light mesons, the radial slope, $\mu^2(q\bar{q}) \sim 1.25(10)$ GeV^2 [34,36], was extracted, while in Refs. [37,38], it was shown that μ^2 slightly depends on how the masses of the resonances are analyzed. If the width of the resonance is taken into account (using the so-called half-width rule [37,38]), then the radial slope, $\mu^2 = 1.34(4)$ GeV^2, is obtained as being approximately (10–15)% larger than in the case in which the width is neglected. This correction is not small in light mesons, since the resonances have large total widths.

Notice that in light mesons with $m_q = 0$ the radial and the angular momentum (orbital) slopes as well as the intercept are fully determined by the $q\bar{q}$ dynamics [40], while in the HQ, the situation changes, first of all, because the radial slope $\mu^2(Q\bar{Q})$, see Eq. (1), also depends on the heavy quark mass taken. Because of that circumstance, in the charmonium family, the radial slope, $\mu^2(c\bar{c}) \sim 3.0$ GeV^2, is very large, and its value varies in the range (2.8–3.4) GeV^2 in different models [25–27,41]. In bottomonium, $\mu^2(b\bar{b}) \sim (5–7)$ GeV^2 is larger and also depends on the static potential $V_0(r)$ and the b-quark mass m_b taken [25,30,31,41].

To escape the dependence of the radial RT on the heavy quark mass, Afonin and Pusenkov [41] have suggested another type of RT (henceforth denoted as ERT), defined for the squared excitation energy $E^2(nl)$, with $E(nl) = M(nl) - 2m_Q$:

$$(M_n(nl) - 2m)^2 = E^2(nl) = a(n + b). \tag{2}$$

Moreover, they also assumed that this ERT is universal and can be applied to all unflavored vector mesons, including $\rho(nS)$, $\phi(nS)$, $\psi(nS)$, and $\Upsilon(nS)$. The advantage of this ERT was discussed in Refs. [42,43], in which this new type of RT was studied and successfully applied to different mesons and also to baryons [44].

With the use of Eq. (2), the ground-state masses of all vector mesons were obtained in reasonable agrement with experiment [41–43]. However, if different masses of the s, c, and b quarks are taken, e.g., in Refs. [41] $m_s = 0.12(8)$ GeV, $m_c = 1.20(7)$ GeV, and $m_b = 4.32(6)$ GeV) and in Ref. [43] $m_s = 0.344(1)$ GeV, $m_c = 1.383(1)$ GeV, and $m_b = 4.561$ GeV, then from the fit to the experimental masses of the vector mesons, $\rho(nS)$, $\phi(nS)$, $\psi(nS)$, and $\Upsilon(nS)$, different values of the parameters a and b are obtained; namely, $a = 1.06(3)$ GeV^2 and $b = 0.63(7)$ GeV^2 in Ref. [41], while in Ref. [43], $a = 0.716(2)$ GeV^2 and $b = 0.153$ GeV^2 are much smaller. This illustrates that the choice of the HQ mass is of special importance for the ERTs.

Now, we rewrite Eq. (2), introducing the intercept b_0 and the radial slope b_n,

$$E^2(nl) = b_0(l) + b_n n, \tag{3}$$

i.e., $b_0 = ab$ and $b_n = a$. Then, the mass of the vector meson $M(n^3S_1; Q\bar{Q})$ with $l = 0$ is

$$M(n^3S_1) = 2m_Q + \sqrt{b_0 + b_n n}. \tag{4}$$

If the slope b_n and the intercept b_0 are assumed to be universal for all mesons [41], then the mass differences,

$$M(2^3S_1) - M(1^3S_1) = \sqrt{b_0 + b_n} - \sqrt{b_0};$$
$$M(3^3S_1) - M(2^3S_1) = \sqrt{b_0 + 2b_n} - \sqrt{b_0 + b_n}, \tag{5}$$

have to be equal in the heavy and the light vector mesons with a given radial quantum number $n = n_r$. However, this strong statement does not agree with the experimental mass differences, which can differ by approximately 100 MeV (see Table I), and therefore there is no strict universality of the ERT for the vector mesons.

In Table I, we give also the experimental values of the mass difference, $M(2^3P_J) - M(1^3P_J)$, for the axial vector ($J^{PC} = 1^{++}$) and the tensor ($J^{PC} = 2^{++}$) states, which will be used later in our analysis of the orbital ERT.

We also pay attention (see Table I) to the fact that the mass difference between the first excited state and the ground state does not change, if instead of the masses $M(n^3S_1)$ one takes the centroid masses $M_{\rm cog}(nS)$, i.e., it does not depend on spin effects. Also, in light mesons, this difference is approximately 100 MeV larger than in charmonium and bottomonium, in which it is small, equal to 26 MeV. Such close values of the mass differences could be partly explained by the existence of the universal potential, $V_0(r) = V_{\rm C}(r) + V_{\rm GE}$, which describes the low-lying states of all mesons with a good accuracy [22–25]; however, the use of the universal potential is not a sufficient condition to obtain equal slopes of ERTs for heavy and light mesons (see below).

TABLE I. The experimental mass differences (in MeV) in light mesons ($n\bar{n}$), charmonium, and bottomonium.

Δ	$n\bar{n}$	$c\bar{c}$	$b\bar{b}$
$M(2^3S_1) - M(1^3S_1)$	690(25)	589(2)	563(1)
$M_{\rm cog}(2S) - M_{\rm cog}(1S)$	700(20)	605(2)	567(1)
$M(3^3S_1) - M(2^3S_1)$	415(55)	353(1)	332(1)
$M(2^3P_1) - M(1^3P_1)$	425(56)	361(2)	363(1)
$M(2^3P_2) - M(1^3P_2)$	388(49)	371(3)	356(2)

Notice that in HQ the mass formula is simpler than in a light meson, in which it includes the self-energy and the string corrections [45,46], which are small and can be neglected in HQ [46]. However, the masses of heavy mesons, which have small sizes, strongly depend on the gluon-exchange (GE) potential used, and for them, the asymptotic freedom (AF) behavior of the strong coupling has to be taken into account, in contrast to light mesons, in which the GE potential can be approximated by the Coulomb potential with the effective coupling $\alpha_{\rm eff} = \text{const}$, while the AF behavior is important only for the $1S$, $2S$, and $1P$ states [40].

One of the main goals of the present paper is to show that the heavy quark mass can be taken not as a fitting parameter but extracted from experimental mass differences, if both the orbital and the radial ERT's in the (E^2, n) and (E^2, nl) planes are used. Also, we have in mind other goals: (i) to extract the radial slope $b_n(Q\bar{Q})$ and the orbital slope $b_l(Q\bar{Q})$ from experimental data and show that they are different, both in charmonium and bottomonium, and smaller than those in light mesons and (ii) to show that in charmonium and bottomonium the generalized (or the joint ERT [43]), $E^2(nl) = b_0 + b_l l + b_n n$, with unequal radial and orbital slopes can be introduced, in correspondence with the predictions in Refs. [39,41–43].

II. REGGE TRAJECTORIES IN THE (E^2, n) AND (E^2, l) PLANES IN BOTTOMONIUM

Bottomonium has the largest number of levels below the open flavor threshold and provides the unique possibility of extracting from experiment the ERT parameters in the (E^2, n) and (E^2, l) planes with high accuracy. For that, it is sufficient to use the mass differences [see Eq. (5)], $M(\Upsilon(2S)) - M(\Upsilon(1S)) = \sqrt{b_0 + b_n} - \sqrt{b_0} = 563.0(6)$ MeV and $M(\Upsilon(3S)) - M(\Upsilon(2S)) = \sqrt{b_0 + 2b_n} - \sqrt{b_0 + b_n} = 331.9(8)$ MeV, and also the definition of the ground-state mass, $M(\Upsilon(1S) = \sqrt{b_0} + 2m_b = 9460.3(3)$ MeV, to define the following values of the radial slope and the intercept of the radial ERT:

$$
\begin{aligned}
b_0(l=0) &= (0.131 \pm 0.001)\ \text{GeV}^2, \\
b_n(l=0) &= (0.724 \pm 0.002)\ \text{GeV}^2.
\end{aligned} \tag{6}
$$

Note that the radial slope $b_n(\Upsilon)$ appears to be almost two times smaller than that of the $\rho(nS)$ trajectory, $b_n(\rho(nS)) = \mu^2 \approx 1.43(5)$ GeV^2 [34,37,39], defined by the experimental masses $M(\rho(nS))$ [1].

Knowing the intercept b_0 and the definition of the ground-state mass,

$$
M(\Upsilon(1S)) = 2m_b + \sqrt{b_0} = 9.4603(3)\ \text{GeV}, \tag{7}
$$

one can extract the quark mass m_b,

TABLE II. The parameters b_0, b_n of ERT (in GeV^2) and the mass m_b (in GeV).

Paper	m_b	b_0	b_n
Ref. [41] set A	4.32(6)	0.63(7)	1.06(3)
Ref. [41] set B	4.59(5)	0	0.67(7)
Ref. [43]	4.561(1)	0.110(1)	0.716(2)
This paper	4.549(1)	0.131(1)	0.724(2)

$$
m_b = 4.549(1)\ \text{GeV}, \tag{8}
$$

the value of which just coincides with the one-loop pole mass, $m_b(1-\text{loop}) = 1.086\bar{m}_b = 4.550$ GeV, if the conventional current mass $\bar{m}_b = 4.18(1)$ GeV and the QCD constant $\Lambda_{\overline{MS}}(n_f = 5) = 200$ MeV (or $\alpha_s(\bar{m}_b) = 0.20$) [47–49] are adopted. In Table II, we compare the calculated parameters b_0, b_n, and m_b with those defined in Refs. [41,43].

It is of interest to notice that the mass m_b and the radial slope b_n, calculated here (see below), have values very close to those from Ref. [43], but the intercept is 20% larger.

Thus, the masses of the $\Upsilon(nS)$ family are defined by the radial ERT,

$$
\begin{aligned}
&E^2(n, l=0) = (0.131(1) + 0.724(2)n)\ \text{GeV}^2; \\
&M(\Upsilon(nS)) = (9.098(2) + \sqrt{0.131(1) + 0.724(2)n})\ \text{GeV},
\end{aligned} \tag{9}
$$

and given in Table III. The masses of the $\Upsilon(nS)$ and the nD states were also calculated, using the spinless Salpeter equation (SSE),

$$
\left(2\sqrt{\boldsymbol{p}^2 + \tilde{m}_b^2} + V_0(r)\right)\varphi_{nl}(r) = M_{\rm cog}(nl)\varphi_{nl}(r), \tag{10}
$$

where, however, the larger two-loop mass $\tilde{m}_b = 4.830(5)$ GeV is used, while the static potential $V_0(r) = \sigma r - \frac{4\alpha_{\rm V}(r)}{3r}$ is defined by the fundamental (not fitting) parameters, taken from Refs. [48,50], $\tilde{m}_b = 4.830$ GeV, $\sigma = 0.18$ GeV^2, $\Lambda_{\rm V}(n_f = 3) = 0.480$ GeV, and $M_{\rm B} = 2\pi\sigma \approx 1.15$ GeV. Therefore, the eigenvalues $M_{\rm cog}(nl)$ of Eq. (10) do not depend on any fitting parameters, while in the mass of $\Upsilon(nS)$, there is one extra parameter, the strong coupling $\alpha_{\rm hf}$, defining the hyperfine correction to the masses: $M(\Upsilon(nS)) = M_{\rm cog}(nS) + 1/4\delta_{\rm hf}$. The value of $\delta_{\rm hf}$ is defined with the universal coupling, $\alpha_{\rm hf} = 0.33(1)$ [46], which produces a small uncertainty, less than or approximately equal to 1 MeV in the mass value.

In our approach, the parameters of the ERT are chosen in such way that the calculated masses of the $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ exactly coincide with the experimental values. But the masses of the $\Upsilon(4S)$, $\Upsilon(5S)$, and $\Upsilon(6S)$ differ from the experimental masses, and this deviation characterizes their hadronic shifts, since in the one-channel approximation with the linear confining potential their ERT has to be linear. To estimate theoretical uncertainties, we will use χ^2, as it is defined in Ref. [38], which takes into account the uncertainties in the mass values and the widths of the resonances,

$$\chi^2 = \sum_n \left(\frac{M_n^2(\mathrm{th}) - M_n^2(\mathrm{exp})}{\Gamma_n M_n(\mathrm{exp})} \right)^2. \qquad (11)$$

From Table III, one can see that the masses of $\Upsilon(nS)$ with $n+1 = 1$, 2, 3, which lie below the $B\bar{B}$ threshold, are exactly equal to the experimental values, if they are defined by the ERT (the uncertainties are approximately (3–5) MeV). On the contrary, the experimental mass of $\Upsilon(4S)$, which lies above the $B\bar{B}$ threshold, is smaller by 37(4) MeV, as compared to the ERT prediction, and this shift may occur due to strong coupling to this threshold. An unusual situation happens with the masses of the $\Upsilon(5S)$, which is shifted up by 53(8) MeV, while the $\Upsilon(6S)$ is shifted down by approximately 50 MeV; i.e., this ERT can be considered an approximately linear ERT, defining higher resonances with accuracy approximately (30–50) MeV. The observed shifts may be also related to the influence of the nearby lying 4^3D_1 and 5^3D_1 resonances (see their masses in Table III and the calculations below) and to coupling to the nearby $B_s\bar{B}_s$ threshold [the threshold mass $M_{\mathrm{thres}} = 10831(1)$ MeV]. Thus, here, we face a many-channel problem, in which a shift of one resonance up and of another resonance down is possible [50]. Also, one can expect that in the region near 11 GeV the $\Upsilon(6S)$ and $\Upsilon(5D)$ resonances, which have large sizes, can be shifted down due to the flattening of the confining potential; calculations in Ref. [51] give these shifts, equal to approximately 40 MeV and 60 MeV for the 6S and 5D states, respectively.

TABLE III. The experimental masses $M(\Upsilon(nS))$ (in MeV) [1]; the masses $M(\Upsilon(nS))$, defined by the ERT, Eq. (9), with the parameters from Eqs. (6) and (8); and the solutions of Eq. (10), $M(\Upsilon(nS) = M_{\mathrm{cog}}(nS) + 1/4\delta_{\mathrm{hf}}(nS)$.

State	From ERT	$M_{\mathrm{cog}}(nS) + 1/4\delta_{\mathrm{hf}}$	Experiment
$\Upsilon(1S)$	9460(5)	9465(5)	9460.3(3)
$\Upsilon(2S)$	10023(5)	10017(3)	10023.3(3)
$\Upsilon(3S)$	10355(4)	10359(2)	10355.2(5)
$\Upsilon(4S)$	10616(4)	10635(2)	10579.4(1.2)
$\Upsilon(5S)$	10838(4)	10884 (2)	10891 (4)
$\Upsilon(6S)$	11035(5)	11093(2)	10987^{+11}_{-3}
$\Upsilon(1D)$	10161(10)	10141 (4)	absent
$\Upsilon(2D)$	10454(8)	10440((4)	absent
$\Upsilon(3D)$	10693(8)	10701(4)	absent
$\Upsilon(4D)$	10901(2)	10933(6)	absent
$\Upsilon(5D)$	11090 (3)	11100 (7)	absent

Thus, in bottomonium and light mesons, the RTs have different features. First, the radial slope of the $\rho(nS)$ RT, approximately $1.43(5)$ GeV^2, is two times larger than the $b_n(\Upsilon(nS)) = 0.724$ GeV^2. Secondly, in bottomonum, its high excitations have characteristic hadronic shifts, about 50 MeV, which has the same order as their widths, and because of that, the ERT reproduces their masses with the accuracy of approximately (30–50) MeV, and therefore the large value of $\chi^2/\mathrm{d.o.f.} = 3.41$ is obtained in bottomonium. In light mesons, which have a lot of open channels and large total widths, approximately (150–200) MeV, the hadronic shifts of the resonances are reproduced well by flattened confining potential, and specifically large shifts are not observed; therefore, in light mesons, their masses are described well by the linear RT with small $\chi^2/\mathrm{d.o.f.}$ [38]. This means that the $\rho(nS)$ mesons have no strong coupling to a specific channel. Later, for the $\psi(nS)$ trajectory, a small $\chi^2/\mathrm{d.o.f.} \sim (0.1\text{–}0.5)$ will be also obtained.

A. Radial ERT of $\chi_{bJ}(nP)$ mesons

At present, there exist a lot of data about the $\chi_{bJ}(nP)$ mesons, and recently, the masses of the high states $\chi_{b1}(3P)$ and $\chi_{b2}(3P)$ were determined [1,52–54]. From the known masses, the radial slope and the intercept of the ERT, describing $\chi_{bJ}(nP)$ mesons, can be extracted,

$$E^2(nJ, l=1) = B(J) + b_n(J)n,$$
$$M(n^3P_J) = 2m_b + \sqrt{B(J) + b_n(J)n}, \qquad (12)$$

where we introduced the factor $B(J) = b_0(J) + b_l(J)$, playing the role of the modified intercept. In general, the parameters $B(J)$ and $b_n(J)$ can depend on the angular momentum J, while the quark mass, $m_b = 4.549(1)$ GeV, is taken from the analysis of the $\Upsilon(nS)$ states, Eq. (8). Using the experimental values of the masses, one can extract both the parameter $B(J)$ and the radial slope $b_n(J)$, for $\mathrm{J} = 0, 1\ 2$. It is of interest that the calculations give equal values of the radial slope $b_n(J, l=1)$ for $J = 0$, 1, 2, i.e., $b_n(J, l=1) = b_n(l=1)$ does not depend on the angular momentum J,

$$b_n(l=1) = 0.708(1)\ \mathrm{GeV}^2, \qquad (J = 0, 1, 2), \qquad (13)$$

while the modified intercept $B(J, l=1)$ depends on J,

$$B(J=0, l=1) = 0.579(1)\ \mathrm{GeV}^2;$$
$$B(J=l=1) = 0.632(2)\ \mathrm{GeV}^2;$$
$$B(J=2, l=1) = 0.663(2)\ \mathrm{GeV}^2, \qquad (14)$$

as it should to provide correct values of the ground-state masses. Notice that for the $\chi_{bJ}(nP)$ mesons the radial slope, $b_n(l=1) = 0.708(1)$ GeV^2, is only 2% smaller

TABLE IV. The masses of the $\chi_{bJ}(nP)$ mesons (in GeV), defined by the ERT (15), and the experimental data.

State	ERT, this paper	Experiment
$\chi_{b0}(1P)$	9.859 (2)	9.8594 (7)
$\chi_{b0}(2P)$	10.232(3)	10.2325(9)
$\chi_{b0}(3P)$	10.510(3)	Absent
$\chi_{b0}(4P)$	10.742(3)	Absent
$\chi b1(1P)$	9.893(3)	9.8928(6)
$\chi_{b1}(2P)$	10.256(2)	10.2555(7)
$chi_{b1}(3P)$	10.529(3)	10.5134(7)
$\chi_{b1}(4P)$	10.758(3)	Absent
$\chi_{b2}(1P)$	9.912(2)	9.9122(6)
$\chi_{b2}(2P)$	10.269(3)	10.2687(7)
$\chi_{b2}(3P)$	10.540(3)	10.5240(8)
$\chi_{b2}(4P)$	10.758(3)	Absent

than $b_n(l=0) = 0.724\ \mathrm{GeV}^2$ of the $\Upsilon(nS)$ ERT. The parameter $B(J, l=1) = b_0(J, l=1) + b_l(J)$ includes the orbital slope $b_l(J)$, which will be defined below.

The masses, defined by the ERT with the use of the calculated $B(J, l=1)$, $b_n(l=1)$ and $m_b = 4.549$ GeV, exactly coincide with the experimental masses of the $1P$ and $2P$ states and are only approximately 10 MeV larger for the $3P$ states, in which experimental data exist now for the states with $J = 1$, 2 (see Table IV). For the $\chi_{bJ}(4P)$ states with $J = 0$, 1, 2, we predict their masses in the region approximately (10.74–10.76) GeV. Notice that *a priori* it was not evident that in bottomonium the radial slope $b_n(l=1)$ does not depend on J, and owing to that, the $\chi_{bJ}(nP)$ ERT can be written as

$$E^2(J, l=1)(\text{in GeV}^2) = B(J) + 0.708(1)n, \quad (l=1), \tag{15}$$

with the values of $B(J, l=1)$ from Eq. (14).

In the $\chi_{bJ}(nP)$ family, the $3P$ states lie below, but very close to, the $B\bar{B}$ threshold, and one cannot exclude that these states are affected by this threshold and shifted down. From the comparison of the experimental and calculated masses, one can see that the masses of the $\chi_{bJ}(3P)$ with $J = 1$, 2 are only 15 MeV larger; therefore, possible hadronic shifts could be approximately 15 MeV. For the $\chi_{bJ}(4P)$ states, which are still not observed, we predict the masses in the region around 10.75 GeV, where as yet no states have been observed either [55].

B. Generalized ERT for the $\chi_{bJ}(nP)$ and $\Upsilon(nD)$ states

In the previous section, we have defined the radial slopes of the $\Upsilon(nS)$ and $\chi_{bJ}(nP)$ trajectories, $b_n(\Upsilon(nS)) = 0.724(2)\ \mathrm{GeV}^2$ and $b_n(\chi_{bJ}(nP) = 0.708(2)\ \mathrm{GeV}^2$, which differ by only 2%, and it allows us to introduce the generalized (the *joint*) ERT,

$$E^2(J, l) = b_0(J) + b_l l + b_n n \quad (l \neq 0), \tag{16}$$

which was suggested in Refs. [41,56] for light mesons with equal radial and orbital slopes, while the trajectories with different b_n and b_l were discussed in Refs. [36–38], in which this representation of the ERT was called the joint RT.

First, we consider the leading ERT, which describes the ground states with $S = 1$, $J = l + 1$, ($n = 0$) and in which the intercept $b_0 = 0.131\ \mathrm{GeV}^2$ is known from the J/ψ mass:

$$E^2(n=0, J=l+1)(\text{in GeV}^2) = 0.131 + b_l l. \tag{17}$$

Then from the mass value, $M(\chi_{b2}(1P)) = 2m_b + \sqrt{0.131 + b_l(n=0)} = 9.912(1)$ GeV, the orbital slope $b_l(n=0) = 0.532(1)\ \mathrm{GeV}^2$ is extracted, and from Eq. (17), the masses of the ground states with $J = l + 1$ (in GeV), 9.460, 9.912, and 10.191, are obtained. Here, the masses of the J/ψ and $\chi_{b2}(1P)$ exactly coincide with the experimental values, while $M(\Upsilon(1^3D_3)) = 10.191(1)$ GeV is significantly larger than the known mass of another member of the $1D$ multiplet, $M(\Upsilon(1^3D_2) = 10.164(1)$ GeV; this indicates that the chosen orbital slope $b_l(n=0) = 0.532\ \mathrm{GeV}^2$ is too large. Therefore, to extract the orbital slope, we will use all known masses with $l \neq 0$: of $\Upsilon_2(1D)$, $\chi_{bJ}(1P)$, and $\chi_{bJ}(2P)$, where the orbital slope enters via the factor $B(J, l=1) = b_0(J) + b_l$, given in Eq. (14). Then, the fit with a smaller χ^2/d.o.f. gives

$$b_l(l) = 0.50(1)\ \mathrm{GeV}^2, \tag{18}$$

which does not depend on J. Then, from the factor $B(J, l=1) = b_0(J) + b_l$ in Eq. (14), different intercepts $b_0(J)$ of the $\chi_{cJ}(nP)$ trajectories,

$$\begin{aligned} b_0(J=0, l=1) &= 0.079(4)\ \mathrm{GeV}^2, \\ b_0(J=l=1) &= 0.132(4)\ \mathrm{GeV}^2, \\ b_0(J=2, l=1) &= 0.163(1)\ \mathrm{GeV}^2, \end{aligned} \tag{19}$$

are found. Thus, in the $\chi_{cJ}(nP)$ ERT, neither the radial nor the orbital slopes depend on J, and one can introduce the generalized ERT,

$$E^2(J)(\text{in GeV}^2) = b_0(J) + 0.50(1)l + 0.708(1)n, \quad (l \neq 0), \tag{20}$$

which also defines the masses of the n^3D_3 states, where the intercept $b_0(J=3, l=2) = 0.145(2)\ \mathrm{GeV}^2$. The calculated masses of the $M(n^3D_3)$ states are given in Table V, together with the solutions of the SSE, Eq. (10), including the fine-structure corrections.

The mass of the 1^3D_3 state, shown in Table V, is close to that of the 1^3D_2 state, $M(1^3D_2) = 10.163(4)$ GeV [1], as it

TABLE V. The masses of the $\Upsilon(n^3D_3)$ (in MeV), calculated from the ERT, Eq. (20), and the solutions of the SSE, Eq. (10).

State	ERT	Solutions of SSE
$\Upsilon(1^3D_3)$	10169(10)	10145
$\Upsilon(2^3D_3)$	10459(8)	10449
$\Upsilon(3^3D_3)$	10698(8)	10710
$\Upsilon(4^3D_3)$	10906(6)	10938
$\Upsilon(5^3D_3)$	11092(5)	11142

should be, if the fine-structure splitting of the $1D$ multiplet is small. It also supports the choice of the orbital $b_l = 0.50(1)$ for the states with $l = 2$. Thus, our analysis of the ERT has shown, first, that in bottomonium the radial slope is approximately 40% larger than the orbital slope and, second, that the intercept of the ERT with $l \neq 0$ depends on J.

III. REGGE TRAJECTORIES IN CHARMONIUM

In charmonium, only three multiplets, $1S$, $2S$, and $1P$, lie below the $D\bar{D}$ threshold, and the available experimental data do not allow us to extract the exact value of the c-quark mass together with the parameters of the ERT. Also, in charmonium, some resonances are shifted, down or up, and the parameters, extracted from the masses of these states, have large uncertainties. Nevertheless, from the masses of J/ψ and $\psi(2S)$, known with an accuracy better than 1 MeV, the radial slope and the intercept of the $\psi(nS)$-trajectory can be extracted,

$$E^2(l=0, c\bar{c}) = b_0(l=0) + b_n(l=0)n,$$
$$M(nS) = 2m_c + \sqrt{b_0(l=0) + b_n(l=0)n}, \qquad (21)$$

if the c-quark mass is fixed. As seen from Eq. (21), the ERT parameters depend on the c-quark mass and varying m_c in the range (1.2–1.4) GeV, the best agreement with the ground-state masses is reached for m_c in the range $m_c = (1.22\text{–}1.28)$ GeV. Here, we take $m_c = 1.24$ GeV, which coincides with the current c-quark mass, $\bar{m}_c(\bar{m}_c) = 1.26(6)$ GeV [47,49]. Then, the intercept of the $\psi(nS)$ trajectory is defined by the mass of J/ψ,

$$b_0(l=0) = (M(J/\psi) - 2.48\ \text{GeV})^2 = 0.381(1)\ \text{GeV}^2. \qquad (22)$$

Using this intercept and $M(\psi(2S)) = 2m_b + \sqrt{b_0(l=0) + b_n(l=0)} = 3.6861(1)$ GeV, one obtains the radial slope,

$$b_n(l=0) = 1.074\ \text{GeV}^2, \qquad (m_c = 1.24\ \text{GeV}), \qquad (23)$$

and the $\psi(nS)$ trajectory,

$$E^2(nS)(\text{in GeV}^2) = 0.381(1) + 1.074n. \qquad (24)$$

These parameters can be compared with those from Ref. [43], in which the radial slope, $b_n = 0.716(2)\text{GeV}^2$, significantly smaller than in our case, is taken; also in Ref. [43], the intercept, $b_0 = 0.110(1)$ GeV^2, is smaller, while the mass, $m_c = 1.383(1)$ GeV, is larger than $m_c = 1.24$ GeV used here. Nevertheless, with the smaller ERT parameters, the masses of J/ψ and $\psi(2S)$ were obtained in full agreement with experiment, and a difference, approximately (70–100) MeV, with our predictions occurs only for the masses of the higher states $\psi(3S)$ and $\psi(4S)$. This happens because the smaller values of the ERT parameters correspond to the larger mass m_c taken. It also shows how important the choice of the quark mass is.

The masses of all $\psi(nS)$ mesons are given in Table VI. For J/ψ and $\psi(2S)$, their values exactly coincide with the experimental values, while $M(\psi(3S)) = 4.070(1)$ GeV is 30 MeV larger and $M(\psi(4S) = 4.378$ MeV is 43 MeV smaller, if the 4^3S_1 state is identified with the $\psi(4415)$. In this case, for the $\psi(nS)$ trajectory, $\chi^2/\text{d.o.f.} = 0.50$ is

TABLE VI. The masses of $\psi(nS)$, $\chi_{cJ}(nP)$, $\psi(n^3D_1)$, and $\psi_3(nD)$ (in mega-electron-volts), defined by the ERT, Eqs. (24), (25), (26), and (31), and the solutions of the SSE.

State	SSE Eq. (10)	ERT	Experiment
J/ψ	3100	3097	3097
$\psi(2S)$	3685	3686	3686
$\psi(3S)$	4100	4070	4039(1)
$\psi(4S)$	4455	4378(1)	4346(6) or 4421(4)
$\psi(5S)$	4760	4643(1)	4643(9)
$\psi(6S)$	5043	4878(1)	Absent
$\chi_{c1}(1P)$	3502	3511(1)	3.510.7(1)
$\chi_{c1}(2P)$	3949	3943(7)	3871.7(2)
$\chi_{c1}(3P)$	4319	4274(11)	4274(8)
$\chi_{c1}(4P)$	4642	4553(15)	Absent
$\chi_{c1}(5P)$	4933	4798(18)	Absent
$\chi_{c0}(1P)$	3435	3415(1)	3414.8(3)
$\chi_{c0}(2P)$	3929	3862(8)	3862^{+82}_{65}
$\chi_{c0}(3P)$	4289	4196(12)	Absent
$\chi_{c0}(4P)$	4622	4475(15)	4506(25)
$\chi_{c0}(5P)$	4920	4720(18)	4704^{+24}_{-38}
$\chi_{c2}(1P)$	3535	3556(1)	3556.2(1)
$\chi_{c2}(2P)$	3970	3928(4)	3927(3)
$\chi_{c2}(3P)$	4335	4223(6)	Absent
$\chi_{c2}(4P)$	4665	4474(8)	Absent
$\psi(1^3D_1)$	3802	3814(26)	3773.1(4)
$\psi(2^3D_1)$	4.190	4162(24)	4191(5)
$\psi(3^3D_1)$	4.533	4527(10)	Absent
$\psi(4^3D_1)$	4.830	4.700(13)	Absent
$\psi_3(1D)$	3.822(2)	3.857(8)	3.843(1)
$\psi_3(2D)$	4.170(2)	4.197(11)	Absent
$\psi_3(3D)$	4.445(2)	4.479(13)	Absent
$\psi_3(4D)$	4.840(2)	4.730(14)	Absent

obtained. If the $\psi(4^3S_1)$ states are identified with $\psi(4360)$, then the smaller value $\chi^2/\mathrm{d.o.f.} = 0.13$ is found. The hadronic shifts of the states $\psi(4040)$ and $\psi(4415)$ can be explained by a possible $S-D$ mixing of the $3S-2D$ and $4S-3D$ resonances via open channels [28].

Notice that in charmonium the radial slope $b_n(n{=}0, c\bar{c}) = 1.074\ \mathrm{GeV}^2$ is larger than the one in bottomonium but smaller than the radial slope of the $\rho(nS)$ trajectory in light mesons, $\beta_n(n\bar{n}, l=0) = 1.43(5)\ \mathrm{GeV}^2$ [36,37,40]; i.e., the radial slope of the ERT depends on the quark flavor.

A. ERT of the $\chi_{cJ}(nP)$ mesons

For the $\chi_{cJ}(nP)$ states, the factors $B(J, l=1)$ can be extracted from the experimental data on the masses of the ground states, known with an accuracy better than 1 MeV: by definition the ground state mass is $M(\chi_{cJ}(1P)) = \sqrt{B(J, l=1)}$, where $B(J, l=1) = b_0(J) + b_l(J)$ plays the role of a modified intercept, which have the values

$$\begin{aligned} B(J=0)) &= 0.874\ \mathrm{GeV}^2, \\ B(J=1) &= 1.063\ \mathrm{GeV}^2, \\ B(J=2) &= 1.158\ \mathrm{GeV}^2, \quad (l=1), \end{aligned} \tag{25}$$

where the difference of approximately (15–30)% corresponds to the differences between the masses of $\chi_{cJ}(1P)$. Knowing $B(J)$ and the masses of higher resonances, one can define the radial slope of the $\chi_{cJ}(nP)$ trajectories; for that, we use the definition

$$M(\chi_{cJ}(nP)) = 2m_c + \sqrt{B(J, l=1) + b_n(J)n} \tag{26}$$

and take for the $\chi_{c0}(nP)$ trajectory $M(\chi_{c0}(2P)) = 3.862^{+66}_{-43}$ GeV, of the recently observed resonance $\chi_{c0}(3863)$ [57]. For the $M(\chi_{c1}(nP))$ trajectory, we use the mass $M(\chi_{c1}(3P)) = 4.274(1)$ GeV [1], considering the $\chi_{c1}(4274)$ as the 3^3P_1 state, and for the $\chi_{c2}(nP)$ trajectory, we take the mass of the $\chi_{c2}(3939) = 3.927(3)$ MeV [1]. Then, the extracted $b_n(J)$,

$$\begin{aligned} b_n(J=0, l=1) &= 1.036^{+15}_{-13} = 1.04(2)\ \mathrm{GeV}^2, \\ b_n(J=l=1) &= 1.078(14)\ \mathrm{GeV}^2, \\ b_n(J=2, l=1) &= 0.94(1)\ \mathrm{GeV}^2, \end{aligned} \tag{27}$$

give their average,

$$\bar{b}_n(l=1) = 1.02(8)\ \mathrm{GeV}^2, \tag{28}$$

with large uncertainty, and therefore the masses of all nP states, given in Table VI, are calculated with the use of the parameters from (27) and (25). Notice that the averaged radial slope is smaller than that of the $\psi(nS)$ ERT, Eq. (24), by only 5%. From the masses, given in Table VI, one can see that the mass of the 2^3P_0 state coincides with that of $\chi_{c0}(3862)$, while the mass $M(\chi_{c1}(2P))) = 3.943(7)$ GeV is larger than the mass of the $\chi_{c1}(3872)$ resonance, probably because of its large hadronic shift, which can be estimated to be approximately 60 MeV. Also, the calculated mass $M(\chi_{cJ}(2P) = 3.928(4)$ GeV is obtained in exact agreement with the mass of $\chi_{c2}(3930)$ (see Table VI).

The parameters of the ERT for the $\psi(nD)$ resonances can be determined in the same way as for the $\chi_{cJ}(nP)$ trajectories, i.e., introducing the factor $B(J, l=2) = b_0(J) + 2b_l(J, l=2)$ and using the ground-state masses, $M(1^3D_1) = 3.773.1(4)$ GeV, $M(1^3D_2) = 3.822(1)$ GeV [1], and $M(1^3D_3) = 3.843(1)$ GeV of the resonance, recently observed by the LHCb [58]. This procedure gives

$$\begin{aligned} B(J=1, l=2) &= 1.672(1)\ \mathrm{GeV}^2, \\ B(J=2, l=2) &= 1.810(1)\ \mathrm{GeV}^2, \\ B(J=3, l=2) &= 1.858(1)\ \mathrm{GeV}^2, \\ \bar{B}(l=2) &= 1.78(7)\ \mathrm{GeV}^2. \end{aligned} \tag{29}$$

The averaged value $\bar{B}(l=2) = 1.78(7)\ \mathrm{GeV}^2$ has a large uncertainty and will be used here only for the $\psi(n^3D_1)$ trajectory, in which it gives the best fit. For the n^3D_2 and n^3D_3 states, a better fit is obtained using the quantities $B(J=l=2)$ and $B(J=3, l=2)$ from Eq. (29). Also, the masses of the radial nD excitations are calculated by taking the averaged radial slope: $b_n(l=2) = 1/2(b_n(l=0) + b_n(l=1)) = 1.05(1)\ \mathrm{GeV}^2$. The use of the averaged slopes allows us to see the physical picture as a whole, but does not always give the best fit.

The calculated mass, $M(1^3D_1) = 3.814(26)$ MeV (see Table V), is larger than that of the $\psi(3770)$, and this result indicates that its possible hadronic shift down is approximately 40 MeV. On the contrary, the calculated mass $M(\psi(2^3D_1)) = 4.162(24)$ GeV has large uncertainty and is smaller than the experimental value, $M(\psi(4160) = 4.191(5)$ GeV. In this case, the hadronic shift up is possible due to the $3S-2D$ mixing via the open channels, $D^*\bar{D}^*$ and $D_s^{*+}D_s^{*-}$ [50]. We can conclude that the radial slope of the $\psi(n^3D_1)$ trajectory with the $\bar{B}$ and $b_n(l=2) = 1.05(1)\ \mathrm{GeV}^2$ provides a rather good description of the $\psi(nD$ states (see Table VI).

To extract the orbital slope, one can use the ground-state masses of J/ψ, $\chi_{c2}(3556)$, and $\psi_3(1^3D_3)$, the mass of which, $\psi_3(1D) = 3.843(3)$ GeV, was recently measured [58]. All these states with spin $S=1$ and $J=l+1$ determine the leading trajectory of the charmonium family. Therefore, the intercept of this ERT is known from the J/ψ mass, $b_0{=}0.381\,\mathrm{GeV}^2$, and then from $B(J{=}2,l{=}1){=}b_0(l{=}0){+}b_l(l{=}1,n{=}0){=}1.158\,\mathrm{GeV}^2$, one finds $b_l(l=1, n=0) = 0.777(1)\ \mathrm{GeV}^2$. On the other hand, the value $b_l(l=2, n=0) = 0.738(4)\ \mathrm{GeV}^2$ is extracted, if the factor $B(l=2, n=0) = b_0 + 2\bar{b}_l(l=2) = 1.858(8)\ \mathrm{GeV}^2$

is used. Taking the averaged value, $\bar{b}_l = 0.758(5)$, the leading ERT of the charmonium family is

$$E^2(J = l + 1, n = 0)(\text{in GeV}^2) = 0.381 + 0.758(5)l. \tag{30}$$

This ERT gives the following masses of the ground states (in GeV): 3.097 (1^3S_1), 3.547(5) (1^3P_2), and 3.857(7) (1^3D_3), where the latter value is larger than the mass of $\psi_3(3843)$) by (16 ± 7) MeV. For the 1^3F_4 state, the mass 4.109(19) GeV is predicted.

For a next step, one can introduce the generalized ERT for the states with $J = l + 1$,

$$E^2(J = l + 1)(\text{in GeV}^2) = 0.381 + 0.758(5)l + 1.05(2)n, \tag{31}$$

and in Table VI, the masses of $\psi_3(n^3D_3)$, corresponding to this ERT, are given, together with the solutions of the SSE (including the fine-structure corrections).

From Table VI, one can see that the calculated masses of the high excitations with $J^{PC} = 0^{++}$, $M(\chi_{c0}(3P)) =$ 4493(15) MeV and $M(\chi_{c0}(4P)) = 4741(18)$ MeV are close to those of the $X(4500)$ and the $X(4700)$ resonances with $J^{PC} = 0^{++}$, observed by the LHCb Collaboration [6]. This coincidence can be considered as an indication that these resonances could have a large $c\bar{c}$ component.

In Table VI, the masses, defined as the solutions of the SSE plus spin-dependent corrections, are also given. In the SSE, the confining potential was taken as linear at all distances; i.e., the flattening effect [51] at large distances was neglected. For that reason, the higher nl resonances with $n = 3$, 4, determined by the SSE, appear to be (100–150) MeV larger than those defined by the ERTs, which evidently take into account the flattening effect.

It is also worth underlining that the physical picture, presented by the ERTs of HQ, clearly shows that the ERT parameters depend on the quark flavor and the quark mass chosen and also weakly depend on the angular momentum J. Also, in charmonium, the orbital slope is significantly smaller than that in light mesons, while the radial slope is smaller only by (10–15)%. In light mesons, the generalized ERT,

$$M^2(nl, n\bar{n})(\text{in GeV}^2) = 0.60 + 1.15(7)n + 1.10(5)l, \quad (l \neq 0, m_q = 0), \tag{32}$$

was derived in Ref. [40]. Notice that in charmonium the averaged radial slope of the ERT, $\bar{b}_n(c\bar{c}) = 1.02(8)$ GeV^2, is about three times smaller than the slope $\mu^2 \sim$ (2.8–3.5) GeV^2 of the conventional radial RT (1).

IV. CONCLUSIONS

In our study of HQ, we have used the Afonin-Pusenkov conception [41] about the RT, defined through the excitation energy, $E(n, l) = M(nl) - 2m_Q$, which later was developed in Refs. [42–44]. Our analysis of the bottomonium and charmonium spectra, performed with the use of ERT and also of the relativistic Hamiltonian with the universal interaction, has shown the following:

(1) The orbital and the radial slopes of the ERTs depend on the flavor and the mass m_Q of the heavy quark.
(2) The mass m_Q can be extracted from experiment, if the masses of the ground and two excited states with the same quantum numbers are known with great accuracy. Such data exist in bottomonium.
(3) The values of the radial and orbital slopes differ by approximately 40% both in charmonium and bottomonium, and therefore their generalized ERTs are not universal.
(4) In bottomonium, the values of the radial and the orbital slopes are significantly smaller, by approximately 50%, than in charmonium, which meanwhile are smaller that those in light mesons.
(5) The intercepts of the ERTs are determined by the masses of the ground states.
(6) In bottomonium, the orbital slope $b_l = 0.50(1)$ GeV^2 and radial slope $b_n = 0.708(2)$ GeV^2 do not depend on the angular momentum J, if $l \neq 0$, while the intercept depends on J.

The parameters of the ERTs and m_Q, extracted from the experimental masses, are collected in Table VII, together with those of the $\rho(^3S_1)$ and $\rho(^3D_1)$ trajectories [40]. From the parameters, given in Table VII, one can see

TABLE VII. The parameters of the generalized ERT (in giga-electron-volts squared) in bottomonium, charmonium, and light vector mesons.

Meson	Quark mass (GeV)	Intercept a	Orbital slope b_l	Radial slope b_n
Bottomonium	4.549(1)	0.131(1)	0.50(1)	0.724(1) ($l = 0$)
Bottomonium	4.549(1)	0.076,0.131,0.166 ($J = 0$, 1, 2)	0.50(1)	0.708(2) ($l \neq 0$)
$\psi(^3S_1)$	1.24	0.381(1)	0	1.074(1)
$\chi_{c2}(nP)$	1.24	0.38(8)	0.76(2)	0.94(1)
$\chi_{c1}(nP)$	1.24	0.364(4)	0.76(2)	1.078(14)
$\psi(n^3D_3)$	1.24	0.381	0.758(5)	1.05(2)
$\rho(^3S_1)$	0	0.60	0	1.45 (5)
$\rho(^3D_1)$	0	0.60	1.10(5)	1.15 (7)

how the intercept and the orbital and the radial slopes are increasing with a decreasing quark mass.

In bottomonium, the resonances $\chi_{b1}(4P)$ with the mass 10758(3) MeV and $\Upsilon(3^3D_1)$ with the mass 10698(8) MeV are predicted, while in charmonium, the masses of the resonances $\chi_{c0}((n+1)P)$ with $J^{PC}=0^{++}$ and $n=3$, 4, equal to 4475(15) MeV and 4720(18) MeV, respectively, are obtained. These mass values are very close to those of the $X(4500)$ and $X(4700)$ resonances [6], and this result can be considered an indication that $X(4500)$ and $X(4700)$ have a large $c\bar{c}$ component in their wave function. Also, in charmonium, the masses of the $\chi_{c0}(2^3P_0)$, $\chi_{c2}(2^3P_2)$, $\chi_{c1}(3^3P_1)$, and 1^3D_2, 1^3D_3 states are in good agreement with the experimental data on $\chi_{c0}(3863)$, $\chi_{c2}(3930)$, $\chi_{c1}(4274)$, $\psi_2(3822)$, and $\psi_3(3843)$.

ACKNOWLEDGMENTS

A. M. B. is very grateful to Yu. A. Simonov for discussions.

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