

Entropy bounds on effective field theory from rotating dyonic black holes

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We derive new bounds on higher-dimension operator coefficients in four-dimensional Einstein-Maxwell theory. Positivity of classically generated corrections to the Wald entropy of thermodynamically stable, rotating dyonic black holes implies a multiparameter family of field basis invariant inequalities that exhibit electromagnetic duality and are satisfied by examples from field and string theory. These bounds imply that effective operators modify the extremality condition of large black holes so as to permit their decay to smaller ones, thus satisfying the weak gravity conjecture.

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I. INTRODUCTION

The swampland program [1–3] is founded on the intriguing possibility that an infrared (IR) consistent effective field theory (EFT) may nevertheless be incompatible with quantum gravitational ultraviolet (UV) completion. An archetype of this approach is the weak gravity conjecture (WGC) [3], which states that any Abelian gauge theory coupled consistently to gravity must contain a state whose charge exceeds its mass in Planck units. Other notable results from the swampland program include the distance conjecture [2], and anti-de Sitter (AdS) [4] and dS [5,6] conjectures.

The very notion of the swampland is predicated on a complete knowledge of the space of IR consistent EFTs. After all, if a putative swampland theory is also pathological from IR considerations, then UV completion has little to do with its sickness. For example, gauge anomalies are absent in grand unified theories, but they are also ruled out by low-energy reasoning. In parallel with the swampland program, there has emerged a bottom-up approach to constraining EFTs using IR properties like unitarity, causality, and analyticity of scattering amplitudes [7]. These tools can be used to derive new positivity conditions

on higher-dimension operator coefficients in EFTs [7–12], in some cases with broad implications, e.g., the proof of the four-dimensional a -theorem [13].

At the same time, the study of black hole thermodynamics has produced many elegant and powerful constraints on entropy with important implications for both the IR and UV [14–17]. In this paper we bring together these three lines of inquiry—the swampland program, EFT bounds, and entropy inequalities—to use black hole entropy to constrain the landscape of consistent EFTs.

We consider rotating dyonic black holes in the EFT of gravitons and photons in four spacetime dimensions, with action $\int d^4x \sqrt{-g}(\tilde{\mathcal{L}} + \Delta\mathcal{L})$, where the Einstein-Maxwell (EM) term is $\tilde{\mathcal{L}} = \frac{1}{2\kappa^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and

$$\begin{aligned} \Delta\mathcal{L} = & c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + c_4 R F^2 + c_5 R^{\mu\nu} F_{\mu}{}^{\rho} F_{\nu\rho} + c_6 R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ & + c_7 F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}, \end{aligned} \quad (1)$$

in the sign conventions of Ref. [18] and where $\kappa^2 = 8\pi G$. This action encodes all leading-order parity-conserving interactions of gravitons and photons at low energies. In the absence of charged currents, any operator with derivatives on $F_{\mu\nu}$ can be recast into those already in Eq. (1) using the Bianchi identities [19].

In Ref. [20] it was proven that when the higher-dimension operators in Eq. (1) are generated at tree level by quantum field theoretic dynamics, they induce a positive shift to the Wald entropy of a thermodynamically stable black hole at fixed mass and charge,

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$$\Delta S > 0. \quad (2)$$

This bound applies for black holes sufficiently small so that the classical Wald entropy shift dominates over quantum corrections, which is always achievable within the regime of validity of a weakly coupled EFT [20]; similar power counting shows that Hawking radiation can be ignored. The physical origin of positivity follows from an intimate linkage between the shift in entropy and the variation of the Euclidean action away from its local minimum, which is positive for thermodynamically stable systems. The extensive details can be found in Ref. [20].

As shown in Ref. [21], higher-derivative operators in the EFT modify the extremality condition for black holes. In Ref. [20] it was shown that $\Delta S > 0$ implies constraints on c_i that *precisely* tip the extremality condition so that large electrically charged black holes are unstable to decay to smaller ones. Consequently, the WGC is automatically satisfied since there exist states—namely, black holes—with charge-to-mass ratio greater than unity. More generally, Ref. [20] showed that $\Delta S > 0$ implies the WGC for any number of Abelian forces and in arbitrary spacetime dimension ≥ 4 .

In this paper we consider four-dimensional rotating dyonic black holes, generalizing the results of Ref. [20] to obtain a family of constraints on the coefficients c_i , labeled by the angular momentum and electric and magnetic charge-to-mass ratios of the black hole. As before, $\Delta S > 0$ exactly guarantees that the extremality condition tips so that large black holes are unstable, thus establishing a dyonic rotating version of the WGC.

II. METRIC AND ACTION

To begin, we compute the Wald entropy for a black hole of fixed electric charge Q , magnetic charge \tilde{Q} , angular momentum J , and mass M at leading order in the higher-dimension operator coefficients c_i in Eq. (1). For later convenience we introduce the natural quantities,

$$\begin{aligned} m &= \kappa^2 M / 8\pi \\ q &= \kappa Q / 4\sqrt{2}\pi \\ \tilde{q} &= \kappa \tilde{Q} / 4\sqrt{2}\pi, \end{aligned} \quad (3)$$

and rescaled coefficients,

$$\begin{aligned} d_{1,2,3} &= \kappa^2 c_{1,2,3} \\ d_{4,5,6} &= c_{4,5,6} \\ d_{7,8} &= \kappa^{-2} c_{7,8}. \end{aligned} \quad (4)$$

The d_i all scale as $1/\Lambda^2$, where Λ is the scale of a weakly coupled UV completion.

The unperturbed Kerr-Newman (KN) metric $\bar{g}_{\mu\nu}$ in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) \\ &+ \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2, \end{aligned} \quad (5)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2 \theta \\ a &= J/M = \kappa^2 J / 8\pi m \\ \Delta &= r^2 - 2mr + a^2 + q^2 + \tilde{q}^2. \end{aligned} \quad (6)$$

It will be convenient to define

$$\xi = \frac{\sqrt{m^2 - (q^2 + \tilde{q}^2 + a^2)}}{m}, \quad (7)$$

so that $\xi \rightarrow 0$ is the extremal limit. The event horizon is located at r -coordinate $r_H = m(1 + \xi)$. We also define parameters $\mu = (q^2 - \tilde{q}^2)/(q^2 + \tilde{q}^2)$ and $\nu = a/r_H$ to characterize the charge and spin, so $\mu = 1$ and $\mu = -1$ correspond to pure electric and pure magnetic black holes, respectively, $\nu = 0$ to nonrotating charged black holes, and $\nu = \sqrt{(1 - \xi)/(1 + \xi)}$ to uncharged rotating black holes.

Next, we employ a relation between entropy and the Euclidean action that was discussed in Ref. [20] and recently generalized and proven in Ref. [22] via a simple argument from basic thermodynamic identities,

$$\Delta S = -\Delta I, \quad (8)$$

where ΔS is the entropy shift at fixed mass, charge, and angular momentum, and ΔI denotes the higher-dimension operator contributions to the action evaluated on the leading-order black hole solution, working in an ensemble with fixed inverse temperature β , angular velocity Ω , and charge.

We will compute the entropy shift ΔS to $\mathcal{O}(d_i)$ using Eq. (8) for a rotating dyonic black hole. Since KN is a stationary spacetime with Killing vector ∂_t , we have

$$\Delta I = -\beta \int d^3x \sqrt{-g} \Delta \mathcal{L}|_{\text{KN}}, \quad (9)$$

with the right-hand side evaluated on the Lorentzian KN black hole background at fixed t , where the integration is over all (r, θ, ϕ) outside the horizon.

Note that the Gauss-Bonnet term $\mathcal{L}_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ corrects black hole entropy [23] even though it is a total derivative in four dimensions. This is not contradictory since the black hole horizon provides a boundary on which total derivatives have support. This is similar in spirit to what happens to $F_{\mu\nu}\tilde{F}^{\mu\nu}$ in electrodynamics, which is a total derivative

but comes into play in the background of a magnetic monopole.

Nevertheless, one can show that there are no other total derivative or boundary operators that correct the entropy at the leading order of interest. For example, $\square R$ and $\square(F_{\mu\nu}F^{\mu\nu})$ do not affect the Wald entropy nor the equations of motion. Meanwhile, higher-derivative boundary terms at asymptotic infinity, analogous to the Gibbons-Hawking-York term, are subleading in $1/r$ at large distances and can therefore be ignored [20,22].

III. BLACK HOLE ENTROPY AND CONSISTENCY CHECKS

Any physical constraint on the action must be invariant under change of field basis, e.g., the metric transformation

$g_{\mu\nu} \rightarrow g_{\mu\nu} + r_1 R_{\mu\nu} + r_2 R g_{\mu\nu} + r_3 F_{\mu\rho} F_{\nu}{}^{\rho} + r_4 F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu}$. Consequently, a given coefficient d_i is not individually meaningful because it mixes with others under a field redefinition. The four field-redefinition invariant linear combinations of coefficients are (d_0, d_3, d_6, d_9) , where [20]

$$\begin{aligned} d_0 &= d_2 + 4d_3 + d_5 + d_6 + 4d_7 + 2d_8 \\ d_9 &= d_2 + 4d_3 + d_5 + 2d_6 + d_8. \end{aligned} \quad (10)$$

Note that Eq. (8) is automatically field-basis invariant because those combinations of higher-dimension operators that are removable by a field redefinition are proportional to equations of motion and thus vanish when evaluated on the leading-order solutions. We find that ΔS is indeed field-basis invariant,

$$\begin{aligned} \Delta S(\xi, \mu, \nu) &= \frac{16\pi^2(2\mu^2 - 1)(\nu^2 - 3)(3\nu^2 - 1)[1 - \xi - \nu^2(1 + \xi)]^2}{15\kappa^2\xi(1 + \xi)(1 + \nu^2)^5} (d_0 + d_6 - d_9) \\ &+ \frac{\pi^2[1 - \xi - \nu^2(1 + \xi)]^2[\nu(3 + 2\nu^2 + 3\nu^4) + 3(\nu^2 - 1)(1 + \nu^2)^2 \arctan \nu]}{2\kappa^2\xi(1 + \xi)\nu^5(1 + \nu^2)} (d_0 + d_6 + d_9) \\ &+ \frac{64\pi^2}{\kappa^2(1 + \nu^2)} d_3 + \frac{32\pi^2\mu[1 - \xi - \nu^2(1 + \xi)][\nu^2(3 + 4\xi) - 1 - 4\xi]}{5\kappa^2\xi(1 + \xi)(1 + \nu^2)^3} d_6. \end{aligned} \quad (11)$$

Note that Eq. (11) is also invariant under the electromagnetic duality transformation, $q \leftrightarrow \tilde{q}$ and $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}/2$. However, individual operators in $\Delta\mathcal{L}$ are not invariant, since $F_{ab} \rightarrow \tilde{F}_{ab}$ sends $d_i \rightarrow \tilde{d}_i$, where

$$\tilde{d}_i = (d_1, d_2, d_3, -d_4 - d_5/2 - d_6, d_5 + 4d_6, -d_6, d_7, d_8), \quad (12)$$

i.e.,

$$\tilde{d}_{0,3,6,9} = (d_0 + 2d_6, d_3, -d_6, d_9). \quad (13)$$

Applying this mapping to the entropy shift $\Delta S(\xi, \mu, \nu)$ should be equivalent to swapping $|q|$ and $|\tilde{q}|$, sending μ to $-\mu$. Indeed, from Eq. (11) we find that

$$\Delta S(\xi, \mu, \nu)|_{d_i \rightarrow \tilde{d}_i} = \Delta S(\xi, -\mu, \nu), \quad (14)$$

so our family of bounds transforms consistently under electromagnetic duality.

While Eq. (11) carries a formal divergence at $\xi = 0$, as shown in Ref. [20] we can take the near-extremal, i.e., small- ξ , limit consistent with perturbative control. Requiring that the near-extremal shifts in entropy and temperature be much smaller than their unperturbed values mandates $\xi \gg \kappa\Lambda$, but $\kappa\Lambda$ is parametrically small for a

field-theoretic completion. For later convenience we will sometimes use $\xi = 0$ to indicate this limiting value.

As an independent check of our calculation, we have used perturbative techniques [21,24] applicable to spherically symmetric black holes to solve for the metric itself at leading order in the d_i , making use of the invertibility of the Ricci tensor for spherical geometries; see the Appendix for details. Inserting the metric into the Wald formula, we obtain ΔS for the $\nu = 0$ case, and we find that it agrees with Eq. (11). As a further consistency check, we have verified that the surface gravity at the horizon in the perturbed metric agrees with the thermodynamic temperature extracted from differentiating the Wald entropy with respect to the mass.

IV. EFT BOUNDS

We now show how $\Delta S > 0$ constrains the EFT. As discussed earlier, this inequality holds for black holes that are thermodynamically stable. For a general KN black hole, the requirement of positive heat capacity at constant J is not enough to guarantee thermodynamic stability. In our ensemble at fixed β and Ω , we must also require positive isothermal moment of inertia [25]. This enhanced stability condition restricts our consideration to

$$0 < \xi < \frac{1 - 3\nu^2}{2(1 + \nu^2)}. \quad (15)$$

In the $\nu = 0$ nonrotating case, this window reduces to $\xi \in (0, \frac{1}{2})$, i.e., $q^2 + \tilde{q}^2 > 3m^2/4$. The stability condition in Eq. (15) cannot be satisfied in the neutral case, where $\nu = \sqrt{(1-\xi)/(1+\xi)}$. Thus, we cannot use neutral Kerr black holes to place a bound because they are thermodynamically unstable. In the $\xi \rightarrow 0$ extremal limit, Eq. (15) reduces to $\nu \in [0, 1/\sqrt{3}]$, i.e., $q^2 + \tilde{q}^2 > 2a^2$.

As discussed earlier, the entropy shift satisfies

$$\Delta S(\xi, \mu, \nu) > 0 \quad (16)$$

for all (ξ, ν) satisfying Eq. (15) and all $\mu \in [-1, 1]$, which is the main result of this work. This condition implies new bounds on the EFT of gravitons and photons mandated by the entropy of rotating dyonic black holes.

Numerical studies indicate that the parameter space excluded by Eq. (16) is already forbidden by the special case $\nu = 0$, i.e., the bounds from rotating black holes are already accounted for by the nonrotating case. See Fig. 1 for examples of excluded regions, which we now discuss.

Two immediate implications of Eq. (16) are that

$$d_0 > 0 \quad \text{and} \quad d_9 > 0, \quad (17)$$

obtained by taking $\Delta S(0, 1, 0) > 0$ and $\Delta S(0, 0, 0) > 0$, respectively. Note that the four-dimensional bounds advocated via IR consistency arguments in Ref. [8] are a subset of Eq. (16). If only $d_{7,8}$ are nonzero, the bounds in this paper can be written simply as $2d_7 + d_8 > 0$ and $d_8 > 0$, both of which are implied by analyticity of four-photon scattering [7,20].

Furthermore, d_3 is bounded from below by d_0 and d_9 . In particular, $\Delta S(\frac{1}{2}, 0, 0) > 0$ implies

$$d_3/d_9 > -1/30, \quad (18)$$

while $7\Delta S(\frac{1}{2}, 1, 0) + 5\Delta S(\frac{1}{2}, -1, 0) > 0$ implies

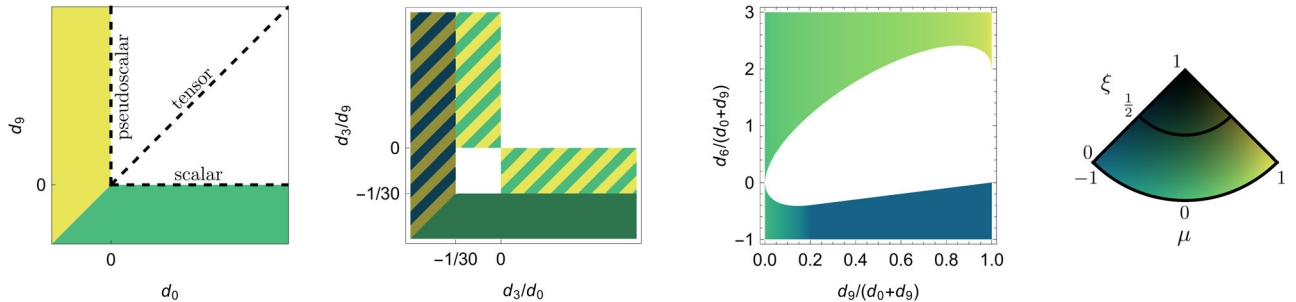


FIG. 1. Entropy bounds on higher-dimension operator coefficients d_i . The shaded regions are excluded since $\Delta S(\xi, \mu, 0) < 0$ for some choice of black hole (yellow for electric, blue for magnetic, with brighter hues indicating larger charge), irrespective of the values of operator coefficients not shown in a given panel. These constraints imply that $d_{0,9}$ are positive, d_3 is bounded from below, and d_6 is bounded by various combinations of $d_{0,9}$. For the scalar, pseudoscalar, and tensor UV completions in Table I, $d_3 = d_6 = 0$ and allowed values of $d_{0,9}$ are indicated by the black dashed lines in the leftmost panel.

$$d_3/d_0 > -1/30, \quad (19)$$

which are the most stringent entropy bounds that can be placed on d_3 in terms of either d_0 or d_9 alone. A lower bound on d_3 is expected in light of the unitarity argument for positivity of d_3 given in Ref. [11].

In addition, Eq. (16) implies that d_6 , which vanishes for supersymmetric theories [26], is bounded by d_0 and d_9 . As illustrated in the last panel of Fig. 1, regions outside of the allowed window in d_6 are excluded since there exist values of ξ and μ such that $\Delta S(\xi, \mu, 0) < 0$ for any value of d_3 . We can obtain this bound analytically by taking $\Delta S(0, \mu, 0)$ and marginalizing over μ , yielding

$$\left[\begin{array}{l} 2(d_9 - \sqrt{d_0 d_9}), \quad 4d_9 \leq d_0 \\ -d_0/2, \quad 4d_9 > d_0 \end{array} \right] < d_6 < 2(d_9 + \sqrt{d_0 d_9}), \quad (20)$$

assuming both d_0 and d_9 are nonzero. If d_0 and d_9 are both generated at Λ , $|d_6|$ larger than $\sim 1/\Lambda^2$ is forbidden. A similar conclusion, disallowing d_6 without the appearance of heavy states at the scale $|d_6|^{-1/2}$, was reached using causality in Ref. [26].

V. UV EXAMPLES

As a consistency check, we calculate the operator coefficients d_i for various tree-level UV completions and verify that they obey the inequalities in Eq. (16). These include UV completions with a massive scalar or pseudo-scalar coupled to the electromagnetic field strength and gravitational curvature tensor in all ways consistent with parity, as shown in Table I. We also consider a UV completion with a massive tensor field with a ghost-free Fierz-Pauli mass term and minimal coupling to the energy-momentum tensor. It is known that such a theory can have a cutoff parametrically higher than the mass of the tensor field [27]. In principle, other interactions with the tensor field could be possible, but such theories have not been well

TABLE I. Higher-dimension operator coefficients d_i and black hole entropy shift ΔS induced at low energies in various field theory UV completions defined by a massive particle ϕ that can be either a scalar, pseudoscalar, or tensor field, as well as the heterotic string. In the former, the Lagrangian is $\mathcal{L}_{\text{UV}} = \tilde{\mathcal{L}} + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}$ where \mathcal{L}_{kin} is the canonical kinetic term for ϕ . The tensor field has a Fierz-Pauli mass term and couples to the energy-momentum tensor, $\tilde{T}_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$. All examples produce positive ΔS for any choice of parameters.

Massive state	\mathcal{L}_{int}	$d_{0,3,6,9}$	$\Delta S(\xi, \mu, 0) \times [5\kappa^2\xi(1+\xi)/32\pi^2]$
Scalar	$\phi(a\kappa^{-1}R + \kappa b F_{\mu\nu}F^{\mu\nu})$	$2b^2/m_\phi^2 \times (1, 0, 0, 0)$	$2(1-\xi)^2 b^2 \mu^2 / m_\phi^2$
Pseudoscalar	$\kappa b \phi F_{\mu\nu} \tilde{F}^{\mu\nu}$	$2b^2/m_\phi^2 \times (0, 0, 0, 1)$	$2(1-\xi)^2 b^2 (1-\mu^2) / m_\phi^2$
Tensor	$\kappa b \phi^{\mu\nu} \tilde{T}_{\mu\nu}$	$b^2/2m_\phi^2 \times (1, 0, 0, 1)$	$(1-\xi)^2 b^2 / 2m_\phi^2$
Heterotic string	...	$\alpha'/16 \times (3, 1, 0, 3)$	$(\alpha'/16)(3+4\xi+13\xi^2)$

studied, and we therefore have no reason to expect a consistent EFT, so such a generalization would not constitute a well-defined check of our entropy bounds. For example, a coupling such as $\phi^\mu{}_\nu F_{\nu\rho} F^{\nu\rho}$ would violate bounds coming from amplitude analyticity as well as entropy. For a tree-level completion of Eq. (1), the massive state must be neutral, since charged lines cannot terminate in Feynman diagrams.

As shown in Table I, the entropy shift is manifestly positive in our example scalar, pseudoscalar, and tensor completions, providing a check of our bounds. In these examples, only d_0 and d_9 are generated. See the first panel of Fig. 1 for the parameter space spanned by these field theory UV completions.

Finally, we considered the low-energy effective theory of the heterotic string [21,28], whose higher-dimension operators are not generated by quantum fields but nonetheless satisfy our bounds, as shown in Table I. Here we have ignored the presence of the dilaton, but if this state is stabilized below the string scale, then its dynamics will be encoded in the massive scalar UV completion.

VI. WEAK GRAVITY CONJECTURE

Our entropy bounds are intimately connected to the WGC. We define the extremality parameter,

$$\zeta = \sqrt{a^2 + q^2 + \tilde{q}^2/m} = \sqrt{1 - \xi^2}. \quad (21)$$

In pure EM theory, black holes are described by the KN metric, which is free from naked singularities provided $\zeta \in [0, 1]$. The higher-dimension operators in Eq. (1) correct the equations of motion, changing the allowed range of physical values to $\zeta \in [0, 1 + \Delta\zeta]$, as shown for the case of nonrotating electric black holes in Ref. [21]. As in Ref. [20], we calculate the extremality shift by writing g^{rr} in terms of \tilde{g}^{rr} and the shifts in various parameters. Since $\partial_\theta \tilde{g}^{rr} = \partial_\mu \tilde{g}^{rr} = \partial_\nu \tilde{g}^{rr} = 0$ on the KN horizon and $\partial_r \tilde{g}^{rr} = 0$ for the extremal horizon, we find that the extremality shift is

$$\Delta\zeta = -\Delta g^{rr} / \partial_\zeta \tilde{g}^{rr} |_{r_H}, \quad (22)$$

evaluated on $r_H = m(1 + \xi)$. Similarly, for a black hole of fixed (m, a, q, \tilde{q}) , the shift in the horizon is

$$\Delta r_H = -\Delta g^{rr} / \partial_r \tilde{g}^{rr} |_{r_H}. \quad (23)$$

In the extremal limit, the Δr_H term in the entropy shift dominates, since $\partial_r \tilde{g}^{rr} \rightarrow 0$, so we have $\Delta S = (2\pi/\kappa^2) \Delta r_H \partial_r \tilde{A}$, where $\tilde{A} = 4\pi(r_H^2 + a^2)$ is the area of the KN black hole. Using the fact that $\partial_\zeta \tilde{g}^{rr} / \partial_r \tilde{g}^{rr} |_{\text{KN}, r=r_H} = m\zeta/\xi$, we directly relate the shift $\Delta\zeta$ in the extremality parameter to the entropy shift,

$$\Delta\zeta(\mu, \nu) = + \frac{\kappa^2}{16\pi^2 m^2} \lim_{\xi \rightarrow 0} [\xi \Delta S(\xi, \mu, \nu)]. \quad (24)$$

Sans rotation, we find that $\Delta\zeta \propto +d_0$ for the electric black hole [20], $\Delta\zeta \propto +d_0 + 2d_6$ for the magnetic black hole, and $\Delta\zeta \propto +d_9$ for $q = \tilde{q}$.

Let us now consider the decay of a black hole to lighter states with masses and charges m_i, q_i, \tilde{q}_i in natural units and angular momenta $j_i = 8\pi m_i a_i / \kappa^2$ including both intrinsic spin and orbital components. Charge conservation implies that $\sum_i (q_i, \tilde{q}_i) = (q, \tilde{q})$, while conservation of energy requires $\sum_i m_i < m$, where the inequality is needed if there is to be nonzero phase space for the decay products. Angular momentum conservation requires $J = aM \leq \sum_i j_i$, where the inequality is due to the possibility of misaligned angular momenta. Defining unitless ratios,

$$\begin{aligned} z &= q/m \\ \tilde{z} &= \tilde{q}/m \\ \alpha &= am\kappa^{-2} = J/8\pi, \end{aligned} \quad (25)$$

the set of KN black holes free of naked singularities forms a spheroid,

$$\zeta^2 = z^2 + \tilde{z}^2 + (\alpha^2 \kappa^4 / m^4) \leq 1, \quad (26)$$

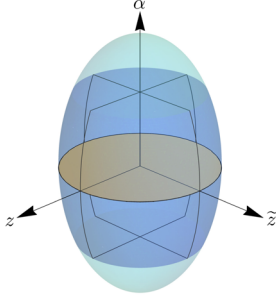


FIG. 2. The cyan spheroid $z^2 + \tilde{z}^2 + \alpha^2 \kappa^4 / m^4 \leq 1$ denotes the space of all possible black holes without naked singularities. The blue barrel-shaped region incorporates the additional criterion $|\alpha| < m^2 / \sqrt{3} \kappa^2$, which defines the subspace of black holes that, due entropy bounds, can decay directly to lighter, extremality-corrected black holes. The gray disk represents all nonrotating dyonic black holes.

depicted in Fig. 2. Defining the analogous ratios for the decay products,

$$(z_i, \tilde{z}_i) = (q_i/m_i, \tilde{q}_i/m_i) \\ \alpha_i = a_i m \kappa^{-2} = j_i / 8\pi \sigma_i, \quad (27)$$

writing $\sigma_i = m_i/m$, we have

$$\sum_i (\sigma_i z_i, \sigma_i \tilde{z}_i) = (z, \tilde{z}) \quad \text{and} \quad \alpha \leq \sum_i \sigma_i \alpha_i. \quad (28)$$

Putting this all together to define $\mathbf{w} = (z, \tilde{z}, \alpha)$, we conclude that a state with vector \mathbf{w} must decay to states \mathbf{w}_i for which \mathbf{w} is either in the convex hull of the \mathbf{w}_i or strictly between that convex hull and the $\alpha = 0$ plane.

The bound in Eq. (16) implies by Eq. (24) that

$$\Delta\zeta(\mu, \nu) > 0 \quad \text{for all } \nu \in [0, 1/\sqrt{3}). \quad (29)$$

Thus, extremal black holes located in the barrel-shaped region with $|a|/m = |\alpha| \kappa^2 / m^2 < 1/\sqrt{3}$ can decay directly to other rotating black holes. The lighter rotating dyonic black holes form a closed surface everywhere outside the barrel-shaped region, so the tower of stable large black holes collapses to the scale at which the EFT breaks down. In this sense, the WGC bounds implied by extremality corrections point to a principle of black hole self-sufficiency in providing their own decay channel. While our $\Delta S > 0$ bounds do not apply to black holes with $\nu > 1/\sqrt{3}$, this is due to the fact that such states are known to be already thermodynamically unstable, even in the extremal limit [25]. In particular, these objects can shed their spin by emitting gravitons in states with nonzero angular momentum, decaying to the $\alpha = 0$ plane. For nonrotating black holes, our entropy bound implies a dyonic version of the WGC: the set of near-extremal dyonic black holes can

decay because they form a closed curve slightly outside the unit circle in the $\alpha = 0$ plane.

If the leading higher-dimension operators in Eq. (1) have vanishing coefficients, the entropy will be corrected by yet-higher-derivative operators, which are still constrained by our reasoning since Eq. (24) implies that $\Delta S \propto \Delta\zeta$ in the extremal limit [29]. While Ref. [17] found R^3 operators generated either sign after integrating out massive matter with various spins at one loop, this does not contradict our results; these massive fields also induce four-derivative operators of the form in Eq. (1), which dominate over the R^3 operators within the regime of validity of the EFT. Moreover, the $\Delta S > 0$ result has been strictly proven only for tree completions [20], though we conjecture it holds more generally. If ΔS strictly vanishes to all orders, then there is a flat direction in the Euclidean action, indicating a symmetry protecting the states. While such a symmetry is not present for EM theory black holes, it would be interesting to see how supersymmetry or massless moduli alter this picture.

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APPENDIX: DYONIC METRIC PERTURBATION

In the nonrotating case, we can use a spherically symmetric ansatz for the black hole metric,

$$ds^2 = -f(r)dt^2 + dr^2/g(r) + r^2 d\Omega^2. \quad (A1)$$

Following Refs. [21,24], we can write $f(r)$ and $g(r)$ as integrals over the perturbed energy-momentum tensor $\tilde{T}_{\mu\nu} + \Delta T_{\mu\nu}$ that appears on the right-hand side of the perturbed Einstein equation, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2(\tilde{T}_{\mu\nu} + \Delta T_{\mu\nu})$. The shift $\Delta T_{\mu\nu}$ can be written as the sum of two contributions, one coming from the metric variation of $\Delta\mathcal{L}$ and one coming from the change to the field strength and metric in the original Maxwell stress tensor.

The shift in field strength is dictated by the perturbed Maxwell's equations, $\nabla_\nu F^{\mu\nu} = \nabla_\nu N^{\mu\nu}$, writing $N^{\mu\nu} = 2\delta\Delta\mathcal{L}/\delta F_{\mu\nu}$. Due to spherical symmetry, the only independent nonzero components of $F^{\mu\nu}$ are F^{tr} and $F^{\theta\phi}$, so the Bianchi identity implies that $\nabla_r F^{\theta\phi} = 0$ in the perturbed solution, which means that $F^{\theta\phi} \propto 1/r^2$. Since $N^{\mu\nu}$ encodes the effects of higher-derivative terms, it cannot have the

Coulomb scaling in r , so solving the perturbed Maxwell equation requires that $\nabla_\theta N^{\theta\phi} = 0$, which implies $N^{\theta\phi} \propto \csc\theta$. Thus, the magnetic component of the field strength is unchanged by the higher-derivative terms,

$F^{\theta\phi} = \bar{F}^{\theta\phi}$ at $\mathcal{O}(d_i)$, while the electric component F^{rr} is indeed modified.

We then jointly solve the modified Einstein and Maxwell equations at $\mathcal{O}(d_i)$,

$$\begin{aligned}
 f(r) &= 1 - \frac{2m}{r} + \frac{q^2 + \tilde{q}^2}{r^2} - \frac{4}{5r^6} \left\{ \begin{aligned} &(d_2 + 4d_3)(q^2 + \tilde{q}^2)(q^2 + \tilde{q}^2 - 5mr + 5r^2) \\ &-10d_4(q^2 - \tilde{q}^2)(2q^2 + 2\tilde{q}^2 - 3mr + r^2) \\ &-d_5[2(q^2 + \tilde{q}^2)(2q^2 - 3\tilde{q}^2) - 5mr(q^2 - 2\tilde{q}^2) - 5\tilde{q}^2 r^2] \\ &+d_6(q^2 + \tilde{q}^2)(q^2 + 3\tilde{q}^2 - 5mr + 5r^2) \\ &+4d_7(q^2 - \tilde{q}^2)^2 + 2d_8(q^4 + \tilde{q}^4) \end{aligned} \right\} \\
 g(r) &= 1 - \frac{2m}{r} + \frac{q^2 + \tilde{q}^2}{r^2} - \frac{4}{5r^6} \left\{ \begin{aligned} &(d_2 + 4d_3)(q^2 + \tilde{q}^2)(6q^2 + 6\tilde{q}^2 - 15mr + 10r^2) \\ &+10d_4(q^2 - \tilde{q}^2)(3q^2 + 3\tilde{q}^2 - 7mr + 4r^2) \\ &+d_5[(q^2 + \tilde{q}^2)(11q^2 - 4\tilde{q}^2) - 5mr(5q^2 - 2\tilde{q}^2) + 5r^2(3q^2 - \tilde{q}^2)] \\ &+d_6[2(q^2 + \tilde{q}^2)(8q^2 - \tilde{q}^2) - 5mr(7q^2 - \tilde{q}^2) + 20q^2 r^2] \\ &+4d_7(q^2 - \tilde{q}^2)^2 + 2d_8(q^4 + \tilde{q}^4) \end{aligned} \right\}. \quad (\text{A2})
 \end{aligned}$$

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