Thermality of the Rindler horizon: A simple derivation from the structure of the inertial propagator

T. Padmanabhan*

IUCAA, Pune University Campus, Ganeshkhind, Pune 411007, India

(Received 29 May 2019; published 28 August 2019)

The Feynman propagator $G(x_1, x_2)$ encodes *all* of the physics contained in a free field and transforms as a covariant biscalar. Therefore, we should be able to discover the thermality of the Rindler horizon just by probing the structure of the propagator, expressed in the Rindler coordinates. We show that the thermal nature of the Rindler horizon is indeed contained—though hidden—in the standard, inertial, Feynman propagator. The probability $P(E)$ for a particle to propagate between two events with energy E can be related to the temporal Fourier transform of the propagator. A strikingly simple computation reveals that (i) $P(E)$ is equal to $P(-E)$ if the propagation is between two events in the same Rindler wedge, while (ii) they are related by a Boltzmann factor with temperature $T = g/2\pi$ if the two events are separated by a horizon. A more detailed computation reveals that the propagator itself can be expressed as a sum of two terms, governing absorption and emission, weighted correctly by the factors $(1 + n_\nu)$ and n_ν, where n_ν is a Planck distribution at the temperature $T = g/2\pi$. In fact, one can *discover* the Rindler vacuum and the alternative (Rindler) quantization just by probing the structure of the inertial propagator. These results can be extended to local Rindler horizons around any event in a curved spacetime. The implications are discussed.

DOI: 10.1103/PhysRevD.100.045024

I. THE MAIN RESULT: INERTIAL PROPAGATOR KNOWS ALL!

The path-integral representation of the (Feynman) propagator is given by the sum-over-paths prescription using the (square-root) action for a relativistic particle:

$$\sum_{\text{paths}} \exp\left[-im\ell(x_1, x_2)\right] = G(x_1, x_2), \tag{1}$$

where $\ell(x_1, x_2)$ is the length of the path. This suggests that one can interpret $G(x_1, x_2)$ as an amplitude for a particle/antiparticle to propagate between two events in the spacetime.[1] This interpretation acquires an operational meaning in the presence of a source $J(x)$ capable of emitting/absorbing the particles [1]. Then, the vacuum persistence amplitude

$$\langle \text{out}|\text{in}\rangle_J = \langle \text{out}|\text{in}\rangle_{J=0} \exp\left\{-\frac{1}{2}\int d^D x_1 \sqrt{-g_1} \times \int d^D x_2 \sqrt{-g_2} J(x_1) G(x_1, x_2) J(x_2)\right\} \tag{2}$$

can be thought of as describing the emission/absorption at the two events [controlled by $J(x_1), J(x_2)$] and the propagation between the events governed by $G(x_1, x_2)$.

We are interested in the stationary situations in which the propagator depends on the time coordinates only through the time difference, so that $G(x_1, x_2) = G(\tau; \boldsymbol{x}_1, \boldsymbol{x}_2)$ with $\tau \equiv (x_1^0 - x_2^0) \equiv (\tau_1 - \tau_2)$. Such stationarity is assured if there exists a Killing vector field ξ_a which, in a suitable coordinate system, can be represented as $\xi_a = \partial/\partial\tau$. One can then interpret the temporal Fourier transform

$$A(\Omega; \boldsymbol{x}_1, \boldsymbol{x}_2) = \int_{-\infty}^{\infty} d\tau G(\tau; \boldsymbol{x}_1, \boldsymbol{x}_2) e^{i\Omega\tau}, \quad \tau = (\tau_1 - \tau_2) \tag{3}$$

as the amplitude for the particle to propagate between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ with energy Ω, introduced as the Fourier conjugate to the time coordinate τ. In what follows we will simplify the notation and write $G(\tau)$ for $G(\tau; \boldsymbol{x}_1, \boldsymbol{x}_2)$ and $A(\Omega)$ for $A(\Omega; \boldsymbol{x}_1, \boldsymbol{x}_2)$, suppressing the spatial coordinates. While evaluating the amplitude $A(\Omega)$ in Eq. (3) it is convenient to assume that $\Omega > 0$ and interpret $A(-\Omega)$ as the expression obtained by replacing Ω with $-\Omega$ in the result of the integral in Eq. (3). My interest lies in comparing $A(-\Omega)$ with $A(\Omega)$. If they are equal, then the amplitudes for the particle to propagate with an energy Ω or $-\Omega$ are the same; however, if they are unequal this indicates some interesting physics.

To probe this issue, let us consider the explicit form of $G(\tau)$ in a D-dimensional flat spacetime given by (with m^2 treated as $m^2 - i\epsilon$)

*paddy@iucaa.in

[1]We use mostly negative signature—except when specified otherwise—and natural units. The propagator in momentum space $G(p) = i(p^2 - m^2 - i\epsilon)^{-1}$ is defined with an i factor, so that $G(x_1, x_2) = \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$.

$$G(\tau) = i\left(\frac{1}{4\pi i}\right)^{D/2}\int_0^\infty \frac{ds}{s^{D/2}}\exp\left[-ism^2 - \frac{i}{4s}\sigma^2(\tau)\right], \quad (4)$$

where $\sigma^2(\tau) \equiv \sigma^2(x_1, x_2)$ is the squared line interval between the two events. The fact that σ^2 depends only on $\tau = \tau_1 - \tau_2$ again arises from the stationarity of the background and the existence of the Killing vector $\partial/\partial\tau$. From the structure of the integral in Eq. (3) it is obvious that, if $G(\tau) = G(-\tau)$, $A(\Omega) = A(-\Omega)$ and thus nothing very interesting happens. This is, of course, trivially true if we take τ to be the standard inertial time coordinate t so that $\sigma^2(t) = t^2 - |\boldsymbol{x}_1 - \boldsymbol{x}_2|^2$. This makes σ^2 and G even functions of the time difference, leading to $A(\Omega) = A(-\Omega)$.

Interestingly enough, the same result holds even when both events x_1 and x_2 are on the right Rindler wedge (R), with τ being the Rindler time coordinate. In R the Rindler coordinates (τ, ρ) can be defined[2] in the usual manner as $t = \rho \sinh\tau$ and $x = \rho\cosh\tau$. The line interval $\sigma^2_{\rm RR}$ for two events in the right wedge has the form

$$\sigma^2_{\rm RR}(\tau) = -L_1^2 + 2\rho_1\rho_2\cosh\tau, \quad (5)$$

where $L_1^2 = (\Delta\mathbf{x}_\perp^2 + 2\xi_1 + 2\xi_2)$, with the ξ coordinate defined through the relation $x^2 - t^2 \equiv 2\xi$. (In R, $2\xi = \rho^2$). The $\sigma^2_{\rm RR}(\tau)$ is clearly an even function of τ and hence we reach the following conclusion: *when a particle propagates between any two events within the right Rindler wedge R, we have* $A(\Omega) = A(-\Omega)$.[3]

Let us next consider what happens when we take one event to be in R and the second event to be in F, where the Rindler-like coordinate system is introduced through $t = \rho\cosh\tau$ and $x = \rho\sinh\tau$. (If one uses the ξ coordinate, then the relation $x^2 - t^2 = 2\xi$ allows the region F to be covered by the range $-\infty < \xi < 0$ and the region R to be covered by the range $0 < \xi < \infty$.) The line interval $\sigma^2_{\rm FR}(\tau)$ between an event $(\tau_{\rm F}, \rho_{\rm F})$ in F and an event $(\tau_{\rm R}, \rho_{\rm R})$ in R is given by

$$\sigma^2_{\rm FR}(\tau) \equiv (t_{\rm F} - t_{\rm R})^2 - (x_{\rm F} - x_{\rm R})^2 - \Delta\mathbf{x}_\perp^2 \quad (6)$$

$$= (\rho_{\rm F}\cosh\tau_{\rm F} - \rho_{\rm R}\sinh\tau_{\rm R})^2 - (\rho_{\rm F}\sinh\tau_{\rm F} - \rho_{\rm R}\cosh\tau_{\rm R})^2 - \Delta\mathbf{x}_\perp^2 \quad (7)$$

$$= \rho_{\rm F}^2 - \rho_{\rm R}^2 - 2\rho_{\rm F}\rho_{\rm R}\sinh(\tau_{\rm R} - \tau_{\rm F}) - \Delta\mathbf{x}_\perp^2 \quad (8)$$

$$\equiv -L_2^2 - 2\rho_{\rm F}\rho_{\rm R}\sinh\tau, \qquad \tau \equiv (\tau_{\rm R} - \tau_{\rm F}), \quad (9)$$

with $L_2^2 \equiv (\Delta\mathbf{x}_\perp^2 + 2\xi_{\rm R} + 2|\xi_{\rm F}|)$. We displayed this calculation in gory detail because there is a bit of algebraic sorcery involved in it. (This is the only nontrivial calculation in this paper!) The line interval $\sigma^2(\mathcal{P}_1, \mathcal{P}_2)$ between any two events in the spacetime, of course, is symmetric with respect to the interchange of events, $\sigma^2(\mathcal{P}_1, \mathcal{P}_2) = \sigma^2(\mathcal{P}_2, \mathcal{P}_1)$. In our case, the two events have the coordinates

$$\mathcal{P}_1 = \mathcal{P}_{\rm F} = (t_{\rm F}, x_{\rm F}, \mathbf{x}^\perp_{\rm F}) = (\rho_{\rm F}\cosh\tau_{\rm F}, \rho_{\rm F}\sinh\tau_{\rm F}, \mathbf{x}^\perp_{\rm F}) \quad (10)$$

and

$$\mathcal{P}_2 = \mathcal{P}_{\rm R} = (t_{\rm R}, x_{\rm R}, \mathbf{x}^\perp_{\rm R}) = (\rho_{\rm R}\sinh\tau_{\rm R}, \rho_{\rm R}\cosh\tau_{\rm R}, \mathbf{x}^\perp_{\rm R}). \quad (11)$$

The symmetry of the line interval is manifest in the inertial coordinates and we have $\sigma^2(t_{\rm F}, \mathbf{x}_{\rm F};, t_{\rm R}, \mathbf{x}_{\rm R}) = \sigma^2(t_{\rm R}, \mathbf{x}_{\rm R};, t_{\rm R}, \mathbf{x}_{\rm F})$. But we *cannot* obtain the same symmetry by interchanging the relevant Rindler coordinates! From Eq. (8) we see that

$$\sigma^2(\tau_{\rm F}, \rho_{\rm F}, \mathbf{x}^\perp_{\rm F}; \tau_{\rm R}, \rho_{\rm R}, \mathbf{x}^\perp_{\rm R}) \neq \sigma^2(\tau_{\rm R}, \rho_{\rm R}, \mathbf{x}^\perp_{\rm R}; \tau_{\rm F}, \rho_{\rm F}, \mathbf{x}^\perp_{\rm F}). \quad (12)$$

Of course, if we introduce arbitrary coordinate labels to events in spacetime, there is no assurance that the interchange of coordinate labels will correspond to the interchange of events when two different coordinate charts are involved. This is precisely what happens here: it is obvious from Eqs. (10) and (11) that the interchange $(\tau_{\rm F}, \rho_{\rm F}) \Leftrightarrow (\tau_{\rm R}, \rho_{\rm R})$ of the coordinate labels that we are using does *not* lead to the interchange of the events $\mathcal{P}_1 \Leftrightarrow \mathcal{P}_2$ because two different coordinate charts[4] are used in R and F.

We will now compute the Fourier transform in Eq. (3) with respect to $\tau \equiv (\tau_{\rm R} - \tau_{\rm F})$. The sign convention in Eq. (3) implies that G picks up a contribution $A(\Omega)\exp -i\Omega(\tau_{\rm R} - \tau_{\rm F})$ which will correspond to a positive

[2]We will work with units such that the acceleration g of the Rindler frame is unity. In the coordinate transformation from $(t, x, \boldsymbol{x}_\perp)$ to $(\tau, \rho, \boldsymbol{x}_\perp)$, the transverse coordinates $\boldsymbol{x}_\perp$ remain invariant and we will not display them unless necessary.

[3]The Unruh-DeWitt detector response [2], for a uniformly accelerated trajectory in R, is computed by a Fourier transform similar to the one in Eq. (3), for events with $\rho_2 = \rho_1$, $\Delta\boldsymbol{x}_\perp = 0$, with the *Wightman function* replacing the propagator. This, of course, leads to $A(-\Omega) \neq A(\Omega)$. The difference arises due to the difference in the structure of the Wightman function and the Feynman propagator. Algebraically, $[\sinh^2(\tau/2) - i\epsilon]$ (which occurs in the propagator) is an even function of τ, while $\sinh^2[(\tau/2) - i\epsilon]$ (which occurs in the Wightman function) is not.

[4]Why does the σ^2 between the events in R and F only depend on the difference in the "time" labels, especially since τ is not even a time variable in F? This has to do with the fact that one can indeed introduce a (Schwarzschild-like) coordinate system covering both R and F in which the two-dimensional metric takes the form $ds^2 = (2\xi)d\tau^2 - (2\xi)^{-1}d\xi^2$. We see that τ retains its Killing character in both R and F, though $\partial/\partial\tau$ is timelike only in R. It is the Killing character which ensures that $\sigma^2_{\rm FR}$ only depends on the difference in the "time" labels.

energy with respect to τ_R when $\Omega > 0$ (and negative energy when $\Omega < 0$). These are defined with respect to τ_R which is a valid time coordinate in R. (Thus, we do not have to worry about the fact that τ_F has no clear meaning as a time coordinate in F; it is an ignorable constant which goes away when we evaluate the integral over the range $-\infty < \tau < \infty$.) The Fourier transform in Eq. (3) requires us to compute the integral

$$I = \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau - \frac{i}{4s}\sigma_{\rm FR}^2(\tau)} = 2e^{\frac{iL^2}{4s}} e^{-\pi\Omega/2} K_{i\Omega}(2\alpha), \quad (13)$$

where $\alpha \equiv (\rho_1\rho_2/2s)$. This is done using the standard integral representation for the McDonald function, leading to

$$\int_0^\infty \frac{dq}{q} q^{i\omega} e^{i\alpha(q-\frac{1}{q})} = 2e^{-\pi\omega/2} K_{i\omega}(2\alpha), \qquad (\alpha > 0). \quad (14)$$

Substituting Eq. (13) into Eq. (3), we find that the relevant amplitude is given by

$$A(\Omega) = e^{-\pi\Omega/2} \int_0^\infty ds F(s) K_{i\Omega}(2\alpha), \quad (15)$$

where

$$F(s) = 2i\left(\frac{1}{4\pi i s}\right)^{D/2} e^{-im^2 s + \frac{iL_2^2}{4s}}. \quad (16)$$

Since $K_{i\Omega} = K_{-i\Omega}$ is an even function of Ω, it follows that

$$A(-\Omega) = e^{\pi\Omega/2} \int_0^\infty ds F(s) K_{i\Omega}(2\alpha) = e^{\pi\Omega} A(\Omega), \quad (17)$$

leading to the familiar Boltzmann factor

$$\frac{|A(\Omega)|^2}{|A(-\Omega)|^2} = e^{-2\pi\Omega} \quad (18)$$

corresponding to the Davis-Unruh [3] temperature $T = g/2\pi = 1/2\pi$ in our units.[5] This result is equivalent to attributing a temperature $T = 1/2\pi$ to the horizon when viewed from R. The propagation of a particle with energy Ω from a spatial location in F to a spatial location in R can be thought of as an emission of a particle by the horizon surface, since an observer confined to R cannot (classically) detect anything beyond the horizon. By the same token, the propagation of a particle with energy $-\Omega$ can be thought of as the absorption of energy Ω by the horizon. Therefore, we have $P_e/P_a = |A(\Omega)|^2/|A(-\Omega)|^2$, where P_e, P_a denote the probabilities for emission and absorption. On the other hand, if we think of the horizon as a hot surface, with fictitious two-level systems in thermal equilibrium, then $P_e \propto N_{\rm up}$ and $P_a \propto N_{\rm down}$, where $N_{\rm up}$ and $N_{\rm down}$ are the population of the upper and lower levels separated by energy Ω. Therefore, our result in Eq. (18) implies that $N_{\rm up}/N_{\rm down} = e^{-2\pi\Omega}$, showing that the level population of a two-level system on the horizon surface satisfies the Boltzmann distribution corresponding to the temperature $T = 1/2\pi$. This is a more concrete, physical interpretation of the result in Eq. (18).

It is particularly gratifying that the propagator can distinguish so nicely between the propagation across the horizon and the propagation within one side of the horizon. Let us stress how this fact prevents us from interpreting ("understanding") Eq. (18) in a trivial manner. We might think, at first sight, that if we are Fourier transforming G with respect to the *Rindler time* τ (and defining positive/negative energies through $\exp \mp i\Omega\tau$) then it is a foregone conclusion that we will get the thermal factor. *This is simply not true.* Recall that, when we do the Fourier transform with respect to Rindler time, etc., but for two events within the right wedge R, we do not get a thermal factor. So the usual suspect, viz., $\exp -i\Omega t$ (being a superposition of $\exp \mp i\Omega\tau$) is *not* responsible for this result. There are two other crucial ingredients which go into it. First, we need a horizon crossing to break the symmetry between $G(\tau)$ and $G(-\tau)$; this is obtained (as stated above) by the only nontrivial calculation in this paper, leading to Eq. (9). Second, it is crucial that the result in Eq. (9) depends only on the difference $\tau \equiv (\tau_R - \tau_F)$. So when we integrate over all τ, we do not have to worry about what τ_F means, since it is not a time coordinate in F. We can stay in R and interpret everything using τ_R. Therefore, it is not just using the Rindler time coordinate which leads to the result. The structure of the propagator is more nontrivial than one would first imagine.

As far as we know, this is the first study to obtain the thermality of the Rindler horizon directly from the *propagation amplitude across the horizon* in a clean, direct manner, without using Rindler modes, Rindler quantization, the Rindler vacuum, etc. To do this, we *have to* use the Feynman propagator which describes the propagation amplitude; other two-point functions can describe *vacuum correlations* but they do *not* describe the *propagation amplitude*. Previous attempts have obtained thermality either by extracting the spectrum of vacuum fluctuations (as in detector response) or by studying the entanglement and correlations between the R and F wedges (see Ref. [4] as well as Ref. [5]). To make the conceptual difference between these attempts clearer, let me emphasize the physical distinction between vacuum correlations [represented by two-point-functions like the Wightman function $G^+(x_1, x_2)$] and propagation [represented by the Feynman propagator $G(x_1, x_2)$].

The fact that the Feynman propagator evolves positive frequencies forward in time and negative frequencies

[5]The analysis leads to similar conclusions for other situations when the events are separated by a horizon, e.g., between regions P and L. We will concentrate on F and R.

backward in time is crucial for describing *relativistic* propagation. (For more details, see, e.g., Sec 1.5.1 of Ref. [1].) The spatial Fourier transform $G(t,\boldsymbol{k})$ [of the Feynman propagator $G(t,\boldsymbol{x})$] has the factor $\exp(-i\omega_{\boldsymbol{k}}|t|)$, where $\omega_{\boldsymbol{k}} = +\sqrt{\boldsymbol{k}^2+m^2}$. [See e.g., Eq. (1.85) of Ref. [1]]. The modulus sign in $|t|$ is crucial for the interpretation which, in turn, is equivalent to the time ordering of $\phi(x)\phi(y)$ in the vacuum expectation value. On the other hand, the spatial Fourier transform of the Wightman function G^+ has the factor $\exp(-i\omega_{\boldsymbol{k}}t)$ (without the modulus on t), and hence it only has forward-in-time evolution; similarly, G^- only has backward-in-time evolution.[6] Therefore, in quantum field theory (QFT), to study the *propagation* of a particle between events (especially across the horizon) we *must* use $G(x_1,x_2)$; the Wightman function $G^+(x_1,x_2)$ is insufficient and inappropriate. This is also obvious from the following two facts stated right at the beginning of the paper: (a) The path integral for a relativistic particle in Eq. (1) sums over paths that go both backward and forward in time and leads naturally to $G(x_1,x_2)$ [and not to $G^+(x_1,x_2)$]. Similarly, the path-integral average of $\phi(x_1)\phi^\dagger(x_2)$ will also lead to $G(x_1,x_2)$. (b) The emission and absorption of particles by a source $J(x)$ in Eq. (2) are described using $G(x_1,x_2)$ [and not $G^+(x_1,x_2)$]. This is linked to the crucial fact that, in QFT, any source which emits particles *must* also absorb them, which forms a cornerstone of Schwinger's source theory. In our case, the emission and absorption of particles by the horizon involves "backward" propagation from F to R and hence *has to be* discussed in terms of $G(x_1,x_2)$.

It is certainly possible to obtain the Rindler temperature using $G^+(x_1,x_2)$ either (i) in the context of the response of particle detectors or (ii) in terms of the entanglement and correlations between R and F (see Ref. [4] as well as Ref. [5]). In approach (i), the horizon plays no role; the detector will respond in several trajectories which do not asymptote to a horizon because it merely records the spectrum of vacuum fluctuations encoded in $G^+(x_1,x_2)$. In addition, in approach (ii) (adopted in Refs. [4,5]), no *propagation* across the horizon is used or computed anywhere; in fact, $G^+(x_1,x_2)$ is incapable of describing propagation. So, while this approach is interesting and provides a different, complementary perspective of the horizon thermality, it is *distinctly* different from the analysis presented here.

In obtaining this result, we worked entirely in the Lorentzian sector with a well-defined causal structure and the horizons at $x^2-t^2=0$. We have also emphasized the key role played by the horizon in obtaining this result. One may wonder what happens to this analysis if it is done with the inertial propagator in the *Euclidean* sector. In the conventional approach, the right wedge (with $t=\rho\sinh\tau, x=\rho\cosh\tau$) itself will fill the *entire* Euclidean plane (t_E,x_E) if we take $it=t_E, i\tau=\tau_E$, leading to $t_E=\rho\sin\tau_E, x=\rho\cos\tau_E$. The horizons $(x^2-t^2=0)$ map to the origin $(x^2+t_E^2=0)$ and the F, P, and L wedges seem to disappear! At first sight, it is not clear how to recover the information contained in the F, P, and L wedges if we start with the Euclidean, inertial propagator. However, it can be done but one needs to use four different types of analytic continuations to proceed from the Euclidean plane to the four Lorentzian sectors (R, F, L, and P). We describe this briefly in the Appendix for the sake of completeness.

II. THE HORIZON THERMALITY HIDING IN THE INERTIAL PROPAGATOR

Given these facts, let us probe the structure of the inertial propagator a little more closely. While obtaining the above result we did not compute the final integral in Eq. (15) because it was unnecessary. However, this can be done both for events in R and for two events separated by a horizon. It is easier to express the relevant integrals if we first get rid of the transverse coordinates by Fourier transforming both sides of Eq. (3) with respect to the transverse coordinate difference $(\boldsymbol{x}_1^\perp-\boldsymbol{x}_2^\perp)$, thereby introducing the conjugate variable $\boldsymbol{k}_\perp$. [As usual, we will simply write $G^{(RR)}(\tau)$ for $G(\tau;\rho,\rho';\mathbf{k}_\perp)$ when both events are in R, etc.] It can be shown that, when both events are located in R, the relevant Fourier transform in Eq. (3) is given by

$$A_{\mathrm{RR}}(\Omega)=\int_{-\infty}^{\infty}d\tau G^{(RR)}(\tau)e^{i\Omega\tau}$$
$$=\frac{i}{\pi}K_{i\Omega}(\mu\rho_2)K_{i\Omega}(-\mu\rho_1),\quad \tau=(\tau_1-\tau_2),\tag{19}$$

with the ordering $\rho_1<\rho_2$. But if the events are in F and R the corresponding Fourier transform is

$$A_{\mathrm{FR}}(\Omega)=\int_{-\infty}^{\infty}d\tau G^{(FR)}(\tau)e^{i\Omega\tau}$$
$$=\frac{1}{2}H_{i\Omega}^{(2)}(\mu\rho_{\mathrm{F}})K_{i\Omega}(\mu\rho_{\mathrm{R}}),\quad \tau=(\tau_{\mathrm{R}}-\tau_{\mathrm{F}}),\tag{20}$$

where $\mu^2\equiv k_\perp^2+m^2$. (We sketch the derivation in the Appendix. The result is also closely related to the form of the Minkowski-Bessel modes [6] in R and F.) The presence

[6] This difference is very apparent in the case of a *complex* scalar field, but it of course exists for the *real* scalar field as well. For a complex field, written as $\phi(x)\equiv A(x)+B^\dagger(x)$ [where $A(x)$ and $B(x)$ are made of positive-frequency modes], $G(x_1,x_2)=\langle M|A(x_1)A^\dagger(x_2)|M\rangle$ if $x_1^0>x_2^0$ while it is $G(x_1,x_2)=\langle M|B(x_1)B^\dagger(x_2)|M\rangle$ if $x_1^0<x_2^0$; here, $|M\rangle$ is the inertial vacuum state. On the other hand, $G^+(x_1,x_2)$ is always $\langle M|A(x_1)A^\dagger(x_2)|M\rangle$ and misses the antiparticle ("backward-in-time") propagation contained in $B(x)$. That piece of information is contained in the complementary function $G^-(x_1,x_2)$ which will always be $\langle M|B(x_1)B^\dagger(x_2)|M\rangle$, thereby missing the particle ("forward-in-time") propagation; the Feynman propagator has both pieces of information.

of the Hankel function $H^{(2)}_{i\Omega}$ in Eq. (20) [in contrast to the McDonald function in Eq. (19)] makes all the difference because—while the McDonald function has even indices—the Hankel function has the property $H^{(2)}_{i\nu} = e^{-\pi\nu} H^{(2)}_{-i\nu}$. This immediately gives

$$\left[\frac{A(\Omega)}{A(-\Omega)}\right]_{\rm FR} = \frac{H^{(2)}_{i\Omega}}{H^{(2)}_{-i\Omega}} = e^{-\pi\Omega}, \tag{21}$$

which is the same as Eq. (17). On the other hand, because $K_{i\Omega} = K_{-i\Omega}$ we trivially get $A_{\rm RR}(\Omega) = A_{\rm RR}(-\Omega)$. So the explicit computation verifies the previous result but—as we will argue later—the original approach offers greater generality.

This is not the only manner in which the inertial propagator hides the thermal nature of the Rindler horizon. We will give one more example which actually takes us to the Rindler quantization—something we have judiciously avoided so far—from the structure of the inertial propagator. To do this, let us start with the *Euclidean* version of the inertial propagator for two events in R:

$$G^{\rm inertial}_{\rm Eu}(\boldsymbol{k}_\perp; \rho_1, \rho_2, \theta - \theta') = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\nu e^{\pi\nu} K_{i\nu}(\mu\rho_2) K_{i\nu}(\mu\rho_1) e^{-\nu|\theta-\theta'|}. \tag{22}$$

As before, we have already Fourier transformed with respect to the transverse coordinate difference $(\boldsymbol{x}^\perp_1 - \boldsymbol{x}^\perp_2)$, thereby introducing the conjugate variable $\boldsymbol{k}_\perp$. Further, $\mu^2 = k^2_\perp + m^2$. [This expression with the term $|\theta - \theta'|$ is well known in the literature and is very easy to derive. Its derivation can be found in the Appendix, as well as its relation with the form of Eq. (19), which is based on another variant with $(\theta - \theta')$; this one is rather nontrivial to derive.] Using just a series of Bessel function identities and *no physics input*, this result can be reexpressed in the following form:

$$G^{\rm inertial}_{\rm Eu}(\theta - \theta') = \sum_{n=-\infty}^{\infty} G^{\rm Rindler}_{\rm Eu}[\theta - \theta' + 2\pi n], \tag{23}$$

where the function $G^{\rm Rindler}_{\rm Eu}$ is given by

$$G^{\rm Rindler}_{\rm Eu} \equiv \frac{1}{\pi^2}\int_0^{\infty} d\omega (\sinh\pi\omega) K_{i\omega}(\mu\rho) K_{i\omega}(\mu\rho') e^{-\omega|\theta-\theta'|}. \tag{24}$$

This result tells us two things. (a) First, the Euclidean version of our standard inertial propagator can be expressed as an infinite, periodic sum in the (Euclideanized) Rindler time. The fact that the inertial propagator is periodic in (Euclideanized) Rindler time is a trivial result; we only need to note that the $\sigma^2_{\rm RR}$ in Eq. (5) is periodic in $i\tau$. But Eqs. (23) and (24) give us a lot more information. They explicitly express $G^{\rm inertial}_{\rm Eu}$ as an infinite periodic sum of *another specific function*, $G^{\rm Rindler}_{\rm Eu}$. (b) From the product structure of $G^{\rm Rindler}_{\rm Eu}$ we learn that, when analytically continued back to the Lorentzian sector, it can be thought of as a propagator built from another set of mode functions,

$$\phi_\nu(\tau, \rho) = \frac{1}{\pi}(\sinh\pi\nu)^{1/2} K_{i\nu}(\mu\rho) e^{-i\nu\tau}, \tag{25}$$

in the standard fashion with time ordering with respect to τ. This allows us to discover the Rindler mode functions, Rindler vacuum, and the Rindler propagator just from analyzing the inertial propagator and rewriting it as in Eqs. (23) and (24). [Of course, the modes in Eq. (25) satisfy the Klein-Gordon equation and are properly normalized.] Thus, just by staring with the inertial propagator, we can discover the Rindler modes and the Rindler vacuum.

There is another closely related feature. To show this, we will introduce a *reflected* wave function $\phi^{(r)}_\nu$ by the definition

$$\phi^{(r)}_\nu(\rho, \tau) = \phi_\nu(-\rho, \tau - i\pi) = \phi_\nu(\rho^r, \tau^r). \tag{26}$$

The adjective "reflected" is justified by the fact that (i) the coordinates ρ and $-\rho$ are obtained by a reflection through the origin, and (ii) the replacement of τ by $\tau - i\pi$ in the Rindler coordinate transformation takes us from R to L. [If we replace ρ by $-\rho$ and *also replace* τ by $\tau - i\pi$ in the coordinate relations ($x = \rho\cosh\tau$, $t = \rho\sinh\tau$), we will get back to the same event in R. But $\phi^{(r)}_\nu(\rho, \tau) \neq \phi_\nu(\rho, \tau)$, making the reflected wave function different from the original one.] It turns out that the propagator for two events *within the right wedge* can be expressed in a very suggestive form as[7]

$$G^{(RR)} = \int_0^{\infty} d\nu[(n_\nu + 1)\phi_\nu\phi^{(r)}_\nu + n_\nu\phi^*_\nu\phi^{(r)*}_\nu], \tag{27}$$

where n_ν is the thermal population:

$$n_\nu = \frac{1}{e^{2\pi\nu} - 1}. \tag{28}$$

Obviously, the second term in Eq. (27) suggests an absorption process weighted by n_ν, while the first term could represent emission with the factor $n_\nu + 1$ coming from a combination of stimulated emission and spontaneous emission. If we think of ϕ_ν and ϕ^r_ν as the wave functions for a

[7]The proofs for all of these claims, like, e.g., Eqs. (22)–(24) are sketched in the Appendix.

fictitious particle, then this structure again encodes the usual thermality.[8]

Since the Rindler frame is just a coordinate transformation of the inertial frame and the propagator $G(x_1, x_2)$ transforms as a biscalar under a coordinate transformation, we can trivially represent it in Rindler coordinates. Further, because $G(x_1, x_2)$ encodes all of the physics contained in a free field we should be able to discover the thermality just by staring at $G(x_1, x_2)$. In other words, it should not be necessary for us to quantize the field in Rindler coordinates, identify positive-frequency modes, construct the Rindler vacuum and particles, etc. Everything should flow out of $G(x_1, x_2)$ expressed in Rindler coordinates, including the alternative, Rindler quantization. This is what we have achieved in the above discussion.

III. DISCUSSION

A. Comparison with other approaches

There are three other main approaches that follow a similar philosophy—viz., to obtain the Davies-Unruh temperature without using an explicit Rindler quantization—as far as thermality of the horizon is concerned. (None of them, however, take us beyond that to the results we obtained in Sec. II.) The first one is through the response of an Unruh-DeWitt detector in which one merely calculates a Fourier transform of the Wightman function. The second is the path-integral approach used in Ref. [7]. Finally, the horizon tunneling approach (see, e.g., Ref. [8]) has some superficial similarity with the ideas presented above.

The approach in Sec. I of this paper is quite different from all three approaches mentioned above. To begin with, it makes use of the Feynman propagator—the central quantity in QFT—and obtains the thermality from it. We stress that the Feynman propagator has a hidden structure which ensures that the notion of thermality arises when events are separated by a horizon but not otherwise. So the "horizon crossing" plays a crucial but hidden role. This is not the case with the calculation of the detector response. In a calculation confined within R, the exact role (if any) played by the horizon is not obvious. In fact, a detector in any nontrivial trajectory will respond—albeit in a complicated and time-dependent manner—even if there is no horizon. So the superficial similarity, i.e., of evaluating the Fourier transform of a two-point function should not mislead us in this matter.

The path-integral approach in Ref. [7], again, has a superficial similarity with what we have done here. However, there are some significant differences. First, the derivation in Ref. [7] suggests that the probability for the absorption of a particle by a region beyond the horizon is related by a thermal factor to the probability for the emission from that region. This is very different from the interpretation we are trying to advocate. We can just look at the propagation amplitude $A(\Omega)$ in the energy domain and ask how $A(\Omega)$ and $A(-\Omega)$ are related, for propagation between the same pair of events. We again have to stress that the nontrivial structure of the Feynman propagator ensures that when the events are separated by a horizon a thermal relationship arises. Second, the analysis in Ref. [7] crucially used the white hole region (P) to arrive at the conclusion. My approach just uses F and R and hence is conceptually clearer.

Finally, my approach is quite distinct from the standard lore of deriving thermality from horizon tunneling. First, the tunneling approach—like the path-integral approach—tries to relate the amplitude for absorption by F to the emission from F and claims that these two are different because of the pole structure in the complex plane. We did not have to resort to procedures like analytic continuation in the main derivation. Further, it is not very clear how the structure of quantum field theory—encoded in the propagator—is incorporated in the tunneling approach. In contrast, it is very clear in what we have done here.

B. Generalizations

The approach and the result have obvious generalizations to more complicated situations, and we concentrated on the Rindler thermality only to keep things simple. To begin with, the result can be extended to de Sitter spacetime in a straightforward manner because the dependence of the propagator on the geodesic distance (see, e.g., Ref. [9]) allows for the same derivation.[9] More generally, one can use this approach to attribute thermality to any *local Rindler horizon* along the following lines.

In an arbitrary spacetime, pick an event $\mathcal{P}$ and introduce the Riemann normal coordinates around $\mathcal{P}$. These coordinates will be valid in a region $\mathcal{V}$ of size L where the typical background curvature is of the order of L^{-2}. Now introduce a local Rindler coordinate system by boosting with an acceleration g with respect to the local inertial frame, defined in $\mathcal{V}$. If we now concentrate on events (x_1, x_2) within $\mathcal{V}$, then the standard Schwinger-DeWitt

[8] In the usual approach, the Bogoliubov transformation between inertial and Rindler modes involves $|\beta|^2 \sim n_\nu, |\alpha|^2 \sim (1+n_\nu)$ and one can transform $G(x_1,x_2) = \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$ expressed in terms of inertial modes to one involving Rindler modes. This is a way of connecting Eq. (27) with something more familiar. The factors multiplying $(1+n)$ and n can be related to bremsstrahlung by an accelerating source. In fact, both terms will correspond to emission when viewed in the inertial frame.

[9] In the case of curved spacetimes with horizons (Schwarzchild, Reissner-Nordström, etc.), we get the same result by explicit computation in $D = 2$. In $D > 2$, we do not have closed expressions for $G(x, x')$, but we can compute it close to the horizon. This is because, close to the horizon, we again get a two-dimensional conformal field theory and we can compute the approximate form of the modes, and through them the propagator $G(x, x')$. This will lead to the same result.

expansion of the propagator tells us that the form in Eq. (4) will be (approximately) valid. The Fourier integral in Eq. (3) can be defined formally, though the range of τ outside the domain $\mathcal{V}$ is not meaningful. To circumvent this, we have to arrange matters such that most of the contribution to the integral in Eq. (3) comes from the range $\tau \lesssim L$. This, in turn, requires us to concentrate on the high frequencies with $\Omega \gg L^{-1}$. In this high-frequency limit everything will go through as before and one will obtain the local Rindler temperature as $T = g/2\pi$. For consistency, we also need to ensure that $gL \gg 1$ which, of course, can be done around any event with finite L. (In fact, this approach suggests a procedure for obtaining the curvature corrections to the temperature systematically, using the Schwinger-DeWitt expansion.) We stress that—in this very general context of a bifurcate Killing horizon, introduced to a local inertial frame—our approach provides what we would reasonably expect. After all, in a curved spacetime one can expect thermality (with approximately constant temperature) only when the modes do not probe the curvature scale; this is what is achieved by concentrating on the Feynman propagator at two events that are localized within $\mathcal{V}$.

C. Future directions

There are three avenues of further work which seem interesting. The first is to probe the uniqueness of the result in Eqs. (23) and (24). We have shown that, starting from just the Euclidean version of the inertial propagator and the coordinate transformation in the right wedge, one can obtain Eqs. (23) and (24). This is just Bessel function gymnastics with *no physics input.* But the resulting structure in Eq. (24)—involving the product of mode functions and time ordering with respect to τ, when analytically continued back into the Lorentzian sector—immediately suggests an alternative set of mode functions (with a positive/negative frequency decomposition with respect to τ), the corresponding Rindler vacuum, and the Rindler propagator. Then, Eq. (23) tells us that the inertial vacuum will appear as a thermal state in the new representation. The only thing missing is a proof that the form of the infinite periodic sum in Eqs. (23) and (24) is unique. We think that this is true, but it might require some analyticity assumptions.

Second, one might like to probe the details of emission/absorption by localized sources (e.g., on two sides of a horizon) using the expression in Eq. (2) and connect it with the structure of Eq. (27). This will shed more light on how such processes appear in inertial coordinates versus Rindler coordinates. In fact, we expect both processes to appear as emission in the inertial frame.

Third, it will be interesting to see whether the path integral in Eq. (1) can be computed from first principles in the Rindler coordinates. It can be done (even with a nonquadratic action) in inertial coordinates by a lattice regularization [10]. But it is not clear how to introduce a suitable lattice, either in polar coordinates in the Euclidean sector or in the Rindler frame in the Lorentzian sector. These and related issues are under investigation.

ACKNOWLEDGMENTS

I thank Karthik Rajeev for discussions and comments on the draft. My research is partially supported by the J. C. Bose Fellowship of the Department of Science and Technology, Government of India.

APPENDIX: THE UNREASONABLE EFFECTIVENESS OF THE EUCLIDEAN CONTINUATION

We will briefly outline the steps involved in obtaining Eqs. (19), (20), (23), (24), and (27) and some related results, postponing their detailed discussion to another publication. We will now use mostly positive signature so that the analytic continuation of the time coordinate leads to a positive-definite metric.

One can obtain Eqs. (19) and (20) by doing the remaining integral in Eq. (15) (and the analogous one for the RR case) but this requires a fairly complicated manipulation of known integrals over Bessel functions. But, since we also want to describe how to do the analytic continuation from the Euclidean sector to get all four wedges (R, F, L, and P), we will follow an alternative route. We will start from the *Euclidean* propagator and obtain all of the relevant results we need by careful analytic continuation.

The Euclidean (inertial) propagator can be expressed in polar coordinates (with $x = \rho\cos\theta, t_E = \rho\sin\theta$) in the following form:

$$G_{\mathrm{Eu}}(\boldsymbol{k}_\perp;\rho_1,\rho_2,\theta) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\nu e^{\pi\nu} K_{i\nu}(\mu\rho_2)K_{i\nu}(\mu\rho_1)e^{-\nu|\theta|}. \tag{A1}$$

In obtaining this propagator, we have already Fourier transformed with respect to the transverse coordinate difference $(\boldsymbol{x}_1^\perp - \boldsymbol{x}_2^\perp)$, thereby introducing the conjugate variable $\boldsymbol{k}_\perp$. Further, $\mu^2 = k_\perp^2 + m^2$. This result is well known in the literature and is trivial to obtain. One begins by noting that if we Fourier transform the transverse coordinates in the Euclidean version of the propagator in Eq. (4) we just get the reduced (two-dimensional) propagator, viz., $K_0(\mu\ell)/2\pi$, where $\ell = |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|$. One can then use the standard identity

$$\frac{1}{2\pi}K_0(\mu\ell) = \frac{1}{\pi^2}\int_0^{\infty} d\nu K_{i\nu}(\mu\rho_1)K_{i\nu}(\mu\rho_2)\cosh[\nu(\pi - |\theta|)] \tag{A2}$$

to express it as an integral over the range $0 < \nu < \infty$. Extending the integration range to $(-\infty < \nu < \infty)$, we obtain Eq. (A1).

To proceed from Eq. (A2) (which has $|\theta_1 - \theta_2|$) to Eq. (19) or Eq. (20) [which have $(\theta_1 - \theta_2)$], one needs to do the analytic continuation of the variables in a specific way. Let me start with the approach to obtain Eq. (19). Usually, one does the analytic continuation by $\theta_1 \to i\tau_1, \theta_2 \to i\tau_2$ and interprets $|\theta_1 - \theta_2|$ as $i|\tau_1 - \tau_2|$, transferring the ordering to the τ coordinate. This, of course, will give the correct Lorentzian propagator but with an $\exp(-i\nu|\tau_1 - \tau_2|)$ factor. To get $(\tau_1 - \tau_2)$ without the modulus, we need to employ the following[10] analytic continuation: $(\rho_>, \theta) \to (\rho_>, i\tau)$ and $(\rho_<, \theta') \to (-\rho_<, \pi + i\tau)$ with the ordering $\rho_> > \rho_<$. For complex numbers, we will interpret the relative ordering in $|z - z'|$ based on the real parts. This leads to the nice result that we now end up replacing

$$e^{\pi\nu - \nu|\theta - \theta'|} \Rightarrow e^{-i\nu(\tau - \tau')}. \tag{A3}$$

Substituting this into Eq. (A1), one immediately obtains

$$.G_{\text{Min}} = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} K_{i\nu}(\mu\rho_>) K_{i\nu}(-\mu\rho_<) e^{-i\nu(\tau - \tau')}, \tag{A4}$$

from which Eq. (19) follows. This is a simple way to get the result.

In case this feels a bit too simplistic, let us show how we can get this result from published tables of integrals. We again begin by recalling that, when we Fourier transform with respect to transverse coordinates in the *Lorentzian* propagator, we get the two-dimensional result $G_{\text{Min}} = iK_0(\mu\ell)/2\pi$, with $\ell^2 = \rho_<^2 + \rho_>^2 - 2\rho_<\rho_> \cosh(\tau_2 - \tau_1)$ where we have ordered the ρ's as $\rho_> > \rho_<$ for future convenience. [The τ ordering is irrelevant; note that, in Eq. (A4), interchanging τ and τ' corresponds to reversing the sign of ν which makes no difference because $K_{i\nu}$ is an even function of ν.] Next, the integral 6.792 (2) of Ref. [12] gives, as a special case, the result

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} e^{-i\omega\tau} K_{i\omega}(a) K_{i\omega}(b) = K_0(\sqrt{a^2 + b^2 + 2ab\cosh\tau}), \quad (|\arg[a]| + |\arg[b]| + |\text{Im}[\tau]| < \pi). \tag{A5}$$

The left-hand side almost looks like what we want, but on the right-hand side the argument of K_0 has a term with $(+\cosh\tau)$ while our ℓ^2 has $(-\cosh\tau)$. We need to take care of this and also ensure that σ^2 comes up as the limit of $\sigma^2 + i\epsilon$ in the Lorentzian sector [i.e., $\text{Im}(\sigma^2) > 0$]. To this end, we make the following identification in Eq. (A5):

$$a = \mu\rho_< e^{i(\pi - \epsilon)}, \qquad b = \mu\rho_>, \tag{A6}$$

with real τ. Then we have $|\arg[a]| + |\arg[b]| + |\text{Im}[\tau]| = \pi - \epsilon < \pi$, which takes care of the condition in Eq. (A5). Further, we can verify that the ordering $\rho_> > \rho_<$ also ensures that $\text{Im}(\ell^2) > 0$, leading to the correct $i\epsilon$ prescription in the Lorentzian sector. [The sign of the imaginary part is decided by the sign of $(\rho_> \cosh(\tau) - \rho_<)$ which remains positive due to our ordering of the ρ's.] We thus get our advertised result,

$$\frac{i}{2\pi^2} \int_{-\infty}^{\infty} d\omega e^{-i\omega\tau} K_{i\omega}(-\mu\rho_<) K_{i\omega}(\rho_>) = \frac{i}{2\pi} K_0(\mu\ell) = G_{\text{Min}}, \tag{A7}$$

which is the same as Eq. (A4). However, we prefer the simpler derivation.

To obtain the structure in Eq. (20) we need to know how to proceed from the Euclidean sector to the wedge F. This is nontrivial because in the usual procedure of analytic continuation $(\theta \to i\tau)$ we go from $(\rho\sin\theta, \rho\cos\theta)$ to $(i\rho\sinh\tau, \rho\cosh\tau)$, which only covers the right wedge! But one can actually get all four wedges from the Euclidean sector by using the following four sets of analytic continuations (this was discussed in greater detail in Ref. [13]):

$$R\colon \rho \to \rho, \theta \to i\tau, \qquad x = \rho\cosh\tau, t = \rho\sinh\tau, \tag{A8}$$

$$F\colon \rho \to i\rho, \theta \to i\tau + \frac{\pi}{2}, \qquad x = \rho\sinh\tau, t = \rho\cosh\tau, \tag{A9}$$

$$L\colon \rho \to \rho, \theta \to i\tau - \pi, \qquad x = -\rho\cosh\tau, t = -\rho\sinh\tau, \tag{A10}$$

$$P\colon \rho \to i\rho, \theta = i\tau - \frac{\pi}{2}, \qquad x = -\rho\sinh\tau, t = -\rho\cosh\tau. \tag{A11}$$

Now, by using $(\rho,\theta) \to (\rho, i\tau)$ in R and $(\rho,\theta) \to (i\rho, i\tau + \pi/2)$ in F, along with the identity

$$K_{i\nu}(iz) = -\frac{i\pi}{2} e^{-\pi\nu/2} H_{-i\nu}^{(2)}(z) = -\frac{i\pi}{2} e^{\pi\nu/2} H_{i\nu}^{(2)}(z), \tag{A12}$$

one obtains a result similar to Eq. (A4) with a Hankel function replacing one McDonald function. This gives us Eq. (20).

In fact, the analytic continuations in Eqs. (A8)–(A11) allow us to obtain the propagator for any pair of points located in any two wedges directly—and rather easily—from the Euclidean propagator. We get a $K_{i\nu}K_{i\nu}$ structure in RR, LL, RL, and LR. (The notation AB corresponds to the first event being in wedge A and the second in wedge B.) In FF, PP, FP, and PF the McDonald functions are replaced by the Hankel functions. In PR, FL, RF, LP, RP, and LF we get a product of a Hankel function and a McDonald function. The interchange of F with P or R with L reverses the sign of ν, as does the interchange of the two events.

[10] It is straightforward to verify that the coordinates transform correctly from the Euclidean-Rindler coordinates to Lorentzian-Rindler coordinates under this transformation. To get the correct $i\epsilon$ prescription in the Lorentzian sector, it is important to interpret $(-\rho_<)$ as the limit of $\rho_< \exp[i(\pi - \epsilon)]$. This aspect has been noticed previously, in a different context, in Ref. [11].

The similarity to the structure of Minkowski-Bessel modes [6] is obvious. (These results agree with those in Ref. [14], obtained using a more complicated procedure, except for some inadvertent typos in Ref. [14]). We discuss this procedure and results in detail in another publication [13].

We can now obtain Eq. (27), working in the Lorentzian sector, by some further straightforward manipulations. One starts with Eq. (A4) and converts it to an integral over the range $(0 < \nu < \infty)$. Then, using the results

$$n_\nu = \frac{e^{-\pi\nu}}{2\sinh\pi\nu} m \qquad 1 + n_\nu = \frac{e^{\pi\nu}}{2\sinh\pi\nu} m \tag{A13}$$

we can rewrite the propagator as

$$\begin{aligned} G^{(RR)} &= \frac{i}{\pi^2}\int_0^\infty d\nu K_{i\nu}(\mu\rho_>)K_{i\nu}(-\mu\rho_<) \\ &\quad\times \sinh\pi\nu[e^{-\pi\nu}(n_\nu+1)e^{-i\nu\tau} + n_\nu e^{\pi\nu}e^{i\nu\tau}] \\ &= \frac{i}{\pi^2}\int_0^\infty d\nu K_{i\nu}(\mu\rho_>)K_{i\nu}(-\mu\rho_<) \\ &\quad\times \sinh\pi\nu[(n_\nu+1)e^{-i\nu(\tau-i\pi)} + n_\nu e^{i\nu(\tau-i\pi)}]. \end{aligned} \tag{A14}$$

The prefactors (outside the square brackets) lead to the product of the wave functions in Eq. (27) and the shift $(\tau - i\pi)$ leads to the reflected coordinate.

However, the thermal factor in Eq. (27) finds a more natural home in the Euclidean sector. Here we will show how this comes about—using again a set of identities related to Bessel functions—when we work in the Euclidean sector. First, the Euclidean propagator $K_0(\mu\ell)/2\pi$ (obtained after transverse coordinates are removed by a Fourier transform) satisfies a Bessel function addition theorem (see p. 351 (8) of Ref. [15]) given by

$$\begin{aligned} G_E &= \frac{1}{2\pi}K_0(\mu\ell) \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} K_m(\mu\rho_>)I_m(\mu\rho_<)\cos m(\theta-\theta'). \end{aligned} \tag{A15}$$

The $K_m I_m$ part of the above result can be rewritten in terms of another identity [see 6.794(10) of Ref. [12]],

$$\frac{2}{\pi^2}\int_0^\infty d\omega\omega\sinh\pi\omega\frac{K_{i\omega}(\mu\rho)K_{i\omega}(\mu\rho')}{\omega^2+m^2} = K_m(\mu\rho_>)I_m(\mu\rho_<), \tag{A16}$$

which gives

$$G_E = \frac{1}{\pi^3}\sum_{m=-\infty}^{\infty}\int_0^\infty d\omega\omega\sinh\pi\omega\frac{K_{i\omega}(\mu\rho)K_{i\omega}(\mu\rho')}{\omega^2+m^2} \times\cos m(\theta-\theta'). \tag{A17}$$

We can look up the sum in the above expression [see 1.445 (2) of Ref. [12]] and find that it is precisely the thermal factor in Eq. (27) written in the Euclidean sector:

$$\begin{aligned} \mathcal{T}_\omega(\theta-\theta') &\equiv \sum_{m=-\infty}^{\infty}\frac{1}{\pi}\frac{\omega}{\omega^2+m^2}\cos m(\theta-\theta') \\ &= \frac{\cosh\omega(\pi-|\theta-\theta'|)}{\sinh\pi\omega} \\ &= (n_\omega+1)e^{-\omega|\theta-\theta'|} + n_\omega e^{\omega|\theta-\theta'|}. \end{aligned} \tag{A18}$$

This will lead to the Euclidean version of Eq. (27), given by

$$G_E = \frac{1}{\pi^2}\int_0^\infty d\omega(\sinh\pi\omega)K_{i\omega}(\mu\rho)K_{i\omega}(\mu\rho')\mathcal{T}_\omega(\theta-\theta'). \tag{A19}$$

The thermal factor in the Euclidean sector can also be expressed as a periodic sum in the Euclidean angle; that is, we can easily show that

$$\mathcal{T}_\omega(\theta-\theta') = \sum_{n=-\infty}^{\infty} e^{-\omega|\theta-\theta'+2\pi n|}, \tag{A20}$$

thereby making the periodicity in the Euclidean, Rindler time obvious. This is yet another hidden thermal feature of the inertial propagator! This allows us to write the Euclidean, inertial propagator as a thermal sum:

$$G_E = \sum_{n=-\infty}^{\infty}\frac{1}{\pi^2}\int_0^\infty d\omega(\sinh\pi\omega)K_{i\omega}(\mu\rho)K_{i\omega}(\mu\rho') \times e^{-\omega|\theta-\theta'+2\pi n|}. \tag{A21}$$

This equation has a simple interpretation (which we explore extensively in Ref. [13]): on the right-hand side the $n = 0$ terms are just the Euclidean propagator *in the Rindler vacuum*. The periodic, infinite sum "thermalizes" it, thereby producing the inertial propagator.

[1] J. Schwinger, *Particles, Sources, and Fields: Vol. 1* (Perseus Books, USA, 1998); For a textbook discussion see, e.g., T. Padmanabhan, *Quantum Field Theory: Why, What and How* (Springer International Publishing, Switzerland, 2016).

[2] B. S. DeWitt, Quantum gravity: The new synthesis, in *General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), pp. 680–745.

[3] P. C. W. Davies, J. Phys. A **8**, 609 (1975); W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[4] A. Higuchi, S. Iso, K. Ueda, and K. Yamamoto, Phys. Rev. D **96**, 083531 (2017).

[5] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, Rev. Mod. Phys. **80**, 787 (2008).

[6] U. H. Gerlach, Phys. Rev. D **38**, 514 (1988).

[7] J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976).

[8] T. Damour and R. Ruffini, Phys. Rev. D **14**, 332 (1976); K. Srinivasan and T. Padmanabhan, Phys. Rev. D **60**, 024007 (1999); For a review with extensive references, see L. Vanzo, G. Acquaviva, and R. Di Criscienzo, Classical Quantum Gravity **28**, 183001 (2011).

[9] K. Lochan, K. Rajeev, A. Vikram, and T. Padmanabhan, Phys. Rev. D **98**, 105015 (2018).

[10] T. Padmanabhan, Eur. Phys. J. C **78**, 563 (2018); arXiv:1901.05008; for a text book derivation, see Sec. I.6.2 of the second reference in Ref. [1] above.

[11] P. Candelas and D. J. Raine, J. Math. Phys. (N.Y.) **17**, 2101 (1976).

[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 7th ed. (Academic Press, New York, 2007).

[13] K. Rajeev and T. Padmanabhan, arXiv:1906.09278.

[14] D. G. Boulware, Phys. Rev. D **11**, 1404 (1975).

[15] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1995).