Generalized Schwinger effect and particle production in an expanding universe

Karthik Rajeev,[1,*] Sumanta Chakraborty,[2,†] and T. Padmanabhan[1,‡]
[1]IUCAA, Post Bag 4, Ganeshkhind, Pune University Campus, Pune 411007, India
[2]School of Physical Sciences, Indian Association for the Cultivation of Science, Kolkata-700032, India

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We discuss several aspects of particle production in (a) a time-dependent electric field and (b) an expanding Friedmann background. In the first part of the paper, we provide an algebraic mapping between the differential equations describing these two phenomena. This mapping allows a direct comparison between a and b, and we highlight several interesting features of both cases using this approach. We determine the form of the (equivalent) electric field corresponding to different Friedmann spacetimes and discover, e.g., a time-dependent electric field, which, in a specific limit, leads to a Planck spectrum of particles. We also discuss the conditions under which the particle production in an expanding background will be nonanalytic in the parameter which encodes the coupling to the curved spacetime, in close analogy with the generalized Schwinger effect. In the second part of the paper, we study the situation in which both a time-dependent electric field and an expanding background are simultaneously present. We compute particle production rate in this context by several different methods, paying special attention to its limiting forms and possible nonanalytic behavior. We also clarify several conceptual issues related to definitions of in vacuum and out vacuum in these systems.

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I. INTRODUCTION

Two examples of particle production in external backgrounds—discussed extensively in the literature—correspond to (a) quantum field theory of a complex scalar field in an external, homogeneous, electric field [1] and (b) quantum field theory of a scalar field in an expanding Friedmann background [2]. We will be dealing with several aspects of these two systems in this paper.

The first of these two examples includes the famous Schwinger effect [1,3], corresponding to a constant electric field, and its generalizations, which deal with time-dependent electric fields (see, e.g., Refs. [4–13]). In the case of constant electric field, the number density of particles produced from the vacuum has a nonanalytic dependence on the coupling constant, which implies that this result cannot be obtained from perturbative QED. The situation changes when the electric field is time dependent (see, e.g., Refs. [14–18]). Broadly speaking, if the electric field is sharply localized in a time interval, the particle production rate exhibits an analytic dependence on the coupling constant. It is possible to construct examples in which the particle production rate makes a smooth transition between nonanalytic to analytic behavior (with respect to the coupling constant), when a parameter which controls the time dependence of the electric field is varied.

In the first part of this paper, we will show that there exists a *purely algebraic* correspondence between the differential equations governing the scalar field in the cases a and b mentioned above. (Though some authors have noticed a parallel between particle production in an expanding universe and the Schwinger effect in the past—one of the earliest works we know being Ref. [19] and a more recent one being [20]—the utility of this parallel has not been adequately exploited in the literature.) Using this correspondence, it is possible to translate the results obtained in the case of a time-dependent electric field to those in an expanding Friedmann background and vice versa. For example, it turns out that the constant electric field case can be mapped to a radiation dominated universe, while the de Sitter universe maps back to a singular electric field. Further, the Milne universe maps to an electric field in flat spacetime, which produces a Planckian spectrum of particles, in a specific limit, thereby providing yet another "black hole analogue" [21].

Just as the particle production in an external electric field vanishes when a coupling constant in the problem goes to zero, the particle production in an expanding universe will vanish when the parameter describing the expansion of the universe goes to zero, thereby reducing the Friedmann universe to a flat spacetime. (For example, the spatially flat de Sitter universe will become flat

*karthik@iucaa.in
†sumantac.physics@gmail.com
‡paddy@iucaa.in

spacetime when $H \to 0$.) This vanishing of the particle production in the expanding universe can, again, have either an analytic or nonanalytic dependence in the relevant coupling constant, just as it happens in the case of a time-dependent electric field. In a previous work [22], we provided a criterion to distinguish between these two types of behavior for a broad class of time-dependent electric fields. The mapping between the two cases a and b allows us to translate the results in Ref. [22] to the case of particle production in Friedmann universes and obtain a criterion for analytic vs nonanalytic dependence of the coupling constant in this context.

In the second part of the paper, we study situations in which both the time-dependent electric field as well as background expansion are present in the form of a time-dependent electric field in the de Sitter universe. (The effects of pair production in such settings are believed to be relevant in the study of inflationary magnetogenesis; see, e.g., Refs. [23–25]). The particle production in this context should reduce to two previously known limits when we switch off the electric field or the de Sitter expansion. To understand these limits properly, we first describe certain peculiar features which arise in the study of particle production in the de Sitter background. The conventional method for studying the particle production in an expanding background is based on calculating the Bogoliubov coefficients between the in modes and the out modes. The straightforward application of this method works in a de Sitter background only when $(M/H) > 3/2$ (where M is the mass of the scalar field quanta and H is the Hubble constant) and fails when $(M/H) < 3/2$. It turns out that similar issues arise when we study particle production due to the combined effect of electric field and de Sitter expansion. We also pay careful attention to the two limits $E \to 0$ and $H \to 0$ and ensure that our results have the correct limiting forms. (This has not been the case in some of the previous literature in this subject.)

The structure of the paper is as follows. In Sec. II, we show that there is a well-defined algebraic correspondence between the differential equations describing the pair production of a massive scalar field in the Friedmann universe and a massive complex scalar in a homogeneous but time-dependent electric field in Minkowski spacetime. We illustrate this correspondence in physically relevant examples and their useful limiting cases. Using the correspondence established in Sec. II and the techniques developed in Ref. [22] to study the generalized Schwinger effect, we analyze the analytic vs nonanalytic dependence of pair production in Friedmann universe in Sec. III. Finally, we discuss the Schwinger effect in de Sitter in Sec. IV using two approximate methods (namely the Landau procedure and the Euclidean action method) and one exact method (using the mode functions) and compare the results. In Sec. V, we complement the results of Sec. II by studying the pair production in a homogeneous but time-dependent electric field in a de Sitter universe.

The last section summarizes the paper and gives a list of new—conceptual and technical—results obtained in this work.

II. CORRESPONDENCE BETWEEN THE GENERALIZED SCHWINGER EFFECT AND PAIR PRODUCTION IN AN EXPANDING UNIVERSE

Two time-dependent quantum systems of importance, in the discussion of quantum fields in nontrivial backgrounds, are (a) a massive scalar field in the Friedmann universe and (b) a massive complex scalar field in a homogeneous but time-dependent electric field. In this section, we will briefly review the algebraic correspondence between these two and study several properties using this correspondence. One of the common features of these systems is that the time dependence of the background fields (viz., the metric and the electric field) leads to the creation of particle pairs from the vacuum. Though these effects are usually studied separately in the literature for these two systems, it is possible to establish an algebraic mapping between them. This allows us to translate the particle production rate in one case to the same in the other. In this section, we will establish this precise correspondence (see, e.g., Ref. [19]) and demonstrate possible applications.

The Friedmann metric describing a spatially flat, isotropic, and homogeneous universe contains a single time-dependent parameter $a(t)$ and is given by

$$ds^2 = -dt^2 + a^2(t)|d\mathbf{x}|^2. \tag{1}$$

It is convenient to transform to the conformal time coordinate η defined by $d\eta = dt/a(t)$, such that the line element takes the following form,

$$ds^2 = a(\eta)^2(-d\eta^2 + |d\mathbf{x}|^2), \tag{2}$$

which is conformally flat in the η coordinate. We will consider a conformally coupled, massive, scalar field in this background with the action

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}\phi\left[\Box - \frac{R}{6} - M^2\right]\phi. \tag{3}$$

In the massless limit, the presence of the Ricci scalar term ensures that this action is invariant under conformal transformations. Thus, for the conformally flat background given in Eq. (7), when $M = 0$, the above action is equivalent to that of a massless field $\Phi = a\phi$, in flat spacetime. However, when $M \neq 0$, the action in Eq. (3) transforms to the form

$$\mathcal{A} = \frac{1}{2}\int d^4x\Phi[\Box_{\mathrm{flat}} - M^2a^2]\Phi, \tag{4}$$

where $\Box_{\mathrm{flat}}$ is the flat spacetime Laplacian and $\Phi = a\phi$. We will Fourier transform the field Φ in the spatial coordinates and introduce the Fourier modes $\Phi_{\mathbf{k}}$. In terms of $\Phi_{\mathbf{k}}$, the action simplifies to that of a bunch of time-dependent harmonic oscillators, each labeled by the wave number $\mathbf{k}$. The corresponding Lagrangian associated with $\Phi_{\mathbf{k}}$ is given by

$$L_{\mathbf{k}} = \left[\frac{1}{2}|\Phi'_{\mathbf{k}}|^2 - \frac{1}{2}(k^2 + a^2M^2)|\Phi_{\mathbf{k}}|^2\right], \tag{5}$$

where the prime denotes a derivative with respect to the conformal time coordinate η. Variation of the above Lagrangian with respect to $\Phi_{\mathbf{k}}$ yields its equation of motion, which can be written as

$$\Phi''_{\mathbf{k}} + \{k^2 + M^2a^2(\eta)\}\Phi_{\mathbf{k}} = 0, \tag{6}$$

with $k^2 = |\mathbf{k}|^2$. This describes a time-dependent harmonic oscillator of unit mass and frequency $\omega_{\mathbf{k}}(\eta)$ given by

$$\omega^2_{\mathbf{k}}(\eta) = k^2 + M^2a(\eta)^2. \tag{7}$$

We will next describe the correspondence between this differential equation and the one which arises in the case of a time-dependent electric field in flat spacetime. To do this, we will consider a complex scalar field Ψ in the background of a homogeneous, but time-dependent, electric field in flat spacetime (with the metric $ds^2 = -d\eta^2 + d\mathbf{x}^2$) and obtain an algebraic one-to-one correspondence between the two systems. Without loss of generality, let us assume that the electric field is along the z direction and choose the vector potential to be $A_a = \{0, 0, 0, A_z(\eta)\}$. The field equation for the complex scalar field, minimally coupled to this background electric field, is given by

$$(\partial_a - iqA_a)(\partial^a - iqA^a)\Psi - m^2\Psi = 0. \tag{8}$$

As in the previous case, we introduce the Fourier transform $\Psi_{\mathbf{p}}$ of the complex scalar field Ψ, defined by

$$\Psi(\mathbf{x}, \eta) = \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}}\Psi_{\mathbf{p}}(\eta), \tag{9}$$

to describe the dynamics. In the above expression, we have separated the momentum integral into that of the longitudinal (p_z) and the transverse ($\mathbf{p}_\perp \equiv \mathbf{p} - p_z\hat{\mathbf{z}}$) components for later convenience. The equation of motion satisfied by $\Psi_{\mathbf{p}}$ takes the form

$$\Psi''_{\mathbf{p}} + \{m^2 + p^2_\perp + (p_z + qA_z)^2\}\Psi_{\mathbf{p}} = 0. \tag{10}$$

Under the one-parameter family of gauge transformations[1] given by $A_a \to \tilde{A}_a = \{0,0,0,\tilde{A}_z(\eta)\} = \{0,0,0,A_z(\eta) + C\}$, where C is a constant, the complex scalar field transforms to $\tilde{\Psi} = e^{iqCz}\Psi$. This, in turn, implies that the Fourier transforms of $\tilde{\Psi}$ and Ψ are related by $\tilde{\Psi}_{(p_z,\mathbf{p}_\perp)} = \Psi_{(\tilde{p}_z,\mathbf{p}_\perp)}$, where $\tilde{p}_z = p_z - qC$. Therefore, the 3-vector $\mathbf{p}$ does not specify a given physical mode of the complex scalar field in a gauge-invariant way. On the other hand, the "physical momentum" 3-vector $\Pi(\eta_0) \equiv (\Pi_z(\eta_0) = p_z + qA_z(\eta_0), \mathbf{p}_\perp)$, where η_0 is an arbitrary time, can be used to specify a physical mode in a gauge-invariant manner, since under a gauge transformation the quantity $p_z + qA_z(\eta_0)$ remains invariant. Once the gauge is fixed, however, we can always work with $\mathbf{p}$, without any ambiguity.

It is evident that Eq. (10) describes a time-dependent harmonic oscillator of unit mass and time-dependent frequency $\Omega_{\mathbf{p}}(\eta)$, where

$$\Omega^2_{\mathbf{p}}(\eta) = m^2 + p^2_\perp + \{p_z + qA_z(\eta)\}^2. \tag{11}$$

Thus, the Fourier modes of both (i) the conformally coupled, massive, scalar field in an expanding background and (ii) the complex scalar field in a homogeneous electric field satisfy the equation for a time-dependent harmonic oscillator with unit mass. Consequently, for appropriate choice of the parameters and functional forms for $a(\eta)$ and $A_z(\eta)$, Eqs. (6) and (10) can be made mathematically identical. That is, a purely algebraic correspondence between the equations of motion of the two systems exists when the following identification is made:

$$k^2 + M^2a^2(\eta) \rightleftarrows m^2 + p^2_\perp + \{p_z + qA_z(\eta)\}^2. \tag{12}$$

This, in turn, implies that the solutions in one case can be mapped to those in the other. We stress that this is just a useful *algebraic* correspondence, at the level of differential equations governing the Fourier modes, using the principle that same equations have the same solutions. (We do not imply any physical equivalence between the two systems; e.g., parameters which appear in the two cases do not share similar physical interpretation.) In the following subsections, we will investigate what this mapping translates to, in terms of the particle production, using some special cases.

A. Sauter-type electric field and its Friedmann analogue

As a warm-up exercise for the above correspondence between a time-dependent electric field and an expanding universe, we start by discussing the Sauter-type electric field. This is a time-dependent electric field and has the following form:

$$\mathbf{E} = (0, 0, E_0\mathrm{sech}^2(\lambda\eta)). \tag{13}$$

The corresponding vector potential, called the Sauter-type potential, is given by

[1] In the following discussion, unless otherwise specified, by the phrases "gauge transformation" and "gauge-invariant," we mean, respectively, this one-parameter family of gauge transformations and invariance under the same.

$$\mathbf{A} = -\left(\frac{E_0}{\lambda}\right)\tanh(\lambda\eta)\hat{\mathbf{z}} \tag{14}$$

where λ is a constant having dimensions of the inverse of time. The differential equation for the Fourier modes of a complex scalar field in this background electric field can be exactly solved, and hence the particle production rate can be explicitly calculated. The result is given by (see, e.g., Refs. [26,27]),

$$n_{\mathbf{k}} = \frac{\cosh^2\left[\frac{\pi}{\lambda}\sqrt{(\frac{qE_0}{\lambda})^2 - 1}\right] - \sinh^2[\frac{\pi}{2\lambda}(\tilde{\omega}_+ - \tilde{\omega}_-)]}{\sinh(\frac{\pi\tilde{\omega}_+}{2\lambda})\sinh(\frac{\pi\tilde{\omega}_-}{2\lambda})}, \tag{15}$$

where the frequencies $\tilde{\omega}_\pm$ are defined as

$$\tilde{\omega}_\pm = \sqrt{m^2 + p_\perp^2 + \left(p_z \mp \frac{qE_0}{\lambda}\right)^2}. \tag{16}$$

We will next discuss how the above expression for the particle number also arises for a conformally coupled massive scalar field in an expanding universe, for a certain choice for the scale factor $a(\eta)$. To keep the discussion somewhat general, we will start with the following form of the scale factor [28],

$$a^2(\eta) = A + B\tanh(\lambda\eta) + C\tanh^2(\lambda\eta), \tag{17}$$

where A, B, and C are dimensionless constants. The number of produced particles in the asymptotic limit, for the scale factor presented in Eq. (17), has been explicitly worked out in Ref. [28] and is given by

$$n_{\mathbf{k}} = \frac{\cosh(2\pi\frac{\omega_-}{\lambda}) + \cosh\left(\pi\sqrt{\frac{4M^2C}{\lambda^2} - 1}\right)}{\cosh(2\pi\frac{\omega_+}{\lambda}) - \cosh(2\pi\frac{\omega_-}{\lambda})}, \tag{18}$$

where the frequencies $\omega_\pm$ are defined in terms of the constants A, B, and C appearing in the scale factor and the mass M as

$$\omega_\pm = \frac{1}{2}\left\{\sqrt{k^2 + M^2(A+B+C)} \pm \sqrt{k^2 + M^2(A-B+C)}\right\}. \tag{19}$$

To algebraically map the time-dependent frequency $\omega_{\mathbf{k}}(\eta)$, with the scale factor $a(\eta)$ as in Eq. (17), to $\Omega_{\mathbf{p}}(\eta)$ for the Sauter-type potential, we start by making the following choice for the arbitrary constants:

$$A = \left(\frac{p_z}{m}\right)^2; \qquad B = -2\sqrt{AC}; \qquad C = \left(\frac{qE_0}{m\lambda}\right)^2. \tag{20}$$

The scale factor, after this identification, becomes $a(\eta) = (p_z/m) - (qE_0/m)\tanh(\lambda\eta)$. (Note that in the expression for scale factor p_z now appears purely as a parameter that describes a family of scale factors.) Given this, one can immediately verify, following Eq. (12), that with the identification $k^2 = M^2 + p_\perp^2$ the corresponding vector potential takes the form as in Eq. (14). Once this correspondence is established, we can use Eqs. (18) and (20), to obtain the particle number in an expanding universe:

$$n_{\mathbf{k}} = \frac{\cosh[\frac{\pi}{\lambda}(\tilde{\omega}_+ - \tilde{\omega}_-)] + \cosh\left[\frac{2\pi}{\lambda}\sqrt{(\frac{qE_0}{\lambda})^2 - 1}\right]}{\cosh[\frac{\pi}{\lambda}(\tilde{\omega}_+ + \tilde{\omega}_-)] - \cosh[\frac{\pi}{\lambda}(\tilde{\omega}_+ - \tilde{\omega}_-)]}. \tag{21}$$

This coincides exactly with the number of particles produced in a Sauter potential given in Eq. (14), and the frequencies $\tilde{\omega}_\pm$ are those presented in Eq. (16). This explicitly demonstrates how one may use the correspondence expressed by Eq. (12) to obtain the particle production rate in a given expanding background using the information about the particle production rate in a homogeneous electric field and vice versa.

Before concluding this discussion, let us briefly discuss some of the limiting cases of the particle number in Eq. (18). First, consider the case $B = 0 = C$, in which case the scale factor is a constant. This yields $\omega_+ = \sqrt{k^2 + M^2 A}$ and $\omega_- = 0$, so that the particle production rate vanishes, as it should:

$$\lim_{B,C\to 0} n_{\mathbf{k}} = \frac{1 + \cosh(i\pi)}{\cosh(2\pi\frac{\omega_+}{\lambda}) - 1} = 0. \tag{22}$$

Second, consider the vanishing λ limit, when the scale factor becomes $a(\eta) \sim A + \lambda B\eta + \lambda^2 C\eta^2$ and hence $a(\eta)$ becomes constant as $\lambda \to 0$. Thus, it is expected that the particle production should also vanish in the $\lambda \to 0$ limit. A key question is whether it vanishes in an analytical fashion or nonanalytically in λ. For positive A, B, and C, we obtain $\omega_+ > \omega_-$, and in the vanishing λ limit, we have $\cosh(2\pi\omega_\pm/\lambda) \sim \exp(2\pi\omega_\pm/\lambda)$, as well as $\sqrt{(4M^2C/\lambda^2) - 1} \sim (2M/\lambda)\sqrt{C}$. Thus, in the $\lambda \to 0$ limit, the particle number presented in Eq. (18) becomes

$$n_{\mathbf{k}} \sim \exp\left\{-\frac{2\pi}{\lambda}(\omega_+ - \omega_-)\right\} + \exp\left\{-\frac{2\pi}{\lambda}(\omega_+ - M\sqrt{C})\right\}. \tag{23}$$

Since ω_+ is greater than ω_- as well as $M\sqrt{C}$, it follows that $n_{\mathbf{k}} \to 0$ in the $\lambda \to 0$ limit. However, due to $(1/\lambda)$ dependence in the exponential, it is *nonanalytic* in λ near $\lambda \sim 0$, as evident from Eq. (23).

Finally, let us consider the other extreme, namely, the $\lambda \to \infty$ limit. The scale factor in this limit takes the following form,

$$a(\eta) = \sqrt{A + B\,\mathrm{Sign}(\eta) + C}, \tag{24}$$

where the Sign function is given by $\mathrm{Sign}(x) = 1$ for $x > 0$ and $\mathrm{Sign}(x) = -1$ for $x < 0$. (This scale factor corresponds to a "sudden" expansion at a single epoch.) Following an identical procedure, the particle number in this limit becomes

$$n_{\mathbf{k}} = \frac{\omega_-^2}{\omega_+^2 - \omega_-^2} + \frac{(\pi^2\omega_+^2\omega_-^2 - 3M^4C^2)}{3\lambda^2(\omega_+^2 - \omega_-^2)} + \mathcal{O}(\lambda^{-4}). \tag{25}$$

As evident from the above expression, the particle number $n_{\mathbf{k}}$ is *analytic* near $\lambda \sim \infty$. Thus, we conclude that the particle production in an expanding universe for the scale factor in Eq. (17) may be either analytic or nonanalytic in λ depending upon the limit of λ under consideration.

The behavior of the particle production rate $n_{\mathbf{k}}$ for the last two cases, namely, $\lambda \to 0$ and $\lambda \to \infty$, is closely related to two well-known limiting cases of the Sauter potential. They are, respectively, the following two limits of Eq. (14): (i) $m\lambda/(qE_0) \to 0$, i.e., when the electric field approaches a constant value (the particle production rate in this case is nonanalytic in qE_0), and (ii) $m\lambda/(qE_0) \to \infty$, i.e., when the electric field approaches a sharply localized function in time, like a pulse (the particle production rate in this case is analytic in qE_0). Our mapping allows us to obtain an expanding universe analogue of these limits.

In general, there is no assurance that the scale factor corresponding to an arbitrary electric field configuration is sourced by a physically acceptable matter distribution; this is the situation, e.g., in the case for a localized pulselike electric field obtained from the $m\lambda/(qE_0) \to \infty$ limit of the Sauter-type field. However, it turns out that the constant electric field—with nonanalytic dependence—actually maps to a radiation dominated universe. We shall briefly discuss this situation next.

B. Radiation dominated universe is equivalent to constant electric field

As a second example, we examine whether a Friedmann universe with a scale factor, which can be generated by a physically acceptable source, can be mapped to a constant electric field. For this purpose, let us again consider the $\lambda \to 0$ limit of the scale factor in Eq. (17), but with the following choices of the parameters B and C:

$$B = \frac{a_0\sqrt{A}}{\lambda}; \qquad C = \frac{a_0^2}{4\lambda^2}. \tag{26}$$

Here, a_0 is a constant with dimension of inverse length. In this particular case, after substitution of the previous expressions for B and C, the square of the scale factor becomes

$$a^2(\eta) = \lim_{\lambda\to 0}\left(A + \frac{a_0\sqrt{A}}{\lambda}\tanh(\lambda\eta) + \frac{a_0^2}{4\lambda^2}\tanh^2(\lambda\eta)\right) = \left(\sqrt{A} + \frac{a_0\eta}{2}\right)^2. \tag{27}$$

Therefore, the scalar factor in the conformal time coordinate has the form $a(\eta) = \sqrt{A} + (a_0\eta/2)$. To see what kind of matter fluid may generate the same, let us consider the scale factor to be sourced by an ideal fluid with the equation of state $p = w\rho$, where p is the pressure and ρ is the energy density; then, the scale factor evolves with the conformal time as

$$a(\eta) = (b_0 + b_1\eta)^{\frac{2}{(1+3w)}}, \tag{28}$$

where b_0 and b_1 are two unknown constants of integration. For a radiation dominated universe, we have $w = 1/3$, so that the scale factor becomes $a(\eta) \propto \eta$, which immediately connects to the scale factor given by Eq. (27). Thus, the choices made in Eq. (26) for the constants B and C correspond to a radiation dominated universe. In this case, we have the following limiting behavior for the particle number in the asymptotic limit (see the Appendix A for details),

$$\lim_{\lambda\to 0} n_{\mathbf{k}} = \exp\left(-\frac{2\pi k^2}{Ma_0}\right), \tag{29}$$

where $k^2 = |\mathbf{k}|^2$. When $a_0M \to 0$, we expect the particle production to vanish because (i) as $a_0 \to 0$ the spacetime is flat and (ii) as $M \to 0$, because of the conformal coupling, the scalar field in the background of a flat Friedmann metric is equivalent to that in Minkowski spacetime. The particle production rate indeed drops to zero as $a_0 \to 0$ in Eq. (29), but in a nonanalytic fashion. An identical scenario arises in the context of constant electric field as well, where the particle number is nonanalytic in the coupling constant. This analogy can in fact be made more precise in light of the correspondence given in Eq. (12), with the following identification,

$$k^2 \rightleftarrows p_\perp^2 + m^2; \qquad M\sqrt{A} \rightleftarrows p_z; \qquad \frac{Ma_0}{2} \rightleftarrows qE_0, \tag{30}$$

so that the particle number in Eq. (29) takes the following familiar form:

$$n_{\mathbf{p}} = \exp\left[-\frac{\pi(p_\perp^2 + m^2)}{qE_0}\right]. \tag{31}$$

The particle number $n_{\mathbf{p}}$, clearly, matches that in the context of a constant electric field and is nonanalytic in qE_0. It is interesting to see that two of the important cases of pair production, namely, the Schwinger effect and pair creation in a radiation dominated universe, which are seemingly different, are related in a very simple manner [19,22,28]. Next, we will seek a time-dependent electric field configuration that corresponds to a de Sitter or quasi–de Sitter spacetime.

C. De Sitter universe is equivalent to a singular electric field

The above analysis shows that the well-known case of the Schwinger effect can be mapped to a radiation dominated universe and the nonanalytic behavior of the particle number holds true in the radiation dominated universe as well. We will next discuss the mapping in the reverse direction; i.e., we will start from a well-known expanding universe, namely, de Sitter, and then study the form of the electric field it maps to. To keep the discussion slightly general, we will start with a generalization of the de Sitter spacetime, described by the following scale factor:

$$a(\eta) = \left(a_0 + \frac{1}{1 - H\eta}\right). \tag{32}$$

This metric approaches (i) the Minkowski metric, except for some rescaling, when $|H\eta| \gg 1$ and (ii) the de Sitter metric as $H\eta \approx 1$ or as $a_0 \to 0$. From the Friedmann equations, we can determine the density ρ and pressure p of the ideal fluid that can act as the source for this geometry. They are given by

$$\begin{aligned} \rho(a) &\equiv \frac{3}{8\pi G}\frac{1}{a^4}\left(\frac{da}{d\eta}\right)^2 = \frac{3H^2}{8\pi G}\frac{(a-a_0)^4}{a^4}; \\ p(a) &\equiv -\frac{1}{8\pi G}\left\{3\frac{(da/d\eta)^2}{a^4} + \frac{2}{a}\frac{d}{d\eta}\left(\frac{(da/d\eta)}{a^2}\right)\right\} \\ &= -\frac{3H^2}{8\pi G}\frac{(a+a_0/3)(a-a_0)^3}{a^4}. \end{aligned} \tag{33}$$

The density and pressure vanish as $a \to a_0$ (equivalently, as $\eta \to -\infty$), as expected, since the spacetime approaches Minkowski space in this limit. On the other hand, as $a \to \infty$ (equivalently, as $\eta \to H^{-1}$), the density and pressure approach constant values such that $\rho = -p = (3H^2)/(8\pi G)$, like in the de Sitter spacetime. Note that the scale factor has a lower bound, namely, a_0, a feature that emerges in theories with certain modifications of general relativity (see, for instance, Ref. [29]).

Now, from Eq. (6), we find that the Fourier modes of a massive conformally coupled scalar field in this background satisfy the following differential equation:

$$\Phi_{\mathbf{k}}'' + \left[k^2 + M^2\left(a_0 + \frac{1}{1 - H\eta}\right)^2\right]\Phi_{\mathbf{k}} = 0. \tag{34}$$

We can simplify this differential equation by introducing two new parameters κ and μ, as well as a new variable z, such that

$$\kappa = \frac{i a_0 M^2}{H\sqrt{a_0^2 M^2 + k^2}}; \qquad \mu = \sqrt{\frac{1}{4} - \frac{M^2}{H^2}};$$
$$z = \frac{2i(Ht - 1)\sqrt{a_0^2 M^2 + k^2}}{H}. \tag{35}$$

In terms of these variables, Eq. (34) takes the form

$$\frac{d^2\Phi_{\mathbf{k}}}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right)\Phi_{\mathbf{k}} = 0, \tag{36}$$

The solutions to this differential equation can be written in terms of Whittaker functions $W_{\kappa,\mu}(z)$ and $M_{\kappa,\mu}(z)$. In particular, the "in" modes, which are the solutions to Eq. (36) that behave as positive frequency functions near $\eta \to -\infty$, are given by $W_{\kappa,\mu}(z)$. One can verify this by looking at the behavior of $\phi_{\mathbf{k}(\mathrm{in})}$ near the asymptotic past:

$$\phi_{\mathbf{k}(\mathrm{in})} \sim e^{-i(a_0^2 M^2 + k^2)^{1/2}\eta}; \quad \eta \to -\infty. \tag{37}$$

The "out" modes, on the other hand, are oscillatory and thus well defined only when $(M^2/H^2) > 1/4$, a result which arises repeatedly in the context of de Sitter spacetime. (We will comment on this feature, in detail, later on.) The parameter μ becomes purely imaginary in this case, so that we can write $\mu = i|\mu|$ and the out modes $\phi_{\mathbf{k}(\mathrm{out})}$ turn out to be proportional to $M_{\kappa,i|\mu|}(z)$. From the asymptotic expansion of the Whittaker function, it follows that, the out modes take the following form at late times,

$$\phi_{\mathbf{k}(\mathrm{out})} \sim e^{-i|\mu|Ht}; \quad \eta \to H^{-1}, \tag{38}$$

where t is the cosmic time and related to the conformal time η through the well-known relation, $dt = a(\eta)d\eta$. The in modes and out modes introduced above are related through a Bogoliubov transformation of the following form,

$$\phi_{\mathbf{k}(\mathrm{out})} = \alpha_{\mathbf{k}}\phi_{\mathbf{k}(\mathrm{in})} + \beta_{\mathbf{k}}\phi^*_{\mathbf{k}(\mathrm{in})}, \tag{39}$$

where $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are the Bogoliubov coefficients. To find the explicit expressions for $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$, we can use the following relation involving the Whittaker functions,

$$\begin{aligned} M_{\kappa,\mu}(z) = &\frac{\Gamma(2\mu+1)e^{i\pi(\kappa-\mu-\frac{1}{2})}}{\Gamma(\mu+\kappa+\frac{1}{2})}W_{\kappa,\mu}(z) \\ &+ \frac{\Gamma(2\mu+1)e^{i\pi\kappa}}{\Gamma(\mu-\kappa+\frac{1}{2})}W_{-\kappa,\mu}(-z) \end{aligned} \tag{40}$$

$$\equiv \mathcal{A}_{\mathbf{k}}W_{\kappa,\mu}(z) + \mathcal{B}_{\mathbf{k}}W_{-\kappa,\mu}(-z), \tag{41}$$

where the last line defines the constants $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{B}_{\mathbf{k}}$, respectively. It is then straightforward to see that the

Bogoliubov coefficients are given in terms of the constants $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{B}_{\mathbf{k}}$ such that

$$\alpha_{\mathbf{k}} = \frac{\mathcal{A}_{\mathbf{k}}}{\sqrt{|\mathcal{A}_{\mathbf{k}}|^2 - |\mathcal{B}_{\mathbf{k}}|^2}}; \qquad \beta_{\mathbf{k}} = \frac{\mathcal{B}_{\mathbf{k}}}{\sqrt{|\mathcal{A}_{\mathbf{k}}|^2 - |\mathcal{B}_{\mathbf{k}}|^2}}. \tag{42}$$

Further, the number (density) of particles produced in the asymptotic future can be expressed as

$$n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2 = e^{-\pi(|\kappa|+|\mu|)} \frac{\cosh\{\pi(|\kappa| - |\mu|)\}}{\sinh\{2\pi|\mu|\}}. \tag{43}$$

Recall that the $a_0 \to 0$ limit of the scale factor in Eq. (32) describes an exact de Sitter spacetime. The number of particles produced in this limit is given by

$$\lim_{a_0\to 0} n_{\mathbf{k}} = \frac{1}{e^{2\pi|\mu|} - 1}. \tag{44}$$

This matches with the well-known result in the literature [30,31] (Also, see Appendix B 1, where we will explore a closely related case in more details). In the limit $H \to 0$, the scale factor becomes constant, and hence we expect zero particle production. This comes out naturally from Eq. (43), since the rhs in the $|\mu| \to \infty$ (equivalent to the $H \to 0$ limit) identically vanishes. We also see that this vanishing occurs through a nonanalytic dependence in the parameter H. On the other hand, the above formula is not applicable for the $M \to 0$ limit, as the mode functions at late times will not be of oscillatory nature in this case. (We will discuss this feature in detail later on.) One can, of course, work out the $M = 0$ case separately and prove that the particle production vanishes. This is consistent, since there cannot be any particle production for a massless, conformally coupled scalar field in a conformally flat spacetime.

We will now determine an electric field configuration that, in accordance with Eq. (12), corresponds to the quasi–de Sitter metric that we have introduced. The form of the scale factor suggests that the vector potential will be of the following form:

$$A_z(\eta) = \frac{E_0}{\omega(1-\omega\eta)} - \frac{E_0}{\omega}. \tag{45}$$

By demanding that the time-dependent frequencies $\omega_{\mathbf{k}}(\eta)$ and $\Omega_{\mathbf{p}}(\eta)$ be algebraically the same, we arrive at the following identification between the parameters of the two scenarios:

$$p_z - \frac{qE_0}{\omega} = Ma_0; \qquad m^2 + p_\perp^2 = k^2; \qquad \omega = H; \qquad \frac{qE_0}{\omega} = M. \tag{46}$$

The time-dependent electric field turns out to be singular at $\eta = \omega^{-1}$, where it diverges quadratically. Its explicit form is given by

$$\mathbf{E} = \left(0, 0, \frac{E_0}{(1-\omega\eta)^2}\right). \tag{47}$$

This describes a family of electric fields parametrized by E_0 and ω. (It is tempting to interpret the parameter ω as the inverse of time at which the electric field diverges. But by virtue of a shift in the time coordinate, the point of divergence can be shifted to any arbitrary instant in time.) It turns out that the above electric field satisfies the following condition:

$$\frac{1}{4}\frac{(\partial_\eta|\mathbf{E}|)^2}{|\mathbf{E}|^3} = \text{constant} = \frac{\omega^2}{E_0} \equiv \sigma. \tag{48}$$

Let us now determine the number of particles $n_{\mathbf{p}}$, associated with a complex scalar field, produced in this electric field background during the period $-\infty < \eta < \omega^{-1}$. First, note that the parameters κ and μ defined in Eq. (35) are replaced by the following expressions,

$$\kappa = \frac{iqE_0(p_z - qE_0\omega^{-1})}{\omega^2\sqrt{(p_z - qE_0\omega^{-1})^2 + p_\perp^2 + m^2}}; \qquad \mu = \sqrt{\frac{1}{4} - \frac{q^2}{\sigma^2}}, \tag{49}$$

and the condition $\mu = i|\mu|$ translates to $\sigma < 2q$. Having identified the parameters that are related to each other in either side of the correspondence, we can use Eq. (43) to show that the particle number for the above time-dependent electric field becomes

$$n_{\mathbf{p}} = \frac{\cosh\left[\pi\left(\frac{qE_0(p_z - qE_0\omega^{-1})}{\omega^2\sqrt{(p_z-qE_0\omega^{-1})^2+p_\perp^2+m^2}} - \sqrt{\frac{(qE_0)^2}{\omega^4} - \frac{1}{4}}\right)\right]}{\sinh\left[2\pi\sqrt{\frac{(qE_0)^2}{\omega^4} - \frac{1}{4}}\right]} \times \exp\left[-\pi\left(\frac{qE_0(p_z - qE_0\omega^{-1})}{\omega^2\sqrt{(p_z - qE_0\omega^{-1})^2 + p_\perp^2 + m^2}} + \sqrt{\frac{(qE_0)^2}{\omega^4} - \frac{1}{4}}\right)\right]. \tag{50}$$

The following aspects are worth noticing as regards this result:

(a) In the $\omega \to 0$ limit, the electric field approaches a constant, and the particle number density $n_{\mathbf{k}}$ approaches the Schwinger result, which is expected.

(b) The $p_z \to qE_0/\omega$ limit (which is the analogue of pure de Sitter spacetime) gives

$$\lim_{p_z \to (qE_0/\omega)} n_{\mathbf{p}} = \frac{1}{\exp\left[2\pi\sqrt{\frac{(qE_0)^2}{\omega^4} - \frac{1}{4}}\right] - 1}. \tag{51}$$

Let us discuss the equivalent of the $H \to 0$ limit in this context. As the correspondence in Eq. (49) shows, this is achieved by taking the following two limits: $\omega \to 0$ as well as $qE_0 \to 0$, keeping $M = (qE_0/\omega)$ finite. As evident from Eq. (51), the particle number identically vanishes in this limit, which is what we expect in the limit of vanishing electric field.

(c) On the other hand, the above estimation for particle number density is not applicable for the $qE_0 \to 0$ limit, as σ diverges, rendering the above analysis inapplicable. This is identical to the massless, conformally coupled, limit of de Sitter. The particle number indeed vanishes in this limit, but this case needs to be worked out separately.

(d) It is instructive to rewrite Eq. (50) in a gauge-invariant manner. In order to do that, we first define the gauge-invariant physical momentum at the asymptotic past by $\Pi = (p_z + qA_z(-\infty), \mathbf{p}_\perp)$. It is easy to see that $\Pi_z = (p_z - qE_0/\omega)$ and $\mathbf{\Pi}_\perp = \mathbf{p}_\perp$. This motivates us to define an "energy" for each mode by $\epsilon_{\mathbf{p}} = \sqrt{\Pi^2 + m^2}$, where $\Pi^2 = |\mathbf{\Pi}|^2$. The particle production rate can then be written as

$$n_{\mathbf{p}} = \frac{\cosh\left(\frac{\pi q \Pi_z}{\sigma \epsilon_{\mathbf{p}}} - \pi|\mu|\right)}{\sinh(2\pi|\mu|)} \exp\left[-\frac{\pi q \Pi_z}{\sigma \epsilon_{\mathbf{p}}} - \pi|\mu|\right]. \tag{52}$$

When $\Pi \gg m$, i.e., in the ultrarelativistic limit, the above expression approximates to

$$n_{\mathbf{p}} \approx \frac{\cosh\left(\frac{\pi q \cos\theta}{\sigma} - \pi|\mu|\right)}{\sinh(2\pi|\mu|)} e^{-\left(\frac{\pi q \cos\theta}{\sigma} + \pi|\mu|\right)}, \quad \Pi \gg m, \tag{53}$$

where θ is the angle between electric and the physical momentum Π. It is interesting to note that, in the ultrarelativistic limit, the leading order particle production depends only on the direction of physical momentum and is independent of its magnitude.

To summarize, we have shown that the particle production by a scalar field in an expanding Friedmann spacetime, that smoothly extrapolates from Minkoswki space to the de Sitter universe, can be algebraically mapped to that of a complex scalar field in the background of a singular electric field that diverges quadratically at a certain instant of time. The special case of particle production in de Sitter spacetime (i.e., $a_0 = 0$), under this map, translates to the case of particle production by the complex scalar field with a certain value of the component of canonical momentum along the electric field (i.e., $p_z = qE_0\omega^{-1}$).

D. Electric field that produces Planck spectrum of particles

The background geometries which produce a Planck spectrum of particles (e.g., black hole spacetimes) are of considerable importance in the study of quantum field theory in curved spacetime. This prompts us to ask if there is a time-dependent electric field in the flat spacetime which produces a Planck spectrum of particles.

It is well known that, for a suitable vacuum choice, the Milne universe does lead to a Planck spectrum of particles at late times [28,32]. Therefore, the corresponding electric field will lead to the same result. We briefly mention this result here, postponing detailed discussion of this "black hole analogue model" to a future work [21]. The scale factor for the Milne universe is given by $a(t) = \mathcal{H}t$, where $\mathcal{H}$ has inverse dimensions of time. [The standard Hubble parameter is $(\dot{a}/a) = 1/t$, and hence $\mathcal{H}$ is *not* the Hubble parameter.] The passage to conformal time is straightforward, and one obtains $\mathcal{H}t = \exp(\mathcal{H}\eta)$, such that $t = 1/\mathcal{H}$ corresponds to $\eta = 0$ and $t = 0$ relates to $\eta = -\infty$. Thus, the scale factor in conformal time reads $a(\eta) = \exp(\mathcal{H}\eta)$. As we will see, it is convenient to generalize the discussion slightly and consider the scale factor of the following form:

$$a(\eta) = a_0 + e^{\mathcal{H}\eta}. \tag{54}$$

Notice that the scale factor reduces to that of the Milne universe in the $a_0 \to 0$ limit. Moreover, the spacetime corresponding to the above scale factor smoothly extrapolates from a Minkowski spacetime (when, $\eta < 0$ and $|\mathcal{H}\eta| \gg 1$) to a Milne universe (when $\eta > 0$ and $|\mathcal{H}\eta| \gg 1$). For a pure Milne universe (i.e., $a_0 = 0$), it can be shown that the density of produced particles [28,32] is Planckian,

$$n_{\mathbf{k}} = \frac{1}{\exp(\frac{2\pi k}{\mathcal{H}}) - 1}, \tag{55}$$

with the temperature given by $T = \mathcal{H}/(2\pi)$.

To study the particle production in the "generalized Milne" universe, corresponding to the scale factor in Eq. (54), let us first consider the equation of motion of the Fourier mode $\Phi_{\mathbf{k}}$ in this background, which is given by

$$\Phi_{\mathbf{k}}'' + [k^2 + M^2(a_0 + e^{\mathcal{H}\eta})^2]\Phi_{\mathbf{k}} = 0. \tag{56}$$

It is convenient at this stage to define a new dependent variable $\xi_{\mathbf{k}}$, such that

$$\Phi_{\mathbf{k}} = e^{-\frac{1}{2}\mathcal{H}\eta}\xi_{\mathbf{k}}, \tag{57}$$

and a new independent variable z by

$$z = \frac{2ie^{\mathcal{H}\eta}M}{\mathcal{H}}, \tag{58}$$

so that Eq. (56) reduces the standard form of Whittaker's differential equation:

$$\frac{d^2\xi_{\mathbf{k}}}{dz^2} + \left(-\frac{1}{4} + \frac{1/4-\mu^2}{z^2}\right)\xi_{\mathbf{k}} = 0. \tag{59}$$

Hence, the general solution to Eq. (56) can be written in terms of the Whittaker functions $W_{\kappa,\mu}(z)$ and $M_{\kappa,\mu}(z)$. It turns out that the out modes $\phi_{\mathbf{k}(\mathrm{out})}$ are given by

$$\phi_{\mathbf{k}(\mathrm{out})} \propto e^{-\frac{1}{2}\mathcal{H}\eta}W_{\kappa,\mu}(z). \tag{60}$$

One can verify this by noting that the late time behavior of $\phi_{\mathbf{k}(\mathrm{out})}$ turns out to be

$$\phi_{\mathbf{k}(\mathrm{out})} \propto e^{-\frac{1}{2}\mathcal{H}\eta}e^{-i\frac{M}{\mathcal{H}}e^{\mathcal{H}\eta}+i\frac{a_0M}{\mathcal{H}}\eta}; \quad \eta\to\infty. \tag{61}$$

Clearly, this mode behaves as a positive frequency solution at late times and hence qualifies as the out modes. On the other hand, in the early times, $\phi_{\mathbf{k}(\mathrm{out})}$ has the following limiting behavior,

$$\phi_{\mathbf{k}(\mathrm{out})} \propto \tilde{\alpha}_{\mathbf{k}}e^{-i\sqrt{k^2+M^2a_0^2}\eta} + \tilde{\beta}_{\mathbf{k}}e^{i\sqrt{k^2+M^2a_0^2}\eta}; \qquad \eta\to-\infty, \tag{62}$$

where

$$\tilde{\alpha}_{\mathbf{k}} = e^{\frac{\pi|\mu|}{2}}\frac{\Gamma(2\mu)}{\Gamma(\mu-\kappa+\frac{1}{2})}; \qquad \tilde{\beta}_{\mathbf{k}} = e^{-\frac{\pi|\mu|}{2}}\frac{\Gamma(-2\mu)}{\Gamma(-\mu-\kappa+\frac{1}{2})}. \tag{63}$$

The asymptotic value of particle production rate can then be evaluated to get

$$n_{\mathbf{k}} = \frac{|\tilde{\beta}_{\mathbf{k}}|^2}{|\tilde{\alpha}_{\mathbf{k}}|^2-|\tilde{\beta}_{\mathbf{k}}|^2} = \frac{e^{\frac{2\pi}{\mathcal{H}}(a_0M+\sqrt{k^2+M^2a_0^2})}+1}{e^{\frac{4\pi}{\mathcal{H}}\sqrt{k^2+M^2a_0^2}}-1}. \tag{64}$$

From Eq. (62), we can see that the in mode labeled by $\mathbf{k}$ has the energy $\epsilon_{\mathbf{k}} = \sqrt{k^2+M^2a_0^2}$. Hence, in the ultrarelativistic limit, given by $\epsilon_{\mathbf{k}} \gg Ma_0$, the particle spectrum in Eq. (64) approximates to

$$n_{\mathbf{k}} \approx \frac{1}{e^{\frac{2\pi}{\mathcal{H}}\epsilon_{\mathbf{k}}}-1}. \tag{65}$$

This corresponds to a Planckian distribution with the temperature $T=\mathcal{H}/(2\pi)$. Note that, in the special case of $a_0=0$, which corresponds to the pure Milne universe, the spectrum is *exactly* Planckian.

The correspondence of scale factor in Eq. (54) with an electric field can be easily achieved, by virtue of Eq. (12), which gives

$$k^2 = m^2 + p_\perp^2; \qquad qA_z(\eta) + p_z = M(a_0 + e^{\mathcal{H}\eta}). \tag{66}$$

Hence, the associated electric field becomes

$$E_z(\eta) = E_1\exp(\mathcal{H}\eta); \qquad E_1 = \frac{M\mathcal{H}}{q}. \tag{67}$$

Thus, at $\eta\to-\infty$, we obtain $E(\eta)=0$, while for $\eta=0$, we have $E(\eta)=E_1$. Therefore, using Eqs. (64) and (66), the number density of the quanta of a complex scalar field, produced due to the coupling with this time-dependent electric field, is given by

$$n_{\mathbf{p}} = \frac{e^{\frac{2\pi}{\mathcal{H}}(\Pi_z+\epsilon_{\mathbf{p}})}+1}{e^{\frac{4\pi}{\mathcal{H}}\epsilon_{\mathbf{p}}}-1}, \tag{68}$$

where we have defined the gauge-invariant physical momentum $\mathbf{\Pi}$ and the energy $\epsilon_{\mathbf{p}}$ for each modes, respectively, by $\Pi \equiv (p_z + qA_z(-\infty), \mathbf{p}_\perp)$ and $\epsilon_{\mathbf{p}} = \sqrt{\Pi^2+m^2}$. For small values of longitudinal physical momentum, i.e., for $\Pi_z \ll \epsilon_{\mathbf{p}}$, the particle spectrum approximates to

$$n_{\mathbf{p}} \approx \frac{1}{e^{\frac{2\pi}{\mathcal{H}}\epsilon_{\mathbf{p}}}-1}, \tag{69}$$

which corresponds to a Planckian distribution with temperature $T=\mathcal{H}/(2\pi)$. The expression for T is reminiscent of that of a fictitious de Sitter spacetime with Hubble parameter $\mathcal{H}$. [We also note that T is the Davies-Unruh temperature corresponding to the asymptotic acceleration $g=(qE_1/M)$. However, M is not the mass of the complex scalar field under consideration but that of the scalar field in the generalized Milne universe; so, this interpretation of T as a Davies-Unruh temperature is incorrect.] The fact that a thermal spectrum can be generated from a homogeneous but time-dependent electric field is an interesting result by itself and definitely needs further study [21].

In the $\mathcal{H}\to 0$ limit, the electric field in Eq. (67) approaches a constant. Therefore, we expect that the particle rate in the $\mathcal{H}\to 0$ limit approaches the Schwinger result. In order to see that this is indeed the case, we choose the vector potential to be of the following form:

$$A_z(\eta) = -\frac{E_1}{\mathcal{H}}(e^{\mathcal{H}\eta}-1). \tag{70}$$

Clearly, the $\mathcal{H}\to 0$ limit of this potential is given by $-E_1\eta$, as is desired. The explicit expression for the longitudinal physical momentum of a mode labeled by $\mathbf{p}$, in this gauge, becomes $\Pi_z = p_z + qE_1/\mathcal{H}$. The particle number density, given by Eq. (68), can then be rewritten as

$$n_{\mathbf{p}} = \frac{\exp\left\{\frac{2\pi}{\mathcal{H}}\left[(p_z + qE_1/\mathcal{H})^2 + \sqrt{m^2 + p_\perp^2 + (p_z + qE_1/\mathcal{H})^2}\right]\right\} + 1}{\exp\left[\frac{4\pi}{\mathcal{H}}\sqrt{m^2 + p_\perp^2 + (p_z + qE_1/\mathcal{H})^2}\right] - 1}. \quad (71)$$

In the small $\mathcal{H}$ limit, the above expression, to the leading order, reduces to $n_{\mathbf{p}} = \exp[-\pi(m^2 + p_\perp^2)/(qE_1)]$, which matches exactly with the Schwinger result.

We will conclude this section with a brief comment regarding the correspondence between the electric field and more general expansion factors of the universe, for the sake of completeness. Consider a universe sourced by matter with the equation of state $p = w\rho$, with constant w ($\neq -1$). Then, the scale factor behaves as $a(t) = (t/t_0)^{\frac{2}{3(1+w)}}$. The conformal time is

$$\eta = \frac{3(1+w)}{1+3w} t_0^{\frac{2}{3(1+w)}} t^{\frac{1+3w}{3(1+w)}} \quad (72)$$

so that $t/t_0 = (\eta/\eta_0)^{3(1+w)/(1+3w)}$. The scale factor, in terms of the conformal time, is then $a(\eta) = (\eta/\eta_0)^{2/(1+3w)}$. The mode functions then satisfy the following differential equation:

$$\Phi_{\mathbf{k}}'' + \left\{\mathbf{k}^2 + M^2\left(\frac{\eta}{\eta_0}\right)^{\frac{4}{(1+3w)}}\right\}\Phi_{\mathbf{k}} = 0. \quad (73)$$

The corresponding vector potential for the equivalent electric field is easy to find using Eq. (12). We get

$$\mathbf{k}^2 = m^2 + p_\perp^2; \qquad M\left(\frac{\eta}{\eta_0}\right)^{\frac{2}{(1+3w)}} = qA_z(\eta) + p_z \quad (74)$$

so that the electric field becomes

$$E_z = -\frac{2M}{(1+3w)q}\frac{1}{\eta_0}\left(\frac{\eta}{\eta_0}\right)^{\frac{1-3w}{1+3w}}. \quad (75)$$

Thus, most of our discussion can be generalized to this case as well, when the mode functions are known. Unfortunately, the closed form solution to Eq. (73) is known only in a few special cases. Thus, we will not pursue this analogy any further. We will next take up more general features suggested by the mapping between the time-dependent electric field and the expanding universe.

III. PERTURBATIVE VS NONPERTURBATIVE LIMITS OF PARTICLE PRODUCTION

In an earlier work [22], we studied pair production in a homogeneous electric field background with the emphasis on analytic vs nonanalytic dependence of the asymptotic particle number on the coupling constant q. In that case, we could obtain two distinct general classes of electric field configurations that exhibit, respectively, analytic and nonanalytic behavior in the coupling constant. In this section, we will explore the implications of these results for particle production in an expanding universe using the correspondence discussed in Sec. II.

Recall that the time-dependent harmonic oscillator equation satisfied by $\Phi_{\mathbf{k}}$ is given by

$$\partial_\eta^2\Phi_{\mathbf{k}} + k^2\left[1 + \frac{a^2(\eta)}{\gamma^2}\right]\Phi_{\mathbf{k}} = 0; \qquad \gamma = \frac{k}{M}. \quad (76)$$

Let us now study the solutions of this equation in two regimes: (i) when a^2/γ^2 is "small" and can be considered as a perturbation and (ii) when a^2/γ^2 cannot be treated as a perturbation. The precise meaning of these conditions will become clear as we proceed.

A. Perturbative limit

Let us consider a regime of expansion in which the scale factor is bounded from above by some value, such that

$$a(\eta) \leq a_{\max} \ll \gamma. \quad (77)$$

In this case, the time-dependent term in Eq. (76) can be treated as a perturbation. For a Friedman universe expanding monotonically from a singularity ($a = 0$), there always exists an epoch in which this condition holds. The particle number obtained may then be interpreted as the instantaneous particle number at the end of this epoch. We can then expand $\Phi_{\mathbf{k}}$ as

$$\Phi_{\mathbf{k}}(\eta) = \Phi_{\mathbf{k}(0)} + \frac{1}{\gamma}\Phi_{\mathbf{k}(1)} + \frac{1}{\gamma^2}\Phi_{\mathbf{k}(2)} + \cdots \quad (78)$$

We seek a solution $\phi_{\mathbf{k}}$ to Eq. (76) that behaves as $e^{-ik\eta}$ as $\eta \to -\infty$. Using standard perturbative analysis techniques, we find that

$$\Phi_{\mathbf{k}}(\eta) = e^{-ik\eta} - \frac{k}{\gamma^2}\int_{-\infty}^{\eta} d\eta' \sin[k(\eta - \eta')]a^2(\eta')e^{-ik\eta'} + \mathcal{O}(\gamma^{-4}). \quad (79)$$

The asymptotic behavior of this solution to leading order in γ^{-1} is then given by

$$\phi_{\mathbf{k}}(\eta) = \begin{cases} e^{-ik\eta}; & \eta \to -\infty \\ \mathcal{A}e^{-ik\eta} + \mathcal{B}e^{ik\eta}; & \eta \to \infty, \end{cases} \quad (80)$$

where $\mathcal{A} = 1 + \mathcal{O}(\gamma^{-2})$ and $\mathcal{B} = (i\pi k/\gamma^2)\chi(2k)$ with

$$\chi(\mu) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} a^2(\eta) e^{-i\mu\eta} \tag{81}$$

being the Fourier transform of the conformal factor. The number of particles $n_{\mathbf{k}}$ produced at the asymptotic future, to leading order in γ^{-1}, can then be calculated as

$$n_{\mathbf{k}} = |\mathcal{B}|^2 = \left(\frac{\pi M^2}{k}\right)^2 |\chi(2k)|^2 + \mathcal{O}(\gamma^{-4}). \tag{82}$$

For an example, let us apply this result to the large λ limit of Eq. (17), in which case the scale factor is given by Eq. (24). In this case, the Fourier transform in Eq. (81) can be easily evaluated to get $\chi(2k) = iB/(2k\pi)$. Hence, the leading order particle number, from Eq. (82), takes the form

$$n_{\mathbf{k}} = \left(\frac{BM^2}{2k^2}\right)^2 + \mathcal{O}(\gamma^{-4}). \tag{83}$$

One can easily verify that this is consistent with leading order behavior of $n_{\mathbf{k}}$ given in Eq. (25).

B. Nonperturbative limit

We will next consider the more interesting case of the nonperturbative limit. The idea is to translate the procedure adopted in Ref. [22], for a time-dependent electric field, to the expanding universe case. This arises when the scale factor is such that $a(\eta) \gg \gamma$ and $|\eta| > \eta_c$. (That is, at some critical value of time $\eta = \eta_c$, the perturbative analysis, discussed in Sec. III A, fails.) We will further assume that the scale factor is changing adiabatically in the asymptotic past and future, i.e.,

$$\left|\frac{a'}{Ma^2}\right| \ll 1; \quad |\eta| > \eta_a > \eta_c, \tag{84}$$

where η_a is another critical time. This condition enables us to perform Wentzel-Kramers-Brillouin (WKB) analysis for finding the asymptotic solution of Eq. (76). The time-dependent frequency of the oscillator $\Phi_{\mathbf{k}}$ from Eq. (76) can now be expanded as

$$\omega_{\mathbf{k}}(\eta) = \frac{ak}{\gamma} + \frac{\gamma k}{2a} + \mathcal{O}(\gamma^2). \tag{85}$$

Motivated by the correspondence—between electric field and expanding universe backgrounds—that we discussed above and the nonperturbative analysis of the electric field case in Ref. [22], we will assume the following asymptotic behavior for the scale factor for $|\eta| \gg \eta_c$,

$$(i)\ a(\eta) \sim \sum_{n=0}^{N} \mathcal{C}_n |\mathcal{H}\eta|^{2n-1} \tag{86}$$

$$(ii)\ \frac{1}{a(\eta)} \sim \sum_{n=-(N-1)}^{\infty} \tilde{\mathcal{C}}_n |\mathcal{H}\eta|^{2n-1}, \tag{87}$$

for some positive integer N. (These correspond to the conditions (35) and (36) in Ref. [22].) The positive frequency modes of the asymptotic past ($\phi_{\mathbf{k}(\mathrm{in})}$) and future ($\phi_{\mathbf{k}(\mathrm{out})}$), in the WKB approximation, can then be written as

$$\phi_{\mathbf{k}(\mathrm{in})} \sim \left(\frac{\gamma}{ak}\right)^{1/2} \exp\left[i\int_{-\eta_0}^{\eta} d\eta' \left(\frac{a(\eta')k}{\gamma}\right) + i\int_{-\eta_0}^{\eta} d\eta' \left(\frac{\gamma k}{2a(\eta')}\right)\right]; \quad \eta \to -\infty \tag{88}$$

$$\phi_{\mathbf{k}(\mathrm{out})} \sim \left(\frac{\gamma}{ak}\right)^{1/2} \exp\left[-i\int_{-\eta_0}^{\eta} d\eta' \left(\frac{a(\eta')k}{\gamma}\right) - i\int_{-\eta_0}^{\eta} d\eta' \left(\frac{\gamma k}{2a(\eta')}\right)\right]; \quad \eta \to \infty, \tag{89}$$

where we have assumed that $a(\eta) > 0$ for $\eta \gg \eta_c$. Let us use Eqs. (86) and (87) to simplify the argument of exponential factors in $\phi_{\mathbf{k}(\mathrm{in})}$ and $\phi_{\mathbf{k}(\mathrm{out})}$, yielding

$$\int_{-\eta_0}^{\eta} d\eta' \left(\frac{a(\eta')k}{\gamma}\right) \sim \frac{k\log(\mathcal{H}\eta)}{\mathcal{H}\gamma} + \left(\frac{k}{\mathcal{H}\gamma}\right)\sum_{n\neq 0} \frac{\mathcal{C}_n (\mathcal{H}\eta)^{2n}}{2n}; \quad \eta > 0 \tag{90}$$

$$\int_{-\eta_0}^{\eta} d\eta' \left(\frac{\gamma k}{2a(\eta')}\right) \sim \frac{\gamma k\log(\mathcal{H}\eta)}{2\mathcal{H}} + \left(\frac{\gamma k}{2\mathcal{H}}\right)\sum_{n\neq 0} \frac{\tilde{\mathcal{C}}_n (\mathcal{H}\eta)^{2n}}{2n}; \quad \eta > 0. \tag{91}$$

Subsequently we can use these expressions to rewrite $\phi_{\mathbf{k}(\mathrm{in})}$ and $\phi_{\mathbf{k}(\mathrm{out})}$ as

$$\phi_{\mathbf{k}(\mathrm{in})} \sim \left(\frac{\gamma}{|a(\eta)|k}\right)^{1/2} \exp\left[i\left(\left\{\frac{k\mathcal{C}_0}{\mathcal{H}\gamma} + \frac{\gamma k\tilde{\mathcal{C}}_0}{2\mathcal{H}}\right\}\log(-\mathcal{H}\eta) + \sum_{n\neq 0}\left\{\frac{k\mathcal{C}_n}{\mathcal{H}\gamma} + \frac{\gamma\tilde{\mathcal{C}}_n k}{2\mathcal{H}}\right\}\frac{(\mathcal{H}\eta)^{2n}}{2n}\right)\right]; \quad \eta \to -\infty \tag{92}$$

$$\phi_{\mathbf{k}(\mathrm{out})} \sim \left(\frac{\gamma}{a(\eta)k}\right)^{1/2} \exp\left[-i\left(\left\{\frac{k\mathcal{C}_0}{\mathcal{H}\gamma} + \frac{\gamma k\tilde{\mathcal{C}}_0}{2\mathcal{H}}\right\}\log(\mathcal{H}\eta) + \sum_{n\neq 0}\left\{\frac{k\mathcal{C}_n}{\mathcal{H}\gamma} + \frac{\gamma\tilde{\mathcal{C}}_n k}{2\mathcal{H}}\right\}\frac{(\mathcal{H}\eta)^{2n}}{2n}\right)\right]; \quad \eta \to \infty. \tag{93}$$

Since both $\{\phi_{\mathbf{k}(\mathrm{in})}, \phi^*_{\mathbf{k}(\mathrm{in})}\}$ and $\{\phi_{\mathbf{k}(\mathrm{out})}, \phi^*_{\mathbf{k}(\mathrm{out})}\}$ are a set of linearly independent solutions of Eq. (76), we can expand $\phi_{\mathbf{k}(\mathrm{in})}$ in terms of $\{\phi_{\mathbf{k}(\mathrm{out})}, \phi^*_{\mathbf{k}(\mathrm{out})}\}$,

$$\phi_{\mathbf{k}(\mathrm{in})} = \mathcal{A}_{\mathbf{k}}\phi_{\mathbf{k}(\mathrm{out})} + \mathcal{B}_{\mathbf{k}}\phi^*_{\mathbf{k}(\mathrm{out})}, \tag{94}$$

where $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{B}_{\mathbf{k}}$ are the Bogoliubov coefficients. To determine the particle production rate, we need to evaluate $\mathcal{B}_{\mathbf{k}}$.

We will now find an approximate expression for $\mathcal{B}_{\mathbf{k}}$ using the asymptotic expressions for the in and out modes. This can be done by a procedure, originally due to Landau, which we will call the Landau procedure. (This was used earlier in Ref. [22] in the case of a time-dependent electric field, wherein more details can be found. We will not repeat the technical details here.) To use the Landau procedure, we will interpret η as a complex variable in Eq. (92). In essence, the procedure amounts to rotating η in the complex plane from $\arg[\eta] = 0$ to $\arg[\eta] = \pi$. We can see that under this transformation the asymptotic expression for $\phi_{\mathbf{k}(\mathrm{in})}$ near $\eta \to -\infty$ transforms to that of $\phi^*_{\mathbf{k}(\mathrm{out})}$ near $\eta \to \infty$, except for a constant factor. In view of Eq. (94), we can immediately interpret this factor as the Bogoliubov coefficient $\mathcal{B}_{\mathbf{k}}$, which reads

$$\mathcal{B}_{\mathbf{k}} \approx e^{i\pi} \exp\left[-\pi\left(\frac{k\mathcal{C}_0}{\mathcal{H}\gamma} + \frac{\gamma k\tilde{\mathcal{C}}_0}{2\mathcal{H}}\right)\right]. \tag{95}$$

The number of particles produced can then be computed in a straightforward manner as

$$n_{\mathbf{k}} = |\mathcal{B}^2_{\mathbf{k}}| = \exp\left[-2\pi\left(\frac{k\mathcal{C}_0}{\mathcal{H}\gamma} + \frac{\gamma k\tilde{\mathcal{C}}_0}{2\mathcal{H}}\right)\right]. \tag{96}$$

Thus, to the leading order, the nonanalytic dependence of particle production rate is controlled by the two constants: $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$. We will now illustrate this result with two examples discussed earlier.

Example 1.—For a first example of this procedure, consider the physically important case of a locally de Sitter metric, with the scale factor, in some appropriate interval being given by

$$a(|\eta|) = \frac{1}{1 + H|\eta|} \qquad \frac{1}{a(\eta)} = 1 + H|\eta| \tag{97}$$

so that the relevant constants appearing in the expression for particle number, taking a cue from Eqs. (86) and (87), yield $\mathcal{C}_0 = 1$, $\tilde{\mathcal{C}}_0 = 0$, and $\mathcal{H} \to H$. Thus, the number of particles produced, according to Eq. (96), is given by

$$n_{\mathbf{k}} \approx \exp\left[-\frac{2\pi M}{H}\right], \tag{98}$$

which, as we will see in the next section, is consistent with the large mass limit of the exact value of $n_{\mathbf{k}}$. If we treat M as the energy, this is just the Boltzmann limit of a Planck spectrum at temperature $T = H/2\pi$.

Example 2.—For a second example, consider the scale factor given by Eq. (27). The asymptotic expansion now reads $a(\eta) \approx (|a_0\eta|/2)$, and hence the inverse scale factor reads $(1/a(\eta)) \approx (2/|a_0\eta|)$. With the identification $a_0 \to \mathcal{H}$, the coefficients $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ in this case are given by 0 and 2, respectively. The particle number then becomes

$$n_{\mathbf{k}} \approx \exp\left[-\frac{2\pi k^2}{Ma_0}\right], \tag{99}$$

which matches exactly with Eq. (29). This explicitly demonstrates the usefulness of this approach.

This concludes the first part of this work, related to the correspondence between a time-dependent electric field and the expanding universe. We next want to study the case of a time-dependent electric field in an expanding de Sitter background wherein both of these effects will be present. (For completeness, in Appendix B, we briefly discuss the case of particle definition and production in the de Sitter universe, in order to clarify/highlight several conceptual and technical features.)

IV. CONSTANT ELECTRIC FIELD IN DE SITTER

In the following sections, we are going to explore pair production when there is both an expanding scale factor as well as a time-dependent electric field. However, for the sake of simplicity, we will be mainly concerned with a scale factor that corresponds to the de Sitter universe.

The particle number in the context of de Sitter spacetime is fairly well studied (see Appendix B for a brief review). There has been some interest in the literature [30,31,33,34] to study the particle number when a constant electric field is present in de Sitter spacetime. Our main aim is to determine the particle number in the context of a time-dependent electric field in de Sitter; however, it is useful to discuss the case of a constant electric field first. We will work exclusively in the spatially flat Friedmann universe, expressed in terms of the conformal time η, in which case the scale factor $a(\eta)$ is given by Eq. (B1). The constant electric field must be defined in a covariant way, and the most natural choice is $F^{\mu\nu}F_{\mu\nu} = \text{constant} \equiv -2E_0^2$. We will assume that the field is along the z direction and is described by the (spatial) vector potential, $\mathbf{A} = \{0, 0, A_z(\eta)\}$. This implies, given the above definition of constant electric field, that $A_z(\eta)$ must satisfy the following differential equation: $\partial_\eta A_z = -a^2 E_0$. This equation can be immediately integrated, yielding the following expression for the vector potential:

$$A_z = -\int \frac{E_0 d\eta}{(1-H\eta)^2} = -\frac{E_0}{H}\frac{1}{(1-H\eta)} + \text{constant}. \quad (100)$$

The constant of integration must be chosen carefully such that in the $\eta \to 0$ limit $A_z(\eta)$ turns out to be finite. This fixes the constant to be (E_0/H). With this choice for the constant, the vector potential turns out to be

$$A_z = -\frac{E_0}{H}\frac{H\eta}{1-H\eta}. \quad (101)$$

Note that when $H = 0$ spacetime becomes flat and this expression reduces to the one in standard flat spacetime Schwinger effect. (If this is not ensured, the final result may not have the correct $H \to 0$ limit, which has happened in some of the previous discussions in the literature.)

As an aside, we will comment on the source for the electromagnetic field which is usually not stressed in the literature. In general, a time-dependent electric field gives rise to a magnetic field. If that is the case, we need to take into account the effect of the magnetic field in the computation of particle production. However, in the present context, the vector potential (assumed to be along the z direction) depends on time alone. Therefore, the only nontrivial component of $F_{\mu\nu}$ corresponds to $F_{0z} = \dot{A}_z = -E_z$ (say, in flat spacetime), and there are no magnetic fields. But for consistency of Maxwell's equations, we *must* have a nonzero current, which is given by $J^\nu = (1/4\pi)\nabla_\nu F^{\mu\nu}$. If we consider time-dependent electric field in a flat spacetime, then it follows that J^z is the only nonzero component, such that $J^z = (1/4\pi)\dot{E}_z$. In the cosmological spacetime as well, we have J^z as the only nonzero component, but its explicit form becomes $J^z = (1/4\pi)a^{-4}(\eta)\partial_\eta E_z$. Hence, in all the cases considered here, there are no magnetic fields, but a nontrivial current must exist to ensure that we have a purely electric field situation. (The role of the current is not usually discussed in the literature—and we will also ignore it—though it may be worth investigating for a more complete picture.)

We now consider a complex scalar field in this background. Its Fourier modes will satisfy the equation of a time-dependent harmonic oscillator, with unit mass and time-dependent frequency. This is essentially a generalization of Eq. (7), which, for this background, yields the time-dependent frequency as

$$\omega_{\mathbf{k}}^2 = k_\perp^2 + \frac{M^2}{(1-H\eta)^2} + \left(k_z + \frac{qE_0}{1-H\eta}\right)^2. \quad (102)$$

It can be easily verified that in the $H \to 0$ limit the frequency becomes $M^2 + k_\perp^2 + (k_z + qE_0\eta)^2$, which is consistent with that of the Schwinger effect discussed in Ref. [22]. On the other hand, in the $qE_0 \to 0$ limit, it immediately follows that the frequency becomes $k^2 + M^2(1-H\eta)^{-2}$, coinciding with the frequency of $\Phi_{\mathbf{k}}$ in the de Sitter spacetime. In what follows, we will study the particle production in this background by three different methods, namely, (i) the Landau procedure, (ii) the Hamilton-Jacobi method, and (iii) using mode functions.

A. Landau procedure

In Sec. III B, we discussed a procedure to calculate the WKB limit of particle production. We will now apply the same techniques—the Landau procedure—to compute particle production for our current case. It turns out that we do not have to redo the whole calculation in Sec. III B to find the generalization of Eq. (96) to the present problem. Instead, we can proceed as follows. Let us first look at the expansion of $\omega_{\mathbf{k}}$ for large $|\eta|$, which takes the form

$$\omega_{\mathbf{k}} \approx \frac{M}{H\eta} + \frac{H\eta}{2M}\{k_\perp^2 + (k_z - qE_0/H)^2\}. \quad (103)$$

Comparing this with Eqs. (86) and (87), we see the following identification of parameters exists:

$$\frac{M}{H} \rightleftarrows \left(\frac{\mathcal{C}_0 k}{\mathcal{H}\gamma} + \frac{\gamma k\tilde{\mathcal{C}}}{2H}\right). \quad (104)$$

Hence, using Eq. (96), the particle number can be immediately written down as

$$n_{\mathbf{k}} = \exp\left[-\frac{2\pi M}{H}\right]. \quad (105)$$

This is, however, only the leading order term, as can be seen from the fact that this expression is independent of the electric field. Thus, the Landau procedure, to the *leading* order, only captures a factor independent of qE_0.

To get the dependence on the electric field and H, then we need to retain more terms in the asymptotic expansion. That is, when $\eta \to \infty$, one must keep $H\eta \to \text{finite} \ll 1$. Then, the expansion in Eq. (103) should be replaced by

$$\omega_{\mathbf{k}} \approx \sqrt{q^2E_0^2 + 3M^2H^2}\,\eta + \frac{M^2H}{\sqrt{q^2E_0^2 + 3M^2H^2}} + \frac{1}{2}\left[\frac{k_\perp^2 + M^2}{\sqrt{q^2E_0^2 + 3M^2H^2}} - \frac{M^4H^2}{(q^2E_0^2 + 3M^2H^2)^{3/2}}\right]\frac{1}{\eta}. \quad (106)$$

Further, the identification in Eq. (104) should be replaced by

$$\frac{1}{2}\left[\frac{k_\perp^2 + M^2}{\sqrt{q^2E_0^2 + 3M^2H^2}} - \frac{M^4H^2}{(q^2E_0^2 + 3M^2H^2)^{3/2}}\right] \equiv \Theta(H, M, qE_0) \rightleftarrows \left(\frac{\mathcal{C}_0 k}{\mathcal{H}\gamma} + \frac{\gamma k\tilde{\mathcal{C}}}{2H}\right). \quad (107)$$

Thus, the particle number density evaluates to

$$n_{\mathbf{k}} = e^{-2\pi\Theta}. \tag{108}$$

This expression has correct limits. In particular, it can easily be verified that, in the $H \to 0$ limit, this expression reduces to

$$n_{\mathbf{k}} = \exp\left[-\frac{\pi(k_\perp^2 + M^2)}{qE_0}\right], \tag{109}$$

which is the standard result in the Schwinger effect.

B. Euclidean action approach

Another elegant method of computing the semiclassical limit of the particle number is by using the Euclidean action. The idea is to first evaluate the action $\mathcal{A}_E$ for an appropriate classical solution of the Euclidean equation of motion for a hypothetical particle. It can be shown that, for the cases which are of interest to us, the particle number, when $\mathcal{A}_E \gg 1$, is given by

$$n \approx \exp(-\mathcal{A}_E). \tag{110}$$

The Euclidean action is most easily computed by solving the Hamilton-Jacobi equation. For our case, let us denote by $\mathcal{A}$ the action for a charged particle in the Friedmann spacetime with the scale factor $a(\eta)$ and constant electric field. The Hamilton-Jacobi equation in this context is given by

$$\frac{1}{a^2}[-\partial_\eta\mathcal{A} + |\partial_{x_\perp}\mathcal{A}|^2 + (\partial_z\mathcal{A} - qA_z)^2] = -M^2a^2. \tag{111}$$

The symmetry of the problem suggests the ansatz, $A_z = k_z z + \mathbf{k}_\perp.\mathbf{x}_\perp + F(\eta)$, where $F(\eta)$ satisfies

$$-(\partial_\eta F)^2 + k_\perp^2 + (k_z - qA_z)^2 = -M^2a^2. \tag{112}$$

This can be integrated to give

$$\mathcal{A} = \int d\eta\sqrt{M^2a^2 + k_\perp^2 + (k_z - qA_z)^2} + k_z z + \mathbf{k}_\perp.\mathbf{x}_\perp. \tag{113}$$

In particular, for the de Sitter spacetime, the classical action evaluates to

$$\mathcal{A} = \int \frac{d\eta}{1 - H\eta} \times \sqrt{M^2 + k_\perp^2(1 - H\eta)^2 + \{qE_0\eta + k_z(1 - H\eta)\}^2} + k_z z + \mathbf{k}_\perp.\mathbf{x}_\perp. \tag{114}$$

We are interested in closed classical trajectories (in the Euclidean sector), for which the last two terms in Eq. (114) vanish. Then, a straightforward computation shows that there are two complex turning points, defined by the vanishing of the square root terms in the integrand in Eq. (114). These turning points are given by

$$H\eta_\pm = \frac{k_\perp^2 + k_z(k_z - qE_0H^{-1}) \pm iM\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2 + (qE_0)^2M^{-2}H^{-2}}}{k_\perp^2 + (k_z - qE_0H^{-1})^2}. \tag{115}$$

The expression for number of particles given in Eq. (110) is good approximation only for a sufficiently large value of the Euclidean action A_E. This, in turn, holds for large values of M. Hence, let us look at the turning points in the leading order in M^{-1}, which are located at

$$H\eta_\pm \approx \frac{k_\perp^2 + k_z(k_z - qE_0H^{-1})}{k_\perp^2 + (k_z - qE_0H^{-1})^2} \pm i\frac{M}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}}. \tag{116}$$

The number of particles is related to the *imaginary* action evaluated for the closed classical trajectory that starts at η_- and comes back to that point through η_+. The following parametrization turns out to be a convenient choice for describing this trajectory:

$$H\eta(\theta) = \frac{k_\perp^2 + k_z(k_z - qE_0H^{-1})}{k_\perp^2 + (k_z - qE_0H^{-1})^2} + \frac{iM}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}}\sin\theta; \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right). \tag{117}$$

Substituting in Eq. (114), we can evaluate the action to get

$$\mathcal{A} = \frac{M^2}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}}\int_{-\pi/2}^{3\pi/2}\frac{\cos^2\theta d\theta}{-iA + B\sin\theta}, \tag{118}$$

where

$$A = \frac{qE_0(k_z - qE_0H^{-1})}{k_\perp^2 + (k_z - qE_0H^{-1})^2}$$
$$B = \frac{MH}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}}. \tag{119}$$

Then, the Euclidean action can be evaluated to get

$$\mathcal{A}_E = -i\mathcal{A} = \frac{M^2}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}} \times \left[\frac{2\pi(-A + \sqrt{A^2 + B^2})}{B^2}\right]$$
$$= 2\pi\frac{M}{H} - 2\pi\frac{qE_0}{H^2}\frac{(k_z - qE_0H^{-1})}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}} + \mathcal{O}(M^{-1}). \tag{120}$$

The number of particles is then given by

$$n_{\mathbf{k}} \approx \exp\left[-2\pi\left\{\frac{M}{H} - \frac{qE_0}{H^2}\frac{(k_z - qE_0H^{-1})}{\sqrt{k_\perp^2 + (k_z - qE_0H^{-1})^2}}\right\}\right]. \tag{121}$$

The $qE_0 \to 0$ limit of the above expression can be easily verified to be consistent our discussions so far on particle production in pure de Sitter, for instance, Eq. (B34).

C. Using mode functions

Finally, we will use the appropriate mode functions to calculate the exact expression for the particle number from the Bogoliubov coefficients. We have deliberately discussed approximate methods first to emphasize the elegance and applicability of these approaches to cases when explicit calculations are impossible. Towards the end of this section, we explicitly verify that the exact expression, in fact, reduces to the results derived in Secs. IV A and IV B in the appropriate limits.

The differential equation satisfied by the Fourier transform $\Phi_{\mathbf{k}}$, in an expanding universe, in the presence of an electric field, is given by

$$\frac{d}{d\eta}\left[a^2\frac{d\Phi_{\mathbf{k}}}{d\eta}\right] + \{(\mathbf{k} - q\mathbf{A})^2 + m^2a^2\}\Phi_{\mathbf{k}}. \tag{122}$$

For a constant electric field in de Sitter, we obtain

$$\frac{d^2\Phi_{\mathbf{k}}}{d\eta^2} + \frac{2H}{1 - H\eta}\frac{d\Phi_{\mathbf{k}}}{d\eta} + \left\{k_\perp^2 + \frac{m^2}{(1 - H\eta)^2} + \left(k_z + \frac{qE_0\eta}{1 - H\eta}\right)^2\right\}\Phi_{\mathbf{k}} = 0. \tag{123}$$

Let $\Phi_{\mathbf{k}} = (1 - H\eta)\psi_{\mathbf{k}}$ so that $\psi_{\mathbf{k}}$ satisfies

$$\psi_{\mathbf{k}}'' + \left[k^2 + \frac{2qE_0k_z\eta}{1 - H\eta} + \frac{m^2 - 2H^2 + q^2E_0^2\eta^2}{(1 - H\eta)^2}\right]\psi_{\mathbf{k}} = 0. \tag{124}$$

We can easily verify that this equation has the correct limits. When $H \to 0$, we obtain

$$\psi_{\mathbf{k}}'' + [k^2 + 2qE_0k_z\eta + m^2 + q^2E_0^2\eta^2]\psi_{\mathbf{k}} = 0; \quad (H \to 0), \tag{125}$$

which matches with the time-dependent frequency of the Fourier mode of a complex scalar field in a constant electric field in flat spacetime, in the time-dependent gauge. On the other hand, for $qE_0 \to 0$, we have

$$\psi_{\mathbf{k}}'' + \left[k^2 + \frac{m^2 - 2H^2}{(1 - H\eta)^2}\right]\psi_{\mathbf{k}} = 0; \quad (qE_0 \to 0), \tag{126}$$

which is perfectly consistent with Eq. (B4). Let us introduce a new variable $z = 2ik\eta$ so that Eq. (124) simplifies to

$$\frac{d^2\psi_{\mathbf{k}}}{dz^2} + \left(-\frac{1}{4} + \frac{\xi}{z} + \frac{\frac{1}{4} - \nu^2}{z^2}\right)\psi_{\mathbf{k}} = 0, \tag{127}$$

where

$$\xi = i\frac{(k_z - qE_0/H)}{\sqrt{k_\perp^2 + (k_z - qE_0/H)^2}}\left(\frac{qE_0}{H^2}\right),$$
$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2} - \frac{q^2E_0^2}{H^4}}. \tag{128}$$

The general solution to this equation can be written in terms of the Whittaker functions as

$$\psi_{\mathbf{k}} = C_1W_{\xi,\nu}\left[2i\sqrt{k_\perp^2 + \left(k_z - \frac{qE_0}{H}\right)^2}\left(\eta - \frac{1}{H}\right)\right] + C_2M_{\xi,\nu}\left[2i\sqrt{k_\perp^2 + \left(k_z - \frac{qE_0}{H}\right)^2}\left(\eta - \frac{1}{H}\right)\right]. \tag{129}$$

From the asymptotic expansion of the Whittaker functions, we get

$$W_{\xi,\nu}\left[2i\sqrt{k_\perp^2 + (k_z - qE_0/H)^2}\left(\eta - \frac{1}{H}\right)\right] \approx (H\eta)^\xi \exp\left(-i\sqrt{k_\perp^2 + (k_z - qE_0/H)^2}\eta\right); \quad \text{for } \eta \to -\infty \tag{130}$$

$$M_{\xi,\nu}\left[2i\sqrt{k_\perp^2+(k_z-qE_0/H)^2}\left(\eta-\frac{1}{H}\right)\right] \approx \left[2i\sqrt{k_\perp^2+(k_z-qE_0/H)^2}\left(\eta-\frac{1}{H}\right)\right]^{\nu+1/2}; \quad \text{for } \eta\approx H^{-1}\propto e^{-\nu Ht}, \tag{131}$$

where the last relation is true for the case when ν is purely imaginary, i.e., when $\nu = i|\nu|$. This is similar to the situation in pure de Sitter discussed earlier. Hence, for $\nu = i|\nu|$, we have $M_{\xi,\nu}$ defining the late time vacuum (in terms of cosmic time t) and $W_{\xi,\nu}$ defining the in vacuum (in terms of the conformal time η).

Some comments regarding the nature of the in-vacuum state are appropriate at this stage. In the $\eta\to-\infty$ limit, which is the appropriate initial Cauchy slice for the de Sitter spacetime, the modes used to define the in-vacuum state behave as $\eta^\xi\exp(-ik'\eta)$, where ξ has been defined in Eq. (128). So, this state is similar to the Bunch-Davies vacuum, with two crucial differences:

(i) First, the wave number appearing in the exponential is not the same as that of the Fourier mode $\Phi_{\mathbf{k}}$, as in the de Sitter spacetime, and rather is modified to $k'=\sqrt{k_\perp^2+(k_z-qE_0)^2}$. This modification is due to the appearance of canonical momenta k_i, through the combination k_i-qA_i. Since at the asymptotic past (when $\eta\to-\infty$) the vector potential becomes $qA_z=qE_0/H$, it is the combination $k_\perp^2+(k_z-qE_0/H)^2$ that appears in the solution of the associated mode function.

(ii) Second, the prefactor depends on η through a term approximately η^ξ, where ξ is proportional to the electric field. The presence of this prefactor η^ξ can be understood using the WKB limit. In this limit, the mode function (in the $\eta\to-\infty$ limit) takes the form $\exp(i\int dz\sqrt{(1/4)-(\xi/z)})$. Since z is very large, one can expand it to the leading order, which upon integration yields a term $\exp(-\xi\ln z)$. This leads to the η^ξ term in the mode function. Defining the in vacuum in terms of the exponential part of the mode functions, one immediately observes that these modes indeed carry positive energy like the Bunch-Davies vacuum state, but with a modified wave number k'.

The problem of particle production is mathematically identical to our discussion in Sec. II C. In particular, an argument similar to the one used to derive Eq. (50) can employed here to arrive at the following expression for the number of particles:

$$n_{\mathbf{k}} = \frac{\cosh\left(\pi|\nu|-\frac{\pi qE_0}{H^2}\frac{(k_z-qE_0/H)}{\sqrt{k_\perp^2+(k_z-qE_0/H)^2}}\right)}{e^{2\pi|\nu|}\cosh\left(\pi|\nu|+\frac{\pi qE_0}{H^2}\frac{(k_z-qE_0/H)}{\sqrt{k_\perp^2+(k_z-qE_0/H)^2}}\right)-\cosh\left(\pi|\nu|-\frac{\pi qE_0}{H^2}\frac{(k_z-qE_0/H)}{\sqrt{k_\perp^2+(k_z-qE_0/H)^2}}\right)}. \tag{132}$$

It can be easily verified that we get the correct limiting forms. For, $qE_0\to 0$, we have

$$n_{\mathbf{k}} = \frac{1}{e^{2\pi|\nu|}-1}, \tag{133}$$

which matches with, say, Eq. (B10). On the other hand, if we demand that $M\gg H$, then, Eq. (132) to the leading order is given by

$$n_{\mathbf{k}} \approx e^{-2\pi\left[\frac{M}{H}-\frac{qE_0}{H^2}\frac{(k_z-qE_0H^{-1})}{\sqrt{k_\perp^2+(k_z-qEH^{-1})^2}}\right]}. \tag{134}$$

This is also in perfect agreement with Eq. (121). Finally, the $H\to 0$ limit reduces to

$$n_{\mathbf{k}} = \exp\left[-\frac{\pi(m^2+k_\perp^2)}{qE_0}\right], \tag{135}$$

which is the correct result for the Schwinger effect.

It is worth mentioning that in most of the previous literature the scale factor as well as the vector potential have been chosen in such a form that they diverge in the $H\to 0$ limit [30,31,33]. This makes the interpretation of the particle number in the limit of the vanishing Hubble constant problematic. This is primarily due to the fact that the scale factor and gauge choice for the vector potential did not have the appropriate limiting behavior. Keeping this in mind, in this paper, we have worked with expressions for the scale factor and vector potential which have appropriate limiting behavior. Then, the particle number as well, naturally, leads to the desired expressions for the pure de Sitter and pure Schwinger effect, in the $qE_0\to 0$ and $H\to 0$ limits, respectively. (For a different view of arriving at the appropriate limits, see Ref. [35]).

As remarked in the beginning of this subsection, we cannot analytically solve for the mode functions for the most general, time-dependent, homogeneous electric field configuration in de Sitter spacetime. In such cases, one plausible strategy is to employ numerical techniques. However, the approximate methods discussed in Secs. IV A and IV B give us an elegant analytic handle. In the following section, we will be using one of these approaches, namely, the Landau procedure, to study the generalized Schwinger effect in a de Sitter background.

V. EXAMPLE OF TIME-DEPENDENT ELECTRIC FIELD IN DE SITTER

In the previous section, we determined the particle production of a complex scalar field in a de Sitter background in the presence of a constant electric field. However, in practical situations, the electric field is often not a constant but depends on time. Keeping this in mind, we would like to understand particle production due to a time-dependent electric field in de Sitter, which may also provide us some insight into the nonanalytic vs analytic behavior of the same.

We start by considering a homogeneous electric field-electric field in the de Sitter background, satisfying the following condition,

$$F^{\mu\nu}F_{\mu\nu} = E^2(\omega_0\eta), \tag{136}$$

where the raising and lowering of indices has been performed using the conformally flat form of the metric ansatz, given by Eq. (B1). Assuming, without any loss of generality, that the electric field is in the z-direction, the above equation provides us with $F_{0z} = E(\omega_0\eta)a(\eta)^2$ as the only nonvanishing component. Given the field tensor, the differential equation governing the vector potential can then be expressed as

$$\frac{dA_z}{d\eta} = -\frac{E(\omega_0\eta)}{(1-H\eta)^2}. \tag{137}$$

Determination of the vector potential from the above differential equation requires an integration, and that requires an explicit expression for the time dependence of the electric field. It also requires an additional condition; namely, the vector potential should be finite in the $H \to 0$ limit. To see what this second condition means, let us consider a power law electric field, such that $E(\omega_0\eta) \sim E_0(\omega_0\eta)^{-s}$; then, the vector potential becomes

$$A_z = -\frac{E_0}{\omega_0^s}\int \frac{d\eta}{\eta^s(1-H\eta)^2} = -\frac{E_0}{\omega_0^s}\frac{1}{\eta^s}\left(1+\frac{1}{H\eta-1}\right)^s \times \frac{{}_2F_1(s,1+s,2+s,\frac{1}{1-H\eta})}{(1+s)H(-1+H\eta)} + \text{constant} \tag{138}$$

so that we obtain

$$\lim_{H\to 0} A_z = -\frac{E_0}{\omega_0^s}\frac{1}{\eta^s}(H\eta)^s \frac{{}_2F_1(s,1+s,2+s,1)}{(1+s)H} + \text{constant} \sim -\frac{E_0}{\omega_0^s}H^{s-1} + \text{constant} \tag{139}$$

Thus, for $s \geq 1$, the vector potential is always finite in the $H \to 0$ limit, and we can choose the constant to be vanishing, while for $s \leq 0$, one must take the constant to be $(E_0/\omega_0)H^{s-1}$ to make the vector potential finite in the $H \to 0$ limit.

In what follows, we will concentrate on the electric field of the form $E(\omega_0\eta) = E_0\{1+f(\omega_0\eta)\}$, where $f(\omega_0\eta)$ is some arbitrary function which decays for large η. That is, the electric field becomes a constant at late times. The corresponding vector potential, having a finite $H \to 0$ limit, can be written as

$$A_z = -\frac{E_0\eta}{1-H\eta} - \frac{E_0}{\omega_0}F(\omega_0\eta; H), \tag{140}$$

where the function $F(\omega_0\eta; H)$ satisfies the following differential equation:

$$\frac{dF(s)}{ds} = \frac{f(s)}{(1-\frac{H}{\omega_0}s)^2} \tag{141}$$

It is, of course, convenient to work with F rather than f, which is what we will do.

A complex massive scalar field, in the background of the time-dependent electric field in the de Sitter universe, will have Fourier modes which again satisfy the equation for a time-dependent harmonic oscillator. In this case, the oscillator associated with the kth wave mode will have unit mass and a time-dependent frequency given by

$$\omega_k^2(\eta) = k_\perp^2 + \frac{m^2}{(1-H\eta)^2} + \left(k_z - \frac{qE_0\eta}{1-H\eta} - \frac{qE_0}{\omega_0}F(\omega_0\eta; H)\right)^2. \tag{142}$$

Here, $k_\perp^2 = k^2 - k_z^2$ is the wave vector component transverse to the direction of the electric field. One cannot, of course, solve for the mode functions for arbitrary F. To illustrate the use of the Landau procedure, we shall confine ourselves to a specific choice, viz., $f(\omega_0\eta) = 2f_2(\omega_0\eta)^{-3}\{1-2H\eta\}\{1-H\eta\}^{-1}$, where f_2 is a constant. With this kind of electric field, the contribution to the vector potential becomes $-f_2(\omega_0\eta)^{-2}(1-H\eta)^{-2}$. Thus, the time-dependent frequency in the large η limit, but with small $H\eta$, can be expanded as

$$\begin{aligned}\omega_k^2(\eta) &\approx k_\perp^2 + m^2(1+2H\eta+3H^2\eta^2) + \left(qE_0\eta(1+H\eta+H^2\eta^2) + \frac{qE_0 f_2}{\omega_0^3\eta^2}(1+2H\eta+3H^2\eta^2)\right)^2 \\ &\approx (q^2E_0^2+3m^2H^2)\eta^2 + \left(2m^2H + 2qE_0\frac{3H^2 f_2 qE_0}{\omega_0^3}\right)\eta + (k_\perp^2+m^2) + \left(\frac{3H^2 f_2 qE_0}{\omega_0^3}\right)^2 \\ &\equiv A\eta^2 + B\eta + C. \end{aligned} \tag{143}$$

Here, the last relation defines the constants A, B, and C in terms of the parameters appearing in this model, e.g., the electric field strength E_0, Hubble constant H, inverse time scale ω_0, etc. The corresponding expansion for ω_k is

$$\begin{aligned}\omega_k &= \sqrt{A}\eta\left(1+\frac{B}{A\eta}+\frac{C}{A\eta^2}\right)^{1/2} \\ &= \sqrt{A}\eta\left(1+\frac{B}{2A\eta}+\frac{C}{2A\eta^2}-\frac{1}{8}\frac{B^2}{A^2\eta^2}\right) \\ &= \sqrt{A}\eta + \frac{B}{2\sqrt{A}} + \frac{C}{2\sqrt{A}\eta} - \frac{1}{8}\frac{B^2}{A^{3/2}\eta}. \end{aligned} \tag{144}$$

Having derived this expression, one can invoke the Landau procedure to extract the nonanalytic part of the particle number. This requires one to analytically continue the range of η from $\{-\infty,(1/H)\}$ to $\{-\infty,\infty\}$. Further, using the WKB method, one can determine the in states and out states associated with the Fourier modes at $\eta = \mp\infty$, respectively. The Bogoliubov coefficient connecting them can be obtained by treating η as a complex variable and rotating it in the complex plane from $\mathrm{Arg}[\eta]=0$ to $\mathrm{Arg}[\eta]=\pi$. This provides the nonanalytic part of the particle number to be dependent on the coefficient of $(1/\eta)$ in the expression for ω_k, which reads

$$\begin{aligned} n_k &= \exp\left[-2\pi\left(\frac{C}{2\sqrt{A}}-\frac{1}{8}\frac{B^2}{A^{3/2}}\right)\right] \\ &= \exp\left[-2\pi\left(\frac{(k_\perp^2+m^2)+\left(\frac{3H^2 f_2 qE_0}{\omega_0^2}\right)^2}{2\sqrt{(q^2E_0^2+3m^2H^2)}} - \frac{1}{8}\frac{\left(2m^2H+2qE_0\frac{3H^2 f_2 qE_0}{\omega_0^2}\right)^2}{(q^2E_0^2+3m^2H^2)^{3/2}}\right)\right]. \end{aligned} \tag{145}$$

Note that in the $H\to 0$ limit the particle number becomes $\exp[-(k_\perp^2+m^2)/qE_0]$ irrespective of the presence of f_2. This is what we expect, as the Landau procedure picks up the nonanalytic part which is given by the coefficient of the constant term irrespective of other terms in the expansion. This assures that Landau procedure works in de Sitter spacetime as well and yields the nonanalytic part of the particle number for time-dependent electric fields in de Sitter, while remaining compatible with the flat spacetime limit.

However, the particle number presented above does not yield the de Sitter particle production as the electric field vanishes. This is due to the fact that for the Landau procedure to work we have analytically extended the de Sitter spacetime to cover the full range of η, namely, $\eta \in (-\infty,\infty)$ and hence the background spacetime is not exactly the de Sitter background we want to work with. Besides, this feature is also present in the context of constant electric field, as evident from Eq. (106). This suggests that, even though the Landau procedure is a useful method to understand nonanalytic behavior of the particle production in the time-dependent electric field, it has its limitations when applied in the context of an expanding universe.

VI. SUMMARY

The previous sections discussed several aspects of particle production in an expanding universe and its possible correspondence with the generalized Schwinger effect. Given the fact that both these phenomena have been investigated extensively in the literature, it is useful to highlight the new—conceptual and technical—results in this paper:

(i) The correspondence between the time-dependent electric field and an expanding universe has been noticed earlier, one of the earliest works being Ref. [19] and a more recent one being Ref. [20]. However, this correspondence was noticed at a formal level and was not adequately exploited. In this work, we have taken this further and applied this formalism to connect some well-known cosmological spacetimes to specific time-dependent electric fields and vice versa.

For example, we studied the cosmological analogue of a Sauter-type electric field and showed that the particle number possess nonanalytic behavior as the scale factor approaches that of the radiation dominated universe. Further, through this correspondence, we could provide estimation of the particle number for a nontrivial electric field configuration using our knowledge of the results for the expanding universe. Starting from the de Sitter (or quasi–de Sitter) spacetime, we determined the corresponding electric field and hence the corresponding particle production. We also discovered a time-dependent electric field in flat spacetime, which can lead, in a specific limit, to a Planck spectrum of particles at late times. (This analogue black hole model deserves further exploration especially with regard to back-reaction. To our knowledge, such electric fields have not been explored earlier.) It will also be interesting to take this correspondence further, to the level of two-point functions, and analyze the analogue of inflationary power spectrum in the context of the generalized Schwinger effect.

(ii) In our earlier work [22], we used an asymptotic expansion of the electric field and identified the terms responsible for nonanalytic behavior of the particle number. Through the correspondence between the generalized Schwinger effect and particle production in an expanding universe, we have determined the corresponding factors responsible for the nonanalytic behavior of the particle number in an expanding background. In particular, we have shown that the coefficients of η^{-1} in the expansion of $a(\eta)$ as well as $a^{-1}(\eta)$ control the nonanalytic behavior of the particle number.

(iii) In the last part, we discussed the case of a constant electric field in de Sitter spacetime, using three different approaches, and compared the results. First, we described how the Landau procedure can be used to infer the nonanalytic part of the particle number, and we showed that it reproduces the correct result. Second, we used the Euclidean action approach to obtain the asymptotic limit of the same result. Finally, we studied this case using the conventional approach based on mode functions. In all the cases, we worked in a gauge which allows taking appropriate limits, and we explicitly verified these limits. (This has been an issue in some of the previous works in the literature.)

(iv) Taking a cue from this discussion and our earlier results in Ref. [22], we described how one may go about studying particle production due to a time-dependent electric field in de Sitter. Using the technique due to Landau, we were able to obtain the nonanalytic part of the particle production in the context of a specific time-dependent field in de Sitter. Even though we could retrieve the desired Schwinger result in the appropriate limit, the general structure of the particle number is more complicated and deserves further attention. We hope to study this in a future work.

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APPENDIX A: DERIVATION FOR RADIATION DOMINATED UNIVERSE

Let us begin by looking at the $\lambda \to 0$ limits of $\tilde{\omega}_{\pm}$. We have

$$\begin{aligned}\sqrt{k^2+M^2(A+B+C)} &= \sqrt{k^2+M^2\left(A+\frac{a_0\sqrt{A}}{\lambda}+\frac{a_0^2}{4\lambda^2}\right)} \\ &\simeq \frac{Ma_0}{2\lambda}\left\{1+4\frac{\sqrt{A}}{a_0}\lambda+4\frac{k^2+M^2A}{M^2a_0^2}\lambda^2\right\}^{1/2} \\ &\simeq \frac{Ma_0}{2\lambda}+M\sqrt{A}+\lambda\frac{k^2}{Ma_0}+\mathcal{O}(\lambda^2) \end{aligned} \tag{A1}$$

as well as

$$\sqrt{k^2+M^2(A-B+C)} = \frac{Ma_0}{2\lambda}-M^2\sqrt{A}+\lambda\frac{k^2}{Ma_0}+\mathcal{O}(\lambda^2). \tag{A2}$$

Therefore, the following limits of the characteristic frequencies $\omega_{\pm}$ are obtained:

$$\omega_+ = \frac{Ma_0}{2\lambda}+\lambda\frac{k^2}{Ma_0}+\mathcal{O}(\lambda^2); \qquad \omega_- = M\sqrt{A}+\mathcal{O}(\lambda^2). \tag{A3}$$

Thus, the $\lambda \to 0$ limit of Eq. (18), with the parameters B and C as given in Eq. (26), becomes

$$\lim_{\lambda\to 0} n_{\mathbf{k}} = \frac{\cosh(\frac{2\pi M\sqrt{A}}{\lambda})+\cosh(\frac{2\pi Ma_0}{2\lambda^2})}{\cosh(\frac{2\pi Ma_0}{2\lambda^2}+\frac{2\pi k^2}{Ma_0})-\cosh(\frac{2\pi M\sqrt{A}}{\lambda})} = \exp\left(\frac{-2\pi k^2}{Ma_0^2}\right). \tag{A4}$$

APPENDIX B: PARTICLE PRODUCTION IN DE SITTER SPACETIME: REVISITED

The purpose of this section is to revisit certain aspects of particle production in a de Sitter universe [36]. We will first recall the standard results using the mode functions to compute the exact Bogoliubov coefficients. Next, we will discuss the instantaneous diagonalization method and compare it with a new technique, explored recently in Ref. [37] (also see Refs. [38,39]), that maps *any* time-dependent oscillator to an oscillator with constant frequency. This comparison gives us a handle on several conceptual issues, related to the definition of particles which is inherently ambiguous in curved spacetime.

Before we begin, we would like to make the following cautionary comments to avoid any misunderstanding. It should be stressed that there is *no* unambiguous way to define particles corresponding to a field that is evolving in an arbitrary time-dependent background (see, e.g., Ref. [40]). The best one can do is to just explore different approaches, each of which comes with its own merits and demerits. For an example, consider three standard approaches used in the literature:

(i) The instantaneous diagonalization method, for instance, has the advantage of being simple and generally makes sense when the time evolution is adiabatic. But it runs into serious problems in its physical interpretation in a general context.

(ii) The particle detector approach is another alternative, which has the advantage of being operationally defined. But the detector approach can give misleading results in certain contexts because the spectrum of fluctuations (measured by the detector) does not match the particle content defined by Bogoliubov transformation. For instance, even though a uniformly rotating observer identifies his/her vacuum state with the Minkowski vacuum, an Unruh-Davies detector in circular motion in the Minkowski vacuum has a nonzero excitation rate [41]. There are other issues which arise in a time-dependent context when the detector is not moving along a Killing trajectory.

(iii) The imaginary part of the effective Lagrangian allows us to determine particle production in some other situations, but this requires the existence of well-defined asymptotic vacuum states, something which does not exist in a generic situation (like, e.g., in a FRW universe).

In short, no single method for defining the particle works in all contexts.

In Appendix B 2, we will concentrate on the instantaneous diagonalization method. The purpose of this choice is not to advocate this particular method as unambiguous or better than the rest; as we said above, each method has its own merits and drawbacks. Our purpose is strictly limited to comparing this method and rephrasing it in the context of some recent work [37] (also see Refs. [38,39]). We believe this connection is interesting even though it does not eliminate the basic issues related to the instantaneous diagonalization method.

1. Particle production using the Bogoliubov coefficients

We start by writing down the de Sitter metric in the conformal flat slicing, which takes the following form:

$$ds^2 = \frac{1}{(1-H\eta)^2}(-d\eta^2 + |d\mathbf{x}|^2); \quad -\infty < \eta < H^{-1}. \tag{B1}$$

We will consider particle production due to a massive quantum field living in this background spacetime, the action of which is given by

$$\mathcal{A} = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi - \frac{1}{2}M^2\Phi^2\right], \tag{B2}$$

where g_{ab} corresponds to the metric given in Eq. (B1). [Note that the above action is different from that in Eq. (3), in that the nonminimal coupling term, $R\Phi^2$ is absent. This is because we are now considering the minimally coupled scalar field, which—since R is a constant for de Sitter spacetime—can be generalized in a straightforward manner to arrive at the results of a scalar field with conformal coupling as well.] Let us introduce the Fourier modes $\Phi_\mathbf{k}$ by standard means, in terms of which the action simplifies to that of a bunch of time-dependent harmonic oscillators, each labeled by $\mathbf{k}$, the time-dependent frequency and mass or which are given by

$$\omega_\mathbf{k}^2(\eta) = k^2 + a^2M^2; \qquad m_\mathbf{k}(\eta) = a^2(\eta). \tag{B3}$$

Hence, the equation of motion satisfied by the Fourier mode functions can then be written as

$$\frac{d^2\Phi_\mathbf{k}}{d\eta^2} + \frac{2H}{1-H\eta}\frac{d\Phi_\mathbf{k}}{d\eta} + \left(k^2 + \frac{M^2}{(1-H\eta)^2}\right)\Phi_\mathbf{k} = 0. \tag{B4}$$

The solution $\phi_\mathbf{k}$ to this equation that corresponds to a positive frequency solution in the asymptotic past is given by

$$\phi_{\mathbf{k}(\mathrm{in})}(\eta) = \left(\frac{\pi}{4Ha^{3/2}(\eta)}\right)^{1/2}\mathrm{H}_\nu^{(1)}\left(\frac{k}{Ha}\right), \tag{B5}$$

where

$$\nu = \sqrt{\frac{9}{4} - \frac{M^2}{H^2}}. \tag{B6}$$

This expression for ν tells us that the situation can be quite different depending on whether (M/H) is greater than or less than $(3/2)$, and we will see that this is indeed the case. One can verify that $\phi_{\mathbf{k}(\mathrm{in})}$ is indeed the positive frequency solution in the early time by noting that as $a \to 0$ (or $\eta \to -\infty$)

$$\phi_{\mathbf{k}}(\eta) \approx \frac{e^{-ik\eta}}{\sqrt{2ka^2(\eta)}} \approx \frac{e^{-i\int d\eta \omega_{\mathbf{k}}(\eta)}}{\sqrt{2m_{\mathbf{k}}(\eta)\omega_{\mathbf{k}}(\eta)}}. \tag{B7}$$

On the other hand, in the late time limit, i.e., as $\eta \to -(1/H)$, $\phi_{\mathbf{k}}$ takes the form

$$\begin{aligned}\phi_{\mathbf{k}}(\eta) \approx &-\frac{i\sqrt{H}2^{\nu-1}}{\sqrt{\pi}}\Gamma(\nu)a^{\nu-\frac{3}{2}}\left(\frac{k}{H}\right)^{-\nu} \\ &+\frac{\sqrt{\pi}\sqrt{H}2^{-\nu-1}}{\Gamma(\nu+1)}\{1+i\cot(\pi\nu)\}a^{-\nu-\frac{3}{2}}\left(\frac{k}{H}\right)^{\nu} \\ &= A_{\mathbf{k}}a^{-\nu-3/2} + B_{\mathbf{k}}a^{\nu-3/2}.\end{aligned} \tag{B8}$$

To determine the particle content at late times, we have to somehow interpret these two terms as positive and negative frequency *oscillations*, which, of course, is possible only if they are oscillatory. This, in turn happens when ν is purely imaginary, so we can write $\nu = i|\nu|$. This corresponds to the situation with $M^2/H^2 > 9/4$, when we can interpret Eq. (B8) as a linear combination of positive and negative frequency modes in the asymptotic future. (The oscillations are with respect to $\ln a \propto t$, the cosmic time, in the asymptotic future, while the oscillations are with respect to conformal time η in the asymptotic past.) In this case, one can read off the Bogoliubov coefficients to be

$$\alpha_{\mathbf{k}} = \frac{A_{\mathbf{k}}}{\sqrt{|A_{\mathbf{k}}|^2 - |B_{\mathbf{k}}|^2}}; \qquad \beta_{\mathbf{k}} = \frac{B_{\mathbf{k}}}{\sqrt{|A_{\mathbf{k}}|^2 - |B_{\mathbf{k}}|^2}}. \tag{B9}$$

The number of particles can then be computed as

$$n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2 = \frac{1}{e^{2\pi|\nu|} - 1}, \tag{B10}$$

which is a *constant*, independent of k. The form of Eq. (B10) is very misleading; it is not a thermal spectrum in the energy of the particle, except when $M \gg H$. Only in this limit, for $k \ll M$, one can interpret Eq. (B10) as a thermal spectrum of particles with a temperature $H/2\pi$.

The situation gets worse for $M^2/H^2 < 9/4$. In this case, there are no solutions to Eq. (B4) that behave as positive/negative frequency oscillatory modes near $\eta \approx -1/H$. This can be seen from the asymptotic behavior of Eq. (B4) in this limit, which takes the following form:

$$\frac{d^2\Phi_{\mathbf{k}}}{d\eta^2} + \frac{2H}{1-H\eta}\frac{d\Phi_{\mathbf{k}}}{d\eta} + \left(\frac{M^2}{(1-H\eta)^2}\right)\Phi_{\mathbf{k}} \approx 0. \tag{B11}$$

The two linearly independent sets of solutions of Eq. (B11), with no restriction on the rage of parameters, are given by

$$\Phi^{\pm}_{\mathbf{k}}(\eta) = (1-H\eta)^{\pm\nu+\frac{3}{2}} = e^{-i\mathcal{E}_{\pm}t}, \tag{B12}$$

where

$$\mathcal{E}_{\pm} = -\frac{3iH}{2} \pm H\sqrt{\frac{M^2}{H^2} - \frac{9}{4}} \tag{B13}$$

and we have once again introduced the cosmic time t defined by $(1-H\eta) = e^{-Ht}$. Once again, we see that the notion of positive and negative frequency oscillatory modes makes sense only when $M^2/H^2 > 9/4$. This implies that, for an arbitrary value of mass M outside this range, we cannot define positive and negative frequency modes in a natural fashion and compute the number of particles produced asymptotically. In the next section, we circumvent this situation by resorting to a different prescription for defining particles.

2. Particle number from constant frequency representation

It is well known that, in the study of the quantum mechanics of a time-dependent oscillator, one can construct an operator, called the Ermakov-Lewis invariant, that is quadratic in the phase space coordinates and has eigenvalues that are constant in time. An important application of this operator is in the study of the particle production of quantum fields in time-dependent backgrounds (see, e.g., Refs. [42,43]). In a recent work [37] (also see Refs. [38,39]), a rather simple and elegant mapping was found between an arbitrary time-dependent harmonic oscillator and a simple harmonic oscillator of unit mass and time-*independent* frequency. This new formalisms "demystifies" the constancy of the Ermakov-Lewis invariant and reveals a natural way of defining particles. We shall first review this approach here for the sake of completeness.

The classical version of this mapping can be summarized as follows. If a dynamical variable q satisfies the time-dependent harmonic oscillator equation with mass and frequency given by $m(\eta)$ and $\omega(\eta)$, respectively, then one can show that the variable $Q = q/f(\eta)$ satisfies the equation of motion of a *constant frequency* oscillator,

$$\frac{dQ}{d\tau^2} + \Omega^2 Q = 0, \tag{B14}$$

where Ω is a constant and we have introduced a new time coordinate τ through $mf^2 d\tau = d\eta$, provided the function f is chosen to be a solution to the differential equation:

$$\{m(\eta)f'\}' + \omega^2(\eta)f = \frac{\Omega}{m(\eta)f^3}. \tag{B15}$$

The quantum mechanical version of this mapping works in a similar way. Let the wave function $\psi(q,\eta)$, for the dynamical system q, satisfy the following time-dependent Schrödinger equation:

$$i\partial_\eta \psi(q,\eta) = \left[-\frac{1}{2m(\eta)}\partial_q^2 + \frac{1}{2}m(\eta)\omega^2(\eta)q^2\right]\psi(q,\eta). \tag{B16}$$

It can then be shown that the new wave function $\Psi(Q,\tau)$, defined by

$$\psi(q,\eta) = \frac{1}{\sqrt{f}}\exp\left(im(\eta)\frac{f'}{2f}q^2\right)\Psi[Q = q/f, \tau(\eta)], \tag{B17}$$

satisfies the Schrödinger equation for a particle of unit mass in the potential of a simple harmonic oscillator of constant frequency Ω, i.e.,

$$i\partial_\tau \Psi(Q,\tau) = \left[-\frac{1}{2}\partial_Q^2 + \frac{1}{2}\Omega^2 Q^2\right]\Psi(Q,\tau), \tag{B18}$$

provided that f satisfies Eq. (B15).

This mapping offers a fresh view of the quantization of a time-dependent harmonic oscillator and definition of vacuum and particle states. Recall that implicit time dependence of the system implies that there is, in general, no stable vacuum state for a time-dependent harmonic oscillator. However, there is a unique vacuum state for the Q system, the wave function of which is given by

$$\Psi_0(Q,\tau) = \left(\frac{\Omega}{\pi}\right)^{1/4} e^{-\frac{\Omega Q^2}{2}} e^{-i\frac{1}{2}\Omega\tau}. \tag{B19}$$

Clearly, being an eigenstate of the Hamiltonian of the Q system, this state is stationary, and hence, once the system is prepared in this state, it continues to be in this state forever. On the other hand, from Eq. (B17), it follows that this vacuum corresponds to a time-dependent state of the q system of which the wave function is given by

$$\psi_0(q;\eta) = \left(\frac{\Omega}{f^2\pi}\right)^{1/4}\exp\left[-\left(\frac{\Omega}{2f^2} - im\frac{f'}{2f}\right)q^2 - \frac{i}{2}\Omega\tau(\eta)\right]. \tag{B20}$$

We can expand this state in terms of the complete set of eigenstates, denoted by $\{\phi_n(\eta); n = 0, 1, 2...\}$, of the instantaneous Hamiltonian of the q system at the instant η. When the oscillator q corresponds to a time-dependent mode function of a physical field in an external background, the average value of the "excitation" parameter n serves as a natural definition for the average number of particles $\bar{n}(\eta)$ produced in that particular mode. A straightforward computation gives

$$\bar{n}(\eta) = \frac{mf^2\omega}{4\Omega}\left[\left(-1 + \frac{\Omega}{mf^2\omega}\right)^2 + \left(\frac{f'}{f\omega}\right)^2\right]. \tag{B21}$$

3. Application to particle production in de Sitter

In this section, we will compute the particle number associated with a certain Fourier mode of a scalar field in de Sitter spacetime, using the formalism presented above. This uses the fact that the Fourier modes of a scalar field in de Sitter spacetime can be transformed into a time-dependent harmonic oscillator, the frequency and mass of which take the following form:

$$\omega_{\mathbf{k}}^2(\eta) = k^2 + M^2 a^2 \tag{B22}$$

$$m(\eta) = a^2(\eta). \tag{B23}$$

Using the redefinition of the dynamical variable through the function $f_{\mathbf{k}}(\eta)$, one can convert the time-dependent oscillator to a constant frequency oscillator with unit mass. The frequency of the constant frequency oscillator has been fixed to be $\omega_{\mathbf{k}(i)}$ associated with some initial time $\eta = \eta_i$. Thus, with the appropriate choice of $f_{\mathbf{k}}$, Eq. (B21) gives particle number $n_{\mathbf{k}}$ for each mode labeled by $\mathbf{k}$. As evident from Eq. (B22), it follows that in this limit $\omega_{\mathbf{k}(i)} = k$ and the function $f_{\mathbf{k}}$ becomes

$$f_{\mathbf{k}} = \left(\frac{k\pi}{2Ha^3(\eta)}\right)^{1/2}\left|\mathrm{H}_\nu^{(1)}\left(\frac{k}{Ha(\eta)}\right)\right| \sim \frac{1}{a(\eta)}, \tag{B24}$$

which, in the asymptotic past, behaves as $f_{\mathbf{k}} \sim 1/a(\eta)$. One can verify that the in vacuum defined with this $f_{\mathbf{k}}$ actually corresponds to the standard Bunch-Davies vacuum. So, the particle number at later times, calculated by this approach, should also correspond to the standard situation when the quantum field starts from the Bunch-Davies vacuum. Finally, let us examine the validity of the adiabatic condition at $\eta_i = -\infty$ and $\eta_f = H^{-1}$,

$$\left|\frac{1}{2[m(\eta)\omega_{\mathbf{k}}^2]}\frac{d}{d\eta}[m(\eta)\omega_{\mathbf{k}}]\right| \approx \begin{cases} |k\eta|^{-1} & ;\eta \to -\infty \\ \frac{3H}{2M} & ;\eta \approx H^{-1}. \end{cases} \tag{B25}$$

The adiabatic condition clearly holds well in the early times. On the other hand, in the asymptotic late times, the adiabatic condition holds only for $M/H \gg 3/2$, and hence the interpretation of $|\beta_{\mathbf{k}}(H^{-1})|^2$ as the number of out particles makes sense only in this limit. This is exactly the reason why we encountered a problem in Appendix B 1 while trying to study particles production for $M/H < 3/2$ using the conventional approach. In what follows, we will study the particle production using the approach discussed in Appendix B 2 for all values of M/H.

a. Massless fields in de Sitter

Application of Eq. (B21) in this case leads to the following expression for the particle number,

$$n_{\mathbf{k}} = \frac{H^2}{4k^2(1-H\eta)^2} = \frac{H^2 a(\eta)^2}{4k^2}, \tag{B26}$$

where we have used the fact that scale factor for de Sitter spacetime behaves as $(1-H\eta)^{-1}$. We see that $n_{\mathbf{k}}$ diverges in the late time limit as $a \to \infty$ [or, equivalently as $\eta \to (1/H)$]. (We note that a similar divergence was also noticed earlier in Ref. [44].) However, it is interesting to note that the proper number density of particles with *physical* momentum $\mathbf{p} = \mathbf{k}/a$ inside a spherical shell in $\mathbf{p}$-space of radius p and thickness dp is finite, constant, and independent of p,

$$n_{\mathbf{k}(\mathbf{p})} \frac{4\pi p^2 dp}{(2\pi)^3} = \left(\frac{H^2}{8\pi^2}\right) dp. \tag{B27}$$

We shall next consider the massive field.

b. Massive field in de Sitter satisfying $(M^2/H^2) < (9/4)$

Using Eq. (B21), the asymptotic particle number for this case takes the following form:

$$n_{\mathbf{k}} \approx a^{2\nu}\left[\frac{2^{2\nu-5}\Gamma(\nu)^2(\frac{k}{H})^{-2\nu}\{H^2(3-2\nu)^2+4M^2\}}{(\pi H M)}\right]. \tag{B28}$$

Thus, in this case as well, we find that $n_{\mathbf{k}} \to \infty$ as $a \to \infty$. However, there is a bit of subtlety in the massless case which we will comment on. We know from the earlier discussion [see Eq. (B26)] that, in the massless limit, corresponding to $\nu \to (3/2)$, the particle number should vary as a^2. But if we naively take the $\nu \to (3/2)$ limit of Eq. (B28), the particle number seems to vary as approximately a^3 rather than as approximately a^2. This arises because the intermediate steps in the calculation involve handling the combination M^2a^2 and its limiting value depends on the order in which the limits $a \to \infty$ and $M \to 0$ are taken. If one takes the $M \to 0$ limit first—at finite a—the combination Ma reduces to zero, but if one first takes $a \to \infty$—with nonzero M—the combination Ma diverges. This requires us to be careful in defining the two limits. We can see this more clearly by defining a function $\mathcal{N}_{\mathbf{k}}(\nu, a)$ given by

$$\mathcal{N}_{\mathbf{k}}(\nu, a) \equiv n_{\mathbf{k}} a^{-2\nu}. \tag{B29}$$

Let us now consider the following two limits of this functions: (i) $\nu \to 3/2$ followed by $a \to \infty$ and (ii) $a \to \infty$ followed by $\nu \to 3/2$. We find that

$$\text{(i)} \lim_{a\to\infty}\lim_{\nu\to 3/2}\mathcal{N}_{\mathbf{k}}(\nu, a) = 0 \tag{B30}$$

$$\text{(ii)} \lim_{\nu\to 3/2}\lim_{a\to\infty}\mathcal{N}_{\mathbf{k}}(\nu, a) = \infty. \tag{B31}$$

c. Massive fields in de Sitter satisfying $(M^2/H^2) > (9/4)$

Finally, application of Eq. (B21) leads to

$$n_{\mathbf{k}} \approx \frac{-8|\nu|^2 + (8|\nu|^2+9)\coth(\pi|\nu|) + 3\mathrm{csch}(\pi|\nu|)(3\cos(2|\nu|\log x) - 4|\nu|\sin(2|\nu|\log x))}{16|\nu|^2}. \tag{B32}$$

The cosine and sine terms in this expression, as $x \to 0$, oscillate fast and average to zero. Thus, the above expression simplifies to

$$n_{\mathbf{k}} \approx \frac{(8|\nu|^2+9)\coth(\pi|\nu|) - 8|\nu|^2}{16|\nu|^2}. \tag{B33}$$

Recall that an adiabatic out vacuum exists only when $M/H \gg 3/2$, in which case we can also assume that $|\nu| \gg 1$. In this limit, Eq. (B33) further simplifies to

$$n_{\mathbf{k}} \approx \frac{1}{e^{2\pi|\nu|}-1} \tag{B34}$$

and matches with the earlier result in Eq. (B10).

4. Discontinuity in asymptotic behavior of $n_{\mathbf{k}}$ at $\nu = 3/2$

Let us first consider the particle number as $\nu = 3/2$. From Eq. (B26), we get

$$\lim_{\nu\to 3/2}\mathcal{N}_{\mathbf{k}}(\nu, a) = \frac{H^2}{4k^2 a}. \tag{B35}$$

One can see that Eq. (B30) easily follows. For the second limit, let us consider the large $a(\eta)$ limit of n_k for a fixed value of ν. We start with the series expansion for J_ν,

$$J_\nu(x) = x^\nu \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(\nu+l+1)} x^{2l}. \tag{B36}$$

The Hankel function can be written in terms of the J_ν as

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-i\pi\nu} J_{-\nu}(x)}{i\sin(\pi\nu)} \tag{B37}$$

$$= \left(\frac{1}{i\sin(\pi\nu)}\right)\left[x^\nu \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(\nu+l+1)} x^{2l} - e^{-i\pi\nu} x^{-\nu} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(-\nu+l+1)} x^{2l}\right]. \tag{B38}$$

Therefore, from Eq. (B24), the function $f_{\mathbf{k}}$ has the following behavior at the leading order in a^{-1} as $a \sim \infty$:

$$f_{\mathbf{k}} \approx \frac{2^{\nu-\frac{1}{2}}}{\sqrt{\pi}} a^{\nu-\frac{3}{2}} \Gamma(\nu) \left(\frac{k}{H}\right)^{\frac{1}{2}-\nu}. \tag{B39}$$

The particle number can then be computed from Eq. (B21) as

$$n_k \approx \frac{a^{2\nu}}{4 f_0^2 k}\left[\sqrt{m^2 + \frac{k^2}{a^2}} + \left\{(5\nu - 3)\nu + \frac{9}{4}\right\} \frac{H^2}{\sqrt{m^2 + \frac{k^2}{a^2}}}\right]; \quad \forall\ \nu > 0, \tag{B40}$$

where

$$f_0 = \sin(\pi\nu)\Gamma(1-\nu)\left(\frac{k}{2H}\right)^{2\nu}. \tag{B41}$$

This implies that

$$\lim_{a\to\infty} \mathcal{N}_{\mathbf{k}}(\nu, a) = \left(\frac{3\sqrt{3}H^4}{2k^4\pi}\right)\left(\nu - \frac{3}{2}\right)^{-\frac{1}{2}} + \mathcal{O}((\nu - 3/2)^{1/2}). \tag{B42}$$

By taking $\nu \to 3/2$ in this equation, we arrive at Eq. (B31).

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