Gravitational perturbations in a cavity: Nonlinearities

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Motivated by recent studies of nonlinear perturbations of asymptotically anti-de Sitter spacetimes, we study gravitational perturbations of $(n+2)$ dimensional Minkowski spacetime with a spherical Dirichlet wall. By considering the tensor, vector, and scalar perturbations on the n sphere, we present simplified nonhomogeneous equations at arbitrary order in a weakly nonlinear perturbation theory for each sector. A suitable choice of perturbative variables is required at higher orders to simplify the expression for the boundary conditions and to expand the variables in terms of linear order eigenfunctions. Finally we comment on the nonlinear stability of the system. Some of the tools used can easily be generalized to study nonlinear perturbations of anti-de Sitter spacetime.

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I. INTRODUCTION

The seminal work of Bizon and Rostworowski [1] demonstrated that four dimensional anti-de Sitter spacetime (AdS) was nonlinearly unstable to spherically symmetric massless scalar field perturbations. The endpoint of the instability was a Schwarzschild-AdS black hole.[1] It was thus concluded that AdS was unstable for black hole formation for a large class of arbitrarily small perturbations and that reflecting boundary conditions played a key role in causing instability [3]. Later, it was demonstrated in [4] that the instability was seen in all dimensions. The instability was also present for complex scalar fields [5] in AdS spacetime. The necessary conditions for an AdS-like instability were analyzed in [6]. In [7], turbulent behavior characterized by a Kolmogorov-Zakharov power spectrum was uncovered for the Klein-Gordon gravity system. Noncollapsing solutions in asymptotically AdS spacetimes were studied in [8–14]. Going beyond Einstein gravity, the system of a scalar field and gravity with a Gauss-Bonnet term was analyzed in [15,16]. The AdS instability problem was also studied in [17–19], using the two-time framework (TTF) and a careful analysis using rigorous renormalization group methods [20,21]. Interacting scalar fields in AdS were studied in [22]. Of particular interest are numerical studies of nonspherically symmetric collapse in the Einstein gravity-scalar field system in asymptotically AdS spacetime [23,24]. A massless scalar field in flat space enclosed in a spherical cavity was studied as a toy model for AdS-like boundary conditions and it was shown to lead to a nonlinear instability [25]. The massive scalar field-gravity system in a cavity in flat space was studied in [26]. A comprehensive review of work on the instability of AdS, particularly the scalar field-gravity system can be found in [27]. Recently, there was also a rigorous proof of the instability of AdS for a specific Einstein-matter system—the Einstein-massless Vlasov system in spherical symmetry [28].[2]

Gravitational turbulent instability was first studied in AdS in [29]. This uncovered geons—time-periodic, asymptotically AdS solutions that were stable [29]; see also [30–37]. Purely gravitational perturbations of AdS satisfying the cohomogeneity-two biaxial Bianchi IX ansatz were studied and black hole formation was observed in [38].

In this work, we depart from the system of gravitational perturbations of AdS spacetime. Instead, we consider gravitational perturbations of Minkowski spacetime with a spherical Dirichlet wall in $(n+2)$ dimensions with $n \geq 2$. The boxlike boundary conditions mimic those in AdS spacetime; however, we find important differences both from the point of view of stability and in the mathematical analysis of higher order equations. Linearized perturbations of this system have been studied in [39] and it has been shown to be linearly stable. Further, linear perturbations have an asymptotically resonant spectrum. We extend this study to the nonlinear regime using weakly nonlinear perturbation theory. We define appropriate variables at any order in perturbation theory. We show that they are expanded in terms of the linear eigenfunctions with time dependent coefficients obeying forced harmonic oscillator equations. The forcing terms are comprised of

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[1]Further, it was shown that the Schwarzchild-AdS black hole was stable for the spherically symmetric Einstein-Klein-Gordon system [2].

[2]We thank the referee for bringing this paper to our attention.

lower order perturbations. This structure enables a description of the system through a perturbation of an integrable Hamiltonian (the Hamiltonian of nonresonant linear harmonic oscillators)—the perturbation leading to the forcing terms at higher orders. It is then possible to use specific results in Hamiltonian perturbation theory to argue that if the linear level oscillators have frequencies satisfying a number-theoretic (Diophantine) condition that characterizes the fact that they are nonresonant(7.1), then the system is stable [6] for generic perturbations. These results from nonlinear dynamics [40] were used in [6] to comment on the stability of the AdS soliton for the gravity-scalar field system. However, what is important for the purpose of application of these results is the structure of the equations for the perturbation variables which is common in both cases—enabling use of Hamiltonian perturbation theory. Further, we have an asymptotically resonant spectrum at the linear level in both cases. Our analysis shows that the system is nonlinearly stable under *arbitrarily small* perturbations. We also see that an indicator of the magnitude of the perturbation that may be required to trigger instability is the deviation of this asymptotically resonant spectrum from a fully resonant one (which can be quantified in number-theoretic terms). If the initial data contains high frequencies, this minimum magnitude of perturbation required to trigger instability could be really small. Thus, numerical studies with a finite amplitude for perturbations could still see a nonlinear instability in the case of systems with asymptotically resonant spectra at the linear level such as the system in this paper. For example, in [25], an instability was observed in numerical studies of a massless scalar field in a cavity in flat space with Neumann boundary conditions—for which the spectrum is asymptotically resonant. Maliborski and Rostworowski [14] repeated the numerical study (with Neumann boundary conditions) for smaller amplitudes and found that there was indeed a threshold amplitude of scalar field below which the instability was not triggered. This implies that there is stability under arbitrarily small perturbations. So also, in [26] a massive scalar field in the cavity with both Dirichlet and Neumann boundary conditions leads to an asymptotically resonant spectrum at the linear level, with numerical studies observing an instability for a finite amplitude of perturbations. However, this does not preclude stability under arbitrarily small perturbations.

In order to write down the perturbation equations, we use the gauge invariant formalism developed by Kodama, Ishibashi, and Seto [41] which we extend to higher order. We present simplified equations for the tensor, vector, and scalar (on the n sphere) perturbations. In all cases, there are subtleties involved in the imposition of Dirichlet wall boundary conditions at higher orders, and in the analysis of solutions to the perturbation equations satisfying these boundary conditions. We analyze these solutions at arbitrary order and apply the results in [6] to comment on the nonlinear stability of this system. Section II describes the methodology of the perturbation theory analysis at arbitrary order. A brief summary of the Ishibashi-Kodama-Seto formalism is also given. The linearized tensor, vector, and scalar perturbations are analyzed in Sec. III. The scalar eigenfunctions at linear order satisfy a modified orthogonality relation owing to the appearance of frequency dependent boundary conditions in the scalar sector. Section IV contains the higher order perturbation equations. By defining shifted perturbation variables (shifted by source terms) at higher order when necessary, we can expand the perturbation variables in terms of the linear order eigenfunctions, with time dependent coefficients obeying a forced harmonic oscillator equation. Section V is a brief section on how the various source terms are obtained from the solutions to perturbation equations at lower orders. In Sec. VI, we analyze special modes with $l_s = 0, 1$ in the case of scalar modes and $l_v = 1$ for vector modes. For these modes, the perturbation equations need appropriate gauge fixing. The modes can be obtained by solving these equations and it can be readily seen that at linear order, these modes reduce to pure gauge as expected. Section VII contains a summary and discussion of the nonlinear stability of this system, by using the equations obtained in the paper and nonlinear dynamics results in [6]. Appendices A and B have detailed computations for some of the equations in the rest of the paper. Appendix C contains an explicit evaluation of the second order source terms in the case when we have only tensor perturbations at linear order.

II. METHODOLOGY

We use perturbation theory to study gravitational perturbations in $(n+2)$ dimensional Minkowski spacetime with a spherical Dirichlet wall. For this we need to find solutions to Einstein's field equations with $\Lambda = 0$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.1}$$

We now follow the analysis and partly the notation in Rostworowski [32]. The "bar" quantities refer to the background Minkowski metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_n^2$, where $d\Omega_n^2$ is the metric for the n sphere, $n \geq 2$. Since we are dealing with weakly nonlinear perturbations, we let $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ where

$$\delta g_{\mu\nu} = \sum_{1\leq i} {}^{(i)}h_{\mu\nu} \epsilon^i. \tag{2.2}$$

Then the inverse metric is given by

$$\begin{aligned} g^{\alpha\beta} &= (\bar{g}^{-1} - \bar{g}^{-1}\delta g \bar{g}^{-1} + \bar{g}^{-1}\delta g \bar{g}^{-1}\delta g \bar{g}^{-1} - \cdots)^{\alpha\beta} \\ &= \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta}. \end{aligned} \tag{2.3}$$

The Christoffel symbol is decomposed as

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} &= \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}(\bar{g}^{-1} - \bar{g}^{-1}\delta g\bar{g}^{-1} + \cdots)^{\alpha\beta} \\ &\quad \times (\bar{\nabla}_{\mu}\delta g_{\beta\nu} + \bar{\nabla}_{\nu}\delta g_{\beta\mu} - \bar{\nabla}_{\beta}\delta g_{\mu\nu}) \\ &= \bar{\Gamma}^{\alpha}_{\mu\nu} + \delta\Gamma^{\alpha}_{\mu\nu}. \end{aligned} \tag{2.4}$$

Similarly the Ricci tensor is decomposed as

$$\begin{aligned} R_{\mu\nu} &= \bar{R}_{\mu\nu} + \bar{\nabla}_{\alpha}\delta\Gamma^{\alpha}_{\mu\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\alpha}_{\alpha\mu} + \delta\Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - \delta\Gamma^{\lambda}_{\mu\alpha}\delta\Gamma^{\alpha}_{\lambda\nu} \\ &= \bar{R}_{\mu\nu} + \delta R_{\mu\nu}. \end{aligned} \tag{2.5}$$

The Lorentzian Lichnerowicz operator Δ_L is defined as

$$\begin{aligned} 2\Delta_L{}^{(i)}h_{\mu\nu} &= -\bar{\nabla}^{\alpha}\bar{\nabla}_{\alpha}{}^{(i)}h_{\mu\nu} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}{}^{(i)}h \\ &\quad + \bar{\nabla}_{\mu}\bar{\nabla}_{\alpha}{}^{(i)}h^{\alpha}_{\nu} + \bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}{}^{(i)}h^{\alpha}_{\mu} \\ &\quad + \bar{R}_{\mu\alpha}{}^{(i)}h^{\alpha}_{\nu} + \bar{R}_{\nu\alpha}{}^{(i)}h^{\alpha}_{\mu} - 2\bar{R}_{\mu\alpha\nu\lambda}{}^{(i)}h^{\alpha\lambda}. \end{aligned} \tag{2.6}$$

We also define a quantity ${}^{(i)}A_{\mu\nu}$ as

$$\begin{aligned} {}^{(i)}A_{\mu\nu} &= [\epsilon^i]\{-\bar{\nabla}_{\alpha}[(-\bar{g}^{-1}\delta g\bar{g}^{-1} + \cdots)^{\alpha\lambda} \\ &\quad \times (\bar{\nabla}_{\mu}\delta g_{\lambda\nu} + \bar{\nabla}_{\nu}\delta g_{\lambda\nu} - \bar{\nabla}_{\lambda}\delta g_{\mu\nu})] \\ &\quad + \bar{\nabla}_{\nu}[(-\bar{g}^{-1}\delta g\bar{g}^{-1} + \cdots)^{\alpha\lambda} \\ &\quad \times (\bar{\nabla}_{\mu}\delta g_{\lambda\alpha} + \bar{\nabla}_{\alpha}\delta g_{\lambda\mu} - \bar{\nabla}_{\lambda}\delta g_{\mu\alpha})] \\ &\quad - 2\delta\Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\mu\nu} + 2\delta\Gamma^{\lambda}_{\mu\alpha}\delta\Gamma^{\alpha}_{\lambda\nu}\}, \end{aligned} \tag{2.7}$$

where $[\epsilon^i]f$ denotes the coefficient of ϵ^i in the expansion of power series $\sum_i \epsilon^i f_i$. Since the background metric is Minkowski, the Ricci and the Riemann tensor $\bar{R}_{\mu\nu}$ and $\bar{R}^{\mu}{}_{\nu\alpha\beta}$ vanish. Moreover, the total metric $g_{\mu\nu}$ is the solution of the vacuum Einstein's equation (2.1). Therefore, by plugging the expressions (2.3), (2.4), and (2.5) in (2.1) and collecting terms in like powers of ϵ, we obtain

$$2\tilde{\Delta}_L{}^{(i)}h_{\mu\nu} = {}^{(i)}S_{\mu\nu}, \tag{2.8}$$

where $\tilde{\Delta}_L{}^{(i)}h_{\mu\nu}$ is defined as

$$2\tilde{\Delta}_L{}^{(i)}h_{\mu\nu} = 2\Delta_L{}^{(i)}h_{\mu\nu} - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\Delta_L{}^{(i)}h_{\alpha\beta} \tag{2.9}$$

and the source ${}^{(i)}S_{\mu\nu}$ is given in terms of ${}^{(i)}A_{\mu\nu}$ as

$${}^{(i)}S_{\mu\nu} = {}^{(i)}A_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta(i)}A_{\alpha\beta}. \tag{2.10}$$

The background Minkowski metric $\bar{g}_{\mu\nu}$ is of the following form:

$$ds^2 = \bar{g}_{\mu\nu}dz^{\mu}dz^{\nu} = g_{ab}(y)dy^a dy^b + r^2(y)d\Omega_n^2, \tag{2.11}$$

where the metric $d\Omega_n^2$

$$d\Omega_n^2 = \gamma_{ij}(w)dw^i dw^j, \tag{2.12}$$

is the metric for the n sphere and has a constant sectional curvature $K = 1$.

One can use the gauge invariant formalism of Ishibashi, Kodama, and Seto [41] to study the perturbations, the difference being that we extend it to higher orders as well. We associate a covariant derivative each with ds^2, $g_{ab}dy^a dy^b$, and $d\Omega_n^2$ which are $\bar{\nabla}_M$, $\bar{D}_a$, and $\bar{D}_i$ respectively. We will also decompose the metric perturbations ${}^{(i)}h_{\mu\nu}$ according to their behavior on the n sphere i.e., into the scalar type, $\mathbb{S}$, the vector type, $\mathbb{V}_i$, and the tensor type, $\mathbb{T}_{ij}$. Note that in the following sections, $\hat{\Delta} = \hat{D}^i\hat{D}_i$ where raising (and lowering) of $\hat{D}_i$ (and $\hat{D}^i$) is done with γ_{ij}. The scalar harmonics $\mathbb{S}$ satisfy

$$(\hat{\Delta} + k_s^2)\mathbb{S} = 0, \tag{2.13}$$

with $k_s^2 = l_s(l_s + n - 1)$ where $l_s = 0, 1, \ldots$ from where one can construct scalar type vector harmonics $\mathbb{S}_i$

$$\mathbb{S}_i = -\frac{1}{k_s}\bar{D}_i\mathbb{S} \tag{2.14}$$

which satisfy

$$\bar{D}^i\mathbb{S}_i = k_s\mathbb{S} \tag{2.15}$$

as well as scalar type tensor harmonics

$$\mathbb{S}_{ij} = \frac{1}{k_s^2}\bar{D}_i\bar{D}_j\mathbb{S} + \frac{1}{n}\gamma_{ij}\mathbb{S} \tag{2.16}$$

which satisfy

$$\mathbb{S}^i_i = 0; \qquad \bar{D}_j\mathbb{S}^j_i = \frac{(n-1)(k_s^2 - nK)}{nk_s}\mathbb{S}_i. \tag{2.17}$$

Vector harmonics $\mathbb{V}_i$ are defined by

$$(\hat{\Delta} + k_v^2)\mathbb{V}_i = 0 \tag{2.18}$$

with $k_v^2 = l_v(l_v + n - 1) - 1$ where $l_v = 1, 2, \ldots$ such that

$$\bar{D}_i\mathbb{V}^i = 0. \tag{2.19}$$

One can construct the following tensor from the vector harmonics:

$$\mathbb{V}_{ij} = -\frac{1}{2k_v}(\bar{D}_i\mathbb{V}_j + \bar{D}_j\mathbb{V}_i), \tag{2.20}$$

which satisfy

$$\mathbb{V}^i_i = 0; \qquad \bar{D}_j \mathbb{V}^j_i = \frac{(k_v^2 - (n-1)K)}{2k_v} \mathbb{V}_i. \tag{2.21}$$

Tensor harmonics, $\mathbb{T}_{ij}$ are defined for $n > 2$ by

$$(\hat{\Delta} + k^2)\mathbb{T}_{ij} = 0 \tag{2.22}$$

with $k^2 = l(l+n-1) - 2$ where $l = 2, 3, \ldots$. They satisfy

$$\mathbb{T}^i_i = 0; \qquad \bar{D}_j \mathbb{T}^j_i = 0. \tag{2.23}$$

Henceforth, we shall consider $n > 2$. The analysis that follows can also be done for $n = 2$, the only change being that we do not have the tensor spherical harmonics in that case. The metric perturbations are decomposed as

$$\begin{aligned} {}^{(i)}h_{ab} &= \sum_{\mathbf{k_s}} {}^{(i)}f_{ab\mathbf{k_s}} \mathbb{S}_{\mathbf{k_s}}; \\ {}^{(i)}h_{ai} &= r\left(\sum_{\mathbf{k_s}} {}^{(i)}f_{a\mathbf{k_s}} \mathbb{S}_{\mathbf{k}_s i} + \sum_{\mathbf{k_v}} {}^{(i)}f^{(v)}_{a\mathbf{k_v}} \mathbb{V}_{\mathbf{k}_v i}\right) \\ {}^{(i)}h_{ij} &= r^2\left(\sum_{\mathbf{k}} {}^{(i)}H_{T\mathbf{k}} \mathbb{T}_{\mathbf{k}ij} + 2\sum_{\mathbf{k}_v} {}^{(i)}H^{(v)}_{T\mathbf{k}_v} \mathbb{V}_{\mathbf{k}_v ij} \right. \\ &\quad \left. + 2\sum_{\mathbf{k}_s} ({}^{(i)}H^{(s)}_{T\mathbf{k}_s} \mathbb{S}_{\mathbf{k}_s ij} + {}^{(i)}H_{L\mathbf{k}_s} \gamma_{ij} \mathbb{S}_{\mathbf{k}_s})\right). \end{aligned} \tag{2.24}$$

In the following sections, we drop the subscripts $\mathbf{k}$, $\mathbf{k}_v$, and $\mathbf{k}_s$ from the metric perturbations to avoid cluttering the equations.

We now consider these perturbations in presence of a spherical Dirichlet wall of radius R. We need to fix the metric induced on a surface of radius $r = \mathrm{R}$, requiring

$$\begin{aligned} {}^{(i)}f_{tt} &= {}^{(i)}f_t = {}^{(i)}H^{(s)}_T = {}^{(i)}H_L = {}^{(i)}f^{(v)}_t = {}^{(i)}H^{(v)}_T \\ &= {}^{(i)}H_T = 0|_{r=\mathrm{R}}. \end{aligned} \tag{2.25}$$

However, the metric components are also gauge dependent. Under an infinitesimal gauge transformation $\bar{\delta} z^\alpha = \sum_i {}^{(i)}\zeta^\alpha$, metric perturbation ${}^{(i)}h_{\mu\nu}$ transforms as

$${}^{(i)}h_{\mu\nu} \to {}^{(i)}h_{\mu\nu} - \bar{\nabla}_\mu {}^{(i)}\zeta_\nu - \bar{\nabla}_\nu {}^{(i)}\zeta_\mu, \tag{2.26}$$

i.e.,

$$\begin{aligned} {}^{(i)}h_{ab} &\to {}^{(i)}h_{ab} - \bar{D}_a {}^{(i)}\zeta_b - \bar{D}_b {}^{(i)}\zeta_a, \\ {}^{(i)}h_{ai} &\to {}^{(i)}h_{ai} - \bar{D}_i {}^{(i)}\zeta_a - r^2 \bar{D}_a\left(\frac{{}^{(i)}\zeta_i}{r^2}\right), \\ {}^{(i)}h_{ij} &\to {}^{(i)}h_{ij} - \bar{D}_i {}^{(i)}\zeta_j - \bar{D}_j {}^{(i)}\zeta_i \\ &\quad - 2r\bar{D}^a r\, {}^{(i)}\zeta_a \gamma_{ij}. \end{aligned} \tag{2.27}$$

Let ${}^{(i)}\zeta_a = {}^{(i)}T_a \mathbb{S}$ and ${}^{(i)}\zeta_i = r\,{}^{(i)}L\mathbb{S}_i + r\,{}^{(i)}L^{(v)}\mathbb{V}_i$. Thus the gauge transformations for ${}^{(i)}f_{ab}$, ${}^{(i)}f_a$, ${}^{(i)}f^{(v)}_a$, ${}^{(i)}H^{(s)}_T$, ${}^{(i)}H^{(v)}_T$, ${}^{(i)}H_L$, and ${}^{(i)}H_T$ are

$${}^{(i)}f_{ab} \to {}^{(i)}f_{ab} - \bar{D}_a {}^{(i)}T_b - \bar{D}_b {}^{(i)}T_a, \tag{2.28}$$

$${}^{(i)}f_a \to {}^{(i)}f_a - r\bar{D}_a\left(\frac{{}^{(i)}L}{r}\right) + \frac{k_s}{r} {}^{(i)}T_a, \tag{2.29}$$

$${}^{(i)}H_L \to {}^{(i)}H_L - \frac{k_s}{nr} {}^{(i)}L - \frac{\bar{D}^a r}{r} {}^{(i)}T_a, \tag{2.30}$$

$${}^{(i)}H^{(s)}_T \to {}^{(i)}H^{(s)}_T + \frac{k_s}{r} {}^{(i)}L, \tag{2.31}$$

$${}^{(i)}f^{(v)}_a \to {}^{(i)}f^{(v)}_a - r\bar{D}_a\left(\frac{{}^{(i)}L^{(v)}}{r}\right), \tag{2.32}$$

$${}^{(i)}H^{(v)}_T \to {}^{(i)}H^{(v)}_T + \frac{k_v}{r} {}^{(i)}L^{(v)}, \tag{2.33}$$

$${}^{(i)}H_T \to {}^{(i)}H_T. \tag{2.34}$$

For all cases except $l_s = 0$, 1 and $l_v = 1$ modes, one can define the following gauge invariant variables:

$${}^{(i)}Z_a = {}^{(i)}f^{(v)}_a + \frac{r}{k_v} \bar{D}_a {}^{(i)}H^{(v)}_T, \tag{2.35}$$

$$\begin{aligned} {}^{(i)}F_{ab} &= {}^{(i)}f_{ab} + 2\bar{D}_{(a} {}^{(i)}X_{b)}; \\ {}^{(i)}F &= {}^{(i)}H_L + \frac{{}^{(i)}H^{(s)}_T}{n} + \frac{1}{r}\bar{D}^a r {}^{(i)}X_a, \end{aligned} \tag{2.36}$$

where ${}^{(i)}X_a$ is defined as

$${}^{(i)}X_a = \frac{r}{k_s}\left({}^{(i)}f_a + \frac{r}{k_s}\bar{D}_a {}^{(i)}H^{(s)}_T\right). \tag{2.37}$$

Since the expressions for $\Delta_L {}^{(1)}h_{\mu\nu}$ have already been given in terms of the gauge invariant variables in [41], one can similarly get $\Delta_L {}^{(i)}h_{\mu\nu}$ in terms of the ith order gauge invariant variables ${}^{(i)}H_T$, ${}^{(i)}Z_a$, ${}^{(i)}F_{ab}$, and ${}^{(i)}F$ by replacing ${}^{(1)}h_{\mu\nu}$ by ${}^{(i)}h_{\mu\nu}$. After we have solved for these variables we will use equations (2.35)–(2.37) to obtain ${}^{(i)}h_{\mu\nu}$. Before we do so, we can use the gauge freedom to put some terms to zero. In subsequent sections, we will work in a gauge in which

$${}^{(i)}f_t = {}^{(i)}H_L = {}^{(i)}H^{(s)}_T = {}^{(i)}H^{(v)}_T = 0. \tag{2.38}$$

(The above gauge choice is the same as used by [42] in case of vector perturbations and [39] in case of scalar perturbations.)

Our strategy will be to first look at the equations for linearized modes. The spectrum of linear perturbations is an important indicator of nonlinear stability. If the spectrum is resonant, and if the higher order perturbation equations have the appropriate form that leads to energy transfer to higher frequency modes, there is likely to be an instability of the kind observed for AdS spacetime [1]. As we showed in earlier work [6] in the context of the Einstein-Klein-Gordon system, results from KAM theory indicate that when the spectrum is not perfectly resonant but only approximately so, it is stable under arbitrarily small perturbations. However a relatively small amplitude of perturbations may trigger instability that can be observed in numerical studies (as opposed to arbitrarily small perturbations for a fully resonant spectrum). It is possible to quantify in number-theoretic terms how close the spectrum is to being resonant. This is an indicator of the magnitude of perturbation that may be required in order to trigger instability. A crucial ingredient in [6] was the structure of the perturbation equations of the Einstein-Klein-Gordon system in weakly nonlinear perturbation theory. The scalar field at third order obeys forced harmonic oscillator equations. The system can then be described by a Hamiltonian that is a perturbation of an integrable Hamiltonian (linear harmonic oscillators) and it is possible to use Hamiltonian perturbation theory to arrive at these conclusions. There are two questions of interest for our system: what is the spectrum of the linearized perturbations, and what is the structure of the higher order perturbation equations? In subsequent sections, we will address both of these questions. By choosing appropriate master variables, we simplify the perturbation equations until the higher order equations for the master variables resemble those of a forced harmonic oscillator (as in weakly nonlinear perturbation theory for the Einstein gravity-scalar field system).

III. LINEARIZED EQUATIONS

We will first have a look at the leading order equations in ϵ,

$$ {}^{(1)}G_{\mu\nu} = \tilde{\Delta}_L {}^{(1)}h_{\mu\nu} = 0. \tag{3.1} $$

Henceforth we drop the superscript "(1)" on ${}^{(1)}h_{\mu\nu}$. Based on the methodology in [41], linear perturbations of flat space in a cavity with Dirichlet boundary conditions at the cavity wall were studied in [39]. In the following subsections, we revisit the linearized perturbations before moving to higher orders. In particular, we are interested in the spectrum of linearized perturbations as well as the linear order eigenfunctions as this is important to study the nonlinear evolution of the perturbations. We use methods of Takahashi and Soda [42] for simplifying the equations for vector and scalar perturbations.

A. Tensor perturbations

In this case,

$$ h_{ab} = 0; \qquad h_{ai} = 0; \qquad h_{ij} = \sum_{\mathbf{k}} r^2 H_{T\mathbf{k}}.\mathbb{T}_{\mathbf{kij}}. \tag{3.2} $$

Upon substituting (3.2) in the leading order in ϵ Einstein's equations, ${}^{(1)}G_{\mu\nu}$, we obtain the equation governing tensor type perturbations as given in [41], wherein upon taking $H_{T\mathbf{k}} = \Phi_{T\mathbf{k}}$ we get

$$ -r^2\ddot{\Phi}_T + r^2\Phi_T'' + nr\Phi_T' - l(l+n-1)\Phi_T = 0. \tag{3.3} $$

Equation (3.3) can be put in the form

$$ \ddot{\Phi}_T + \hat{L}\Phi_T = 0, \tag{3.4} $$

where

$$ \hat{L} = -\frac{1}{r^n}\partial_r(r^n\partial_r) + \frac{l(l+n-1)}{r^2}. \tag{3.5} $$

The solution to (3.4) is given by

$$ \Phi_{T\mathbf{k}} = \sum_{p=1}^{\infty} a_{p,\mathbf{k}} \cos(\omega_{p,l}t + b_{p,\mathbf{k}}) e_{p,l}(r), \tag{3.6} $$

where $e_{p,l}(r)$ is given by

$$ e_{p,l}(r) = d_{p,l}\frac{J_\nu(\omega_{p,l}r)}{r^{(n-1)/2}}; \qquad \nu = l + \frac{(n-1)}{2}. \tag{3.7} $$

Constants $a_{p,\mathbf{k}}$ and $b_{p,\mathbf{k}}$ are determined from initial conditions and $d_{p,l}$ is the normalization constant given by $\frac{\sqrt{2}}{\mathrm{R}J_\nu'(\omega_{p,l}\mathrm{R})}$. The eigenfrequencies $\omega_{p,l}$ are discrete and associated with a mode number p for each l. They are determined from the Dirichlet boundary condition: $\Phi_T = 0$ at $r = \mathrm{R}$,

$$ \Rightarrow \omega_{p,l} = \frac{j_{\nu,p}}{\mathrm{R}}, \tag{3.8} $$

where $j_{\nu,p}$ is the pth zero of a Bessel function of order ν. For large values of p, eigenfrequencies approach $(p + \nu/2 - 1/4)\frac{\pi}{\mathrm{R}}$; therefore it is an asymptotically resonant spectrum.

The eigenfunctions $e_{p,l}$ form a complete set and are orthonormal in the space of functions $\hat{L}^2([0,\mathrm{R}], r^n dr)$. The inner product $\langle f, g\rangle_T$ is defined as $\int_0^{\mathrm{R}} f(r)g(r)r^n dr$.

B. Vector perturbations

The following equations govern the linear vector perturbations [41]:

$$\bar{D}_a(r^{n-1}Z^a) = 0, \tag{3.9}$$

$$-\frac{1}{r^n}\bar{D}^b\left[r^{n+2}\left[\bar{D}_b\left(\frac{Z_a}{r}\right) - \bar{D}_a\left(\frac{Z_b}{r}\right)\right]\right] + \frac{(k_v^2-(n-1))}{r}Z_a = 0. \tag{3.10}$$

These, when expanded yield three equations. Of them, two are independent. The third equation can be obtained from the first two through a suitable combination. Here we will use only two of them. One is obtained by expanding equation (3.9):

$$\dot{Z}_t = (n-1)\frac{Z_r}{r} + Z_r'. \tag{3.11}$$

The other is obtained by making the substitution $a = r$ in (3.10):

$$-\partial_t^2 Z_r + r\partial_t\partial_r\left(\frac{Z_t}{r}\right) - \frac{(k_v^2-(n-1))}{r^2}Z_r = 0. \tag{3.12}$$

Next we substitute the expression for $\dot{Z}_t$ from (3.11) in (3.12), so that we get a second order equation in Z_r:

$$-\ddot{Z}_r + Z_r'' + \frac{(n-2)}{r}Z_r' - \frac{(l_v(l_v+n-1)+(n-2))}{r^2}Z_r = 0. \tag{3.13}$$

We rewrite $Z_{r\mathbf{k}_v}$ as

$$Z_{r\mathbf{k}_v} = r\Phi_{v\mathbf{k}_v}. \tag{3.14}$$

Hence upon writing (3.13) in terms of the new variable $\Phi_{v\mathbf{k}_v}$, we obtain the following master equation:

$$\ddot{\Phi}_v + \hat{L}_v\Phi_v = 0, \tag{3.15}$$

where $\hat{L}_v$ is defined as

$$\hat{L}_v = -\frac{1}{r^n}\partial_r(r^n\partial_r) + \frac{l_v(l_v+n-1)}{r^2}. \tag{3.16}$$

Let $\Phi_{v\mathbf{k}_v} = \cos(\omega t + b)\phi_{v\mathbf{k}_v}(r)$, so that Eq. (3.15) could be rewritten as

$$\hat{L}_v\phi_v = \omega^2\phi_v. \tag{3.17}$$

Demanding Dirichlet boundary condition is equivalent to imposing $Z_t = 0$ at $r = \mathrm{R}$. In order to impose this condition, we will write (3.11) in terms of $\Phi_{v\mathbf{k}_v}$ to get

$$\dot{Z}_t = r\Phi_v' + n\Phi_v. \tag{3.18}$$

Then using the ansatz $\Phi_{v\mathbf{k}_v} = \cos(\omega t + b)\phi_{v\mathbf{k}_v}(r)$ and then integrating with respect to t, we get

$$Z_t = \frac{1}{\omega}\{r\phi_v{}' + n\phi_v\}\sin(\omega t + b). \tag{3.19}$$

Any r-dependent integration constant in the above expression is put to zero. Hence, if Dirichlet condition on Z_t needs to be satisfied at all times, we require

$$r\phi_v' + n\phi_v = 0|_{r=\mathrm{R}}. \tag{3.20}$$

The linear stability of the vector modes has been shown in [39] so we will not repeat the argument here. The eigenfrequencies are discrete and hence can be associated with a mode number p for each l_v. Hence, we find that

$$\phi_{v\mathbf{k}_v} = e^{(v)}_{p,l_v} = d^{(v)}_{p,l_v}\frac{J_{\nu_v}(\omega_{p,l_v}r)}{r^{(n-1)/2}}; \qquad \nu_v = l_v + \frac{(n-1)}{2}, \tag{3.21}$$

where $d^{(v)}_{p,l_v}$ is the normalization constant given by

$$d^{(v)}_{p,l_v} = \frac{\sqrt{2}\omega_{p,l_v}}{J_{\nu_v}(\omega_{p,l_v}\mathrm{R})}[(n+1)^2/4 + (\omega_{p,l_v}\mathrm{R})^2 - \nu_v^2]^{-1/2}. \tag{3.22}$$

The eigenfunctions $\phi_{v\mathbf{k}_v} = e^{(v)}_{p,l_v}(r)$ are complete and form an orthonormal basis in the space of functions $\hat{L}_v^2([0,\mathrm{R}], r^n dr)$. Therefore, the general solution to (3.15) is given by

$$\Phi_{v\mathbf{k}_v} = \sum_{p=1}^{\infty} a^{(v)}_{p,\mathbf{k}_v}\cos(\omega_{p,l_v}t + b_{p,\mathbf{k}_v})e^{(v)}_{p,l_v}(r), \tag{3.23}$$

where $a^{(v)}_{p,\mathbf{k}_v}$ and $b_{p,\mathbf{k}_v}$ are determined from initial conditions and ω_{p,l_v} satisfies (3.20). The inner product $\langle f, g\rangle_v$ is defined as $\int_0^{\mathrm{R}} f(r)g(r)r^n dr$. Upon substituting for ϕ_v from (3.21) in (3.20) we obtain

$$\omega r J_{\nu_v}'(\omega r) + \frac{(n+1)}{2}J_{\nu_v}(\omega r) = 0|_{r=\mathrm{R}}. \tag{3.24}$$

Now we will look at the asymptotic nature of the frequencies associated with vector modes by considering the large argument expansion of the Bessel functions, which is given by

$$J_{\nu_v}(z) \sim \sqrt{\frac{2}{z\pi}}\cos\left(z - \frac{\nu_v\pi}{2} - \frac{\pi}{4}\right) \quad \text{as } z \to \infty. \tag{3.25}$$

This tells us that for large modes

$$\tan\left(z-\frac{\nu_v\pi}{2}-\frac{\pi}{4}\right)=\frac{(n+1)/2}{z}, \tag{3.26}$$

where $z=\omega\mathrm{R}$. It can be seen that the frequencies tend to $(p+\frac{\nu_v}{2}-\frac{3}{4})\frac{\pi}{\mathrm{R}}$.

C. Scalar perturbations

Using the Ishibashi-Kodama-Seto formalism [41], we get the scalar perturbation equations. In order to obtain the master equation, we will use the method followed by Takahashi and Soda [42]. Scalar perturbations satisfy the following identity:

$$[2(n-2)F+F_c^c]=0, \tag{3.27}$$

which is obtained from (traceless part of) the ${}^{(1)}G_{ij}=0$ equation. From ${}^{(1)}G_{rt}=0$ one gets

$$\frac{n}{r}\dot{F}_{rr}+\frac{k_s^2}{r^2}F_{rt}-2n\dot{F}'-\frac{2n}{r}\dot{F}=0. \tag{3.28}$$

Similar to [42], we choose

$$F_{rt}=2r(\dot{\Phi}_s+\dot{F}), \tag{3.29}$$

where Φ_s is our master variable. This helps us to integrate (3.28) with respect to t and get an expression for F_{rr} in terms of F and Φ_s which is

$$F_{rr}=2rF'+2F-\frac{2k_s^2}{n}F-\frac{2k_s^2}{n}\Phi_s. \tag{3.30}$$

The extra integration constant, which would be a function of r, is absorbed in the definition of Φ_s.

From ${}^{(1)}G_{tt}=0$ one gets

$$-2nF''+\frac{n}{r}F'_{rr}+\left(\frac{k_s^2}{r^2}+\frac{n(n-1)}{r^2}\right)F_{rr}-\frac{2n(n+1)}{r}F' +\left(\frac{2k_s^2(n-1)}{r^2}-\frac{2n(n-1)}{r^2}\right)F=0. \tag{3.31}$$

Substituting the expression for F_{rr} from (3.30) into (3.31) gives us an expression for F solely in terms of Φ_s and its derivative:

$$F=-\frac{n}{k_s^2-n}\left[r\Phi'_s+\left(\frac{k_s^2}{n}+n-1\right)\Phi_s\right]. \tag{3.32}$$

In the scalar component of ${}^{(1)}G_{ir}=0$ we will substitute for F_c^c from (3.27). This gives

$$-\frac{(n-2)}{r}F_{rr}+\dot{F}_{rt}-F'_{rr}+2F'+\frac{2(n-2)}{r}F=0. \tag{3.33}$$

Next, by using ${}^{(1)}G_{rr}=0$ and (3.27), one gets

$$-2n\ddot{F}+\frac{2n}{r}F'-\frac{2k_s^2}{r^2}F+\frac{2n(n-1)}{r^2}F-\frac{n}{r}F'_{rr} +\left(\frac{k_s^2}{r^2}-\frac{n(n-1)}{r^2}\right)F_{rr}+\frac{2n}{r}\dot{F}_{rt}=0. \tag{3.34}$$

We eliminate F'_{rr} from (3.34) by using (3.33). This gives

$$-2n\ddot{F}+\frac{(k_s^2-n)}{r^2}F_{rr}-\frac{2(k_s^2-n)}{r^2}F+\frac{n}{r}\dot{F}_{rt}=0. \tag{3.35}$$

Next, using the expressions for F_{rt}, F_{rr}, and F from (3.29), (3.30), and (3.32) in (3.35) leads to the second order master equation for $\Phi_{s\mathbf{k}_s}$,

$$\ddot{\Phi}_s-\Phi''_s-\frac{n}{r}\Phi'_s+\frac{l_s(l_s+n-1)}{r^2}\Phi_s=0. \tag{3.36}$$

We can then rewrite (3.36) as

$$\ddot{\Phi}_s+\hat{L}_s\Phi_s=0, \tag{3.37}$$

where $\hat{L}_s=-\frac{1}{r^n}\partial_r(r^n\partial_r)+\frac{l_s(l_s+n-1)}{r^2}$.

We substitute the ansatz $\Phi_s=\cos(\omega t+b)\phi_s$ in (3.37) and get

$$\hat{L}_s\phi_s=\omega^2\phi_s. \tag{3.38}$$

The eigenfrequencies ω must satisfy the mixed frequency dependent boundary condition obtained by requiring f_{tt} (or equivalently F_{tt}) to vanish at the boundary $r=\mathrm{R}$. This is given by

$$(n-1)r\phi'_s+\left(-\omega^2r^2+\frac{(n-1)}{n}(k_s^2+n(n-1))\right)\phi_s =0|_{r=\mathrm{R}}. \tag{3.39}$$

In [39], the stability under scalar perturbations has been demonstrated. The eigenfrequencies are discrete and can be associated with the mode number, say p for each l_s. Numerically, it can be seen that the spectrum is asymptotically resonant and high frequencies approach $(p+\frac{\nu_s}{2}-\frac{5}{4})\frac{\pi}{\mathrm{R}}$.

The eigensolutions of Eq. (3.38) are

$$\phi_{s\mathbf{k}_s}=e^{(s)}_{p,l_s}(r)=d^{(s)}_{p,l_s}\frac{J_{\nu_s}(\omega_{p,l_s}r)}{r^{(n-1)/2}};\qquad \nu_s=l_s+\frac{(n-1)}{2}. \tag{3.40}$$

where the constant $d^{(s)}_{p,l_s}$ is given by

$$d_{p,l_s}^{(s)} = \left[\int_0^{\mathrm{R}} |J_{\nu_s}(\omega_{p,l_s} r)|^2 r dr + \frac{\mathrm{R}^2}{(n-1)} |J_{\nu_s}(\omega_{p,l_s}\mathrm{R})|^2\right]^{-1/2}. \tag{3.41}$$

Although the eigenfunctions $e_{p,l_s}^{(s)}$ are not orthogonal, they satisfy the modified orthogonality relation, $\langle e_{p,l_s}^{(s)}, e_{q,l_s}^{(s)}\rangle_s$, given by

$$\langle e_{p,l_s}^{(s)}, e_{q,l_s}^{(s)}\rangle_s = \int_0^{\mathrm{R}} e_{p,l_s}^{(s)} e_{q,l_s}^{(s)} r^n dr + \frac{\mathrm{R}^{n+1}}{(n-1)} e_{p,l_s}^{(s)}(\mathrm{R}) e_{q,l_s}^{(s)}(\mathrm{R}) = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q \end{cases}. \tag{3.42}$$

We can write the general solution to (3.38) in terms of a series expansion of the discrete modes. We use the work of Zecca [43] which deals with the Bessel equation in a finite interval with singularity at one end and a eigenvalue dependent boundary condition, similar to ours, at the regular end point. He shows that the general solution can be expanded in a series of Bessel functions within this finite interval. Therefore, the solution to Eq. (3.37) is

$$\Phi_{s\mathbf{k}_s} = \sum_{p=0}^{\infty} a_{p,\mathbf{k}_s}^{(s)} \cos(\omega_{p,l_s} t + b_{p,\mathbf{k}_s}) e_{p,l_s}^{(s)}(r), \tag{3.43}$$

where $a_{p,\mathbf{k}_s}^{(s)}$ and $b_{p,\mathbf{k}_s}$ are constants set by initial conditions. An expansion theorem in his paper then implies that a function $f(r)$ in $C^1[0,1]$ with square integrable second derivative, and which satisfies the same boundary conditions as $J_{\nu_s}(\omega_{p,l_s} r)$ (or $e_{p,l_s}^{(s)}$) can be expanded in a series of these Bessel functions (or eigenfunctions $e_{p,l_s}^{(s)}$). Hence we can write $f(r)$ as

$$f = \sum_p \sigma_p e_{p,l_s}^{(s)} \tag{3.44}$$

where

$$\sigma_p = \langle f, e_{p,l_s}^{(s)}\rangle_s = \int_0^{\mathrm{R}} f r^n e_{p,l_s}^{(s)} dr + \frac{\mathrm{R}^{n+1}}{(n-1)} f(\mathrm{R}) e_{p,l_s}^{(s)}(\mathrm{R}). \tag{3.45}$$

We will be using the above results when dealing with higher order perturbations.

IV. HIGHER ORDER EQUATIONS

The higher order perturbed equations have to be solved for ${}^{(i)}h_{\mu\nu}$ given a source which is composed of $1,\ldots,(i-1)$th order metric perturbations. For example, the second order perturbed equations ${}^{(2)}G_{\mu\nu} = 0$ become

$${}^{(2)}G_{\mu\nu} = \tilde{\Delta}_L {}^{(2)}h_{\mu\nu} - \frac{1}{2} {}^{(2)}S_{\mu\nu} = 0, \tag{4.1}$$

where ${}^{(2)}S_{\mu\nu}$ is

$$\begin{aligned} {}^{(2)}S_{\mu\nu} = &-\frac{1}{2}\bar{\nabla}_\lambda h(-\bar{\nabla}^\lambda h_{\mu\nu} + \bar{\nabla}_\nu h^\lambda_\mu + \bar{\nabla}_\mu h^\lambda_\nu) - \frac{1}{2}\bar{\nabla}_\nu h^{\lambda\sigma}\bar{\nabla}_\mu h_{\sigma\lambda} - \bar{\nabla}^\sigma h_{\mu\lambda}\bar{\nabla}_\sigma h^\lambda_\nu \\ &+ \bar{\nabla}_\lambda h^\sigma_\mu \bar{\nabla}_\sigma h^\lambda_\nu + \bar{\nabla}_\lambda h^{\lambda\sigma}(-\bar{\nabla}_\sigma h_{\mu\nu} + \bar{\nabla}_\nu h_{\mu\sigma} + \bar{\nabla}_\mu h_{\sigma\nu}) \\ &+ h^{\lambda\sigma}(-\bar{\nabla}_\lambda\bar{\nabla}_\sigma h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h_{\lambda\sigma} + \bar{\nabla}_\lambda\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\sigma\bar{\nabla}_\nu h_{\mu\lambda}) \\ &+ \frac{1}{2}\eta_{\mu\nu}\bigg(-\frac{1}{2}\bar{\nabla}_\alpha h\bar{\nabla}^\alpha h + 2\bar{\nabla}_\alpha h^{\alpha\beta}\bar{\nabla}_\beta h + \frac{3}{2}\bar{\nabla}_\alpha h_{\beta\sigma}\bar{\nabla}^\alpha h^{\beta\sigma} \\ &- \bar{\nabla}_\alpha h^\sigma_\beta \bar{\nabla}_\sigma h^{\alpha\beta} - 2\bar{\nabla}_\alpha h^\alpha_\sigma \bar{\nabla}_\beta h^{\beta\sigma} + h^{\lambda\sigma}\bar{\nabla}_\lambda\bar{\nabla}_\sigma h + h^{\lambda\sigma}\bar{\nabla}^\alpha\bar{\nabla}_\alpha h_{\lambda\sigma} - 2h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\sigma h^\sigma_\beta\bigg). \end{aligned} \tag{4.2}$$

The details of the calculation are given in Appendix A. Now we look at a general ith order equation where $i \geq 2$.

Equation (2.8) can be written in terms of $\bar{D}_a$ and $\bar{D}_i$ operators. Using the expansion of $\Delta_L h_{\mu\nu}$ given in the appendix of [41] we obtain the perturbed equations, ${}^{(i)}G_{\mu\nu} = 0$ in terms of ${}^{(i)}H_T$, ${}^{(i)}F$, ${}^{(i)}F_{ab}$, and ${}^{(i)}Z_a$. For $\mu = i$, $\nu = j$ one obtains

$$\begin{aligned} &\sum_{\mathbf{k}}[-r^2\bar{D}^a\bar{D}_a{}^{(i)}H_T - nr\bar{D}^a r\bar{D}_a{}^{(i)}H_T + (k^2+2K){}^{(i)}H_T]_{\mathbf{k}}\mathbb{T}_{\mathbf{k}ij} \\ &\quad + \sum_{\mathbf{k}_v}\left[-\frac{2k_v}{r^{n-2}}\bar{D}_a(r^{n-1}{}^{(i)}Z^a)\right]_{\mathbf{k}_v}\mathbb{V}_{\mathbf{k}_v ij} + \sum_{\mathbf{k}_s}[-k_s^2[2(n-2){}^{(i)}F + {}^{(i)}F^c_c]]_{\mathbf{k}_s}\mathbb{S}_{\mathbf{k}_s ij} \\ &= {}^{(i)}S_{ij} - \sum_{\mathbf{k}_s}[Q_4]_{\mathbf{k}_s}\gamma_{ij}\mathbb{S}_{\mathbf{k}_s}. \end{aligned} \tag{4.3}$$

We do not write the explicit form of $[Q_4]$ as it is not required in our calculations and it does not contribute when we finally project to individual tensor components of each type.

Similarly from the ${}^{(i)}G_{ai} = 0$ equation one gets

$$\sum_{\mathbf{k}_v}\left[-\frac{1}{r^n}\bar{D}^b\left\{r^{n+2}\left[\bar{D}_b\left(\frac{{}^{(i)}Z_a}{r}\right)-\bar{D}_a\left(\frac{{}^{(i)}Z_b}{r}\right)\right]\right\}+\frac{k_v^2-(n-1)K}{r}{}^{(i)}Z_a\right]_{\mathbf{k}_v}\mathbb{V}_{\mathbf{k}_v i}$$
$$+\sum_{\mathbf{k}_s}\left[-k_s\left(\frac{1}{r^{n-2}}\bar{D}_b(r^{n-2\,(i)}F_a^b)-r\bar{D}_a\left(\frac{1}{r}{}^{(i)}F_b^b\right)-2(n-1)\bar{D}_a{}^{(i)}F\right)\right]_{\mathbf{k}_s}\mathbb{S}_{\mathbf{k}_s i}={}^{(i)}S_{ai}. \tag{4.4}$$

In order to decompose the various sectors we use the fact that

$$\int\mathbb{T}^{ij}\mathbb{V}_{ij}d^n\Omega=\int\mathbb{T}^{ij}\mathbb{S}_{ij}d^n\Omega=\int\mathbb{V}^{ij}\mathbb{S}_{ij}d^n\Omega=\int\mathbb{V}^{i}\mathbb{S}_{i}d^n\Omega=0. \tag{4.5}$$

We obtain the tensor equation from (4.3) which is

$$-r^2\bar{D}^a\bar{D}_a{}^{(i)}H_T-nr\bar{D}^a r\bar{D}_a{}^{(i)}H_T+(k^2+2K){}^{(i)}H_T=\int\mathbb{T}_{\mathbf{k}}^{ij}\,{}^{(i)}S_{ij}d^n\Omega. \tag{4.6}$$

Similarly using (4.5) we obtain the two vector equations from (4.3) and (4.4)

$$-\frac{1}{r^n}\bar{D}^b\left\{r^{n+2}\left[\bar{D}_b\left(\frac{{}^{(i)}Z_a}{r}\right)-\bar{D}_a\left(\frac{{}^{(i)}Z_b}{r}\right)\right]\right\}+\frac{k_v^2-(n-1)K}{r}{}^{(i)}Z_a=\int\mathbb{V}_{\mathbf{k}_v}^{i}\,{}^{(i)}S_{ai}d^n\Omega, \tag{4.7}$$

$$-\frac{2k_v}{r^{n-2}}\bar{D}_a(r^{n-1\,(i)}Z^a)=\int\mathbb{V}_{\mathbf{k}_v}^{ij}\,{}^{(i)}S_{ij}d^n\Omega. \tag{4.8}$$

For the scalar case we will use the following three equations, which are

$$\begin{aligned}&-\bar{D}^c\bar{D}_c{}^{(i)}F_{ab}+\bar{D}_a\bar{D}_c{}^{(i)}F_b^c+\bar{D}_b\bar{D}_c{}^{(i)}F_a^c+n\frac{\bar{D}^c r}{r}(-\bar{D}_c{}^{(i)}F_{ab}+\bar{D}_a{}^{(i)}F_{cb}+\bar{D}_b{}^{(i)}F_{ca})\\&+\frac{k_s^2}{r^2}{}^{(i)}F_{ab}-\bar{D}_a\bar{D}_b{}^{(i)}F_c^c-2n\left(\bar{D}_a\bar{D}_b{}^{(i)}F+\frac{1}{r}\bar{D}_a r\bar{D}_b{}^{(i)}F+\frac{1}{r}\bar{D}_b r\bar{D}_a{}^{(i)}F\right)\\&-\left(\bar{D}_c\bar{D}_d{}^{(i)}F^{cd}+\frac{2n}{r}\bar{D}^c r\bar{D}^{d\,(i)}F_{cd}+\frac{n(n-1)}{r^2}\bar{D}^c r\bar{D}^d r^{(i)}F_{cd}-2n\bar{D}^c\bar{D}_c{}^{(i)}F\right.\\&-\frac{2n(n+1)}{r}\bar{D}^c r\bar{D}_c{}^{(i)}F+2(n-1)\frac{(k_s^2-nK)}{r^2}{}^{(i)}F-\bar{D}^c\bar{D}_c{}^{(i)}F_d^d\\&\left.-\frac{n}{r}\bar{D}^c r\bar{D}_c{}^{(i)}F_d^d+\frac{k_s^2}{r^2}{}^{(i)}F_d^d\right)\eta_{ab}=\int\mathbb{S}_{\mathbf{k}_s}{}^{(i)}S_{ab}d^n\Omega,\end{aligned} \tag{4.9}$$

$$-k_s\left(\frac{1}{r^{n-2}}\bar{D}_b(r^{n-2\,(i)}F_a^b)-r\bar{D}_a\left(\frac{1}{r}{}^{(i)}F_b^b\right)-2(n-1)\bar{D}_a{}^{(i)}F\right)=\int\mathbb{S}_{\mathbf{k}_s}^{i}\,{}^{(i)}S_{ai}d^n\Omega, \tag{4.10}$$

$$-k_s^2[2(n-2){}^{(i)}F+{}^{(i)}F_c^c]=\int\mathbb{S}_{\mathbf{k}_s}^{ij}\,{}^{(i)}S_{ij}d^n\Omega. \tag{4.11}$$

A. Tensor perturbations at higher orders

Let ${}^{(i)}H_{T\mathbf{k}}={}^{(i)}\Phi_{T\mathbf{k}}$. Thus (4.6) becomes

$${}^{(i)}\ddot{\Phi}_{T\mathbf{k}}+\hat{L}\,{}^{(i)}\Phi_{T\mathbf{k}}=\frac{1}{r^2}\int\mathbb{T}_{\mathbf{k}}^{ij}\,{}^{(i)}S_{ij}d^n\Omega. \tag{4.12}$$

Condition (2.25) implies that at each order $\Phi_{T\mathbf{k}}$ should vanish at the boundary $r = \mathrm{R}$. The set of eigenfunctions $e_{p,l}(r)$ are complete and also satisfy condition (2.25). Hence, one can expand $\Phi_{T\mathbf{k}}$ as

$$^{(i)}\Phi_{T\mathbf{k}} = \sum_{p=1}^{\infty} {}^{(i)}c_{p,\mathbf{k}}(t)e_{p,l}(r). \tag{4.13}$$

Hence (4.12) can be written as

$$^{(i)}\ddot{c}_{p,\mathbf{k}}(t) + \omega_{p,l}^2\,{}^{(i)}c_{p,\mathbf{k}}(t) = \left\langle \frac{1}{r^2}\int \mathbb{T}_{\mathbf{k}}^{ij}\,{}^{(i)}S_{ij}, e_{p,l}(r) \right\rangle_T. \tag{4.14}$$

B. Vector perturbations at higher orders

Before we proceed, we will define the following two quantities:

$$^{(i)}V_{s1} = \int \mathbb{V}_{\mathbf{k}_v}^{ij}\,{}^{(i)}S_{ij}d^n\Omega, \tag{4.15}$$

$$^{(i)}V_{s2} = \int \mathbb{V}_{\mathbf{k}_v}^{i}\,{}^{(i)}S_{ri}d^n\Omega. \tag{4.16}$$

We first expand Eq. (4.8) and obtain the following:

$$^{(i)}\dot{Z}_t = (n-1)\frac{^{(i)}Z_r}{r} + {}^{(i)}Z_r' + \frac{1}{2k_v r}\,{}^{(i)}V_{s1}. \tag{4.17}$$

(Note that in the preceding equation as well as in the equations which follow, we drop the subscript $\mathbf{k}_v$ in $Z_{a\mathbf{k}_v}$ for convenience.) Next, by making the substitution $a = r$ in (4.7), we obtain

$$-{}^{(i)}\ddot{Z}_r + r\partial_t\partial_r\left(\frac{^{(i)}Z_t}{r}\right) - \frac{(k_v^2-(n-1))}{r^2}\,{}^{(i)}Z_r = -\frac{^{(i)}V_{s2}}{r}. \tag{4.18}$$

Now we substitute the expression for $^{(i)}\dot{Z}_t$ from (4.17) in (4.18) to get

$$\begin{aligned} &-{}^{(i)}\ddot{Z}_r + {}^{(i)}Z_r'' + \frac{(n-2)}{r}\,{}^{(i)}Z_r' \\ &-\frac{(l_v(l_v+n-1)+(n-2))}{r^2}\,{}^{(i)}Z_r \\ &= -\left[\frac{^{(i)}V_{s2}}{r} + r\left(\frac{^{(i)}V_{s1}}{2k_v r^2}\right)'\right]. \end{aligned} \tag{4.19}$$

Now we rewrite $^{(i)}Z_{r\mathbf{k}_v}$ in terms of a new variable $^{(i)}\Phi_{s\mathbf{k}_s}$ as

$$^{(i)}Z_{r\mathbf{k}_v} = r\left({}^{(i)}\Phi_{v\mathbf{k}_v} - \frac{1}{2k_v r^n}\int {}^{(i)}V_{s1}r^{(n-2)}dr\right). \tag{4.20}$$

The above definition is crucial, since it enables us to expand the higher order perturbations $^{(i)}\Phi_{v\mathbf{k}_v}$ in terms of the complete set of eigenfunctions $e_{p,l_v}^{(v)}$. This will be made clear in a while. Substitution of (4.20) in (4.19) leads to the following equation in terms of the variable $^{(i)}\Phi_{v\mathbf{k}_v}$:

$$\begin{aligned} {}^{(i)}\ddot{\Phi}_{v\mathbf{k}_v} + \hat{L}_v\,{}^{(i)}\Phi_{v\mathbf{k}_v} = &\frac{1}{r}\left[\frac{^{(i)}V_{s2}}{r} + r\left(\frac{^{(i)}V_{s1}}{2k_v r^2}\right)'\right] \\ &+ \frac{\int {}^{(i)}\ddot{V}_{s1}r^{(n-2)}dr}{2k_v r^n} \\ &+ \hat{L}_v\left[\frac{\int {}^{(i)}V_{s1}r^{(n-2)}dr}{2k_v r^n}\right]. \end{aligned} \tag{4.21}$$

Further simplification of the above equation can be done by expanding $^{(i)}\Phi_{v\mathbf{k}_v}$ in the basis of a complete set of functions (which also satisfy the appropriate boundary condition) say $^{(i)}\phi_{v\mathbf{k}_v}$ as follows:

$$^{(i)}\Phi_{v\mathbf{k}_v} = \sum_{p=1}^{\infty} {}^{(i)}c_{p,\mathbf{k}_v}^{(v)}(t)\,{}^{(i)}\phi_{v\mathbf{k}_v}(r). \tag{4.22}$$

We substitute for $^{(i)}Z_r$ from (4.20) in (4.17) and use the expansion (4.22). The expression for $^{(i)}Z_{t\mathbf{k}_v}$ then becomes

$$^{(i)}Z_{t\mathbf{k}_v} = \sum_{p=1}^{\infty}\int {}^{(i)}c_{p,\mathbf{k}_v}(t)dt\{r\,{}^{(i)}\phi_{v\mathbf{k}_v}' + n\,{}^{(i)}\phi_{v\mathbf{k}_v}\}. \tag{4.23}$$

In the above equation any r-dependent integration constant is put to zero. Then we apply the Dirichlet boundary condition that requires $^{(i)}Z_t$ to vanish at $r = \mathrm{R}$ for all times. This means

$$r\,{}^{(i)}\phi_v' + n\,{}^{(i)}\phi_v = 0|_{r=\mathrm{R}}. \tag{4.24}$$

Note that the ansatz (4.20) has specifically been chosen so that the boundary condition takes the above form. Since the eigenfunctions corresponding to the linear perturbation $\phi_v = e_{p,l_v}^{(v)}(r)$ (which form a complete set) also satisfy (4.24), we can choose to expand metric perturbations in this basis, i.e., choose $^{(i)}\phi_v = \phi_v$. Taking the projection of (4.21) on $e_{p,l_v}^{(v)}$ one gets a forced harmonic oscillator equation of the form

$$\begin{aligned} {}^{(i)}\ddot{c}_{p,\mathbf{k}_v}^{(v)} + \omega_{p,l_v}^2\,{}^{(i)}c_{p,\mathbf{k}_v}^{(v)} = \Bigg\langle &\left[\frac{^{(i)}V_{s2}}{r^2} + \left(\frac{^{(i)}V_{s1}}{2k_v r^2}\right)'\right] \\ &+ \frac{\int {}^{(i)}\ddot{V}_{s1}r^{(n-2)}dr}{2k_v r^n} \\ &+ \hat{L}_v\left[\frac{\int {}^{(i)}V_{s1}r^{(n-2)}dr}{2k_v r^n}\right], e_{p,l_v}^{(v)}\Bigg\rangle_v. \end{aligned} \tag{4.25}$$

C. Scalar perturbations at higher order

Before we consider the scalar equations we will define the following quantities:

$$^{(i)}S_{s0} = \int \mathbb{S}^{ij}_{\mathbf{k}_s}\, {}^{(i)}S_{ij} d^n\Omega, \tag{4.26}$$

$$^{(i)}S_{s1} = \int \mathbb{S}_{\mathbf{k}_s}\, {}^{(i)}S_{rt} d^n\Omega, \tag{4.27}$$

$$^{(i)}S_{s2} = \int \mathbb{S}_{\mathbf{k}_s}\, {}^{(i)}S_{tt} d^n\Omega, \tag{4.28}$$

$$^{(i)}S_{s3} = \left(\frac{k_s^2}{nr} + \frac{(n-1)}{r}\right)\int^t {}^{(i)}S_{s1}dt - {}^{(i)}S_{s2} + \frac{n}{r}\left(\frac{r}{n}\int^t {}^{(i)}S_{s1}dt\right)', \tag{4.29}$$

$$^{(i)}S_{s4} = \frac{\int \mathbb{S}^{i}_{\mathbf{k}_s}\, {}^{(i)}S_{ri} d^n\Omega}{k_s} + \left(\frac{{}^{(i)}S_{s0}}{k_s^2}\right)' - \left(\frac{{}^{(i)}S_{s0}}{rk_s^2}\right), \tag{4.30}$$

$$^{(i)}S_{s5} = \int \mathbb{S}_{\mathbf{k}_s}\, {}^{(i)}S_{rr} d^n\Omega + \frac{n}{rk_s^2}\, {}^{(i)}S'_{s0} - \frac{{}^{(i)}S_{s0}}{r^2}, \tag{4.31}$$

$$^{(i)}S_{s6} = {}^{(i)}S_{s5} - \frac{n}{r}\, {}^{(i)}S_{s4}, \tag{4.32}$$

$$^{(i)}S_{s7} = \frac{{}^{(i)}S_{s3}}{2n} - \frac{(k_s^2-n)}{2n^2 r}\int^t {}^{(i)}S_{s1}dt + \frac{{}^{(i)}S_{s6}}{2n} - \frac{1}{2rk_s^2}[r^2\, {}^{(i)}S_{s3}]'. \tag{4.33}$$

Similar to the linear case, we need only five equations to get a master equation governing scalar perturbations at higher orders. First, we have equation (4.11) which relates variables ${}^{(i)}F_{tt}$, ${}^{(i)}F_{rr}$ (through the trace ${}^{(i)}F^c_c$) and ${}^{(i)}F$:

$$-k_s^2[2(n-2)\, {}^{(i)}F + {}^{(i)}F^c_c] = {}^{(i)}S_{s0}. \tag{4.34}$$

For $a = r$, $b = t$ in (4.9) one gets

$$\frac{n}{r}\, {}^{(i)}\dot{F}_{rr} + \frac{k_s^2}{r^2}\, {}^{(i)}F_{rt} - 2n\, {}^{(i)}\dot{F}' - \frac{2n}{r}\, {}^{(i)}\dot{F} = {}^{(i)}S_{s1}. \tag{4.35}$$

Now we write ${}^{(i)}F_{rt}$ in terms of the variable ${}^{(i)}\Psi_{s\mathbf{k}_s}$ as

$$^{(i)}F_{rt} = 2r({}^{(i)}\dot{\Psi}_s + {}^{(i)}\dot{F}), \tag{4.36}$$

where ${}^{(i)}\Psi_s$ itself is defined in terms of our master variable ${}^{(i)}\Phi_s$ as

$$^{(i)}\Psi_s = {}^{(i)}\Phi_s - {}^{(i)}S_{s8}. \tag{4.37}$$

The expression ${}^{(i)}S_{s8}$ is defined as

$$^{(i)}S_{s8} = -\frac{1}{2}\left(\frac{k_s^2}{n} - 1\right) r^{-(\frac{k_s}{\sqrt{n}}+n-1)} \times \int^r \left[r^{(\frac{2k_s}{\sqrt{n}}-1)}\int^r r'^{(-\frac{k_s}{\sqrt{n}}+n-2)}\, {}^{(i)}\mathcal{B}dr'\right]dr, \tag{4.38}$$

where

$$^{(i)}\mathcal{B}(t,r) = \frac{n}{k_s^2-n}\left[r\left(\frac{r^2}{k_s^2}\, {}^{(i)}S_{s3}\right)' + \frac{r^2}{k_s^2}\left((n-1) - \frac{k_s^2}{n}\right){}^{(i)}S_{s3}\right] + \frac{r}{n}\int^t {}^{(i)}S_{s1}dt + \frac{{}^{(i)}S_{s0}}{k_s^2}. \tag{4.39}$$

The above ansatz ensures that the boundary condition is devoid of the products of the lower order metric perturbation contributed by the source terms. The details are given in Appendix B.

We integrate (4.35) with respect to t and get an expression for ${}^{(i)}F_{rr}$ in terms of ${}^{(i)}F$ and ${}^{(i)}\Psi_s$ which is

$$^{(i)}F_{rr} = 2r\, {}^{(i)}F' + 2\, {}^{(i)}F - \frac{2k_S^2}{n}\, {}^{(i)}F - \frac{2k_s^2}{n}\, {}^{(i)}\Psi_s + \frac{r}{n}\int^t {}^{(i)}S_{s1}dt. \tag{4.40}$$

For $a = b = t$ in (4.9) one gets

$$-2n\, {}^{(i)}F'' + \frac{n}{r}\, {}^{(i)}F'_{rr} + \left(\frac{k_s^2}{r^2} + \frac{n(n-1)}{r^2}\right){}^{(i)}F_{rr} - \frac{2n(n+1)}{r}\, {}^{(i)}F' + \left(\frac{2k_s^2(n-1)}{r^2} - \frac{2n(n-1)}{r^2}\right){}^{(i)}F = {}^{(i)}S_{s2}. \tag{4.41}$$

Substitution of (4.40) in (4.41) leads to an expression for ${}^{(i)}F$ in terms of ${}^{(i)}\Psi_s$ and its derivatives:

$$^{(i)}F = -\frac{n}{k_s^2-n}\left[r\, {}^{(i)}\Psi'_s + \left(\frac{k_s^2}{n} + n - 1\right){}^{(i)}\Psi_s - \frac{r^2}{2k_s^2}\, {}^{(i)}S_{s3}\right]. \tag{4.42}$$

Consider the expansion of (4.10) for $a = r$ in which we substitute for ${}^{(i)}F^c_c$ from (4.34). This gives

$$-\frac{(n-2)}{r}\, {}^{(i)}F_{rr} + {}^{(i)}\dot{F}_{rt} - {}^{(i)}F'_{rr} + 2\, {}^{(i)}F' + \frac{2(n-2)}{r}\, {}^{(i)}F = {}^{(i)}S_{s4}. \tag{4.43}$$

By making the substitution $a = b = r$ in (4.9) and using (4.34), one gets

$$-2n\, {}^{(i)}\ddot{F} + \frac{2n}{r}\, {}^{(i)}F' - \frac{2k_s^2}{r^2}\, {}^{(i)}F + \frac{2n(n-1)}{r^2}\, {}^{(i)}F - \frac{n}{r}\, {}^{(i)}F'_{rr} + \left(\frac{k_s^2}{r^2} - \frac{n(n-1)}{r^2}\right){}^{(i)}F_{rr} + \frac{2n}{r}\, {}^{(i)}\dot{F}_{rt} = {}^{(i)}S_{s5}. \tag{4.44}$$

Now, we eliminate ${}^{(i)}F'_{rr}$ from (4.44) by using (4.43), to get

$$-2n{}^{(i)}\ddot{F} + \frac{(k_s^2-n)}{r^2}{}^{(i)}F_{rr} - \frac{2(k_s^2-n)}{r^2}{}^{(i)}F + \frac{n}{r}{}^{(i)}\dot{F}_{rt} = {}^{(i)}S_{s6}. \tag{4.45}$$

Substituting the expression for ${}^{(i)}F_{rt}$, ${}^{(i)}F_{rr}$, and ${}^{(i)}F$ from (4.36), (4.40). and (4.42) in (4.45) we obtain the following equation in terms of variable ${}^{(i)}\Phi_{s\mathbf{k}_s}$

$$ {}^{(i)}\ddot{\Phi}_s + \hat{L}_s{}^{(i)}\Phi_s = {}^{(i)}S_{s9}, \tag{4.46}$$

where ${}^{(i)}S_{s9}$ is defined as

$$ {}^{(i)}S_{s9} = {}^{(i)}S_{s7} + {}^{(i)}\ddot{S}_{s8} + \hat{L}_s{}^{(i)}S_{s8}. \tag{4.47}$$

We can now expand ${}^{(i)}\Phi_s$ in the basis of the eigenfunctions of the linear perturbation $e^{(s)}_{p,l_s}(r)$ as follows:

$$ {}^{(i)}\Phi_s = \sum_{p=0}^{\infty} {}^{(i)}c^{(s)}_{p,l_s}(t)e^{(s)}_{p,l_s}(r). \tag{4.48}$$

According to condition (2.25), we require ${}^{(i)}F_{tt}$ to vanish at the boundary $r = \mathrm{R}$, which implies ${}^{(i)}\Phi_{s\mathbf{k}_s}$ should satisfy (see Appendix B for further details)

$$ r^2{}^{(i)}\Phi''_s + (2n-1)r{}^{(i)}\Phi'_s + \left((n-1)^2 - \frac{k_s^2}{n}\right){}^{(i)}\Phi_s = 0\Big|_{r=\mathrm{R}}. \tag{4.49}$$

The expansion (4.48) ensures that this boundary condition is automatically satisfied; this has been shown in Appendix B. One can now use (3.42) to show that the ${}^{(i)}c^{(s)}_{q,l_s}(t)$ satisfy

$$ {}^{(i)}\ddot{c}^{(s)}_{q,l_s} + \omega^2_{q,l_s}{}^{(i)}c^{(s)}_{q,l_s} = \langle {}^{(i)}S_{s7}, e^{(s)}_{q,l_s}\rangle_s \tag{4.50}$$

where $\langle \ldots \rangle_s$ is defined by (3.45).

V. CALCULATING THE SOURCE TERMS

The source terms ${}^{(i)}S_{\mu\nu}$ depend on ${}^{(1)}h_{\mu\nu}, {}^{(2)}h_{\mu\nu}, \ldots {}^{(i-1)}h_{\mu\nu}$. Hence, once we calculate ${}^{(i)}\Phi_{T\mathbf{k}}$, ${}^{(i)}\Phi_{v\mathbf{k}_v}$, and ${}^{(i)}\Phi_{s\mathbf{k}_s}$, we need to use them to get back ${}^{(i)}h_{\mu\nu}$. Since we have chosen our gauge choice to be (2.38), determining ${}^{(i)}f_{ab\mathbf{k}_s}$, ${}^{(i)}f^{(s)}_{r\mathbf{k}_s}$, ${}^{(i)}f^{(v)}_{a\mathbf{k}_v}$, and ${}^{(i)}H_{T\mathbf{k}}$, completely fixes the various components of ${}^{(i)}h_{\mu\nu}$.

Tensor components.—Since by definition, ${}^{(i)}H_{T\mathbf{k}} = {}^{(i)}\Phi_{\mathbf{k}}$, determining $\Phi_{\mathbf{k}}$ determines $H_{T\mathbf{k}}$.

Vector components.—By definition, ${}^{(i)}Z_{r\mathbf{k}_v}$ is related to ${}^{(i)}\Phi_{v\mathbf{k}_v}$ through (3.14) and (4.20) for linear order and higher orders respectively. ${}^{(i)}Z_{t\mathbf{k}_v}$ is related to ${}^{(i)}Z_{r\mathbf{k}_v}$ through (3.11) and (4.17) for linear and higher orders respectively. Hence the vector components are given by ${}^{(i)}Z_a = {}^{(i)}f^{(v)}_a$.

Scalar components.—Once the quantities ${}^{(i)}F_{\mathbf{k}_s}$ and components of ${}^{(i)}F_{ab\mathbf{k}_s}$ are determined in terms of ${}^{(i)}\Phi_{s\mathbf{k}_s}$, the scalar components are given by

$$ {}^{(i)}f_r = k_s{}^{(i)}F, \tag{5.1}$$

$$ {}^{(i)}f_{tt} = {}^{(i)}F_{tt}, \tag{5.2}$$

$$ {}^{(i)}f_{rr} = {}^{(i)}F_{rr} - \frac{2}{k_s}(r{}^{(i)}f_r)', \tag{5.3}$$

$$ {}^{(i)}f_{rt} = {}^{(i)}F_{rt} - \frac{r}{k_s}{}^{(i)}\dot{f}_r. \tag{5.4}$$

VI. SPECIAL MODES

A. Scalar perturbations $l_s = 0$, 1 modes

1. $l_s = 0$ mode

In this case, $\mathbb{S}$ is constant and hence, $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ vanish. This means, only ${}^{(i)}f_{ab}$ and ${}^{(i)}H_L$ exist. We will use gauge freedom to put

$$ {}^{(i)}H_L = {}^{(i)}f_{tt} = 0. \tag{6.1}$$

Let ${}^{(i)}\tilde{S}_{0\mu\nu} = \int \mathbb{S}_{l_s=0}{}^{(i)}S_{0\mu\nu}d^n\Omega$. We get the following equations from ${}^{(i)}G_{rt} = 0$, ${}^{(i)}G_{tt} = 0$, and ${}^{(i)}G_{rr} = 0$ respectively.

$$ \frac{n}{r}{}^{(i)}\dot{f}_{rr} = {}^{(i)}\tilde{S}_{0\,rt}, \tag{6.2}$$

$$ \frac{n}{r}{}^{(i)}f'_{rr} + \frac{n(n-1)}{r^2}{}^{(i)}f_{rr} = {}^{(i)}\tilde{S}_{0\,tt}, \tag{6.3}$$

$$ \frac{2n}{r}{}^{(i)}\dot{f}_{rt} - \frac{n(n-1)}{r^2}{}^{(i)}f_{rr} = {}^{(i)}\tilde{S}_{0\,rr}. \tag{6.4}$$

From (6.2), we can obtain ${}^{(i)}f_{rr}$ as

$$ {}^{(i)}f_{rr} = \int_{t_1}^{t} \frac{r}{n}{}^{(i)}\tilde{S}_{0\,rt}dt + {}^{(i)}f_{rr}(t_1, r). \tag{6.5}$$

${}^{(i)}f_{rr}(t_1, r)$ can be obtained from (6.3):

$$ {}^{(i)}f_{rr}(t_1, r) = \frac{1}{r^{n-1}}\int_0^r \frac{\bar{r}^n}{n}{}^{(i)}\tilde{S}_{0\,tt}(t_1, \bar{r})d\bar{r}. \tag{6.6}$$

Whereas, ${}^{(i)}f_{rt}$ is given by

$$ {}^{(i)}f_{rt} = \int^t \left[\frac{(n-1)}{2r} {}^{(i)}f_{rr} + \frac{r}{2n} {}^{(i)}\tilde{S}_{0\,rr} \right] dt. \tag{6.7} $$

2. $l_s = 1$ ($k_s^2 = n$) mode

Let ${}^{(i)}\tilde{S}_{1\,\mu\nu} = \int \mathbb{S}_{l_s=1} {}^{(i)}S_{1\,\mu\nu} d^n\Omega$. Since $\mathbb{S}_{ij}$ vanishes for this mode, only ${}^{(i)}f_{ab}$, ${}^{(i)}f_a$, and ${}^{(i)}H_L$ exist. We will use gauge freedom to put ${}^{(i)}f_{tt}$, ${}^{(i)}f_t$, and ${}^{(i)}H_L$ to zero. Now we define the following quantities, composed solely of source terms:

$$ {}^{(i)}S_1 = \frac{1}{r^n} \int_0^r \bar{r}^n \left[\frac{\bar{r}}{\sqrt{n}} {}^{(i)}\tilde{S}_{1\,tt} - \frac{1}{\sqrt{n}} \left(\bar{r} \int^t {}^{(i)}\tilde{S}_{1\,rt} dt \right)' - \sqrt{n} \left(\int^t {}^{(i)}\tilde{S}_{1\,rt} dt \right) \right] d\bar{r}, \tag{6.8} $$

$$ {}^{(i)}S_2 = \left[\frac{1}{2\sqrt{n}} {}^{(i)}\tilde{S}_{1\,rr} + \frac{(n-1)}{2\sqrt{n}r} \int^t {}^{(i)}\tilde{S}_{1\,rt} dt + \frac{(n-1)}{2r^2} {}^{(i)}S_1 \right]. \tag{6.9} $$

We will use the following four equations, namely ${}^{(i)}G_{rt} = 0$, ${}^{(i)}G_{tt} = 0$, ${}^{(i)}G_{rr} = 0$, and ${}^{(i)}G^i_i = 0$:

$$ \frac{n}{r} {}^{(i)}\dot{f}_{rr} + \frac{n}{r^2} {}^{(i)}f_{rt} + \frac{\sqrt{n}}{r} {}^{(i)}\dot{f}_r = {}^{(i)}\tilde{S}_{1\,rt}, \tag{6.10} $$

$$ \frac{n}{r} {}^{(i)}f'_{rr} + \frac{n^2}{r^2} {}^{(i)}f_{rr} + \frac{2\sqrt{n}}{r} {}^{(i)}f'_r + \frac{n^{3/2}}{r^2} {}^{(i)}f_r = {}^{(i)}\tilde{S}_{tt}, \tag{6.11} $$

$$ \frac{2n}{r} {}^{(i)}\dot{f}_{rt} - \frac{n(n-1)}{r^2} {}^{(i)}f_{rr} - \frac{2\sqrt{n}(n-1)}{r^2} {}^{(i)}f_r = {}^{(i)}\tilde{S}_{rr}, \tag{6.12} $$

$$ {}^{(i)}\dot{f}'_{rt} + \frac{(n-1)}{r} {}^{(i)}\dot{f}_{rt} - \frac{1}{2} {}^{(i)}\ddot{f}_{rr} - \frac{(n-1)}{2r} {}^{(i)}f'_{rr} - \frac{(n-1)}{\sqrt{n}r} {}^{(i)}f'_r - \frac{(n-1)^2}{2r^2} {}^{(i)}f_{rr} + \frac{(n-1)^2}{\sqrt{n}r^2} {}^{(i)}f_r = \frac{1}{n} {}^{(i)}\tilde{S}^i_{1\,i}. \tag{6.13} $$

We will redefine ${}^{(i)}f_{rt}$ as

$$ {}^{(i)}f_{rt} = \frac{r}{\sqrt{n}} {}^{(i)}\dot{\phi}_0. \tag{6.14} $$

Substituting this ansatz in (6.10) and then integrating with respect to t gives

$$ {}^{(i)}f_{rr} = -\frac{1}{\sqrt{n}} {}^{(i)}\phi_0 - \frac{1}{\sqrt{n}} {}^{(i)}f_r + \int^t \frac{r}{n} {}^{(i)}\tilde{S}_{1\,rt} dt. \tag{6.15} $$

The extra r-dependent integration function can be absorbed in the definition of ${}^{(i)}\phi_0$. Substituting the expression for ${}^{(i)}f_{rr}$ from (6.15) in (6.11) allows us to obtain ${}^{(i)}f_r$ in terms of ${}^{(i)}\phi_0$:

$$ {}^{(i)}f_r = {}^{(i)}\phi_0 + {}^{(i)}S_1. \tag{6.16} $$

Now, by substituting (6.16) in (6.12), one obtains

$$ {}^{(i)}\ddot{\phi}_0 = {}^{(i)}S_2. \tag{6.17} $$

Hence from (6.13), we can obtain the following expression for ${}^{(i)}\phi_0$:

$$ {}^{(i)}\phi_0 = \frac{\sqrt{n}r^2}{2(n-1)^2} \left[\frac{1}{n} {}^{(i)}\tilde{S}^i_{1\,i} + \frac{r}{2n} {}^{(i)}\dot{\tilde{S}}_{1\,rt} + \frac{(n-1)}{2r} \left(\frac{r}{n} \int^t {}^{(i)}\tilde{S}_{1\,rt} dt \right)' + \frac{(n-1)^2}{2nr} \int^t {}^{(i)}\tilde{S}_{1\,rt} dt + \frac{(n-1)}{2\sqrt{n}r} {}^{(i)}S'_1 - \frac{3(n-1)^2}{2\sqrt{n}r^2} {}^{(i)}S_1 - \frac{r}{\sqrt{n}} {}^{(i)}S'_2 - \frac{(n+1)}{\sqrt{n}} {}^{(i)}S_2 - \frac{1}{2\sqrt{n}} \ddot{S}_1 \right]. \tag{6.18} $$

Once ${}^{(i)}\phi_0$ is obtained, ${}^{(i)}f_r$, ${}^{(i)}f_{rr}$, and ${}^{(i)}f_{rt}$ can be determined using (6.16), (6.15), and (6.14) respectively.

B. Vector perturbations $l_v = 1$ ($k_v^2 = n - 1$) mode

Let ${}^{(i)}\tilde{S}^{(v)}_{1\,ia} = \int \mathbb{V}^i_{l_v=1} {}^{(i)}S_{1\,ia} d^n\Omega$ be the source associated with these modes. Since $\mathbb{V}_{ij}$ vanishes, only ${}^{(i)}f^{(v)}_a$ exist. Through a suitable gauge choice, one can put ${}^{(i)}f^{(v)}_t$ to zero. Thus, from ${}^{(i)}G_{ir} = 0$ one can obtain ${}^{(i)}f^{(v)}_r$ as

$$ \partial_t {}^{(i)}f^{(v)}_r = \frac{1}{r} \int_{t_1}^t {}^{(i)}\tilde{S}^{(v)}_{1\,ir} dt' + {}^{(i)}\dot{f}^{(v)}_r(t_1, r), \tag{6.19} $$

where ${}^{(i)}\dot{f}^{(v)}_r(t_1, r)$ is obtained from ${}^{(i)}G_{it} = 0$ equation,

$$ {}^{(i)}\dot{f}^{(v)}_r(t_1, r) = \frac{1}{r^{n+1}} \int_0^r \bar{r}^n \, {}^{(i)}\tilde{S}^{(v)}_{1\,it}(t_1, \bar{r}) d\bar{r}. \tag{6.20} $$

VII. SUMMARY AND DISCUSSION

In this article, we have analyzed perturbations of Minkowski spacetime with a spherical Dirichlet wall beyond linear order. This is a model where it is possible to simplify the perturbation equations at arbitrary order, and the tools and techniques we use can be generalized to study perturbations of AdS spacetime. We work in weakly nonlinear perturbation theory and decompose the perturbations into scalar, vector, and tensor spherical harmonics using the formalism of Ishibashi, Kodama, and Seto. The system has already been shown to be stable at linear order [39]. Further, the spectrum for the linear tensor, scalar and vector perturbations is asymptotically resonant as opposed to a resonant spectrum in the case of AdS spacetime. Therefore, in contrast to AdS where weakly nonlinear perturbation theory is expected to break down due to irremovable secular terms, we do not expect secular terms in this model. We argue that it is possible to make a stronger prediction—that of nonlinear stability under arbitrarily small perturbations. This requires writing the perturbation equations at any order in a vastly simplified form.

Even at linear order, the scalar sector of perturbations requires careful analysis and we use techniques in [42] to analyze the equations. This is because the Dirichlet wall boundary conditions lead to a frequency-dependent boundary condition for the scalar master variable (which depends on the scalar perturbations), a fact noted in [39]. Due to these boundary conditions, the scalar eigenfunctions are not orthogonal with respect to the usual inner product. We define a modified orthogonality relation which the eigenfunctions satisfy. Going beyond linear order, by fixing gauge appropriately, we present the (nonhomogeneous) perturbation equations at arbitrary order in a simplified form. The source terms are made of lower order perturbations. At any order, the perturbation consists of scalar, vector, and tensor-type parts. The equation for each of these is derived by projecting onto the space of perturbations of each type. Once these equations are obtained, we analyze each type separately at arbitrary order. The tensor perturbations are straightforward to analyze. The perturbation at arbitrary order can be written in terms of the eigenbasis of linear tensor perturbations with time dependent coefficients. These time dependent coefficients satisfy a simple forced harmonic oscillator equation. In the case of vector and scalar perturbations, we define new shifted master variables (shifted by source terms) such that these new variables obey the same boundary conditions as the linear perturbations. They are then expanded in an eigenbasis of linear perturbations with time dependent coefficients satisfying a forced harmonic oscillator equation. This (forced harmonic oscillator) structure of the equations is important in predicting its nonlinear stability. The system can then be described by a Hamiltonian that is a perturbation of the integrable Hamiltonian of linear harmonic oscillators which leads to forcing terms at a given order from perturbations of lower order. A similar structure was observed in [6] in the analysis of the AdS scalar field and the AdS soliton-scalar field systems. The authors of [6] used specific results from Hamiltonian perturbation theory (a theorem of Benettin and Gallavotti [40]) to comment on stability of the perturbed Hamiltonian. These results are for generic perturbations of nonresonant linear harmonic oscillators and hence apply for a wide range of examples, ranging from the AdS soliton-scalar field system discussed in [6] to the problem of interest in this paper. The discussion in [6] is lengthy, but let us summarize some important results: if the spectrum of the linear harmonic oscillators is nonresonant then there is longtime stability under arbitrarily small perturbations. Let ω_i, $i = 1, \ldots, n$ denote the frequencies of the linear oscillators n (assumed large but finite) participating significantly in the dynamics and let $\boldsymbol{\omega} \in R^n$ denote the frequency vector with components ω_i. Let $\mathbf{k} \in Z^n - \{0\}$ denote a vector of integers. The condition for a resonance is that $\boldsymbol{\omega} \cdot \mathbf{k} = 0$ for some $\mathbf{k} \in Z^n - \{0\}$. Let us now consider a spectrum that is not resonant. This can be quantified by a *Diophantine condition* on $\boldsymbol{\omega}$ for all $k \in Z^n - \{0\}$ and some $\gamma > 0$, namely:

$$|\boldsymbol{\omega} \cdot \mathbf{k}| \geq \frac{\gamma}{|k|^n}; \tag{7.1}$$

$|k|$ denoting supremum (over i) of $|k_i|$. By choosing large integers for k_i we can get arbitrarily close to the resonance condition. γ quantifies how close the frequencies are to being perfectly resonant—asymptotically resonant or nearly resonant spectra will satisfy this condition for small γ. The theorem of Benettin and Gallavotti implies that the magnitude of perturbation required to trigger a possible instability depends on γ. The more "nonresonant" the system is, the more magnitude of perturbation is required to trigger a possible instability. An asymptotically resonant spectrum approaches the resonant one for high frequencies. While the asymptotically resonant spectrum (as opposed to a fully resonant one) guarantees stability under *arbitrarily small* perturbations, in a numerical study, it is possible to see an instability for perturbations of finite magnitude. This magnitude could be quite small if the initial perturbation involved high frequencies. This analysis also explains the result of previous numerical studies of the Einstein-scalar field system where an instability was seen in a cavity with both Dirichlet and Neumann boundary conditions. Neumann boundary conditions resulted in an asymptotically resonant spectrum, yet an instability was seen [25]. However, a careful analysis by Maliborski and Rostworowski [14] by decreasing the amplitude of perturbations revealed that a certain minimum threshold amplitude of the scalar field was required to trigger instability in the Neumann case where the spectrum is asymptotically resonant as opposed to the Dirichlet case where the spectrum is perfectly resonant and perturbations, however small, result in black hole formation. This is exactly as per the predictions of Hamiltonian perturbation theory. Similarly, for massive scalar fields in a cavity with both Dirichlet and Neumann boundary conditions, where the spectrum is asymptotically resonant, an instability is seen for finite magnitude of perturbations [26]. However, our analysis only predicts stability for arbitrarily small perturbations, thus there is no contradiction. We can thus use these results to predict that the system we study in this paper will exhibit similar behavior due to an asymptotically resonant spectrum.

Finally, we analyze certain special modes separately. These are the scalar modes with $l_s = 0$, $l_s = 1$ and the vector mode with $l_v = 1$ for which the equations become gauge dependent. By a choice of gauge fixing, we analyze these perturbations at arbitrary order. It is possible to integrate the equations and write the form of the perturbations. As expected, at linear order, these are pure gauge.

One of the interesting questions we have not addressed and indeed, can be answered only numerically is the fate of the system for gravitational perturbations of appropriate magnitude that may trigger instability—whether a rotating black hole is the result.

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APPENDIX A: GENERAL SECOND ORDER PERTURBATIONS

We use (2.3) and (2.4) to get the expansions of metric perturbation $\delta g_{\mu\nu}$ and Christoffel's symbol to second order. In the following the superscript on the left-hand side of a quantity denotes the order of ϵ,

$$\delta\Gamma^{\alpha}_{\mu\nu} = {}^{(1)}\Gamma^{\alpha}_{\mu\nu} + {}^{(2)}\Gamma^{\alpha}_{\mu\nu}\ldots \tag{A1}$$

where

$$^{(1)}\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}(\bar{\nabla}_\mu h^{\alpha}_{\nu} + \bar{\nabla}_\nu h^{\alpha}_{\mu} - \bar{\nabla}^{\alpha} h_{\mu\nu}) \tag{A2}$$

$$\begin{aligned} ^{(2)}\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}(\bar{\nabla}_\mu {}^{(2)}h^{\alpha}_{\nu} + \bar{\nabla}_\nu {}^{(2)}h^{\alpha}_{\nu} - \bar{\nabla}^{\alpha(2)} h_{\mu\nu}) \\ &\quad - \frac{1}{2}h^{\alpha\lambda}(\bar{\nabla}_\mu h_{\lambda\nu} + \bar{\nabla}_\nu h_{\lambda\mu} - \bar{\nabla}_\lambda h_{\mu\nu}) \end{aligned} \tag{A3}$$

$$\delta R_{\mu\nu} = {}^{(1)}R_{\mu\nu} + {}^{(2)}R_{\mu\nu}\ldots \tag{A4}$$

where

$$^{(1)}R_{\mu\nu} = \Delta_L h_{\mu\nu} = 0 \tag{A5}$$

$$\begin{aligned} 2^{(2)}R_{\mu\nu} &= 2\Delta_L {}^{(2)}h_{\mu\nu} + \frac{1}{2}\bar{\nabla}_\alpha h(-\bar{\nabla}^{\alpha} h_{\mu\nu} + \bar{\nabla}_\nu h^{\alpha}_{\mu} + \bar{\nabla}_\mu h^{\alpha}_{\nu}) \\ &\quad - h^{\lambda\alpha}(-\bar{\nabla}_\lambda \bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h_{\lambda\alpha} \\ &\quad + \bar{\nabla}_\lambda \bar{\nabla}_\mu h_{\alpha\nu} + \bar{\nabla}_\alpha \bar{\nabla}_\nu h_{\mu\lambda}) \\ &\quad + \frac{\bar{\nabla}_\nu h^{\lambda\alpha}\bar{\nabla}_\mu h_{\lambda\alpha}}{2} - \bar{\nabla}_\lambda h^{\alpha}_{\mu}\bar{\nabla}_\alpha h^{\lambda}_{\nu} + \bar{\nabla}^{\alpha} h_{\mu\lambda}\bar{\nabla}_\alpha h^{\lambda}_{\nu} \\ &\quad - \bar{\nabla}_\lambda h^{\lambda\alpha}(-\bar{\nabla}_\alpha h_{\mu\nu} + \bar{\nabla}_\nu h_{\mu\alpha} + \bar{\nabla}_\mu h_{\alpha\nu}) \end{aligned} \tag{A6}$$

$$\begin{aligned} ^{(2)}R &= \bar{g}^{\mu\nu}\,{}^{(2)}R_{\mu\nu} - h^{\mu\nu}\,{}^{(1)}R_{\mu\nu} \\ &= \bar{g}^{\mu\nu}\,{}^{(2)}R_{\mu\nu}. \end{aligned} \tag{A7}$$

The second order in ϵ Einstein's equation is

$$2^{(2)}R_{\mu\nu} - {}^{(2)}h_{\mu\nu}\bar{R} - \bar{g}_{\mu\nu}{}^{(2)}R - h_{\mu\nu}{}^{(1)}R = 0 \tag{A8}$$

where the last term vanishes because of condition (A5). Substituting the appropriate expressions in (A8) one finds the second order equation to be

$$\tilde{\Delta}_L {}^{(2)}h_{\mu\nu} = {}^{(2)}S_{\mu\nu}. \tag{A9}$$

APPENDIX B: BOUNDARY CONDITIONS FOR HIGHER ORDER SCALAR PERTURBATIONS

From using (4.40) and (4.34) we obtain the following expression for ${}^{(i)}F_{tt}$:

$$^{(i)}F_{tt} = 2r^{(i)}F' + 2(n-1)^{(i)}F - \frac{2k_s^2}{n}{}^{(i)}F - \frac{2k_s^2}{n}{}^{(i)}\Psi + \frac{r}{n}\int^t {}^{(i)}S_{s1}dt + \frac{^{(i)}S_{s0}}{k_s^2}. \tag{B1}$$

Now we substitute for ${}^{(i)}F$ and ${}^{(i)}\Psi_s$ from (4.42) and (4.37) in the above equation

$$\begin{aligned} ^{(i)}F_{tt} &= -\frac{n}{k_s^2 - n}\left[2r^{2(i)}\Phi''_s + 2r(2n-1)^{(i)}\Phi'_s + 2\left((n-1)^2 - \frac{k_s^2}{n}\right)^{(i)}\Phi_s\right] \\ &\quad + \frac{n}{k_s^2 - n}\left[2r^{2(i)}S''_{s8} + 2r(2n-1)^{(i)}S'_{s8} + 2\left((n-1)^2 - \frac{k_s^2}{n}\right)^{(i)}S_{s8}\right] + {}^{(i)}\mathcal{B} \end{aligned} \tag{B2}$$

where ${}^{(i)}\mathcal{B}$ is given by (4.39). We wish to choose a form for ${}^{(i)}S_{s8}$ which will ensure that the terms in second line of (B2) vanish. Define

$$^{(i)}S_{s8} = {}^{(i)}\chi f \tag{B3}$$

where

$$f = r^{-(\frac{k_s}{\sqrt{n}}+n-1)}; \qquad {}^{(i)}\chi = -\frac{k_s^2 - n}{2n}\int^r \left[r^{(\frac{2k_s}{\sqrt{n}}-1)}\int^r r'^{(-\frac{k_s}{\sqrt{n}}+n-2)}\,{}^{(i)}\mathcal{B}\,dr'\right]dr. \tag{B4}$$

Substituting (B3) in (B2) one obtains

$$ {}^{(i)}F_{tt} = -\frac{n}{k_s^2 - n}\left[2r^{2\,(i)}\Phi_s'' + 2r(2n-1)^{(i)}\Phi_s' + 2\left((n-1)^2 - \frac{k_s^2}{n}\right){}^{(i)}\Phi_s\right] + \frac{2n}{k_s^2 - n}\left\{{}^{(i)}\chi\left[r^2 f'' + r(2n-1)f' + \left((n-1)^2 - \frac{k_s^2}{n}\right)f\right] + r^2 f\left[{}^{(i)}\chi'' + {}^{(i)}\chi'\left(\frac{(2n-1)}{r} + 2\frac{f'}{f}\right)\right]\right\} + {}^{(i)}\mathcal{B} \tag{B5} $$

One can easily see that for the choice of f and ${}^{(i)}\chi$ given by (B4), the last two lines in (B5) vanish. Hence the expression for ${}^{(i)}F_{tt}$ is

$$ {}^{(i)}F_{tt} = -\frac{n}{k_s^2 - n}\left[2r^{2\,(i)}\Phi_s'' + 2r(2n-1)^{(i)}\Phi_s' + 2\left((n-1)^2 - \frac{k_s^2}{n}\right){}^{(i)}\Phi_s\right]. \tag{B6} $$

Applying Dirichlet condition (2.25) then gives

$$ r^{2\,(i)}\Phi_s'' + r(2n-1)^{(i)}\Phi_s' + \left((n-1)^2 - \frac{k_s^2}{n}\right){}^{(i)}\Phi_s = 0\Big|_{r=\mathrm{R}}. \tag{B7} $$

The expansion (4.48) ensures that condition (B7) is automatically satisfied. This can be seen as follows:

In terms of expansion (4.48), ${}^{(i)}F_{tt}$ is

$$ {}^{(i)}F_{tt} = -\frac{2n}{k_s^2 - n}\sum_{p=0}^{\infty} {}^{(i)}c^{(s)}_{p,l_s}\left[r^2 e^{(s)}_{p,l_s}{}'' + (2n-1)re^{(s)}_{p,l_s}{}' + \left((n-1)^2 - \frac{k_s^2}{n}\right)e^{(s)}_{p,l_s}\right]. \tag{B8} $$

Since $e^{(s)}_{p,l_s}$ satisfies (3.38), $r^2 e^{(s)}_{p,l_s}{}'' = (-r^2\omega^2 + k_s^2)e^{(s)}_{p,l_s} - nre^{(s)}_{p,l_s}{}'$. Hence by the use of this expression in (B8) one obtains

$$ {}^{(i)}F_{tt} = \sum_{p=0}^{\infty}\frac{2n^{(i)}c^{(s)}_{p,l_s}}{n - k_s^2}\left[(n-1)re^{(s)}_{p,l_s}{}' + \left(-\omega^2 r^2 + \frac{(n-1)}{n}(k_s^2 + n(n-1))\right)e^{(s)}_{p,l_s}\right], \tag{B9} $$

which vanishes at $r = \mathrm{R}$ because of (3.39).

APPENDIX C: SECOND ORDER SOURCE TERMS

The expansion of source terms in general is tedious. Nevertheless, here we give an example by considering a simple case. Suppose we start out with only tensor-type perturbations at the linear level. Then ${}^{(2)}A_{ij}$ is given by

$$ {}^{(2)}A_{ij} = \sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2} H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\left(T^{kl}_{\mathbf{k}_1}(-D_iD_jT_{kl_{\mathbf{k}_2}} + D_kD_iT_{jl_{\mathbf{k}_2}} + D_kD_jT_{li_{\mathbf{k}_2}} - D_kD_lT_{ij_{\mathbf{k}_2}}) - \frac{D_iT^{kl}_{\mathbf{k}_1}D_jT_{kl_{\mathbf{k}_2}}}{2} + D_kT^l_{i_{\mathbf{k}_1}}D_lT^k_{j_{\mathbf{k}_2}} - D^kT_{il_{\mathbf{k}_1}}D_kT^l_{j_{\mathbf{k}_2}}\right) - rD^a rD_a H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\gamma_{ij}T^{kl}_{\mathbf{k}_1}T_{kl_{\mathbf{k}_2}} - r^2 D^a H_{T\mathbf{k}_1}D_a H_{T\mathbf{k}_2}T_{ik_{\mathbf{k}_1}}T^k_{j_{\mathbf{k}_2}}. \tag{C1} $$

Similarly, ${}^{(2)}A_{ai}$ and ${}^{(2)}A_{ab}$ are

$$ {}^{(2)}A_{ai} = \sum_{\mathbf{k}_1,\mathbf{k}_2}\left\{H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}\mathbb{T}^{kl}_{\mathbf{k}_1}(-\bar{D}_i\mathbb{T}_{\mathbf{k}_2kl} + \bar{D}_k\mathbb{T}_{\mathbf{k}_2il}) - \frac{1}{2}\bar{D}_a H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_i\mathbb{T}_{\mathbf{k}_2kl}\right\} \tag{C2} $$

$$ {}^{(2)}A_{ab} = \sum_{\mathbf{k}_1,\mathbf{k}_2}\left\{\left(-H_{T\mathbf{k}_1}\bar{D}_a\bar{D}_b H_{T\mathbf{k}_2} - \bar{D}_a H_{T\mathbf{k}_1}\bar{D}_b H_{T\mathbf{k}_2} - \frac{1}{r}\bar{D}_a rH_{T\mathbf{k}_1}\bar{D}_b H_{T\mathbf{k}_2} - \frac{1}{r}\bar{D}_b r\bar{D}_a H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\right)\mathbb{T}^{ij}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2ij}\right\}. \tag{C3} $$

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