Properties of expansion-free dynamical stars

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(Received 24 June 2019; published 20 August 2019)

We study the geometrical and dynamical features of expansion-free dynamical stars in general relativity. Such stars can exist only if particular physical and geometric conditions are satisfied. First, for trapping to exist in an expansion-free dynamical star, the star must accelerate and radiate simultaneously. If either is zero, then the shear (Σ) must be zero throughout the star, in which case the star is static ($\Theta = \Sigma = 0$). Second, we prove that with nonzero acceleration and radiation expansion-free dynamical stars must be conformally flat.

DOI: 10.1103/PhysRevD.100.044039

I. INTRODUCTION

Models of radiating stars in general relativity are important to describe astrophysical processes and to study gravitational collapse. Some recent examples of exact models that are physically reasonable were obtained by Tewari and Charan [1], Tewari [2], and Ivanov [3–5]. Anisotrophy and dissipative effects have been shown to influence the collapse rate and temperature profiles in radiating stars by Reddy *et al* [6]. It has been demonstrated that classes of exact solutions exist in general relativity, referred to as Euclidean stars, which regain Newtonian stars in the appropriate limit [7–9]. The Lie analysis of differential equations using symmetry invariance has proven to be a systematic method to produce general categories of exact solutions to the boundary condition of radiating objects [10–12]. An important class of radiating stars that are expansion free was introduced by Herrera *et al* [13]. Expansion-free dynamical models imply the existence of a cavity or void. Matter distributions with a vanishing expansion scalar have to be inhomogeneous. These physical features should have important astrophysical consequences for spherically symmetric distributions. Studies containing the description of physical properties of expansion-free dynamical radiating stars are contained in several treatments [14–16]. Therefore, it is important to study the geometrical properties of expansion-free dynamical stars and find general conditions for their existence.

The aim of this paper is to investigate under what conditions there can be trapping in a relativistic expansion-free dynamical star. This analysis falls in the scope of stability analysis of self-gravitating systems (some of the references are given in Refs. [17–22]). We will consider the conditions on the acceleration and radiation quantities that allow for trapping in such stars. It is also an interesting exercise, with all the different quantities acting on such stars, to determine the geometry as these structures evolve. We will make use of the equivalent forms of the field equations from the $1+1+2$ semitetrad covariant formulation of general relativity [23–28]. The semitetrad formalism has been a useful approach in displaying geometrical features, which are difficult to find using other approaches, in a transparent fashion.

Various authors have explored expansion-free dynamical models with different considerations. The central theme of interest in such models is the possibility that they could help explain the existence of voids on cosmological scales. In 2008, Herrera and coauthors [13] studied such models with nonzero shear and showed that the appearance of a cavity (see Ref. [29] for more discussion), with matter that is anisotropic and dissipative, undergoing explosion is inevitable. The same authors followed this result with Ref. [30] in 2009, in which they ruled out the Skripkin expansion-free dynamical model (see Ref. [31]) with constant energy density and isotropic pressure. Another study in Ref. [32] involved the study of models collapsing adiabatically and showed that the instability was independent of the star's stiffness. In particular, it was shown that the instability is entirely governed by the pressures and the radial profile of the energy density.

In Sec. II, we give a short overview of the $1+1+2$ semitetrad formulation. In Sec. III, we present the results of the paper. We conclude with a discussion of the results in Sec. IV.

II. PRELIMINARIES

We provide some background material in this section, covering the $1+1+2$ semitetrad covariant formalism as

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well as notes on and calculations of useful quantities, utilized in this paper.

We start by explicitly defining *locally rotationally symmetric* class II spacetimes [33,34].

Definition 1: A locally rotationally symmetric class II (LRS II) spacetime is an evolving and vorticity-free (zero rotation) and spatial twist-free spacetime with a one-dimensional isotropy group of spatial rotations defined at each point of the spacetime. It is given by the general line element

$$ds^2 = -A^2(t,\chi) + B^2(t,\chi) + C^2(t,\chi)(dy^2 + D^2(y,k)dz^2), \tag{1}$$

where t, χ are parameters along integral curves of the timelike vector field $u^a = A^{-1}\delta^a_0$ of a timelike congruence and the preferred spacelike vector $e^a = B^{-1}\delta^a_\nu$, respectively. The constant k fixes the function $D(y,k)$ ($k = -1$ corresponds to $\sinh y$, $k = 0$ corresponds to y, and $k = 1$ corresponds to $\sin y$) [34].

LRS II spacetimes generalize spherically symmetric spacetimes and can be used to study astrophysical bodies such as stars and their evolution. From the line element in Eq. (1), it is clear that most physically realistic and interesting spacetimes fall within the LRS II class.

Let us next introduce the $1+1+2$ covariant splitting of spacetime and the resulting field equations for LRS II spacetimes [27,28].

To start with, let (M, g_{ab}) be a spacetime manifold. To any timelike congruence, we associate a unit vector field u^a tangent to the congruence for which $u^a u_a = -1$. Given any 4-vector U^a in the spacetime, the projection tensor $h_a{}^b \equiv g_a{}^b + u_a u^b$ projects U^a onto the 3-space as

$$U^a = U u^a + U^{\langle a \rangle},$$

where U is the scalar along u^a and $U^{\langle a \rangle}$ is the projected 3-vector [35]. This naturally gives rise to two derivatives:

(i) The *covariant time derivative* (or simply the dot derivative) along the observers' congruence. For any tensor $S^{a..b}{}_{c..d}$, $\dot{S}^{a..b}{}_{c..d} \equiv u^e \nabla_e S^{a..b}{}_{c..d}$.

(ii) Fully orthogonally *projected covariant derivative D* with the tensor h_{ab}, with the total projection on all the free indices. For any tensor $S^{a..b}{}_{c..d}$, $D_e S^{a..b}{}_{c..d} \equiv h^a{}_f h^p{}_c \ldots h^b{}_g h^q{}_d h^r{}_e \nabla_r S^{f..g}{}_{p..q}$.

This $1+3$ splitting irreducibly splits the covariant derivative of u^a as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab}\Theta + \sigma_{ab}. \tag{2}$$

In Eq. (2), $A_a = \dot{u}_a$ is the acceleration vector, $\Theta \equiv D_a u^a$ is the expansion, and $\sigma_{ab} = D_{\langle b} u_{a \rangle}$ is the shear tensor (wherever used in this paper, angle brackets will denote the projected symmetric trace-free part of the tensor). LRS II spacetimes also have the property that the Weyl tensor is purely electric as the magnetically part of the Weyl tensor is identically zero (see Ref. [27] for details).

The splitting also allows for the energy-momentum tensor to be decomposed as

$$T_{ab} = \rho u_a u_b + 2q_{(a}u_{b)} + p h_{ab} + \pi_{ab}, \tag{3}$$

where $\rho \equiv T_{ab}u^a u^b$ is the energy density, $q_a = -h_a{}^c T_{cd} u^d$ is the 3-vector defining the heat flux, $p \equiv (1/3)h^{ab}T_{ab}$ is the isotropic pressure, and π_{ab} is the anisotropic stress tensor.

If there is a preferred unit normal spatial direction e^a as is the case with LRS II spacetimes, the metric g_{ab} can be split into terms along the u^a and e^a directions (the vector field e^a splits the 3-space), as well as on the 2-surface, i.e.,

$$g_{ab} = N_{ab} - u_a u_b + e_a e_b, \tag{4}$$

where the projection tensor N_{ab} projects any two vectors orthogonal to u^a and e^a onto the 2-surface defined by the sheet $N_a{}^a = 2$ ($u^a N_{ab} = 0, e^a N_{ab} = 0$) and e^a is defined such that $e^a e_a = 1$ and $u^a e_a = 0$. This is referred to as the $1+1+2$ splitting. This splitting of the spacetime additionally gives rise to the splitting of the covariant derivatives along the e^a direction and on the 2-surface:

(i) The *hat derivative* is the spatial derivative along the vector e^a: for a 3-tensor $\psi_{a..b}{}^{c..d}$, $\hat{\psi}_{a..b}{}^{c..d} \equiv e^f D_f \psi_{a..b}{}^{c..d}$.

(ii) The *delta derivative* is the projected spatial derivative on the 2-sheet by $N_a{}^b$ and projected on all the free indices: for any 3-tensor $\psi_{a..b}{}^{c..d}$, $\delta_e \psi_{a..b}{}^{c..d} \equiv N_a{}^f..N_b{}^g N_h{}^c..N_i{}^d N_e{}^j D_j \psi_{f..g}{}^{h..i}$.

For LRS II spacetimes, the $1+1+2$ covariant scalars fully describing the LRS II spacetimes are [27]

$$\{A, \Theta, \phi, \Sigma, \mathcal{E}, \rho, p, \Pi, Q\}.$$

The quantity $\phi \equiv \delta_a e^a$ is the sheet expansion, $\Sigma \equiv \sigma_{ab}e^a e^b$ is the scalar associated to the shear tensor σ_{ab}, $\mathcal{E} \equiv E_{ab}e^a e^b$ is the scalar associated with the electric part of the Weyl tensor E_{ab}, $\Pi \equiv \pi_{ab}e^a e^b$ is the anisotropic stress scalar, and $Q \equiv -e^a T_{ab} u^b = q_a e^a$ is the scalar associated to the heat flux vector q_a.

The full covariant derivatives of the vector fields u^a and e^a are given by [27]

$$\nabla_a u_b = -A u_a e_b + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right), \tag{5}$$

$$\nabla_a e_b = -A u_a u_b + \left(\frac{1}{3}\Theta + \Sigma\right) e_a u_b + \frac{1}{2}\phi N_{ab}. \tag{6}$$

We also note the useful expression

$$\hat{u}^a = \left(\frac{1}{3}\Theta + \Sigma\right)e^a. \tag{7}$$

When acting on a scalar ψ, the dot $(\dot{})$ and hat $(\hat{})$ derivatives satisfy the commutation relation [27]

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -A\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi}. \tag{8}$$

This is a useful relation that will be utilized often in our calculations.

The evolution and propagation equations may be obtained from using the Ricci identities of the vectors u^a and e^a as well as the doubly contracted Bianchi identities [27,28]. The evolution and propagation equations are given as follows (for full derivation of the equations, see Ref. [27]):

(i) *Evolution (LRS II):*

$$\frac{2}{3}\dot{\Theta} - \dot{\Sigma} = A\phi - 2\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 - \frac{1}{3}(\rho + 3p) + \mathcal{E} - \frac{1}{2}\Pi, \tag{9}$$

$$\dot{\phi} = \left(\frac{2}{3}\Theta - \Sigma\right)\left(A - \frac{1}{2}\phi\right) + Q, \tag{10}$$

$$\dot{\mathcal{E}} - \frac{1}{3}\dot{\rho} + \frac{1}{2}\dot{\Pi} = -\frac{3}{2}\left(\frac{2}{3}\Theta - \Sigma\right)\mathcal{E} - \frac{1}{4}\left(\frac{2}{3}\Theta - \Sigma\right)\Pi + \frac{1}{2}\phi Q + \frac{1}{2}(\rho + p)\left(\frac{2}{3}\Theta - \Sigma\right). \tag{11}$$

(ii) *Propagation (LRS II):*

$$\frac{2}{3}\hat{\Theta} - \hat{\Sigma} = \frac{3}{2}\phi\Sigma + Q, \tag{12}$$

$$\hat{\phi} = \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{1}{2}\phi^2 - \frac{2}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi, \tag{13}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\rho} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) - \frac{1}{2}\left(\frac{2}{3}\Theta - \Sigma\right)Q. \tag{14}$$

(iii) *Propagation/evolution (LRS II):*

$$\hat{A} - \dot{\Theta} = -(A + \phi)A + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 + \frac{1}{2}(\rho + 3p), \tag{15}$$

$$\hat{Q} + \dot{\rho} = -\Theta(\rho + p) - (\phi + 2A)Q - \frac{3}{2}\Sigma\Pi, \tag{16}$$

$$\hat{p} + \hat{\Pi} + \dot{Q} = -\left(\frac{3}{2}\phi + A\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\rho + p)A. \tag{17}$$

The outgoing null expansion, the vanishing of which necessitates trapping, has been calculated in Refs. [28,36] as

$$\Theta_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma + \phi\right). \tag{18}$$

The equation of the outgoing null expansion scalar here corresponds to Eq. (32) of Ref. [28], but we have unitized the energy function for our choice of the outgoing null normal vector field k^a, the divergence of which gives Θ_k. It is clear from Eq. (18) that, even with the vanishing of Θ, it is still possible to have trapping. This is the main focus of this work and is investigated in the next section.

III. RESULTS

In this section, we state and prove the results of the paper.

A. Dynamics of expansion-free stars

We state and prove the following theorem.

Theorem 1: An expansion-free dynamical star must accelerate and radiate simultaneously.

Proof.—We establish this by fixing both the acceleration and the heat flux to zero and then by fixing either the acceleration or the heat flux to zero.

1. Case 1

First, suppose $A = 0$ and $Q = 0$. From Eq. (15), we have the algebraic constraint equation

$$0 = \frac{3}{2}\Sigma^2 + \frac{1}{2}(\rho + 3p). \tag{19}$$

Here, we note that, since $\Sigma^2 > 0$, Eq. (19) implies that the strong energy condition must be violated, i.e., $\rho + 3p < 0$. For $A = \Theta = 0$, Eq. (8) is simply

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = \Sigma\hat{\psi}. \tag{20}$$

Taking the hat derivative of Eq. (10) and the dot derivative of Eq. (13), we obtain, respectively,

$$\hat{\dot{\phi}} = \frac{1}{2}\hat{\phi}\Sigma + \frac{1}{2}\phi\hat{\Sigma} = -\left(\phi^2 + \frac{1}{2}\Sigma^2 + \frac{1}{3}\rho + \frac{1}{2}\mathcal{E} + \frac{1}{4}\Pi\right)\Sigma \tag{21}$$

and

$$\dot{\hat{\phi}} = -2\Sigma\dot{\Sigma} - \phi\dot{\phi} - \frac{2}{3}\dot{\rho} - \dot{\mathcal{E}} - \frac{1}{2}\dot{\Pi}$$
$$= -\left(\frac{1}{2}\phi^2 + \Sigma^2 + \frac{1}{6}\rho - \frac{1}{2}\mathcal{E} + \frac{3}{2}p\right)\Sigma. \quad (22)$$

Using the commutation relation on Eqs. (21) and (22), we obtain

$$\left[\frac{1}{4}\Pi + \frac{3}{2}\Sigma^2 + \frac{1}{2}(\rho + 3p)\right]\Sigma = 0. \quad (23)$$

So, either $\Sigma = 0$ or

$$\frac{1}{4}\Pi + \frac{3}{2}\Sigma^2 + \frac{1}{2}(\rho + 3p) = 0.$$

If $\Sigma = 0$, then the star must be static ($\Theta = \Sigma = 0$), so we assume that $\Sigma \neq 0$ and that

$$\frac{1}{4}\Pi + \frac{3}{2}\Sigma^2 + \frac{1}{2}(\rho + 3p) = 0. \quad (24)$$

From Eq. (19), Eq. (24) implies that $\Pi = 0$. Now, if we take the dot derivative of Eq. (19) and substitute for Eqs. (9) and (16), we obtain the evolution of p:

$$\dot{p} = (\Sigma^2 + 2\mathcal{E})\Sigma. \quad (25)$$

Taking the hat derivative of Eq. (25) and the dot derivative of Eq. (17), we obtain, respectively,

$$\dot{\hat{p}} = \left(-\frac{9}{2}\phi\Sigma^2 - 6\phi\mathcal{E} + \frac{2}{3}\hat{\rho}\right)\Sigma \quad (26)$$

and

$$\dot{\hat{p}} = 0. \quad (27)$$

Using the commutation relation on Eqs. (26) and (27), we obtain the propagation of ρ,

$$\hat{\rho} = 9\phi\left(\frac{3}{4}\Sigma^2 + \mathcal{E}\right). \quad (28)$$

Now, taking the hat derivative of Eq. (9) and the dot derivative of Eq. (12), we obtain, respectively,

$$\hat{\dot{\Sigma}} = \Sigma\hat{\Sigma} - \hat{\mathcal{E}} = -\frac{3}{2}\phi\Sigma^2 - \hat{\mathcal{E}} \quad (29)$$

and

$$\dot{\hat{\Sigma}} = -\frac{3}{2}(\dot{\phi}\Sigma + \phi\dot{\Sigma}) = -\frac{3}{2}\phi\mathcal{E}. \quad (30)$$

Using the commutation relation on Eqs. (29) and (30), we obtain

$$\hat{\mathcal{E}} = \frac{3}{2}\phi(\Sigma^2 - \mathcal{E}), \quad (31)$$

and upon substituting in Eq. (14), we obtain

$$\hat{\rho} = \frac{9}{2}\phi\Sigma^2. \quad (32)$$

Comparing Eqs. (28) and (32), we get

$$\phi\left(\mathcal{E} + \frac{1}{4}\Sigma^2\right) = 0. \quad (33)$$

Therefore, we must have either $\phi = 0$ or

$$\mathcal{E} = -\frac{1}{4}\Sigma^2. \quad (34)$$

We show that either case yields $\Sigma = 0$, in which case the star is static. First, suppose $\phi = 0$. Then, Eq. (13) gives the constraint equation

$$\Sigma^2 = -\frac{2}{3}\rho - \mathcal{E}, \quad (35)$$

and comparing Eqs. (19) and (35), we obtain

$$\mathcal{E} = -\frac{1}{3}(\rho - 3p). \quad (36)$$

Taking the dot derivative of Eqs. (36), using Eqs. (11) and (25), we obtain

$$\Sigma\left[\Sigma^2 + \frac{1}{2}(\mathcal{E} + \rho + p)\right] = 0. \quad (37)$$

Again, assuming $\Sigma \neq 0$, we must have

$$\Sigma^2 = -\frac{1}{2}(\mathcal{E} + \rho + p). \quad (38)$$

Taking the dot derivative of Eq. (38) and using Eq. (9), we obtain

$$\dot{p} = \Sigma(4\mathcal{E} - 2\Sigma^2), \quad (39)$$

and upon comparing to Eq. (25), we obtain

$$\Sigma(2\mathcal{E} - 3\Sigma^2) = 0. \quad (40)$$

Since $\Sigma \neq 0$, we have

$$\mathcal{E} = \frac{3}{2}\Sigma^2, \quad (41)$$

and upon using Eqs. (19) and (36), we obtain

$$\rho = -\frac{5}{3}p. \tag{42}$$

Taking the dot derivative of Eq. (42), we have $\dot{p} = 0$. Setting Eq. (25) to zero, while using Eq. (41) to substitute for $\mathcal{E}$, gives $\Sigma^2 = 0$, which gives $\Sigma = 0$.

Next, assume $\phi \neq 0$ and that Eq. (34) is satisfied. Now, taking the dot derivative of Eq. (34) and using Eqs. (11), (9), and (34) to simplify, we obtain

$$\left[\frac{3}{4}\Sigma^2 + \rho + p\right]\Sigma = 0. \tag{43}$$

Assume $\Sigma \neq 0$. We must have

$$\frac{3}{4}\Sigma^2 + \rho + p = 0. \tag{44}$$

Using Eq. (19), Eq. (44) simplifies to

$$\rho = -\frac{14}{5}p. \tag{45}$$

Finally, taking the dot derivative of Eq. (45), we have $\dot{p} = 0$, and upon comparing to Eq. (25) and substituting for $\mathcal{E}$ using Eq. (34), we obtain $\Sigma^2 = 0$, which gives $\Sigma = 0$.

2. Case 2

Let us next consider the case $A \neq 0$ and $Q = 0$. The commutation relation (8) now becomes

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -A\dot{\psi} + \Sigma\hat{\psi}. \tag{46}$$

Taking the hat derivative of Eq. (11) and the dot derivative of Eq. (14), we obtain, respectively,

$$\begin{aligned}\hat{\dot{\Sigma}} &= -\hat{A}\phi - A\hat{\phi} + \Sigma\hat{\Sigma} + \frac{1}{3}\hat{\rho} + \hat{p} - \hat{\mathcal{E}} + \frac{1}{2}\hat{\Pi} \\ &= A^2\phi + \frac{3}{2}A\phi^2 - 3\phi\Sigma^2 - \frac{1}{3}A\rho - \frac{3}{4}\phi\Pi - \frac{1}{2}\phi\rho \\ &\quad - \frac{3}{2}\phi p + A\mathcal{E} - \frac{1}{2}A\Pi - Ap + \frac{3}{2}\phi\mathcal{E}\end{aligned} \tag{47}$$

and

$$\begin{aligned}\dot{\hat{\Sigma}} &= -\frac{3}{2}(\dot{\phi}\Sigma + \phi\dot{\Sigma}) \\ &= \frac{3}{2}A\Sigma^2 - \frac{3}{2}\phi\Sigma^2 + \frac{3}{2}A\phi^2 - \frac{1}{2}\phi\rho - \frac{3}{2}\phi p \qquad (48) \\ &\quad + \frac{3}{2}\phi\mathcal{E} - \frac{3}{4}\phi\Pi. \qquad (49)\end{aligned}$$

Using the commutation relation on Eqs. (47) and (48), we obtain

$$A\Sigma^2 = 0. \tag{50}$$

Since $A \neq 0$, we must have $\Sigma = 0$, and thus the star is static.

3. Case 3

Finally, we consider the case $A = 0$ and $Q \neq 0$. From Eq. (15), we have the constraint equation as Eq. (19). The commutation relation in this case is Eq. (20). Taking the hat derivative of Eq. (10) and the dot derivative of Eq. (13), we obtain, respectively,

$$\begin{aligned}\hat{\dot{\phi}} &= \frac{1}{2}\hat{\Sigma}\phi + \frac{1}{2}\Sigma\hat{\phi} \\ &= -\phi^2\Sigma - \frac{1}{2}\phi Q - \frac{1}{2}\Sigma^3 - \frac{1}{3}\Sigma\rho - \frac{1}{2}\Sigma\mathcal{E} - \frac{1}{4}\Sigma\Pi + \hat{Q}\end{aligned} \tag{51}$$

and

$$\begin{aligned}\dot{\hat{\phi}} &= -2\Sigma\dot{\Sigma} - \phi\dot{\phi} - \frac{2}{3}\dot{\rho} - \left(\dot{\mathcal{E}} + \frac{1}{2}\dot{\Pi}\right) \\ &= -\Sigma^3 - \frac{1}{6}\Sigma\rho - \frac{5}{4}\Sigma\Pi + \frac{1}{2}\Sigma\mathcal{E} + \frac{1}{2}\Sigma p - \dot{\rho}.\end{aligned} \tag{52}$$

Using the commutation relation on Eqs. (51) and (52), we obtain

$$\left[\frac{3}{2}\Sigma^2 + \frac{1}{2}(\rho + 3p)\right]\Sigma = \phi Q, \tag{53}$$

which, upon using Eq. (19), reduces to

$$\phi Q = 0. \tag{54}$$

Since $Q \neq 0$, we must have $\phi = 0$. But from Eq. (10), this gives $Q = 0$.

From the three cases considered, we therefore must have $A \neq 0$ and $Q \neq 0$ to have an expansion-free star that is evolving. ■

B. Geometry of expansion-free stars

We state and prove the following theorem on the geometry of expansion-free dynamical stars.

Theorem 2: An expansion-free dynamical star must be conformally flat.

Proof.—We prove this by checking for additional constraints from the field equations with $A \neq 0$ and $Q \neq 0$. The commutation relation in this case is given by Eq. (46). Taking the hat derivative of Eq. (9), we obtain

$$\begin{aligned}\hat{\dot{\Sigma}} &= A^2\phi + \frac{3}{2}A\phi^2 - 3\phi\Sigma^2 + A\Sigma^2 - \frac{1}{2}\phi\rho - \frac{3}{2}\phi p \\ &\quad + \frac{2}{3}A\rho + \frac{1}{2}A\Pi - \frac{3}{2}\Sigma Q + \frac{3}{2}\phi\mathcal{E} + \frac{3}{4}\phi\Pi + \hat{p} + \hat{\Pi},\end{aligned} \tag{55}$$

and taking the dot derivative of Eq. (12), we have

$$\dot{\hat{\Sigma}} = \frac{3}{2}A\Sigma^2 - \frac{3}{2}\phi\Sigma^2 - \frac{3}{2}\Sigma Q + \frac{3}{2}A\phi^2 - \frac{1}{2}\phi\rho - \frac{1}{2}\phi p + \frac{3}{2}\phi\mathcal{E} - \frac{3}{4}\phi\Pi - \dot{Q}. \quad (56)$$

Taking the difference of Eqs. (55) and (56) and employing the commutation relation (46), we obtain

$$-A\dot{\Sigma} + \Sigma\hat{\Sigma} = A^2\phi - \frac{3}{2}\phi\Sigma^2 - \frac{1}{2}A\Sigma^2 + \frac{3}{2}\phi\Pi + \frac{2}{3}A\rho + \frac{1}{2}A\Pi + \dot{Q} + \hat{p} + \hat{\Pi}, \quad (57)$$

which, upon using Eq. (17) and simplifying, gives

$$A\mathcal{E} = 0. \quad (58)$$

Since $A \neq 0$, we must have $\mathcal{E} = 0$ so that the electric part of the Weyl tensor is vanishing. Fixing $\mathcal{E} = 0$ in the field equations for $A \neq 0$, $Q \neq 0$, all other commutation relations on pairs of evolution and propagation equations return identities. ■

IV. DISCUSSION

The expansion-free condition in general relativity has received considerable attention in recent years and has been applied to describe physical features of radiating stars. We have utilized the $1 + 1 + 2$ semitetrad covariant formalism to study such stars in general in spherical symmetry. The analysis shows that expansion-free dynamical stars are severely constrained and can only exist under very particular conditions. From the set of field equations, we have explicitly shown that a necessary condition for a star with zero expansion to evolve is that the star has nonzero radiation and acceleration. With further analysis of the field equations with $A \neq 0$ and $Q \neq 0$, it is shown that the star is necessarily conformally flat. Proving these results amounts to the analysis of the field equation via commutation relations, through which we obtain additional constraints, which further give us additional evolution equations that can be matched against the original set of equations. These results add to the literature on expansion-free dynamical stars, which have been developed over the last decade and half, most notably through works of Herrera and coauthors.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee for the very kind and valuable comments that have helped this paper to be clearer. A. S. and R. G. is supported by the National Research Foundation, South Africa. S. D. M. acknowledges that this work is based on research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation.

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