Nonlinear perturbations of Reissner-Nordström black holes

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We develop a nonlinear perturbation theory of Reissner-Nordström black holes. We show that, at each perturbation level, Einstein-Maxwell equations can be reduced to four inhomogeneous wave equations, two for the polar and two for the axial sector. The gravitational part of these equations is similar to Regge-Wheeler and Zerilli equations with source and additional coupling to the electromagnetic sector. We construct solutions to the inhomogeneous part of wave equations in terms of sources for Einstein-Maxwell equations. We discuss the $\ell = 0$ and $\ell = 1$ cases separately.

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I. INTRODUCTION

Perturbative methods play an important role in General Relativity. They find application to stability analysis, gravitational radiation, cosmology, rotating stars, the accretion disk, self-force, etc. Sometimes linear analysis gives sufficient insight into physical phenomena, but sometimes going beyond linear order can change qualitatively linear predictions (e.g., the Bizoń-Rostworowski conjecture of instability of anti-de Sitter spacetime [1]). In this paper, we study nonlinear perturbations of spherically symmetric solutions to Maxwell-Einstein equations. Linear perturbation theory of the Schwarzschild solution was formulated by Regge and Wheeler [2] and Zerilli [3] and then generalized to a Reissner-Nordström black hole by Zerilli [4] (see also Refs. [5–8], and [9]). Perturbations of Reissner-Nordström have also been recently discussed in the context of stability of the Cauchy horizon (issue crucial for the *strong cosmic censorship conjecture*; see Refs. [10,11], and [12]). All of these calculations are, however, only linear (or numerical only), and there was no robust procedure to move beyond linearity. Master equations from the present article provide a straightforward procedure to move beyond first-order estimates; at higher orders, there are still only wave equations (now inhomogeneous) to solve.

Taking into account higher-order perturbation terms makes the computations significantly more difficult; equations at each order beyond linear include all the previous-order terms. This issue has been treated by some authors—e.g., second-order perturbations of Schwarzschild were studied by Tomita and Tajima [13], Garat and Price [14], Gleiser *et al.* [15], Nakano and Ioka [16], and Brizuela *et al.* [17]. Recently, Rostworowski [18] provided a robust framework to deal with nonlinear (in principle of any order) gravitational perturbations of spherically symmetric spacetimes. The present article is an extension of Ref. [18] to both gravitational and electromagnetic nonlinear perturbations of Reissner-Nordström black holes. It also generalizes Zerilli's master equations [4] to any perturbation order.

Our approach is based on assumptions similar to those from Ref. [18]. We rewrite them explicitly here, since there are some differences:

(1) At each perturbation level, there are four master scalar variables, two in the polar and two in the axial sector. In each sector, they fulfill a system of two linearly coupled inhomogeneous (homogeneous at the linear order) wave equations with potentials.
(2) At each perturbation level, Regge-Wheeler variables and electromagnetic tensor components are linear combinations of master scalar variables from the suitable sector and their derivatives up to the second order. At the nonlinear orders, one needs to include additional functions to fulfill Maxwell-Einstein equations.
(3) At the linear level, relations from the previous point can be inverted to express master scalars as combinations of Regge-Wheeler variables and electromagnetic tensor components. At the nonlinear level, we take the same expressions for the master scalar functions.

In our considerations, we restrict ourselves to axially symmetric perturbations only (going beyond axial symmetry is a straightforward procedure, that conceptually adds little to this paper). During calculations, we stick to the Regge-Wheeler (RW) gauge. For practical implementations, after finding a solution in the RW gauge, one should move to an asymptotically flat gauge to ensure regularity of higher-order source functions (see Brizuela *et al.* [17]).

The paper is organized as follows. In Sec. II, we briefly introduce the Reissner-Nordström metric, and in Sec. III, we discuss the general form of perturbation expansion of

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Einstein-Maxwell equations. In Secs. IV, V, and VI, we remind the reader of polar expansion in axial symmetry, discuss gauge choice, and present source identities. The main result of this paper, namely providing inhomogeneous wave equations for Einstein-Maxwell equations of any perturbation order, is contained in Sec. VII.

II. BACKGROUND METRIC

The Reissner-Nordström solution describes a static, spherically symmetric black hole with an electric charge. In static coordinates $(t\in(-\infty,\infty), r\in(r_+,\infty), \theta\in(0,\pi), \phi\in[0,2\pi))$, its metric is given by (we use $G=c=4\pi\epsilon_0=1$)

$$\bar{g} = -Adt^2 + \frac{1}{A}dr^2 + r^2 d\Omega^2, \tag{1}$$

where $A = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, $r_+ = M + \sqrt{M^2 - Q^2}$, and M and Q are mass and charge of a black hole, respectively (we assume $|Q| < M$). Together with an electromagnetic tensor $\bar{F}$ with only nonzero terms $\bar{F}_{tr} = -\bar{F}_{rt} = \frac{Q}{r}$, metric (1) solves Einstein-Maxwell equations,

$$\bar{R}_{\mu\nu} = 8\pi\bar{T}_{\mu\nu}, \tag{2}$$

$$\bar{\nabla}_\mu \bar{F}^{\mu\nu} = 0, \tag{3}$$

$$\bar{F}_{[\mu\nu,\lambda]} = 0, \tag{4}$$

where $\bar{\nabla}$ and $\bar{R}_{\mu\nu}$ are, respectively, the covariant derivative and Ricci tensor with respect to the metric $\bar{g}$ and the comma denotes a partial derivative. $\bar{T}_{\mu\nu}$ is given by

$$\bar{T}_{\mu\nu} = \frac{1}{4\pi}\left(\bar{F}_{\mu\alpha}\bar{F}^{\alpha}_{\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}\right) \tag{5}$$

III. GRAVITATIONAL AND ELECTROMAGNETIC PERTURBATIONS OF EINSTEIN-MAXWELL SYSTEMS

Let us assume that metric $\bar{g}$ and electromagnetic tensor $\bar{F}$ solve Einstein-Maxwell equations (2)–(4). Now, we seek for new solutions g and F that we expand around $\bar{g}$ and $\bar{F}$ with respect to the perturbation parameter ϵ:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i>0} {}^{(i)}h_{\mu\nu}\epsilon^i, \tag{6}$$

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + \sum_{i>0} {}^{(i)}f_{\mu\nu}\epsilon^i. \tag{7}$$

We plug (6) and (7) into Einstein-Maxwell equations, to obtain their perturbative form of order i,

$$\Delta_L {}^{(i)}h_{\mu\nu} - 8\pi {}^{(i)}t_{\mu\nu} = {}^{(i)}S^G_{\mu\nu}, \tag{8}$$

$$\bar{\nabla}^{\mu(i)}f_{\mu\nu} - {}^{(i)}\Theta_\nu = {}^{(i)}S^M_\nu, \tag{9}$$

$${}^{(i)}f_{[\mu\nu,\lambda]} = 0, \tag{10}$$

where

$$\Delta_L {}^{(i)}h_{\mu\nu} = \frac{1}{2}(-\bar{\nabla}^\alpha\bar{\nabla}_\alpha {}^{(i)}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu {}^{(i)}h^\alpha_\alpha - 2\bar{R}_{\mu\alpha\nu\beta} {}^{(i)}h^{\alpha\beta} + \bar{\nabla}_\mu\bar{\nabla}^{\alpha(i)}h_{\nu\alpha} + \bar{\nabla}_\nu\bar{\nabla}^{\alpha(i)}h_{\mu\alpha}), \tag{11}$$

$$\begin{aligned}{}^{(i)}t_{\mu\nu} &= 2{}^{(i)}f_{\alpha(\mu}\bar{F}^\alpha{}_{\nu)} - \frac{1}{2}{}^{(i)}f_{\alpha\beta}\bar{F}^{\alpha\beta}\bar{g}_{\mu\nu} \\ &+ \left(\frac{1}{2}\bar{F}_{\alpha\sigma}\bar{F}_\beta{}^\sigma\bar{g}_{\mu\nu} - \bar{F}_{\mu\alpha}\bar{F}_{\nu\beta}\right){}^{(i)}h^{\alpha\beta} \\ &- \frac{1}{4}\bar{F}^{2(i)}h_{\mu\nu} - {}^{(i)}h_{\alpha(\mu}\bar{T}^\alpha{}_{\nu)},\end{aligned} \tag{12}$$

$${}^{(i)}\Theta_\nu = \bar{g}^{\alpha\beta}(\bar{F}_{\sigma\nu} {}^{(i)}\delta\Gamma^\sigma_{\alpha\beta} + \bar{F}_{\beta\sigma} {}^{(i)}\delta\Gamma^\sigma_{\alpha\nu}), \tag{13}$$

$${}^{(i)}\delta\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}\bar{g}^{\sigma\delta}(-\bar{\nabla}_\delta {}^{(i)}h_{\alpha\beta} + \bar{\nabla}_\alpha {}^{(i)}h_{\beta\delta} + \bar{\nabla}_\beta {}^{(i)}h_{\delta\alpha}). \tag{14}$$

Tensor sources ${}^{(i)}S^G_{\mu\nu}$ and vector sources ${}^{(i)}S^M_\nu$ are expressed by ${}^{(j<i)}h_{\mu\nu}$ and ${}^{(j<i)}f_{\mu\nu}$; therefore, perturbative Einstein equations should be solved order by order (see Appendix A for the construction of sources). For $i=1$, both sources vanish.

IV. POLAR EXPANSION

In a spherically symmetric background, in 3 + 1 dimensions, vector and tensor components split into two sectors that transform differently under rotations: polar and axial (for the details, see, e.g., Refs. [2,3,19,20]). Symmetric tensors have seven polar and three axial components, and antisymmetric tensors have three polar and three axial components. Below, we list expansions of all the components of both symmetric and antisymmetric tensors and of vectors in axial symmetry (P_ℓ denotes ℓth Legendre polynomial).

The symmetric tensor, polar sector is

$$S_{ab}(t,r,\theta) = \sum_{0\le\ell} S_{\ell ab}(t,r)P_\ell(\cos\theta), \quad a,b=t,r, \tag{15}$$

$$S_{a\theta}(t,r,\theta) = \sum_{1\le\ell} S_{\ell a\theta}(t,r)\partial_\theta P_\ell(\cos\theta), \quad a=t,r, \tag{16}$$

$$\frac{1}{2}\left(S_{\theta\theta}(t,r,\theta) + \frac{S_{\phi\phi}(t,r,\theta)}{\sin^2\theta}\right) = \sum_{0\le\ell} S_{\ell+}(t,r)P_\ell(\cos\theta), \tag{17}$$

$$\frac{1}{2}\left(S_{\theta\theta}(t,r,\theta)-\frac{S_{\phi\phi}(t,r,\theta)}{\sin^2\theta}\right) = \sum_{2\le\ell} S_{\ell-}(t,r)(-\ell(\ell+1)P_\ell(\cos\theta)-2\cot\theta\partial_\theta P_\ell(\cos\theta)). \tag{18}$$

The symmetric tensor, axial sector is

$$S_{a\phi}(t,r,\theta)=\sum_{1\le\ell} S_{\ell a\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta), \quad a=t,r, \tag{19}$$

$$S_{\theta\phi}(t,r,\theta)=\sum_{2\le\ell} S_{\ell\theta\phi}(t,r)(-\ell(\ell+1)\sin\theta P_\ell(\cos\theta) - 2\cos\theta\partial_\theta P_\ell(\cos\theta)). \tag{20}$$

The antisymmetric tensor, polar sector is

$$A_{tr}(t,r,\theta)=\sum_{0\le\ell} A_{\ell tr}(t,r)P_\ell(\cos\theta), \tag{21}$$

$$A_{a\theta}(t,r,\theta)=\sum_{1\le\ell} A_{\ell a\theta}(t,r)\partial_\theta P_\ell(\cos\theta), \quad a=t,r. \tag{22}$$

The antisymmetric tensor, axial sector is

$$A_{a\phi}(t,r,\theta)=\sum_{1\le\ell} A_{\ell a\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta), \quad a=t,r, \tag{23}$$

$$A_{\theta\phi}(t,r,\theta)=\sum_{0\le\ell} A_{\ell\theta\phi}(t,r)\sin\theta P_\ell(\cos\theta). \tag{24}$$

The vector, polar sector is

$$V_a(t,r,\theta)=\sum_{0\le\ell} V_{\ell a}(t,r)P_\ell(\cos\theta), \quad a=t,r, \tag{25}$$

$$V_\theta(t,r,\theta)=\sum_{1\le\ell} V_{\ell\theta}(t,r)\partial_\theta P_\ell(\cos\theta). \tag{26}$$

The vector, axial sector is

$$V_\phi(t,r,\theta)=\sum_{1\le\ell} V_{\ell\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta). \tag{27}$$

Since the background is spherically symmetric, differential operators acting on ${}^{(i)}h_{\mu\nu}$ and ${}^{(i)}f_{\mu\nu}$ do not mix axial and polar sectors; therefore, Einstein-Maxwell equations split into two sectors as well: there are seven Einstein and three Maxwell equations in the polar sector and three Einstein and one Maxwell equation in the axial sector. In our paper, we consider separately $\ell \ge 2$, $\ell = 1$, and $\ell = 0$.

V. GAUGE CHOICE

Under a gauge transformation induced by a vector X^μ, tensors transform as $t_{\mu\nu} \to t_{\mu\nu} + \mathcal{L}_X t_{\mu\nu}$ (see Appendix B for the explicit form of transformations). Throughout the paper, we use Regge-Wheeler gauge [2]; namely, we set ${}^{(i)}h_{\ell tr}$, ${}^{(i)}h_{\ell r\theta}$, and ${}^{(i)}h_{\ell-}$ to zero in the polar sector and ${}^{(i)}h_{l\theta\phi}=0$ in the axial sector. It turns out that variables we use correspond exactly to RW gauge invariants; therefore, a result for $\ell \ge 2$ can be read as expressions for RW gauge invariants. However, throughout the paper, we stick to fixed RW gauge because the discussion of $\ell = 0$ and $\ell = 1$ cases is more straightforward then. When the background quantities $\bar{g}$ and $\bar{F}$ fulfill Einstein equations, the left-hand sides of perturbation equations (8)–(10) of order i do not feel gauge transformations of order i, but source functions ${}^{(i)}S^G_{\mu\nu}$ and ${}^{(i)}S^G_{\mu\nu}$ depend on the gauge transformations of order $j < i$ explicitly, so such a formulation is not fully gauge invariant. This, however, is not a problem, since equations are solved order by order and for the practical implementations one goes to the asymptotically flat gauge before moving to the next order anyway.

VI. SOURCES FOR EINSTEIN-MAXWELL EQUATIONS

Sources ${}^{(i)}S^G_{\ell\,\mu\nu}$ and ${}^{(i)}S^M_{\ell\ \nu}$ are built of ${}^{(j)}h_{\ell\mu\nu}$ and ${}^{(j)}f_{\ell_{\mu\nu}}$ with $j < i$. These sources are not independent but fulfill five identities:

$$\bar{\nabla}^{\mu\,(i)}S^G{}_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\nu{}^{(i)}S^{G^\mu}{}_\mu - 2\bar{F}^\mu{}_\nu{}^{(i)}S^M{}_\mu = 0, \tag{28}$$

$$\bar{\nabla}^{\mu\,(i)}S^M{}_\mu = 0, \tag{29}$$

which come from the Bianchi identity and contracted Jacobi identity for tensor $F_{\mu\nu}$. One can check that they hold using (8)–(10) directly. The explicit form of identities (28) and (29) for polar-expanded sources in the polar sector reads [we introduce $\tau = \sqrt{(\ell-1)(\ell+2)}$]

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\,tr} + \frac{2QA}{r^2}{}^{(i)}S^M_{\ell\ r} + A\partial_r{}^{(i)}S^G_{\ell\,tr} - \frac{1}{2A}\partial_t{}^{(i)}S^G_{\ell\,tt} - \frac{1}{2}A\partial_t{}^{(i)}S^G_{\ell\,rr} - \frac{\ell(\ell+1)}{r^2}{}^{(i)}S^G_{\ell\,t\theta} - \frac{1}{r^2}\partial_t{}^{(i)}S^G_{\ell+} = 0, \tag{30}$$

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\,rr} + \frac{2Q}{r^2A}{}^{(i)}S^M_{\ell\,t} + \frac{1}{2A}\partial_r{}^{(i)}S^G_{\ell\,tt} + \frac{1}{2}A\partial_r{}^{(i)}S^G_{\ell\,rr} - \frac{1}{A}\partial_t{}^{(i)}S^G_{\ell\,tr} - \frac{\ell(\ell+1)}{r^2}{}^{(i)}S^G_{\ell\,r\theta} - \frac{\partial_r}{r^2}{}^{(i)}S^G_{\ell\,+} = 0, \tag{31}$$

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\,r\theta} + \frac{1}{2A}{}^{(i)}S^G_{\ell\,tt} - \frac{1}{2}A{}^{(i)}S^G_{\ell\,rr} + A\partial_r S_{r\theta} - \frac{1}{A}\partial_t{}^{(i)}S^G_{\ell\,t\theta} - \frac{\tau^2}{r^2}{}^{(i)}S^G_{\ell\,-} = 0, \tag{32}$$

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^M_{\ell\,r} + A\partial_r{}^{(i)}S^M_{\ell\,r} - \frac{1}{A}\partial_t{}^{(i)}S^M_{\ell\,t} - \frac{\ell(\ell+1)}{r^2}{}^{(i)}S^M_{\ell\,\theta} = 0, \tag{33}$$

and in the axial sector reads

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\,r\phi} + A\partial_r{}^{(i)}S^G_{\ell\,r\phi} - \frac{\partial_t{}^{(i)}S^G_{\ell\,t\phi}}{A} + - \frac{\tau^{2\,(i)}S^G_{\ell\,\theta\phi}}{r^2} = 0. \tag{34}$$

VII. GRAVITATIONAL AND ELECTROMAGNETIC PERTURBATIONS

Now, we polar expand Eqs. (8)–(10):

$${}^{(i)}E_{\ell\mu\nu} = \Delta_L{}^{(i)}h_{\ell\mu\nu} - 8\pi{}^{(i)}t_{\ell\mu\nu} = {}^{(i)}S^G_{\ell\,\mu\nu}, \tag{35}$$

$${}^{(i)}J_{\ell\nu} = \bar{\nabla}^{\mu(i)}f_{\ell\mu\nu} - {}^{(i)}\Theta_{\ell\nu} = {}^{(i)}S^M_{\ell\,\nu}, \tag{36}$$

$${}^{(i)}f_{\ell(\mu\nu,\alpha)} = 0. \tag{37}$$

A. Polar sector, $\ell \geq 2$

First, from (37), we have

$${}^{(i)}f_{\ell\,tr} = \partial_r{}^{(i)}f_{\ell\,t\theta} - \partial_t{}^{(i)}f_{\ell\,r\theta}, \tag{38}$$

and from ${}^{(i)}E_{\ell-}$,

$$\frac{1}{4}\left(\frac{1}{A}{}^{(i)}h_{\ell\,tt} - A{}^{(i)}h_{\ell\,rr}\right) - {}^{(i)}S_{\ell-} = 0. \tag{39}$$

We use relations (38) and (39) to eliminate ${}^{(i)}f_{\ell_{tr}}$ and ${}^{(i)}h_{\ell\,tt}$ from Eqs. (35)–(37). Then, we are left with five variables: ${}^{(i)}h_{\ell\,tt}$, ${}^{(i)}h_{\ell\,tr}$, ${}^{(i)}h_{\ell\,+}$, ${}^{(i)}f_{\ell_{t\theta}}$, and ${}^{(i)}f_{\ell_{r\theta}}$.

Remaining equations can all be fulfilled by introducing two master scalar variables ${}^{(i)}\Phi^{\mathcal{P}}_\ell$ and ${}^{(i)}\Psi^{\mathcal{P}}_\ell$ which solve a system of two coupled inhomogeneous (homogeneous at the linear order) wave equations [21]:

$$r(-\bar{\Box} + V^{\mathcal{P}}_{G\ell})\frac{{}^{(i)}\Phi^{\mathcal{P}}_\ell}{r} + V^{\mathcal{P}}_{MG\ell}{}^{(i)}\Psi^{\mathcal{P}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}, \tag{40}$$

$$r(-\bar{\Box} + V^{\mathcal{P}}_{M\ell})\frac{{}^{(i)}\Psi^{\mathcal{P}}_\ell}{r} + V^{\mathcal{P}}_{MG\ell}{}^{(i)}\Phi^{\mathcal{P}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell}. \tag{41}$$

Following the idea of Ref. [18], we express leftover variables by linear combinations of master scalar functions, their partial derivatives up to the second order (to solve homogeneous part of Einstein-Maxwell equations), and additional source functions (to solve the inhomogeneous part of equations). These combinations and potentials $V^{\mathcal{P}}_{G\ell}$, $V^{\mathcal{P}}_{M\ell}$, and $V^{\mathcal{P}}_{MG\ell}$ are defined uniquely:

$$V^{\mathcal{P}}_{G\ell} = \tau^2\hat{V}^{\mathcal{P}}_{G\ell} = \frac{\tau^2(-r^2A'^2 - 2A(-2A + \ell(\ell+1) + 2) + \ell^2(\ell+1)^2)}{r^2(rA' - 2A + \ell(\ell+1))^2} + \frac{8Q^2\tau^2A}{r^4(rA' - 2A + \ell(\ell+1))^2}, \tag{42}$$

$$V^{\mathcal{P}}_{M\ell} = \frac{-rA' + \ell(\ell+1)}{r^2} + \frac{4Q^2(2A(2r^3A' + \tau^2r^2 + 4Q^2) - r^4A'^2 - 4r^2A^2 + (\ell(\ell+1))^2r^2)}{r^6(rA' - 2A + \ell(\ell+1))^2}, \tag{43}$$

$$V^{\mathcal{P}}_{MG\ell} = \tau\hat{V}^{\mathcal{P}}_{MG\ell} = \frac{2\tau Q(2A(r^3A' + 4Q^2 - 2r^2) - r^4A'^2 + (\ell(\ell+1))^2r^2)}{r^5(rA' - 2A + \ell(\ell+1))^2}, \tag{44}$$

$${}^{(i)}h_{\ell\,tr} = -r\partial_{tr}{}^{(i)}\Phi^{\mathcal{P}}_\ell + \left(\frac{rA'}{2A} - \frac{\tau^2}{rA' - 2A + \ell(\ell+1)}\right)\partial_t{}^{(i)}\Phi^{\mathcal{P}}_\ell - \frac{2\tau Q\partial_t}{r(rA' - 2A + \ell(\ell+1))}{}^{(i)}\Psi^{\mathcal{P}}_\ell + {}^{(i)}\alpha_\ell, \tag{45}$$

$${}^{(i)}h_{\ell\,tt} = -r\partial_{rr}{}^{(i)}\Phi^{\mathcal{P}}_\ell + \left(-\frac{\tau^2}{rA' - 2A + \ell(\ell+1)} - \frac{rA'}{2A}\right)\partial_r{}^{(i)}\Phi^{\mathcal{P}}_\ell + \frac{r}{2A}\left(\frac{A'}{r} + V^{\mathcal{P}}_{G\ell}\right){}^{(i)}\Phi^{\mathcal{P}}_\ell + - \frac{2\tau Q}{r(rA' - 2A + \ell(\ell+1))}\partial_r{}^{(i)}\Psi^{\mathcal{P}}_\ell + \frac{r}{2A}V^{\mathcal{P}}_{MG\ell}{}^{(i)}\Psi^{\mathcal{P}}_\ell + {}^{(i)}\beta_\ell, \tag{46}$$

$$ {}^{(i)}h_{\ell +} = -A\partial_r {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{\left(\frac{2A(rA'-2A+2)}{rA'-2A+\ell(\ell+1)} - \ell(\ell+1)\right)}{2r} {}^{(i)}\Phi^{\mathcal{P}}_{\ell} - \frac{2\tau QA}{r^2(rA'-2A+\ell(\ell+1))} {}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\gamma_{\ell}, \tag{47} $$

$$ {}^{(i)}f_{\ell\, t\theta} = \frac{A\tau}{4}\partial_r {}^{(i)}\Psi^{\mathcal{P}}_{\ell} - \frac{QA}{2r}\partial_r {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{QA}{2r^2} {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\lambda_{\ell}, \tag{48} $$

$$ {}^{(i)}f_{\ell\, r\theta} = \frac{\tau}{4A}\partial_t {}^{(i)}\Psi^{\mathcal{P}}_{\ell} - \frac{Q}{2rA}\partial_t {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\kappa_{\ell}. \tag{49} $$

At the linear level, ${}^{(1)}\alpha_{\ell} = {}^{(1)}\beta_{\ell} = {}^{(1)}\gamma_{\ell} = {}^{(1)}\lambda_{\ell} = {}^{(1)}\kappa_{\ell} = {}^{(1)}\tilde{S}^{\mathcal{P}}_{G\ell} = {}^{(1)}\tilde{S}^{\mathcal{P}}_{M\ell} = 0$ and relations (45)–(46) can be inverted to express ${}^{(1)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(1)}\Psi^{\mathcal{P}}_{\ell}$ as functions of ${}^{(i)}h_{\ell\mu\nu}$ and ${}^{(i)}f_{\ell\mu\nu}$. At higher orders, we treat linear level expressions for ${}^{(1)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(1)}\Psi^{\mathcal{P}}_{\ell}$ as definitions of ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$:

$$ {}^{(i)}\Phi^{\mathcal{P}}_{\ell} = \frac{4rA(r\partial_r {}^{(i)}h_{\ell +} - A {}^{(i)}h_{\ell\, rr})}{\ell(\ell+1)(rA'-2A+\ell(\ell+1))} - \frac{2r {}^{(i)}h_{\ell +}}{\ell(\ell+1)}, \tag{50} $$

$$ {}^{(i)}\Psi^{\mathcal{P}}_{\ell} = \frac{4r^2(\partial_r {}^{(i)}f_{\ell\, t\theta} - \partial_t {}^{(i)}f_{\ell\, r\theta})}{\ell(\ell+1)\tau} + \frac{8QA(r\partial_r {}^{(i)}h_{\ell +} - A {}^{(i)}h_{\ell\, rr})}{\ell(\ell+1)\tau(rA'-2A+\ell(\ell+1))}. \tag{51} $$

Having these definitions, we may express the left-hand side of (40) and (41) as combinations of ${}^{(i)}E_{\ell\mu\nu}$, ${}^{(i)}J_{\ell\nu}$, and their derivatives. Finding these combinations, we use (35) and (36) to build sources for wave equations:

$$ \begin{aligned} {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell} = & -\frac{4A^2(\tau^2 r^2 + 4Q^2) {}^{(i)}S^{G}_{\ell\, rr}}{\ell(\ell+1)r(rA'-2A+\ell(\ell+1))^2} + \frac{4 {}^{(i)}S^{G}_{\ell\, tt}(2r^3A' - 4r^2A + (\ell(\ell+1)+2)r^2 - 4Q^2)}{\ell(\ell+1)r(rA'-2A+\ell(\ell+1))^2} \\ & + \frac{8A\partial_r {}^{(i)}S^{G}_{\ell +}}{\ell(\ell+1)(rA'-2A+\ell(\ell+1))} + \frac{8A {}^{(i)}S^{G}_{\ell\, r\theta}}{rA'-2A+\ell(\ell+1)} - \frac{4rV^{\mathcal{P}}_{G\ell} {}^{(i)}S^{G}_{\ell +}}{\ell(\ell+1)\tau^2} \\ & + \frac{4 {}^{(i)}S^{G}_{\ell -}\left(\frac{Q^2(8A)}{r^3(rA'-2A+\ell(\ell+1))} - A' - rV^{\mathcal{P}}_{M\ell}\right)}{\ell(\ell+1)} - \frac{16Q {}^{(i)}S^{M}_{\ell\, t}}{\ell(\ell+1)(rA'-2A+\ell(\ell+1))}, \end{aligned} \tag{52} $$

$$ \begin{aligned} \frac{\ell(\ell+1)}{4}\tau {}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell} = & \frac{\ell(\ell+1)}{4} {}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell} \\ = & r^2\partial_r {}^{(i)}S^{M}_{\ell\, t} - r^2\partial_t {}^{(i)}S^{M}_{\ell\, r} - {}^{(i)}S^{M}_{\ell\, t}\left(2r - \frac{8Q^2}{r(rA'-2A+\ell(\ell+1))}\right) + \frac{8Q\left(\frac{r^2(rA'-2A+2)}{4} - Q^2\right) {}^{(i)}S^{G}_{\ell\, tt}}{r^2(rA'-2A+\ell(\ell+1))^2} \\ & + \frac{8QA^2 {}^{(i)}S^{G}_{\ell\, rr}\left(\frac{r^2(rA'-2A+2(\ell(\ell+1)-1))}{4} + Q^2\right)}{r^2(rA'-2A+\ell(\ell+1))^2} + \frac{4\ell(\ell+1)QA {}^{(i)}S^{G}_{\ell\, r\theta}}{r(rA'-2A+\ell(\ell+1))} + \frac{4QA\partial_r {}^{(i)}S^{G}_{\ell +}}{r(rA'-2A+\ell(\ell+1))} \\ & + 2QA'\partial_r {}^{(i)}S^{G}_{\ell -} + -\frac{2Q {}^{(i)}S^{G}_{\ell -}(A' + rV^{\mathcal{P}}_{M\ell})}{r} + 2QA\partial_r^2 {}^{(i)}S^{G}_{\ell -} - \frac{2Q\partial_t^2 {}^{(i)}S^{G}_{\ell -}}{A} - \frac{rV^{\mathcal{P}}_{MG\ell} {}^{(i)}S^{G}_{\ell +}}{\tau}. \end{aligned} \tag{53} $$

${}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}$ and ${}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell}$ are given uniquely up to the source identities (30)–(33). We introduced auxiliary quantities $\hat{V}^{\mathcal{P}}_{G\ell}$, $\hat{V}^{\mathcal{P}}_{MG\ell}$, and ${}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell}$, which are nonzero (or nonsingular) for $\ell = 1$.

At the nonlinear level, part of the solution (45)–(49) consisting of master scalars ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$ and their derivatives fulfills the homogeneous part of Einstein-Maxwell equations (35), (36), and (37), whereas part of (45)–(49) consisting of functions ${}^{(i)}\alpha_{\ell}$, ${}^{(i)}\beta_{\ell}$, ${}^{(i)}\gamma_{\ell}$, ${}^{(i)}\lambda_{\ell}$, and ${}^{(i)}\kappa_{\ell}$ is responsible for the inhomogeneous part of the Einstein-Maxwell equations. To find these functions, we plug (45)–(49) into Eqs. (35) and (36) and into definitions (50) and (51) to ensure consistency. Then, we solve these equations for ${}^{(i)}\alpha_{\ell}$, ${}^{(i)}\beta_{\ell}$, ${}^{(i)}\gamma_{\ell}$, ${}^{(i)}\lambda_{\ell}$, and ${}^{(i)}\kappa_{\ell}$. These functions, as well as scalar sources for wave equations ${}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}$ and ${}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell}$, are defined uniquely [up to the source identities (30)–(33)]:

$$ \begin{aligned} {}^{(i)}\alpha_{\ell} = & -\frac{2r^2(r^2A^2 {}^{(i)}S^{G}_{\ell\, rr} + r^2 {}^{(i)}S^{G}_{\ell\, tt} + 2A {}^{(i)}S^{G}_{\ell +})}{\ell(\ell+1)r^2(rA'-2A+\ell(\ell+1))} + \\ & -\frac{16Q^2A {}^{(i)}S^{G}_{\ell -}}{\ell(\ell+1)r^2(rA'-2A+\ell(\ell+1))}, \end{aligned} \tag{54} $$

$$ {}^{(i)}\beta_\ell = r\left(\frac{2r\,{}^{(i)}S^G_{\ell\,tr}}{\ell(\ell+1)} + \frac{\partial_t{}^{(i)}\alpha_\ell}{A}\right), \tag{55} $$

$$ {}^{(i)}\gamma_\ell = \frac{r\partial_r{}^{(i)}\alpha_\ell + {}^{(i)}\alpha_\ell}{A} - \frac{{}^{(i)}\alpha_\ell(rA' + \ell(\ell+1))}{2A^2}, \tag{56} $$

$$ {}^{(i)}\kappa_\ell = \frac{r^{2\,(i)}S^M_{\ell\,r}}{\ell(\ell+1)} + \frac{2Q\partial_t{}^{(i)}S^G_{\ell-}}{A\ell(\ell+1)}, \tag{57} $$

$$ {}^{(i)}\lambda_\ell = \frac{r^{2\,(i)}S^M_{\ell\,t}}{\ell(\ell+1)} + \frac{2QA\partial_r{}^{(i)}S^G_{\ell-}}{\ell(\ell+1)}. \tag{58} $$

B. Polar sector, $\ell = 1$

For $\ell = 1$, there is no $S_{\ell-}$ coefficient in a symmetric tensor decomposition; therefore, we do not have a ${}^{(i)}h_{\ell-}$ metric coefficient, and we lose the algebraic Einstein equation (39). However, since one of the gauge conditions was ${}^{(i)}h_{\ell-} = 0$, we gain additional gauge freedom, which we can use to keep algebraic relation (39). That means our $\ell \geq 2$ results are directly applicable to $\ell = 1$ as well. The only obstacle is that for the $\ell = 1$ coefficient $\tau = 0$ and singular terms appear in the source for wave equation (41) and in the definition (51). We can deal with it by introducing ${}^{(i)}\hat{\Psi}^{\mathcal{P}}_\ell = \tau{}^{(i)}\Psi^{\mathcal{P}}_\ell$, which, together with ${}^{(i)}\Phi^{\mathcal{P}}_\ell$, fulfills a set of wave equations,

$$ r(-\bar{\Box} + \tau^2\hat{V}^{\mathcal{P}}_{G\ell})\frac{{}^{(i)}\Phi^{\mathcal{P}}_\ell}{r} + \hat{V}^{\mathcal{P}}_{MG\ell}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}, \tag{59} $$

$$ r(-\bar{\Box} + V^{\mathcal{P}}_{M\ell})\frac{{}^{(i)}\hat{\Psi}^{\mathcal{P}}_\ell}{r} + \tau^2\hat{V}^{\mathcal{P}}_{MG\ell}{}^{(i)}\Phi^{\mathcal{P}}_\ell = {}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell}, \tag{60} $$

where $\hat{V}^{\mathcal{P}}_{G\ell}$, $\hat{V}^{\mathcal{P}}_{MG\ell}$, and ${}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell}$ are defined in (42), (44), and (53). For $\ell = 1$, the system is simpler—there is no coupling to the gravitational master scalar in (60). Now, scalar sources for both equations are regular for $\ell = 1$. Metric and electromagnetic tensor perturbations are then given by

$$ {}^{(i)}h_{1\,tr} = -r\partial_{tr}{}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{rA'}{2A}\partial_t{}^{(i)}\Phi^{\mathcal{P}}_1 - \frac{2Q\partial_t}{r(rA'-2A+2)}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\alpha_1, \tag{61} $$

$$ {}^{(i)}h_{1\,rr} = -r\partial_{rr}{}^{(i)}\Phi^{\mathcal{P}}_1 - \frac{rA'}{2A}\partial_r{}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{A'}{2A}{}^{(i)}\Phi^{\mathcal{P}}_1 + - \frac{2Q}{r(rA'-2A+2)}\partial_r{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + \frac{r}{2A}\hat{V}^{\mathcal{P}}_{MG1}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\beta_1, \tag{62} $$

$$ {}^{(i)}h_{1+} = -A\partial_r{}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{A-1}{r}{}^{(i)}\Phi^{\mathcal{P}}_1 - \frac{2QA}{r^2(rA'-2A+2)}{}^{(i)}\Psi^{\mathcal{P}}_1 + {}^{(i)}\gamma_1, \tag{63} $$

$$ {}^{(i)}f_{1\,t\theta} = \frac{A}{4}\partial_r{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 - \frac{QA}{2r}\partial_r{}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{QA}{2r^2}{}^{(i)}\Phi^{\mathcal{P}}_1 + {}^{(i)}\lambda_1, \tag{64} $$

$$ {}^{(i)}f_{1\,r\theta} = \frac{1}{4A}\partial_t{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 - \frac{Q}{2rA}\partial_t{}^{(i)}\Phi^{\mathcal{P}}_1 + {}^{(i)}\kappa_1. \tag{65} $$

Since there is no ${}^{(i)}S^G_{\ell-}$ source term, ${}^{(i)}\alpha_1$, ${}^{(i)}\beta_1$, ${}^{(i)}\gamma_1$, ${}^{(i)}\lambda_1$, and ${}^{(i)}\kappa_1$ for $\ell = 1$ are given by

$$ {}^{(i)}\alpha_1 = -\frac{r^2A^{2\,(i)}S^G_{1\,rr} + r^{2\,(i)}S^G_{1\,tt} + 2A{}^{(i)}S^G_{1\,+}}{(rA'-2A+2)}, \tag{66} $$

$$ {}^{(i)}\beta_1 = r^{2\,(i)}S^G_{1\,tr} + \frac{r\partial_t{}^{(i)}\alpha_1}{A}, \tag{67} $$

$$ {}^{(i)}\gamma_1 = \frac{r\partial_r{}^{(i)}\alpha_1 + {}^{(i)}\alpha_1}{A} - \frac{{}^{(i)}\alpha_1(rA'+2)}{2A^2}, \tag{68} $$

$$ {}^{(i)}\kappa_1 = \frac{r^2}{2}{}^{(i)}S^M_{1\,r}, \tag{69} $$

$$ {}^{(i)}\lambda_1 = \frac{r^2}{2}{}^{(i)}S^M_{1\,t}. \tag{70} $$

Although direct implementation of previous results provides a general solution to $\ell = 1$ equations, it can be misleading; it looks like there are two dynamical variables, whereas there should be only one [7] (for the Schwarzschild case $\ell = 1$, gravitational modes are pure gauge [22]). However, by the following gauge transformation, one can get rid of ${}^{(i)}\Phi^{\mathcal{P}}_1$ from (75)–(79),

$$ {}^{(i)}\zeta_{1\,t} = -\partial_t{}^{(i)}\zeta_{1\,\theta}, \tag{71} $$

$$ {}^{(i)}\zeta_{1\,r} = \frac{2{}^{(i)}\zeta_{1\,\theta}}{r} - \partial_r{}^{(i)}\zeta_{1\,\theta}, \tag{72} $$

$$ {}^{(i)}\zeta_{1\,\theta} = -\frac{r}{2}{}^{(i)}\Phi^{\mathcal{P}}_1, \tag{73} $$

and the solution reads

$$ {}^{(i)}h_{1\,tt} = -\frac{2A^2Q}{r(rA'-2A+2)}\partial_r{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 - \frac{rA}{2}\hat{V}^{\mathcal{P}}_{MG1}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + A^{2\,(i)}\beta_1 + rA{}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell} \tag{74} $$

$$ {}^{(i)}h_{1\,tr} = -\frac{2Q\partial_t}{r(rA'-2A+2)}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\alpha_1, \tag{75} $$

$$ {}^{(i)}h_{1\,rr} = -\frac{2Q}{r(rA'-2A+2)}\partial_r{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + \frac{r}{2A}\hat{V}^{\mathcal{P}}_{MG\,1}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\beta_1, \tag{76} $$

$$ {}^{(i)}h_{1\,+} = -\frac{2QA}{(rA'-2A+2)}{}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + r^{2\,(i)}\gamma_1, \tag{77} $$

$$ {}^{(i)}f_{1\,t\theta} = \frac{A}{4}\partial_r {}^{(i)}\hat{\Psi}_1^{\mathcal{P}} + {}^{(i)}\lambda_1, \tag{78} $$

$$ {}^{(i)}f_{1\,r\theta} = \frac{1}{4A}\partial_t {}^{(i)}\hat{\Psi}_1^{\mathcal{P}} + {}^{(i)}\kappa_1. \tag{79} $$

The cost of performing this transformation is the loss of algebraic relation (39). From our results, one can also move to a gauge used by some authors (Refs. [6,7]) in which ${}^{(i)}h_{1+} = 0$.

C. Polar sector, $\ell = 0$

In this case, we follow Rostworowski [18]. Using gauge freedom, we set ${}^{(i)}h_{0\,+} = 0$ and ${}^{(i)}h_{0\,tr=0}$, and leftover nonzero variables are ${}^{(i)}h_{0\,tt}$, ${}^{(i)}h_{0\,rr}$, and ${}^{(i)}f_{0\,tr}$. From ${}^{(i)}E_{0\,01}$, ${}^{(i)}E_{0\,00} + A^{2(i)}E_{0\,11}$, and ${}^{(i)}J_{0\,1}$ (the only independent equations), we have, respectively,

$$ \frac{A}{r}\partial_t {}^{(i)}h_{0\,rr} = {}^{(i)}S^G_{0\,tr}, \tag{80} $$

$$ \frac{A}{r}\partial_r\left(A^{(i)}h_{0\,rr} - \frac{{}^{(i)}h_{0\,tt}}{A}\right) = \frac{{}^{(i)}S^G_{0\,tt}}{A} + A^{(i)}S^G_{0\,rr}, \tag{81} $$

$$ \partial_t\left({}^{(i)}f_{0\,tr} + \frac{Q}{2r^2}\left(\frac{{}^{(i)}h_{0\,tt}}{A} - A^{(i)}h_{0\,rr}\right)\right) = -A^{(i)}S^M_{0\,r}. \tag{82} $$

These equations can be therefore integrated directly, starting from (80).

D. Axial sector, $\ell \geq 2$

First, we use (37) to obtain

$$ {}^{(i)}f_{\ell\,t\phi} = -\frac{\partial_t f_{\theta\phi}}{\ell(\ell+1)}, \tag{83} $$

$$ {}^{(i)}f_{\ell\,r\phi} = -\frac{\partial_r f_{\theta\phi}}{\ell(\ell+1)}. \tag{84} $$

We are left with three variables ${}^{(i)}h_{\ell\,t\phi}$, ${}^{(i)}h_{\ell\,r\phi}$, and ${}^{(i)}f_{\ell\,\theta\phi}$. In the same manner as before, we can fulfill equations (35) and (36) by introducing two master scalar variables ${}^{(i)}\Phi^{\mathcal{A}}_\ell$ and ${}^{(i)}\Psi^{\mathcal{A}}_\ell$, which solve a system of two coupled wave equations:

$$ r(-\bar{\Box} + V^{\mathcal{A}}_{G\,\ell})\frac{{}^{(i)}\Phi^{\mathcal{A}}_\ell}{r} + V^{\mathcal{A}}_{MG\,\ell}{}^{(i)}\Psi^{\mathcal{A}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell}, \tag{85} $$

$$ r(-\bar{\Box} + V^{\mathcal{A}}_{M\,\ell})\frac{{}^{(i)}\Psi^{\mathcal{A}}_\ell}{r} + V^{\mathcal{A}}_{MG\,\ell}{}^{(i)}\Phi^{\mathcal{A}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell}. \tag{86} $$

Following the procedure described in the previous section, we find three potentials and express $h_{t\phi}$, $h_{r\phi}$, and $f_{\theta\phi}$ by master scalars and their derivatives:

$$ V^{\mathcal{A}}_{G\,\ell} = \frac{r^2(A - 3rA') + (\tau^2+1)r^2 - Q^2}{r^4}, \tag{87} $$

$$ V^{\mathcal{A}}_{M\,\ell} = \frac{-A'r^3 + \ell(\ell+1)r^2 + 4Q^2}{r^4}, \tag{88} $$

$$ V^{\mathcal{A}}_{MG\,\ell} = -\frac{2\tau Q}{r^3}, \tag{89} $$

$$ {}^{(i)}h_{\ell\,t\phi} = A\partial_r(r^{(i)}\Phi^{\mathcal{A}}_\ell) + {}^{(i)}\sigma_\ell, \tag{90} $$

$$ {}^{(i)}h_{\ell\,r\phi} = \frac{r}{A}\partial_t {}^{(i)}\Phi^{\mathcal{A}}_\ell + {}^{(i)}\chi_\ell, \tag{91} $$

$$ {}^{(i)}f_{\ell\,\theta\phi} = \frac{1}{2}\ell(\ell+1)\tau^{(i)}\Psi^{\mathcal{A}}_\ell + {}^{(i)}\delta_\ell. \tag{92} $$

Now, we invert the above relations for linear order and treat the following expressions as definitions of ${}^{(i)}\Phi^{\mathcal{A}}_\ell$ and ${}^{(i)}\Psi^{\mathcal{A}}_\ell$ at the nonlinear order:

$$ {}^{(i)}\Phi^{\mathcal{A}}_\ell = \frac{(r(\partial_r {}^{(i)}h_{\ell\,t\phi} - \partial_t {}^{(i)}h_{\ell\,r\phi}) - 2^{(i)}h_{\ell\,t\phi})}{\ell(\ell+1)\tau^2 r} + \frac{4Q^{(i)}f_{\ell\,\theta\phi}}{\tau^2}, \tag{93} $$

$$ {}^{(i)}\Psi^{\mathcal{A}}_\ell = \frac{2^{(i)}f_{\ell\,\theta\phi}}{\tau\ell(\ell+1)}. \tag{94} $$

Finally, we find inhomogeneous functions ${}^{(i)}\sigma_\ell$, ${}^{(i)}\chi_\ell$, and ${}^{(i)}\delta_\ell$,

$$ {}^{(i)}\sigma_\ell = \frac{2r^2}{\tau^2}{}^{(i)}S^G_{\ell\,t\phi}, \tag{95} $$

$$ {}^{(i)}\chi_\ell = \frac{2r^2}{\tau^2}{}^{(i)}S^G_{\ell\,r\phi}, \tag{96} $$

$$ {}^{(i)}\delta_\ell = 0, \tag{97} $$

and scalar sources ${}^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell}$ and ${}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell}$,

$$ {}^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell} = \frac{2r(\partial_r {}^{(i)}S^G_{\ell\,t\phi} - \partial_t {}^{(i)}S^G_{\ell\,r\phi})}{\tau^2}, \tag{98} $$

$$ {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell} = \frac{2^{(i)}S^M_{\ell\,\phi}}{\tau}. \tag{99} $$

E. Axial sector, $\ell = 1$

Since ${}^{(i)}h_{\ell\,\theta\phi}$ does not appear for $\ell = 1$, we can use gauge freedom to set ${}^{(i)}h_{1\,r\phi} = 0$. From (37), we have

$$ {}^{(i)}f_{1\,t\phi} = -\frac{\partial_t {}^{(i)}f_{1\,\theta\phi}}{2}, \tag{100} $$

$$ {}^{(i)}f_{1\ r\phi} = -\frac{\partial_r {}^{(i)}f_{1\ \theta\phi}}{2}. \tag{101} $$

Remaining equations contain ${}^{(i)}h_{1\ t\phi}$ and ${}^{(i)}f_{1\ \theta\phi}$ only. From ${}^{(i)}E_{1\ r\phi} = {}^{(i)}S^G_{1\ r\phi}$, we find

$$ -\frac{r^2}{2A}\partial_r\left(\frac{{}^{(i)}h_{1\ t\phi}}{r^2}\right) - \frac{Q\,{}^{(i)}f_{1\ \theta\phi}}{Ar^2} + \eta(r) = \int^t {}^{(i)}S^G_{1\ r\phi}dt', \tag{102} $$

where $\eta(r)$ is some function of r. It is not arbitrary—from ${}^{(i)}E_{1\ t\phi} = {}^{(i)}S^G_{1\ t\phi}$ and source identity (34), we find $\eta = \frac{C_1}{Ar^2}$, C_1 being an arbitrary constant.

Let us introduce ${}^{(i)}\Psi^{\mathcal{A}}_1$ such that ${}^{(i)}f_{1\ \theta\phi} = {}^{(i)}\Psi^{\mathcal{A}}_1 + \frac{4C_1Q}{3r^2(rA'+2A-2)}$. From (36), we find that ${}^{(i)}\Psi^{\mathcal{A}}_1$ fulfills an inhomogeneous (homogeneous at the linear level) wave equation,

$$ r(-\bar{\Box} + V^{\mathcal{A}}_{M\ 1})\frac{{}^{(i)}\Psi^{\mathcal{A}}_1}{r} = {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\ 1}, \tag{103} $$

where

$$ V^{\mathcal{A}}_{M\ 1} = \frac{4Q^2 - r^3A' + 2r^2}{r^4}, \tag{104} $$

$$ {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\ 1} = 2\,{}^{(i)}S^M_{1\phi} - \frac{4AQ\int^t {}^{(i)}S^G_{1\ r\phi}dt'}{r^2}. \tag{105} $$

We note that at the linear level setting ${}^{(i)}\Psi^{\mathcal{A}}_1 = 0$ corresponds to the linearized Kerr-Newman metric.

VIII. SUMMARY

Nonlinear perturbation theory of the Reissner-Nordström solution has not been present in the literature so far, and the present article fills this gap. Basing on a systematic approach to gravitational perturbations by Rostworowski [18], we have shown that one can fulfill perturbative Einstein-Maxwell equations at any perturbation order by solving two inhomogeneous master wave equations at each sector (cases $\ell = 0$, 1 needed special treatment). This makes treatment of higher-order perturbations of Reissner-Nordström clear and would be especially useful for numerical purposes. To summarize, a complete order by order algorithm of solving Einstein-Maxwell equations within our formalism would be:

(1) Solve wave equations (40), (41), (85), and (86), and calculate RW variables and electromagnetic tensor components according to (45)–(49) and (90)–(92),
(2) Move to asymptotically flat gauge, and calculate sources to Einstein-Maxwell equations (Appendix A),
(3) Construct sources to wave equations [Eqs. (52), (53), (98), and (99)], and move to the next order.

Applications of presented calculations possibly include nonlinear studies on the strong cosmic censorship conjecture and on astrophysical systems, where electromagnetic field is taken into account.

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APPENDIX A: SOURCES FOR EINSTEIN-MAXWELL EQUATIONS

Let us fix index i and assume that we already know the solution to Einstein-Maxwell equations (35)–(37) up to ith order:

$$ \tilde{g}_{\mu\nu} = \sum_{j=1}^{i}\sum_{\ell} {}^{(j)}h_{\ell\,\mu\nu}, \tag{A1} $$

$$ \tilde{F}_{\mu\nu} = \sum_{j=1}^{i}\sum_{\ell} {}^{(j)}f_{\ell\,\mu\nu}. \tag{A2} $$

Using this solution, we can calculate the Einstein tensor $G_{\mu\nu}(\tilde{g})$ and the energy-momentum tensor $T_{\mu\nu}(\tilde{g},\tilde{F})$. Although these tensors fulfill Einstein-Maxwell equations up to order i, they contribute to the $i+1$ (and higher) perturbation equations. Finally, tensor and vector sources of order $i+1$ are given by

$$ {}^{(i+1)}S^G{}_{\mu\nu} = [i+1](-G_{\mu\nu}(\tilde{g}) + 8\pi T_{\mu\nu}(\tilde{g},\tilde{F})), \tag{A3} $$

$$ {}^{(i+1)}S^E{}_{\nu} = [i+1](-\nabla^\mu(\tilde{g}_{\alpha\beta})\tilde{F}_{\mu\nu}), \tag{A4} $$

where $[k](\ldots)$ denotes the kth-order expansion in ϵ of a given quantity.

Although in most cases expressions for the sources ${}^{(i+1)}S^G{}_{\mu\nu}$ and ${}^{(i+1)}S^G{}_{\mu\nu}$ are complicated, their construction is a purely algebraic task and can be easily performed using computer algebra.

APPENDIX B: GAUGE TRANSFORMATIONS

Under a gauge transformation $x^\mu \to x^\mu + X^\mu$, tensors transform as $t_{\mu\nu} \to t_{\mu\nu} + \mathcal{L}_X t_{\mu\nu}$. For $X^\mu = {}^{(i)}\zeta^\mu\epsilon^i$, perturbation functions of order i transform in the following way:

$$ {}^{(i)}h_{\ell\,\mu\nu} \to {}^{(i)}h_{\ell\,\mu\nu} + \mathcal{L}_{{}^{(i)}\zeta_\ell}\bar{g}_{\mu\nu}, \tag{B1} $$

$$ {}^{(i)}f_{\ell\,\mu\nu} \to {}^{(i)}f_{\ell\,\mu\nu} + \mathcal{L}_{{}^{(i)}\zeta_\ell}\bar{F}_{\mu\nu}. \tag{B2} $$

The explicit form of these transformations in a polar sector is

$$ {}^{(i)}h_{\ell tt} \to {}^{(i)}h_{\ell tt} + 2\partial_t {}^{(i)}\zeta_{\ell t} - AA' {}^{(i)}\zeta_{\ell r}, \tag{B3} $$

$$ {}^{(i)}h_{\ell tr} \to {}^{(i)}h_{\ell tr} + \partial_r {}^{(i)}\zeta_{\ell t} + \partial_t {}^{(i)}\zeta_{\ell r} - \frac{A'}{A} {}^{(i)}\zeta_{\ell t}, \tag{B4} $$

$$ {}^{(i)}h_{\ell t\theta} \to {}^{(i)}h_{\ell t\theta} + \partial_t {}^{(i)}\zeta_{\ell\theta} + {}^{(i)}\zeta_{\ell t}, \tag{B5} $$

$$ {}^{(i)}h_{\ell rr} \to {}^{(i)}h_{\ell rr} + 2\partial_r {}^{(i)}\zeta_{\ell\theta} + \frac{A'}{A} {}^{(i)}\zeta_{\ell r}, \tag{B6} $$

$$ {}^{(i)}h_{\ell r\theta} \to {}^{(i)}h_{\ell r\theta} + \partial_r {}^{(i)}\zeta_{\ell\theta} - \frac{2}{r} {}^{(i)}\zeta_{\ell\theta} + {}^{(i)}\zeta_{\ell r}, \tag{B7} $$

$$ {}^{(i)}h_{\ell +} \to {}^{(i)}h_{\ell +} + 2A\frac{{}^{(i)}\zeta_{\ell r}}{r} - \ell(\ell+1)\frac{{}^{(i)}\zeta_{\ell\theta}}{r^2}, \tag{B8} $$

$$ {}^{(i)}h_{\ell -} \to {}^{(i)}h_{\ell -} + {}^{(i)}\zeta_{\ell\theta}, \tag{B9} $$

$$ {}^{(i)}f_{\ell t\theta} \to {}^{(i)}f_{\ell t\theta} + \frac{AQ}{r^2} {}^{(i)}\zeta_{\ell r}, \tag{B10} $$

$$ {}^{(i)}f_{\ell r\theta} \to {}^{(i)}f_{\ell r\theta} + \frac{Q}{Ar^2} {}^{(i)}\zeta_{\ell t}, \tag{B11} $$

$$ {}^{(i)}f_{\ell tr} \to {}^{(i)}f_{\ell tr} + \frac{Q}{Ar^2} {}^{(i)}\zeta_{\ell t}, \tag{B12} $$

$$ {}^{(i)}f_{\ell tr} \to {}^{(i)}f_{\ell tr} + Q\partial_r\left(\frac{A}{r^2} {}^{(i)}\zeta_{\ell r}\right) - \frac{Q}{r^2 A}\partial_t {}^{(i)}\zeta_{\ell t} \tag{B13} $$

and in the axial sector is

$$ {}^{(i)}h_{\ell t\phi} \to {}^{(i)}h_{\ell t\phi} + \partial_t {}^{(i)}\zeta_{\ell\phi}, \tag{B14} $$

$$ {}^{(i)}h_{\ell r\phi} \to {}^{(i)}h_{\ell r\phi} + \partial_r {}^{(i)}\zeta_{\ell\phi} - 2\frac{{}^{(i)}\zeta_{\ell\phi}}{r}, \tag{B15} $$

$$ {}^{(i)}h_{\ell\theta\phi} \to {}^{(i)}h_{\ell\theta\phi} + {}^{(i)}\zeta_{\ell\phi}. \tag{B16} $$

$$ {}^{(i)}f_{\ell t\phi} \to {}^{(i)}f_{\ell t\phi}, \tag{B17} $$

$$ {}^{(i)}f_{\ell r\phi} \to {}^{(i)}f_{\ell r\phi}, \tag{B18} $$

$$ {}^{(i)}f_{\ell\theta\phi} \to {}^{(i)}f_{\ell\theta\phi}. \tag{B19} $$

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