Dynamical system analysis at background and perturbation levels: Quintessence in severe disadvantage comparing to ΛCDM

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We perform for the first time a dynamical system analysis of both the background and perturbation equations, of ΛCDM cosmology, and quintessence scenario with an exponential potential. In the former case the perturbations do not change the stability of the late-time attractor of the background equations, and the system still results in the dark-energy-dominated, de Sitter solution, having passed from the correct dark-matter era with $\gamma \approx 6/11$. However, in the case of quintessence the incorporation of perturbations changes the stability and properties of the background evolution, and the only conditionally stable points present an exponentially increasing matter clustering, not favored by observation; thus, this situation is not physically interesting. This result is a severe disadvantage of quintessence cosmology compared to the ΛCDM paradigm.

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I. INTRODUCTION

The dynamical system approach is a powerful tool that allows us to extract information on the evolution of a cosmological model, independently of the initial conditions or its specific behavior at intermediate times [1]. In particular, although a general cosmological scenario may exhibit an infinite number of possible evolutions, its asymptotic behavior, namely, its behavior at late times, can be classified in a few different classes, which correspond to the stable critical points of the autonomous-form transformed cosmological equations. Thus, through such an analysis, one obtains information of the late-time universe, bypassing the complications of the cosmological equations, which prevent complete analytical treatments, as well as the ambiguity of the initial conditions.

The dynamical system approach has been applied to numerous cosmological scenarios since the late 1990s (see [2] and references therein); nevertheless, up to now it remained only at the background level, namely, examining the behavior of the background equations and calculating, at the critical points, the values of background-related quantities such as the density parameters, the equation-of-state parameter, etc. Although this analysis was important and adequate for the earlier cosmology advance, significantly advancing cosmological progress, and especially the huge amount of data related to perturbations (such as the growth index and the large scale structure), leads to the need to extend the dynamical system approach in order to investigate cosmological scenarios at both the background and perturbation levels.

II. DYNAMICAL ANALYSIS AT THE BACKGROUND LEVEL

Let us briefly review the phase-space analysis of the ΛCDM paradigm, as well as of the basic dynamical dark-energy scenario, namely, the quintessence one with an exponential potential, which is the archetype quintessence scenario due to the well-posed theoretical justification of exponential potentials. Considering a flat Friedmann-Robertson-Walker (FRW) metric $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$, the equations of a general cosmological scenario read as

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_d), \qquad (1)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_m + p_m + \rho_d + p_d), \qquad (2)$$

with $\kappa^2 = 8\pi G$, and where ρ_m, p_m are, respectively, the energy density and pressure of the matter fluid, while ρ_d, p_d

are the energy density and pressure of the (effective) dark-energy fluid. Finally, assuming that interactions do not take place among the cosmic fluid components, the system of equations closes with the conservation equations

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0, \tag{3}$$

$$\dot{\rho}_d + 3H(1 + w_d)\rho_d = 0, \tag{4}$$

where we have introduced the equation-of-state parameters $w_i \equiv p_i/\rho_i$. Note that only three out of four equations (1)–(4) are independent.

The above framework provides ΛCDM cosmology for $\rho_d = -p_d = \Lambda/\kappa^2$, with Λ the cosmological constant, and in this case, Eq. (4) becomes trivial. Additionally, for the case of the basic quintessence scenario, in which a scalar field ϕ is introduced, we have $\rho_d = \dot{\phi}^2/2 + V$ and $p_d = \dot{\phi}^2/2 - V$, with $V(\phi)$ its potential, and then Eq. (4) becomes the Klein-Gordon equation $\ddot{\phi} + 3H\dot{\phi} + V' = 0$, with $V'(\phi) \equiv \partial V/\partial\phi$.

The essence of the dynamical system approach is to transform the equations into an autonomous system, using $\tau \equiv \ln a$ as the dynamical variable, extract its critical points, perturbing around them, and investigate their stability by examining the eigenvalues of the involved perturbation matrix [1,2].

For ΛCDM cosmology, the cosmological equations can be transformed into an autonomous form by simply using the matter density parameter $\Omega_m \equiv \kappa^2\rho_m/(3H^2)$ as the auxiliary variable. Thus, Eqs. (1) and (3) give rise to the one-dimensional system

$$\Omega_m' = 3(\Omega_m - 1)\Omega_m, \tag{5}$$

where primes denote derivatives with respect to τ. The system has two critical points, characterized by $\Omega_m = 1$ and $\Omega_m = 0$, and one can see that the former is unstable while the latter is stable. Therefore, for ΛCDM cosmology, the cosmological-constant-dominated [$\Omega_m = 0$ according to (1) implies that $\Omega_d \equiv (\kappa^2\rho_d/3H^2) = 1$], de Sitter solution is the stable late-time attractor, and thus the universe will result in it independently of the initial conditions and its evolutions at intermediate times. We mention that the dynamical system analysis is actually not needed in this scenario, since the equations are integrable, with the solution

$$\Omega_m = \frac{\Omega_{m0}}{e^{3(1+w_m)\tau}(1 - \Omega_{m0}) + \Omega_{m0}}, \tag{6}$$

with Ω_{m0} the value of Ω_m at $a = 1$. Hence, we can immediately see that at late times the system always reaches the de Sitter solution (for matter sectors that do not violate the null energy condition).

In the case of the quintessence scenario, and focusing on the basic model where an exponential potential $V = V_0 e^{-\lambda\kappa\phi}$ for the scalar field is imposed, introducing the auxiliary variables [3]

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \qquad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \tag{7}$$

we obtain the dynamical system

$$x' = \frac{3}{2}x(2x^2 + \gamma_m(1 - x^2 - y^2)) - 3x + \sqrt{\frac{3}{2}}\lambda y^2, \tag{8}$$

$$y' = \frac{3}{2}y(2x^2 + \gamma_m(1 - x^2 - y^2)) - \sqrt{\frac{3}{2}}\lambda xy, \tag{9}$$

in terms of which the various density parameters are expressed as $\Omega_d = x^2 + y^2$, $\Omega_m = 1 - \Omega_d$, $\gamma_m \equiv w_m + 1$, while $w_d = \frac{x^2 - y^2}{x^2 + y^2}$. The critical points of the system (8) and (9), along with their stability conditions and the corresponding values of Ω_d and w_d, are shown in Table I. As we observe, the scenario possesses two stable late-time attractors, with the scalar-field-dominated solution D being the most physically interesting.

TABLE I. The critical points, their stability conditions (the corresponding eigenvalues are given in [3]), and the values of Ω_d and w_d, for the quintessence scenario with exponential potential, with $\gamma_m \equiv w_m + 1$.

C.P.	x	y	Existence	Stability	Ω_d	w_d
A	0	0	Always	Saddle for $0 < \gamma_m < 2$	0	Undefined
B	1	0	Always	Unstable node for $\lambda < \sqrt{6}$ Saddle for $\lambda > \sqrt{6}$	1	1
C	-1	0	Always	Unstable node for $\lambda > -\sqrt{6}$ Saddle for $\lambda < -\sqrt{6}$	1	1
D	$\lambda/\sqrt{6}$	$[1 - \lambda^2/6]^{1/2}$	$\lambda^2 < 6$	Stable node for $\lambda^2 < 3\gamma_m$ Saddle for $3\gamma_m < \lambda^2 < 6$	1	$\frac{\lambda^2}{3} - 1$
E	$(3/2)^{1/2}\gamma_m/\lambda$	$[3(2 - \gamma_m)\gamma_m/2\lambda^2]^{1/2}$	$\lambda^2 > 3\gamma_m$	Stable node for $3\gamma_m < \lambda^2 < 24\gamma_m^2/(9\gamma_m - 2)$ Stable spiral for $\lambda^2 > 24\gamma_m^2/(9\gamma_m - 2)$	$3\gamma_m/\lambda^2$	w_m

III. DYNAMICAL ANALYSIS AT THE PERTURBATION LEVEL

The investigation of scalar perturbations is crucial in every cosmological scenario since they are connected to perturbation-related observables such as the growth index γ and σ_8 [4]. From now on, and for the convenience of calculation, we focus on the most interesting case of dust matter; namely, we set $\gamma_m = 1$ ($w_m = 0$) since a nonzero w_m does not qualitatively affect our results.

In a general noninteracting scenario, which includes dust matter and dynamical dark energy, the scalar perturbations in the Newtonian gauge are determined by the equations [5]

$$\dot{\delta}_m + \frac{\theta_m}{a} = 0, \tag{10}$$

$$\dot{\delta}_d + (1 + w_d)\frac{\theta_d}{a} + 3H(c_{\rm eff}^2 - w_d)\delta_d = 0, \tag{11}$$

$$\dot{\theta}_m + H\theta_m - \frac{k^2\psi}{a} = 0, \tag{12}$$

$$\dot{\theta}_d + H\theta_d - \frac{k^2 c_{\rm eff}^2 \delta_d}{(1 + w_d)a} - \frac{k^2\psi}{a} = 0, \tag{13}$$

where k is the wave number of Fourier modes and ψ the scalar metric perturbation assuming zero anisotropic stress. Additionally, $\delta_i \equiv \delta\rho_i/\rho_i$ are the density perturbations, and θ_i are the velocity perturbations [5]. Furthermore, $c_{\rm eff}^2$ is the effective sound speed of the dark-energy perturbations (the corresponding quantity for matter is zero in the dust case), which determines the amount of dark-energy clustering. Note that the above equations can be simplified by considering the Poisson equation, which in subhorizon scales becomes [5]

$$-\frac{k^2}{a^2}\psi = \frac{3}{2}H^2[\Omega_m\delta_m + (1 + 3c_{\rm eff}^2)\Omega_d\delta_d]. \tag{14}$$

Finally, we mention that the above perturbation equations must be considered alongside the background evolution equations (1)–(4).

In general, the fact that Λ does not change in space and time implies that the cosmological constant cannot cluster like dark matter. On the other hand, dynamical dark energy may cluster, and the amount of clustering is affected by the effective sound speed. Specifically, in the case of $c_{\rm eff}^2 = 1$, pressure suppresses any dark-energy fluctuation at subhorizon scales. Therefore, for homogeneous dark energy, the quantities δ_d and θ_d vanish. On the other hand, for $c_{\rm eff}^2 = 0$ dark energy clusters similar to dark matter, and perturbations will grow with time. The clustering of dark energy modifies the evolution of dark matter fluctuation perturbations; hence, it affects the structure formation rate of the universe (for more discussion see [6] and references therein).

Let us first investigate the case of the ΛCDM paradigm, which is obtained by the above general framework for $w_d = -1$ and $\rho_d = \Lambda/\kappa^2$, alongside $\delta_d = 0$ and $\theta_d = 0$ (i.e., dark energy is not clustering and thus its perturbation equations can be completely ignored). As auxiliary variables we introduce Ω_m, as well as the variable

$$U_m \equiv \frac{\delta_m'}{\delta_m}. \tag{15}$$

Hence, in terms of Ω_m, U_m, Eqs. (1)–(4) and (10)–(13) become

$$\Omega_m' = 3(\Omega_m - 1)\Omega_m, \tag{16}$$

$$U_m' = \frac{3}{2}(U_m + 1)\Omega_m - U_m(U_m + 2). \tag{17}$$

The critical points of the system (16) and (17), along with the corresponding eigenvalues and their stability conditions, are presented in Table II. The system admits four critical points, with P_3 being the stable one. It corresponds to the cosmological-constant-dominated, de Sitter solution, which moreover has $\delta_m = \text{const}$ (since $U_m = 0$). Similarly, one can observe the saddle point P_4, which is a matter-dominated universe in which the perturbations increase as $\delta_m \propto e^\tau = a$ exactly at the critical point. Thus, for ΛCDM cosmology the incorporation of perturbations does not change the late-time attractor of the background evolution.

For completeness we must examine the possibility of critical points that exist at "infinity" and hence that are missed through the above basic analysis. Introducing the transformation $\{\Omega_m, U_m\} \to \{\Omega_m, \bar{U}_m\}$ with $\bar{U}_m = \frac{2}{\pi}\arctan(U_m)$, we find that such critical points at infinity do not exist, since $\bar{U}_m'|_{\bar{U}_m=\pm 1} = -2/\pi \neq 0$.

Finally, we note that in the literature it is standard to consider that in the matter-dominated phase, in which the large scale structure builds up due to the increase of matter perturbations, we have the relation $d\ln\delta_m/d\ln a \simeq \Omega_m^\gamma$, where γ is the growth index [7], which in our notation becomes just $U_m \simeq \Omega_m^\gamma$. Inserting this into (17) we obtain

TABLE II. The critical points and their stability conditions, of both background and perturbation equations, in the case of the ΛCDM paradigm.

C.P.	Ω_m	U_m	Existence	Eigenvalues	Stability
P_1	0	-2	Always	$\{-3, 2\}$	Saddle
P_2	1	$-\frac{3}{2}$	Always	$\{3, \frac{5}{2}\}$	Unstable
P_3	0	0	Always	$\{-3, -2\}$	Stable
P_4	1	1	Always	$\{3, -\frac{5}{2}\}$	Saddle

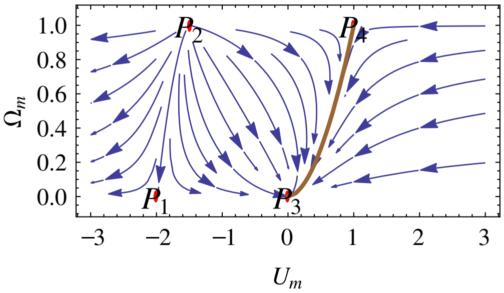

FIG. 1. The phase-space diagram for ΛCDM cosmology, at both background and perturbation levels. At late times the system is attracted by the de Sitter point P_3. The thick line is the curve (18), which coincides with the unstable manifold of the matter-dominated solution P_4, and which for Ω_m close to 1 gives $\gamma = \frac{6}{11}$ analytically as expected (see text).

$$3\gamma(\Omega_m - 1)\Omega_m^\gamma + (\Omega_m^\gamma + 2)\Omega_m^\gamma - \frac{3}{2}\Omega_m(\Omega_m^\gamma + 1) = 0, \tag{18}$$

which, expanded around $\Omega_m = 1$, leads to

$$-\left(\frac{11\gamma}{2} - 3\right)(1 - \Omega_m) + O((1 - \Omega_m)^2) = 0. \tag{19}$$

As expected, the asymptotic value of the growth index is $\gamma = \frac{6}{11}$. The curve (18) is depicted in Fig. 1 with a thick (brown) line, and as we observe, it coincides with the unstable manifold of the matter-dominated solution P_4.

We mention here that the dynamical system analysis is not needed for ΛCDM cosmology since, even including the perturbations, the system remains integrable. In particular, the general solution reads

$$\Omega_m(\tau) = \frac{\Omega_{m0}}{e^{3\tau}(1 - \Omega_{m0}) + \Omega_{m0}}, \tag{20}$$

$$\begin{aligned} U_m(\tau) = &\{2\Omega_m(2U_{m0} + 3\Omega_{m0})(\Omega_{m0}^{2/3} g_0 - \Omega_m^{2/3} g) \\ &+ 8(1 - \Omega_{m0})^{5/6}\Omega_{m0}^{2/3}\}^{-1} \\ &\cdot \{3\Omega_m(2U_{m0} + 3\Omega_{m0})(\Omega_m^{2/3} g - \Omega_{m0}^{2/3} g_0) \\ &+ 4\Omega_m^{2/3}[(2U_{m0} + 3\Omega_{m0})(1 - \Omega_m)^{5/6} \\ &- 3\Omega_{m0}^{2/3}\Omega_m^{1/3}(1 - \Omega_{m0})^{5/6}]\}, \end{aligned} \tag{21}$$

where $g(\tau) = {}_2F_1[\frac{1}{6}, \frac{2}{3}, \frac{5}{3}, \Omega_m(\tau)]$ and $g_0 = {}_2F_1[\frac{1}{6}, \frac{2}{3}, \frac{5}{3}, \Omega_{m0}]$, with Ω_{m0} and U_{m0} the values of Ω_m and U_m at $\tau = 0$ (i.e., at $a = 1$). From the analytical solutions (20) and (21) we can easily see that for $\tau \to \infty$ we have $\Omega_m \to 0$ and $U_m \to 0$; i.e., the system results in the de Sitter point P_3.

We now proceed to the investigation of perturbations in quintessence with an exponential potential. As we mentioned above, this simple dark-energy scenario has $c_{\mathrm{eff}}^2 = 1$, which implies that dark energy is nonclustering, and hence one should consider only the perturbation equations (10) and (12), alongside the background ones (1)–(4). In order to transform them into autonomous form, we use the variables x, y of (7), as well as the additional variable

TABLE III. The physical (real with $0 \leq \Omega_m \leq 1$ and expanding) critical points, their stability conditions, and their properties, of both background and perturbation equations, in the case of quintessence with exponential potential. The stability conditions arise from the examination of the sign of the eigenvalues of the involved perturbation matrix.

C.P.	$\{x, y\}$	U_m	Ω_m	w_d	Existence	Eigenvalues	Stability
A_1	$\{0, 0\}$	$-\frac{3}{2}$	1	Undefined	Always	$\frac{5}{2}, -\frac{3}{2}, \frac{3}{2}$	Saddle
A_2	$\{0, 0\}$	1	1	Undefined	Always	$-\frac{5}{2}, -\frac{3}{2}, \frac{3}{2}$	Saddle
B_1	$\{1, 0\}$	0	0	1	Always	$3, 1, 3 - \sqrt{\frac{3}{2}}\lambda$	Unstable for $\lambda < \sqrt{6}$ Saddle for $\lambda > \sqrt{6}$
B_2	$\{1, 0\}$	1	0	1	Always	$3, -1, 3 - \sqrt{\frac{3}{2}}\lambda$	Saddle
C_1	$\{-1, 0\}$	0	0	1	Always	$3, 1, 3 + \sqrt{\frac{3}{2}}\lambda$	Unstable for $\lambda > -\sqrt{6}$ Saddle for $\lambda < -\sqrt{6}$
C_2	$\{-1, 0\}$	1	0	1	Always	$3, -1, \sqrt{\frac{3}{2}}\lambda + 3$	Saddle
D_1	$\left\{\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}\right\}$	1	0	$-1 + \frac{\lambda^2}{3}$	$\lambda^2 \leq 6$	$\lambda^2 - 3, \frac{1}{2}(\lambda^2 - 6), \frac{1}{2}(\lambda^2 - 8)$	Stable for $\lambda^2 < 3$ Saddle for $\lambda^2 > 3$
D_2	$\left\{\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}\right\}$	$\frac{\lambda^2}{2} - 3$	0	$-1 + \frac{\lambda^2}{3}$	$\lambda^2 \leq 6$	$\lambda^2 - 3, \frac{1}{2}(\lambda^2 - 6), -\frac{1}{2}(\lambda^2 - 8)$	Saddle
E_1	$\left\{\frac{\sqrt{\frac{3}{2}}}{\lambda}, \frac{\sqrt{\frac{3}{2}}}{\lambda}\right\}$	1	$1 - \frac{3}{\lambda^2}$	0	$\lambda^2 \geq 3$	$-\frac{5}{2}, -\frac{3}{4}\left(1 \pm \frac{\sqrt{24 - 7\lambda^2}}{\lambda}\right)$	Stable for $\lambda^2 > 3$
E_2	$\left\{\frac{\sqrt{\frac{3}{2}}}{\lambda}, \frac{\sqrt{\frac{3}{2}}}{\lambda}\right\}$	$-\frac{3}{2}$	$1 - \frac{3}{\lambda^2}$	0	$\lambda^2 \geq 3$	$\frac{5}{2}, -\frac{3}{4}\left(1 \pm \frac{\sqrt{24 - 7\lambda^2}}{\lambda}\right)$	Saddle

$$U_m = \frac{\delta_m'}{\delta_m}. \tag{22}$$

Therefore, the autonomous dynamical system consists of Eqs. (8) and (9) and

$$U_m' = -U_m^2 - \frac{U_m}{2}(1 - 3x^2 + 3y^2) + \frac{3}{2}(1 - x^2 + y^2); \tag{23}$$

i.e., it is now 3 dimensional, in contrast to the 2-dimensional one of the background equations. Since the first two equations are decoupled from the third one, the system admits the five critical points of the background analysis of Table I, each of which is now split into two points due to the additional variable U_m. The physical critical points and their stability conditions are presented in Table III. Finally, the analysis at infinity shows that stable critical points do not exist.

The crucial feature, which lies in the center of the analysis of this work, is that the stability and properties of the points change due to the existence of extra dimensions (reflecting the incorporation of perturbation equations) in the phase space. In particular, we can see that the only two points that can be conditionally stable, namely, D_1 and E_1, have $U_m = 1$, which implies that δ_m increases exponentially in an expanding universe; hence, they are not physically interesting. Therefore, the incorporation of perturbation ruins the dark-energy-dominated, de Sitter solution, which is the physically interesting late-time attractor of the background equations, since it induces an asymptotically infinite clustering in an expanding universe.

IV. CONCLUSIONS

We performed for the first time a dynamical system analysis of both the background and perturbation equations, in ΛCDM cosmology and a quintessence scenario with exponential potential. In the former case, the incorporation of perturbations does not change the stability of the late-time attractor of the background equations, and the system still results in the dark-energy-dominated, de Sitter solution, having passed the correct dark matter era with $\gamma \approx 6/11$ (actually, in this scenario one extracts analytical solutions). However, in the case of quintessence, the incorporation of perturbation changes the stability and properties of the background evolution, and the only conditionally stable points present an exponentially increasing matter clustering, not favored by observation; thus, this situation is not physically interesting. In summary, the above results have a severe disadvantage of quintessence with exponential potential (which is the scenario archetype due to the well-posed theoretical justification of exponential potentials) compared to the ΛCDM paradigm.

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