# Hodograph solutions of the wave equation of nonlinear electrodynamics in the quantum vacuum

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The process of photon-photon scattering in vacuum is investigated analytically in the long-wavelength limit within the framework of the Euler-Heisenberg Lagrangian. In order to solve the nonlinear partial differential equations (PDEs) obtained from this Lagrangian use is made of the hodograph transformation. This transformation makes it possible to turn a system of quasilinear PDEs into a system of linear PDEs. Exact solutions of the equations describing the nonlinear interaction of electromagnetic waves in vacuum in a one-dimensional configuration are obtained and analyzed.

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## I. INTRODUCTION

Perturbation theory has proven to be extremely successful in obtaining a number of prominent results in quantum field theories (QFTs) [1–4]. In spite of these achievements, as is well known, perturbation theory is only valid provided the interaction is weak and thus it cannot provide a full description of a QFT [5,6]. For this reason the nonperturbative behavior of QFTs has attracted a great deal of attention for decades [7]. As examples of physical objects typical for QFTs and classical mechanics of continuous media whose theoretical description cannot be obtained within the framework of perturbation theory we may list the breaking of nonlinear waves, solitons, instantons, etc., [8–12].

In quantum electrodynamics (QED) perturbation theory breaks in the limit of strong electric fields, when the electric field $E$ approaches the critical field of quantum electrodynamics [13,14]

$$E_S = m_e^2 c^3 / e\hbar \tag{1}$$

and/or the photon energy becomes substantially large, i.e., for $\alpha\chi_\gamma^{2/3} \geq 1$ [6] where $\alpha = e^2/\hbar c$ is the fine structure constant, $\chi_\gamma = \hbar\sqrt{(F_{\mu\nu}k^\mu)^2}/m_e c E_S$ is the so called nonlinear quantum parameter (see Refs. [2,6]), $F_{\mu\nu}$ is the electromagnetic field tensor, and $\hbar k^\mu$ is the four-momentum of the photon. The electron mass and electric charge are $m_e$ and $e$, respectively, $c$ is the speed of light in vacuum, and $\hbar$ is the Planck constant. The critical field corresponds to the electric field that, acting on the electron charge $e$, would produce a work equal to the electron rest mass energy $m_e c^2$ over a distance equal to the Compton wavelength $\lambdabar_C = \hbar/m_e c$. Here $\hbar$ is the reduced Planck constant, $e$ and $m_e$ are the electron electric charge and mass, and $c$ is the speed of light in vacuum (see for details Refs. [2, 13–15]). The corresponding wavelength and intensity of electromagnetic radiation are $\lambda_S = 2\pi\lambdabar_C \approx 2\times 10^{-10}$ cm and $I_S = cE_S^2/4\pi \approx 10^{29}$ W/cm$^2$, respectively.

One of the most remarkable effects predicted in QED is the vacuum polarization connected with light-light scattering and pair production from vacuum. In classical electrodynamics electromagnetic waves do not interact in vacuum. On the contrary, in QED photon-photon scattering can take place in vacuum via the generation of virtual electron-positron pairs. This interaction gives rise to vacuum polarization and birefringence, to the Lamb shift, to a modification of the Coulomb field, and to many other phenomena [2]. Photon-photon scattering was observed in collisions of heavy ions accelerated in standard particle accelerators (see review article [16] and the results of the experiments obtained with the ATLAS detector at the Large Hadron Collider [17]).

Photon-photon interaction provides a tool for the search for new physics [16,18]: further studies of this process will make it possible to test extensions of the Standard Model in which new particles contribute to the interaction loop diagrams [19]. Using the Euler-Heisenberg Lagrangian [14,20], which describes the vacuum polarization and electron-positron pair generation by super-strong electromagnetic field in vacuum [15,21] also provides one of the most developed approaches for studying nonperturbative processes in QFT, when finding exact solutions of nonlinear problems cannot be underestimated.

The increasing availability of high power lasers has stimulated a growing interest towards the experimental observation of photon-photon scattering processes [22–24] and electron positron pair creation [25]. In addition it has provided strong motivation for their theoretical study in processes such as the scattering of a laser pulse by a laser pulse [26–34], the scattering of $XFEL$ emitted photons [19], and the interaction of relatively long-wavelength, high intensity, laser light pulses with short-wavelength $X$-ray photons [35].

The process of vacuum polarization can be described within the framework of the approximation using the Euler-Heisenberg Lagrangian [14,20]. Although this approximation is valid in the limit of colliding photons with relatively low energy and of low amplitude electromagnetic pulses, it allows one to extend consideration over the nonperturbative theory. Its applicability requires the colliding photon energy to be below the electron rest-mass energy, $\mathcal{E}_\gamma = \hbar\omega < m_e c^2$, and the electric field of the colliding electromagnetic waves to be below the critical field given by Eq. (1). When writing the condition for the validity of the long-wavelength approximation given above it was assumed that the frequencies of the colliding photons are equal. If the frequencies are different, say $\omega$ and $\Omega$ with $\Omega \neq \omega$, the low-frequency approximation requires that

$$\omega\Omega < m_e^2 c^4/\hbar^2. \tag{2}$$

In the limit of electromagnetic fields with extremely large amplitudes approaching the QED critical field $E_S$, the nonlinear modification of the vacuum refraction index via the polarization of virtual electron-positron pairs leads to the decrease of the propagation velocity of counterpropagating electromagnetic waves [36–39] while, on the contrary, copropagating waves do not change their propagation velocity because copropagating photons do not interact, see, e.g., Ref. [40].

The nonlinear properties of the QED vacuum in the long-wavelength, low frequency limit can find a counterpart in those of nonlinear dispersionless media, keeping however in mind that in QED there is no preferred frame where the nonlinear medium is at rest. In a material nonlinear medium with a refraction index that depends on the electromagnetic field amplitude an electromagnetic wave can evolve into a configuration with singularities [41,42]. The evolution of a finite amplitude wave is accompanied by the steepening of its wave front, by the formation of shocklike waves, i.e., it is characterized by a processes leading to gradient catastrophes [9]. In the case of the quantum vacuum, corresponding phenomena have been investigated in Refs. [21,43,44] and [38]. The occurrence of singularities in the Euler-Heisenberg electrodynamics has been noticed in Refs. [21,43], indicated in computer simulations presented in Ref. [44], and thoroughly studied in Ref. [38].

In the present paper, we analyze the interaction of finite amplitude, counterpropagating electromagnetic (e.m.) waves in a one dimensional (1-D) configuration. The interacting waves are assumed to be linearly polarized and to have the same polarization direction. In such a configuration the propagation directions of the two colliding plane waves are collinear, and this collinearity is preserved by Lorentz boosts along the propagation direction. However, the Euler-Heisenberg Lagrangian is invariant under the full Lorentz group. This makes it possible to use the solutions that will be derived in the following sections to construct solutions that describe the interaction of plane waves colliding at an angle, e.g., by considering Lorentz boosts in the direction perpendicular to the direction of the polarization vector of the two colliding waves. This extension of the results presented below may be of interest in an experimental setting.

The hodograph transformation [45] is a useful tool in the study of nonlinear waves as it allows us to obtain a linear system of second order partial differential equations (PDEs) instead of a system of second order quasilinear PDEs. In the case of the e.m. 1-D configuration under study, this transformation makes the electric and the magnetic fields play the role of the independent coordinates. The hodograph transform has been adopted for a *nondispersive* formulation of the electromagnetic field equations in a nonlinear material medium, see, e.g., Refs. [46,47] that focused on determining constitutive relations for which exact solutions can be derived analytically. It has also been used in the context of relativistic 1-D magnetohydrodynamics in Ref. [48] where a linear equation is derived for a "potential function" $\chi(r, w)$ where $r$ is the relativistic rapidity and $w$ the proper enthalpy of the cold magnetized plasma. This equation corresponds to the equation for the "potential" $\Phi$, function now of the electromagnetic fields, that will be derived in Sec. IV. In the context of nonlinear weaves the hodograph transformation has been applied to the Born-Infeld equation in Ref. [9].

The analysis described in the following sections allows us to find exact solutions describing the nonlinear interaction of electromagnetic waves in vacuum both in the space-time coordinates and in the hodograph variables, to formulate a perturbative approach that, in the limit of monochromatic waves, does not lead to secularities and to derive the dispersion relation of e.m. waves propagating in vacuum in the presence of steady and uniform, strong e.m. fields.

This article is organized as follows. In Sec. II the Euler-Heisenberg Lagrangian is recalled and in Sec. II A it is specialized to the case of counterpropagating e.m. waves in a 1-D configuration and the corresponding nonlinear wave equation is derived using the so-called light cone coordinates. As an illustration, higher order terms that depend on the sixth power of the e.m. fields are included in the Euler-Heisenberg Lagrangian but, for the sake of algebraic simplicity, the contribution of these terms is neglected in some of the formulae in the present text. In Sec. II B the conservations that arise from the translational and from the invariance of the 1-D Euler-Heisenberg Lagrangian under Lorentz boosts along the propagation direction are presented. In Sec. II C the linear case of noninteracting waves is briefly described and in Sec. II D perturbative solutions are obtained in light cone coordinates. In Sec. II E the derivation of the characteristics of the nonlinear wave equations is outlined, while in Sec. II F exact self-similar solutions are derived. In Sec. II G the dispersion equation of e.m. waves propagating in vacuum perpendicularly to large, steady and uniform, e.m. fields is presented. In Sec. III the hodograph transform of the equations of nonlinear electrodynamics in vacuum is derived and in Sec. IV it is applied to the study the nonlinear interaction of electromagnetic waves in the QED vacuum. In Sec. IV A symmetries and conservations are reformulated in the hodograph framework, while in Sec. IV B the expression of noninteracting waves in hodograph variables is shown. In Sec. IV C perturbative solutions are derived and in Sec. IV D an exact self-similar solution is obtained. In Sec. IV E the reduction of the hodograph equations to standard form is derived. This reduction makes possible the use of well known expansion techniques for the solution of linear PDEs with constant coefficients. An explicit inversion of a hodograph solution is then derived. Finally in Sec. V a summary of the main results obtained is given, while in the Appendices extended proofs of some results given in the main text are provided.

## II. EQUATIONS OF NONLINEAR VACUUM ELECTRODYNAMICS

The Euler–Heisenberg Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}', \tag{3}$$

where

$$\mathcal{L}_0 = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \tag{4}$$

is the Lagrangian in classical electrodynamics, $F_{\mu\nu}$ is the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5}$$

with $A_\mu$ being the 4-vector of the electromagnetic field and $\mu = 0, 1, 2, 3$. Here and below a summation over repeating indices is assumed.

In the Euler–Heisenberg theory, the QED radiation corrections are described by $\mathcal{L}'$ on the right-hand side of Eq. (3), which can be written as [2]

$$\mathcal{L}' = -\frac{m^4}{8\pi^2}\int_0^\infty \frac{\exp(-\eta)}{\eta^3}\left[-(\eta\mathfrak{a}\cot\eta\mathfrak{a})(\eta\mathfrak{b}\coth\eta\mathfrak{b}) + 1 - \frac{\eta^2}{3}(\mathfrak{a}^2 - \mathfrak{b}^2)\right] d\eta. \tag{6}$$

Here the invariants $\mathfrak{a}$ and $\mathfrak{b}$ can be expressed in terms the Poincaré invariants

$$\mathfrak{F} = F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad \mathfrak{G} = F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{7}$$

as

$$\mathfrak{a} = \sqrt{\sqrt{\mathfrak{F}^2 + \mathfrak{G}^2} + \mathfrak{F}} \ \text{and}\ \mathfrak{b} = \sqrt{\sqrt{\mathfrak{F}^2 + \mathfrak{G}^2} - \mathfrak{F}}, \tag{8}$$

respectively, where dual tensor $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ contains $\varepsilon^{\mu\nu\rho\sigma}$ being the Levi-Civita symbol in four dimensions. Here and in the following text, we use the units $c = \hbar = 1$, and the electromagnetic field is normalized on the QED critical field $E_S$.

As explained in Ref. [2] the Euler–Heisenberg Lagrangian in the form given by Eq. (6) should be used for obtaining an asymptotic series over the invariant electric field $\mathfrak{a}$ assuming its smallness.

In the weak field approximation the Lagrangian $\mathcal{L}'$ is given by (e.g., see [49])

$$\mathcal{L}' = \frac{\kappa}{4}\left[\mathfrak{F}^2 + \frac{7}{4}\mathfrak{G}^2 + \frac{90}{315}\mathfrak{F}\left(\mathfrak{F}^2 + \frac{13}{16}\mathfrak{G}^2\right)\right] + \cdots \tag{9}$$

with the constant $\kappa = (e^4/360\pi^2)m^4$. In the Lagrangian (9) the first two terms on the right-hand side and the last two correspond respectively to four and to six photon interaction.

### A. Counterpropagating electromagnetic waves

In the following we consider the interaction of counterpropagating electromagnetic waves with the same linear polarization, in which case the invariant $\mathfrak{G}$ vanishes identically. Such a field configuration can be described in a transverse gauge by a vector potential having a single component, $\mathbf{A} = A\mathbf{e}_z$, with $\mathbf{e}_z$ the unit vector along the $z$ axis. In terms of the light cone coordinates (see, e.g., Ref. [50])

$$x_+ = (x + t)/\sqrt{2}, \qquad x_- = (x - t)/\sqrt{2}, \tag{10}$$

the vector potential $\mathbf{A}$ can be written as

$$A = a(x_+, x_-). \tag{11}$$

Here and in the following we use natural units $c = 1, \hbar = 1$ with the electromagnetic field normalized on the QED critical field (the so-called Schwinger field) $E_S = (m_e^2 c^3)/(\hbar e)$ and $x$ and $t$ normalized on the Compton length $\lambdabar_C = \hbar/(m_e c)$ and $\lambdabar_C/c$, respectively. Accordingly, the potential $A$ is normalized on $m_e c^2/e$, as is conventional in the nonquantum case. In these variables the Lagrangian (3) takes the form

$$\mathcal{L}(a) = -\frac{1}{4\pi}[wu - \epsilon_2 (wu)^2 - \epsilon_3 (wu)^3] \tag{12}$$

where the field variables $u$ and $w$ are defined as

$$u = \partial_{x_-} a \quad \text{and} \quad w = \partial_{x_+} a \tag{13}$$

and are related to the electric field $E = -\partial_t A$ (along $z$) and to the magnetic field $B = -\partial_x A$ (along $y$) by

$$w = -(E + B)/\sqrt{2}, \quad u = (E - B)/\sqrt{2} \quad \text{and} \\ uw = (B^2 - E^2)/2. \tag{14}$$

The dimensionless parameters $\epsilon_2$ and $\epsilon_3$ in Eq. (12) are given by

$$\epsilon_2 = \frac{2e^2}{45\pi} = \frac{2}{45\pi}\alpha \quad \text{and} \quad \epsilon_3 = \frac{32e^2}{315\pi} = \frac{32}{315\pi}\alpha, \tag{15}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant, i.e., $\epsilon_2 \approx 10^{-4}$ and $\epsilon_3 \approx 2 \times 10^{-4}$, respectively. The field equations can be found by varying the e.m. action

$$\mathcal{S}(a) = \int dx_+ \int dx_- \mathcal{L}(a),$$

with respect to the vector potential $a(x_+, x_-)$ which gives

$$\partial_{x_-}(\partial_u \mathcal{L}) + \partial_{x_+}(\partial_w \mathcal{L}) = 0. \tag{16}$$

As a result, we obtain the system of equations (see also Appendix A)

$$\partial_{x_-} w = \partial_{x_+} u, \tag{17}$$

$$\partial_{x_+}[u(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)] \\ + \partial_{x_-}[w(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2)] = 0. \tag{18}$$

Equation (17), is simply a consequence of the symmetry of the second derivatives, $\partial_{x_- x_+} a = \partial_{x_+ x_-} a$ and it expresses the vanishing of the 4-divergence of the dual e.m. tensor $\tilde{F}^{\mu\nu}$. By rearranging terms and by inserting Eqs. (13), (17), Eq. (18) can be rewritten in the form of a second order partial differential equation for the potential $a(x_+, x_-)$:

$$[1 - uw(4\epsilon_2 + 9\epsilon_3 uw)]\partial_{x_- x_+} a \\ = w^2(\epsilon_2 + 3\epsilon_3 uw)\partial_{x_- x_-} a + u^2(\epsilon_2 + 3\epsilon_3 uw)\partial_{x_+ x_+} a, \tag{19}$$

where $u(x_+, x_-)$ and $w(x_+, x_-)$ are defined by Eq. (13).

### B. Symmetries and conservations

The Lagrangian (12), and thus Eq. (19), are invariant under time reversal i.e., under the discrete transformation $x_+ \leftrightarrow x_-$ that interchanges $u$ and $w$. The Lagrangian (12) is also invariant under translations along $x$ and $t$ and under Lorentz boosts along $x$. In fact the four-vector potential component $a$ is transverse to the boost and the field product $uv$ is proportional to the Lorentz invariant $\mathfrak{F}$. In terms of the light cone coordinates the corresponding infinitesimal transformations can be written with obvious notation as (see also Ref. [50])

$$x_+ \to x_+ + \delta_+, \qquad x_- \to x_- + \delta_-, \\ x_+ \to (1 - \beta)x_+ \quad \text{and} \quad x_- \to (1 + \beta)x_-, \tag{20}$$

and the product $x^+ x^-$ is invariant under Lorentz boosts along $x$. According to Noether's theorem these continuous symmetries imply the local conservation of the electromagnetic energy momentum tensor and of the "barycenter" (center of the energy-momentum distribution) which in light cone coordinates takes the form

$$\partial_{x_+} T_{ww} + \partial_{x_-} T_{uw} = 0, \qquad \partial_{x_+} T_{wu} + \partial_{x_-} T_{uu} = 0 \\ \partial_{x_+}(T_{ww} x_+ - T_{wu} x_-) + \partial_{x_-}(T_{uw} x_+ - T_{uu} x_-) = 0, \tag{21}$$

where

$$T_{ij} = \frac{\partial \mathcal{L}}{\partial(\partial_i a)}(\partial_j a) - \delta_{ij}\mathcal{L}, \qquad i, j = \pm, \quad \text{and} \\ T_{++} \equiv T_{ww}, \qquad T_{+-} \equiv T_{wu}, \quad \text{etc.} \tag{22}$$

Neglecting for simplicity the $\epsilon_3$ term, from $\mathcal{L} = -(uw - \epsilon_2 u^2 w^2)/4\pi$ we have

$$T_{ww} = T_{uu} = \epsilon_2 u^2 w^2/4\pi, \\ T_{wu} = -u^2(1 - 2\epsilon_2 uw)/4\pi, \\ T_{uw} = -w^2(1 - 2\epsilon_2 uw)/4\pi, \tag{23}$$

where the trace and the determinant are Lorentz invariants. The corresponding expression for the energy momentum tensor in $x$, $t$ coordinates is given in Appendix B and expressed in terms of the fields $E$ and $B$. Note that, independently of the chosen coordinates, the trace of the energy momentum tensor of counterpropagating beams does not vanish, as would instead be the case for the e.m.

fields in vacuum in the linear limit. This is consistent with the fact that the propagation velocities of interacting counterpropagating beams is smaller than the speed of light in vacuum, see Eq. (33) below.

### C. Linear approximation and noninteracting waves

In linear approximation where the Euler-Heisenberg correction $\mathcal{L}'$ to the e.m. Lagrangian in Eq. (3) is neglected, Eqs. (18) and (19) take the form

$$\partial_{x_+} u = -\partial_{x_-} w, \qquad \partial_{x_- x_+} a = 0, \tag{24}$$

where Eq. (19) has reduced to the standard linear wave equation in the light cone coordinates.

The first of Eqs. (24), together with Eq. (17), leads to the general solution $u = f(x_-)$ and $w = g(x_+)$ with $f$ and $g$ arbitrary functions that are determined by the initial conditions. For these solutions the vector potential $a(x_+, x_-)$ takes the factorized form $a(x_+, x_-) = a_+(x_+) + a_-(x_-)$ with $u = \partial_{x_-} a$ and $w = \partial_{x_+} a$. These solutions describe noninteracting electromagnetic waves propagating toward positive and negative directions along the $x$ axis, respectively.

Equations (17), (18) allow for particular solutions for which either $u = 0$ or $w = 0$, in which case $w$ (or $u$) is an arbitrary function depending on the light cone variable $x_+$ (or $x_-$). These solutions describe finite amplitude electromagnetic waves propagating along $x$ from right to left (from left to right) with propagation velocity equal to the speed of light in vacuum. Their shape does not change in time and the electric and magnetic field components are equal $E = B = -w/\sqrt{2}$ and $T_{wu} = -w^2/(4\pi) = -E^2/(2\pi)$, or equal and opposite $E = -B = u/\sqrt{2}$ and $T_{uw} = -u^2/(4\pi) = -E^2/(2\pi)$.

### D. Perturbative solutions

In the case of small but finite field amplitudes $u$, $w$ we can solve Eqs. (17) and (18) [or equivalently Eq. (19)] perturbatively by expanding in powers of the field amplitudes, seeking solutions of the form $u(x_-, x_+) = u_0(x_-) + u_1(x_-, x_+)$, $w(x_-, x_+) = w_0(x_+) + w_1(x_+, x_-)$ (or equivalently of the form $a(x_+, x_-) = a_{0-}(x_-) + a_{0+}(x_+) + a_1(x_-, x_+)$). Keeping only cubic terms in the fields we obtain

$$\begin{aligned} u_1(x_-, x_+) &= \epsilon_2 u_0^2(x_-) w_0(x_+) \\ &\quad + \epsilon_2[\partial_{x_-} u_0(x_-)] \int^{x_+} dx'_+ w_0^2(x'_+), \\ w_1(x_+, x_-) &= \epsilon_2 w_0^2(x_+) u_0(x_-) \\ &\quad + \epsilon_2[\partial_{x_+} w_0(x_+)] \int^{x_-} dx'_- u_0^2(x'_-), \end{aligned} \tag{25}$$

where the two integral terms give the net effect of the interaction between two finite length counterpropagating waves after the end of the interaction. Note that the lower limits in the integrals in Eq. (25) need not be stated explicitly since a change in the lower limits simply amounts to a redefinition of the zero order solution where $u$ depends only on $x_-$ and $w$ on $x_+$. The lower integration limits will be determined when assigning the initial condition on $a_o$ in the full wave equation.

Corresponding results can be obtained by integrating directly the wave equation for $a_1(x_+ x_-)$ up to cubic terms

$$\begin{aligned} \partial_{x_- x_+} a_1(x_+, x_-) = \epsilon_2 [(\partial_{x_+} a_{0+})^2 \partial_{x_- x_-} a_{0-} \\ + (\partial_{x_-} a_{0-})^2 \partial_{x_+ x_+} a_{0+}]. \end{aligned} \tag{26}$$

#### 1. Phase shift induced by the interaction with a localized pulse

Taking as an example a monochromatic wave $u_0(x_-) = U_0 \cos k(x - t)$ interacting with a localized counterpropagating pulse $w_0$, such that $w_0(x_+) = 0$ both for $x_+ > L$ and for $x_+ < -L$, we find

$$\begin{aligned} u(x_-, x_+ < -L) &= u_0(x_-) = U_0 \cos k(x - t), \quad \text{and} \\ u(x_-, x_+ > L) &= u_0(x_-) + \epsilon_2[\partial_{x_-} u_0(x_-)] \int_{-L}^{L} dx'_+ w_0^2(x'_+) \\ &= U_0 \Big[ \cos(kx - t) - k\epsilon_2 \sin k(x - t) \\ &\quad \times \int_{-L}^{L} dx'_+ w_0^2(x'_+) \Big] \end{aligned} \tag{27}$$

which, to the considered expansion order, corresponds to a phase shift [51].

#### 2. Interaction between monochromatic waves and propagation velocity

In the case of two interacting monochromatic waves (independently of their relative frequencies) Eqs. (25) would lead to a secular behavior due to the quadratic terms in the integrands Eqs. (25). In order to cancel such a secular behavior, we may uplift an $\epsilon_2$ term in the expansion of the vector potential $a(x_+, x_-)$ and define the zeroth order solution as

$$\bar{a}_{+0}(x_+ + \epsilon_2 s_+(x_+, x_-)), \quad \bar{a}_{-0}(x_- + \epsilon_2 s_-(x_+, x_-)). \tag{28}$$

To leading order we recover Eq. (24), while two counterterms are added to Eq. (26) that is changed into

$$\begin{aligned} \frac{\partial^2 a_1(x_+, x_-)}{\partial x_+ \partial x_-} &= \epsilon_2 [(\partial_{x_+} \bar{a}_{0+})^2 \partial_{x_- x_-} \bar{a}_{0-} + (\partial_{x_-} \bar{a}_{0-})^2 \partial_{x_+ x_+} \bar{a}_{0+}]. \\ &\quad - \epsilon_2 \frac{\partial}{\partial x_+} \left[ (\partial_{x_+} \bar{a}_{0+}) \frac{\partial s_+(x_+, x_-)}{\partial x_-} \right] \\ &\quad - \epsilon_2 \frac{\partial}{\partial x_-} \left[ (\partial_{x_-} \bar{a}_{0-}) \frac{\partial s_-(x_+, x_-)}{\partial x_+} \right]. \end{aligned} \tag{29}$$

Neglecting higher order terms in $\epsilon_2$ we have

$$\frac{\partial^2 a_1(x_+,x_-)}{\partial x_+\partial x_-} = \epsilon_2 \frac{\partial}{\partial x_+}\left[(\partial_{x_+}\bar{a}_{0+})\left((\partial_{x_-}\bar{a}_{0-})^2 - \frac{\partial s_+(x_+,x_-)}{\partial x_-}\right)\right] + \epsilon_2 \frac{\partial}{\partial x_-}\left[(\partial_{x_-}\bar{a}_{0-})\left((\partial_{x_+}\bar{a}_{0+})^2 - \frac{\partial s_-(x_+,x_-)}{\partial x_+}\right)\right] \tag{30}$$

where we take

$$s_+(x_+,x_-) = \int^{x_-} dx'_-(\partial_{x'_-}\bar{a}_{0-})^2 \sim \int^{x_-} dx'_-(\partial_{x'_-}a_{0-})^2, \quad \to s_+(x_+,x_-) \sim s_+(x_-)$$
$$s_-(x_+,x_-) = \int^{x_+} dx'_+(\partial_{x'_+}\bar{a}_{0+})^2 \sim \int^{x_+} dx'_+(\partial_{x'_+}a_{0+})^2, \quad \to s_-(x_+,x_-) \sim s_-(x_+) \tag{31}$$

where in the expression of $s_\pm$ only the leading coordinate dependence is retained and, without loss of generality, we can set $a_1 = 0$. Then to first order in $\epsilon_2$ the renormalized solutions read

$$a(x_+,x_-) = a_+\left(x_+ + \epsilon_2\int^{x_-} dx'_-(\partial_{x'_-}a_{0-})^2\right) + a_-\left(x_- + \epsilon_2\int^{x_+} dx'_+(\partial'_{x_+}a_{0+})^2\right). \tag{32}$$

The integrals in the arguments lead to two amplitude dependent, inhomogeneous, propagation velocities with absolute values smaller than the speed of light [52]

$$v_-(x_+) = 1 - \epsilon_2(\partial_{x_+}a_{0+})^2, \qquad v_+(x_-) = 1 - \epsilon_2(\partial_{x_-}a_{0-})^2, \tag{33}$$

and, for localized pulses, to a phase shift at the end of the interaction in agreement with Eq. (27). This amplitude dependent slowing of the wave propagation velocity may lead to self-lensing and wave collapse of two counter-propagating pulses [27,53].

### 3. *Perturbed light cone variables*

Referring to Eq. (32), we note that the variables

$$X_+ = x_+ + \epsilon_2\int^{x_-} dx'_-(\partial_{x'_-}a_{0-})^2,$$
$$X_- = x_- + \epsilon_2\int^{x_+} dx'_+(\partial_{x'_+}a_{0+})^2 \tag{34}$$

are "gauge invariant" and transform properly under 1-D Lorentz transformations, see the second line in Eqs. (20). Thus the condition $X_+X_- = 0$ defines a Lorentz invariant *perturbed light cone*. It is interesting to notice that the causal cone of a wave event is "shrunk" by its interaction with a counterpropagating wave.

## E. Full solutions

The characteristics $x_\pm = \xi_\pm(s)$ of Eq. (19), neglecting for the sake of notational simplicity the $\epsilon_3$ term, are given by the quadratic equation

$$\epsilon_2 u^2(s)\left(\frac{d\xi_+}{ds}\right)^2 + \epsilon_2 w^2(s)\left(\frac{d\xi_-}{ds}\right)^2 + [1 - 4\epsilon_2 u(s)w(s)]\left(\frac{d\xi_+}{ds}\right)\left(\frac{d\xi_-}{ds}\right) = 0, \tag{35}$$

and are used in Ref. [43] in order to construct "simple wave" solutions of Eq. (19) and to prove that it admits the formation of discontinuities.

In the following instead we will seek for self-similar (scale invariant) solutions of Eq. (19) by reducing it to an ordinary nonlinear differential equation.

## F. Lorentz invariant solutions

We look for solutions of the form $a(x_+,x_-) = a(\rho)$, with $\rho \equiv x_+x_-$ i.e., for solutions that are constant along the curves $x_+x_- = \text{const}$ which are invariant under Lorentz boosts along $x$. Then, from Eq. (19) we obtain

$$\left[1 - 4\epsilon_2\rho\left(\frac{da}{d\rho}\right)^2\right]\frac{d}{d\rho}\left(\rho\frac{da}{d\rho}\right) = 2\epsilon_2\rho^2\left(\frac{da}{d\rho}\right)^2\frac{d^2a}{d\rho^2}, \tag{36}$$

which can be rewritten as

$$\frac{d}{d\rho}\left(\rho\frac{da}{d\rho}\right) = 2\epsilon_2\frac{d}{d\rho}\left[\rho^2\left(\frac{da}{d\rho}\right)^3\right] \tag{37}$$

and yields the algebraic equation

$$\frac{da}{d\rho} - 2\epsilon_2\rho\left(\frac{da}{d\rho}\right)^3 = \frac{C_2}{\rho}. \tag{38}$$

In the limit $\epsilon_2 \to 0$ we obtain (with $C_1$, $C_2$ arbitrary constants)

$$a = C_1 + C_2\ln|\rho|, \qquad w = C_2/x_+, \qquad u = C_2/x_- \tag{39}$$

In these solutions the electric and the magnetic fields "cumulate" at the light cone $x = \pm t$ where their amplitude diverges. The renormalization approach of Eq. (28) gives

$$a = C_1 + C_2 \ln|\bar{\rho}|, \quad \text{with}$$
$$\bar{\rho} = X_+ X_- = x_+ x_- + \epsilon_2 C_2^2 (x_+/x_{+0} + x_-/x_{-0} - 2). \quad (40)$$

which amounts to an amplitude dependent shift in the cumulation coordinates with

$$w = \frac{C_2(x_- + \epsilon_2 C_2^2/x_{+0})}{x_+ x_- + \epsilon_2 C_2^2 (x_+/x_{+0} + x_-/x_{-0} - 2)},$$
$$u = \frac{C_2(x_+ + \epsilon_2 C_2^2/x_{-0})}{x_+ x_- + \epsilon_2 C_2^2 (x_+/x_{+0} + x_-/x_{-0} - 2)}. \quad (41)$$

Here $x_{+0}$ and $x_{-0}$ are the lower integration limits in the integrals in Eq. (25) and are determined by the initial condition on $a_o$. These Lorentz invariant solutions represent a special case of solutions obtained in the hyperbolic coordinates

$$\rho = x_+ x_-, \qquad \psi = (1/2)\ln(x_+/x_-) \quad (42)$$

that are briefly discussed in Appendix D.

### G. Waves in finite amplitude, uniform electric and magnetic fields in vacuum

Let us set

$$a(x_+, x_-) = W_0 x_+ + U_0 x_- + \tilde{a}(x_+, x_-) \quad (43)$$

with $W_0$, $U_0$ uniform background fields and assume the a finite amplitude field ordering

$$\hat{W}_0 = \epsilon_2^{1/2} W_0 \sim \hat{U}_0 = \epsilon_2^{1/2} U_0 \sim \mathcal{O}(1),$$
$$W_0, U_0 \gg \partial_{x_+}\tilde{a}(x_+, x_-), \partial_{x_-}\tilde{a}(x_+, x_-). \quad (44)$$

Then Eq. (19) (with $\epsilon_3 = 0$ for the sake of simplicity) becomes

$$(1 - 4\hat{U}_0\hat{W}_0)\partial_{x_- x_+}\tilde{a} = \hat{W}_0^2 \partial_{x_- x_-}\tilde{a} + \hat{U}_0^2 \partial_{x_+ x_+}\tilde{a}, \quad (45)$$

which is hyperbolic, and thus describes waves, for $(1 - 4\hat{U}_0\hat{W}_0)^2 > 4U_0^2 W_0^2$, i.e., for $\hat{U}_0\hat{W}_0 < 1/6$ and for $\hat{U}_0\hat{W}_0 > 1/2$. Taking for the sake of simplicity

$$\tilde{a} = \tilde{a}_0 \exp[i(k_+ x_+ + k_- x_-)] = \tilde{a}_0 \exp[i(kx - \omega t)], \quad (46)$$

with $k = (k_+ + k_-)/\sqrt{2}$ and $\omega = -(k_+ - k_-)/\sqrt{2}$, we obtain the dispersion equation

$$(1 - 4\hat{U}_0\hat{W}_0)k_+ k_- = \hat{W}_0^2 k_-^2 + \hat{U}_0^2 k_+^2, \quad \text{i.e.}$$
$$(1 - 4\hat{U}_0\hat{W}_0 + \hat{W}_0^2 + \hat{U}_0^2)\omega^2$$
$$= (1 - 4\hat{U}_0\hat{W}_0 - \hat{W}_0^2 - \hat{U}_0^2)k^2 - 2(\hat{W}_0^2 - \hat{U}_0^2)\omega k, \quad (47)$$

In the two interesting limits of a purely electric ($W_0 = -U_0$, $E = \sqrt{2}W_0$) and purely magnetic ($W_0 = U_0$, $B = -\sqrt{2}W_0$) background fields we obtain

$$\omega_e^2 = [(1 + \epsilon_2 E^2)/(1 + 3\epsilon_2 E^2)]k_e^2$$
$$\omega_b^2 = [(1 - 3\epsilon_2 B^2)/(1 - \epsilon_2 B^2)]k_b^2, \quad \text{for } \epsilon_2 B^2 < 1/3, \quad (48)$$

that correspond to phase velocities smaller than the speed of light in vacuum (see reviews [37,54] and references therein). Note that when both $E$ and $B$ do not vanish the dispersion relation depends on the sign of $\omega/k$, i.e., it differs for propagation in the positive and in the negative $x$ directions. This can be interpreted as a special limit of copropagation and counterpropagation of e.m. fields.

## III. HODOGRAPH TRANSFORM OF THE EQUATIONS OF NONLINEAR ELECTRODYNAMICS IN VACUUM

A system of quasilinear partial differential equations, i.e., a system linear with respect to the highest order terms in the partial derivatives $\partial_{x_-}$ and $\partial_{x_+}$ with coefficients nonlinearly dependent on variables $u$ and $w$, admits the hodograph transformation [45]. Assuming that both $u$ and $w$ are not constant, we perform the hodograph transformation by treating them as coordinates, i.e., we consider $x_-$ and $x_+$ as functions of $u$ and $w$:

$$x_- = x_-(u, w) \quad \text{and} \quad x_+ = x_+(u, w). \quad (49)$$

To transform the system of Eqs. (17) and (18) to the new coordinates $u$ and $w$ we need to express the partial derivatives with respect to $x_-$ and $x_+$ in terms of derivatives with respect to $u$ and $w$. For a function $\Upsilon(x_-, x_+)$, using the chain rule, we have

$$\partial_u \Upsilon = \partial_{x_-}\Upsilon \partial_u x_- + \partial_{x_+}\Upsilon \partial_u x_+,$$
$$\partial_w \Upsilon = \partial_{x_-}\Upsilon \partial_w x_- + \partial_{x_+}\Upsilon \partial_w x_+. \quad (50)$$

Solving this system of equations with respect to $\partial_{x_-}\Upsilon$ and $\partial_{x_+}\Upsilon$ we obtain

$$\partial_{x_-}\Upsilon = J^{-1}(\partial_u \Upsilon \partial_w x_+ - \partial_w \Upsilon \partial_u x_+),$$
$$\partial_{x_+}\Upsilon = J^{-1}(\partial_w \Upsilon \partial_u x_- - \partial_u \Upsilon \partial_w x_-). \quad (51)$$

Here $J = (\partial_u x_- \partial_w x_+ - \partial_w x_- \partial_u x_+)$ is the Jacobian of the coordinate transformation, which is assumed not to vanish. Taking $\Upsilon$ equal either $u$ or $w$ we find

$$\partial_{x_-} u = J^{-1}\partial_w x_+, \qquad \partial_{x_+} u = -J^{-1}\partial_w x_-,$$
$$\partial_{x_-} w = -J^{-1}\partial_u x_+, \qquad \partial_{x_+} w = J^{-1}\partial_u x_+. \tag{52}$$

Substitution of these relationships to Eqs. (17) and (18) yields

$$\partial_u x_+ = \partial_w x_-, \tag{53}$$

$$[1 - uw(4\epsilon_2 + 9\epsilon_3 uw)]\partial_w x_- = -w^2(\epsilon_2 + 3\epsilon_3 uw)\partial_w x_+ - u^2(\epsilon_2 + 3\epsilon_3 uw)\partial_u x_+. \tag{54}$$

From the system (17) and (18) with coefficients nonlinearly dependent on $u$ and $w$ we have obtained a system of linear equations for $x_-$ and $x_+$. Equations (53), (54) are the hodograph transform of Eqs. (17) and (18). As is well known the nonlinearity of the original system is shifted from the field equation to the coordinate transformation. Note that $J^{-1}$ vanishes for purely copropagating solutions when either $u = 0$ or $w = 0$.

## IV. NONLINEAR INTERACTION OF ELECTROMAGNETIC WAVES IN QED VACUUM

Equation (53) makes it possible to define a potential function $\Phi(u, w)$ such that the functions $x_-$ and $x_+$ are given by

$$x_- = \partial_u \Phi, \quad \text{and} \quad x_+ = \partial_w \Phi. \tag{55}$$

Thus we can write Eq. (54) in the form

$$[1 - uw(4\epsilon_2 + 9\epsilon_3 uw)]\partial_{uw}\Phi = -w^2(\epsilon_2 + 3\epsilon_3 uw)\partial_{ww}\Phi - u^2(\epsilon_2 + 3\epsilon_3 uw)\partial_{uu}\Phi. \tag{56}$$

In Appendix C an equivalent derivation of Eq. (56) involving the momenta of the Lagrangian $\mathcal{L}$ is presented. The potential $\Phi$ is invariant under Lorentz boosts along $x$.

### A. Symmetries and conservations in the hodograph representation

When applying the hodograph transformation $x_\pm = x_\pm(u,w)$ a conservation equation of the form

$$\partial_{x_+}\mathcal{A}_+(x_+, x_-) + \partial_{x_-}\mathcal{A}_-(x_+, x_-) = 0, \tag{57}$$

becomes (see Appendix C)

$$\{\mathcal{A}_+(u, w), x_-\}_{u,w} = \{\mathcal{A}_-(u, w), x_+\}_{u,w}, \tag{58}$$

where $\mathcal{A}_\pm(u, w) = \mathcal{A}_\pm(x_+(u, w), x_+(u, w))$, and

$$\{X, Y\}_{u,w} = (\partial X/\partial u)(\partial Y/\partial w) - (\partial Y/\partial u)(\partial X/\partial w),$$

denotes Poisson brackets with respect to $u$ and $w$. Inserting the potential $\Phi(u, w)$, Eq. (58) can be rewritten as

$$\{\mathcal{A}_+(u, w), \partial_u\Phi\}_{u,w} = \{\mathcal{A}_-(u, w), \partial_w\Phi\}_{u,w}. \tag{59}$$

Taking either $\mathcal{A}_+(u, w) = T_{ww}$ and $\mathcal{A}_-(u, w) = T_{uw}$ or $A(u, w) = T_{wu}$ and $B(u, w) = T_{uu}$ as given by the expression of the energy-momentum tensor in Eqs. (23) we recover Eq. (56), here for the sake of simplicity we have set $\epsilon_3 = 0$. Finally we note that Eq. (59) can be rewritten as a conservation law in $u$, $w$ space as

$$\partial_w[(\partial_u\mathcal{A}_+)(\partial_u\Phi) - (\partial_u\mathcal{A}_-)(\partial_w\Phi)] + \partial_u[(\partial_w\mathcal{A}_-)(\partial_w\Phi) - (\partial_w\mathcal{A}_+)(\partial_u\Phi)] = 0. \tag{60}$$

The conservation equation obtained by inserting the components of the energy-momentum tensor in Eqs. (23) into Eq. (60) is related to the invariance of Eq. (56) under the transformation

$$\Phi(u, w) \to \Phi(u, w) + \delta_+ w + \delta_- u, \tag{61}$$

which is the hodograph counterpart of the coordinate translations in Eq. (20). Inserting again either $\mathcal{A}_+(u, w) = T_{ww}$ and $\mathcal{A}_-(u, w) = T_{uw}$ or $A(u, w) = T_{wu}$ and $B(u, w) = T_{uu}$ into Eq. (60) we see that the hodograph equation (56) can be written as a conservation equation in $u - w$ space. A similar procedure shows that the hodograph counterpart of the conservation of the "barycenter" that is given in Eq. (21) and that arises from the Lorentz invariance under boosts along $x$, yields a conserved quantity that is quadratic in $\partial_u\Phi, \partial_u\Phi$, see later Eq. (82).

### B. Hodograph transformation in the linear limit

In the linear limit, $\epsilon_2$, $\epsilon_3 \to 0$, Eq. (56) reduces to

$$\frac{\partial^2\Phi(u, w)}{\partial u \partial w} = 0, \quad \text{i.e.,} \quad \Phi(u, w) = \mathcal{U}(u) + \mathcal{W}(w). \tag{62}$$

Here $\mathcal{U}(u)$ and $\mathcal{W}(w)$ correspond to counterpropagating noninteracting electromagnetic waves with

$$x_- = \frac{\partial\Phi(w, u)}{\partial u} = \frac{\partial\mathcal{U}(u)}{\partial u}, \quad x_+ = \frac{\partial\Phi(w, u)}{\partial w} = \frac{\partial\mathcal{W}(w)}{\partial w}. \tag{63}$$

The choice that corresponds to counterpropagating monochromatic waves is

$$\mathcal{U}_{k_u}(u) = \int_0^u du'[-\psi_u + \arcsin(u'/\mathfrak{A}_u)]/k_u + \text{const},$$
$$\mathcal{W}_{k_w}(w) = \int_0^w dw'[-\psi_w + \arcsin(w'/\mathfrak{A}_w)]/k_w + \text{const}, \tag{64}$$

where $\mathfrak{A}_{w,u}$ are amplitudes, $k_{u,w}$ "frequencies", $\psi_{u,w}$ are phases and

$$\int_0^y dy' \arcsin(y') = y \arcsin y + (1-y^2)^{1/2} - 1.$$

The definition domain is limited by $|u/\mathfrak{A}_u|, |w/\mathfrak{A}_w| \leq 1$. By properly extending the image domain of the arcsin function, Eqs. (64) can be inverted as

$$u(x_-) = \mathfrak{A}_u \sin(k_u x_- + \psi_u),$$
$$w(x_+) = \mathfrak{A}_w \sin(k_w x_+ + \psi_w), \tag{65}$$

where the expressions inside each oscillation half-periods have been joined smoothly so as to cross over the points where the Jacobian of the hodograph transformation vanishes. By redefining the origin of $x_\pm$ we can set $\psi_u = \psi_w = 0$ in agreement with Eq. (61).

### C. Perturbative hodograph solutions

In analogy to the perturbative approach in $(x_+ - x_-)$ space we can search for solutions of Eq. (56) in the form of power series $\Phi = \Phi_0 + \Phi_1 + \cdots$, where $\Phi_0$ satisfies Eq. (62). To the first order to small parameters $\epsilon_2$ and $\epsilon_3$ we obtain

$$\partial_{uw}\Phi_1 = -w^2(\epsilon_2 + 3\epsilon_3 uw)\partial_{ww}\mathcal{W}(w) - u^2(\epsilon_2 + 3\epsilon_3 uw)\partial_{uu}\mathcal{U}(u), \tag{66}$$

which yields

$$\Phi_1 = -\epsilon_2 u \int^w (w')^2 \partial_{w'w'}\mathcal{W}(w')dw' - \frac{3}{2}\epsilon_3 u^2 \int^w (w')^3 \partial_{w'w'}\mathcal{W}(w')dw' - \epsilon_2 w \int^u (u')^2 \partial_{u'u'}\mathcal{U}(u')du' - \frac{3}{2}\epsilon_3 w^2 \int^u (u')^3 \partial_{u'u'}\mathcal{U}(u')du'. \tag{67}$$

For the choice of $\mathcal{W}(w)$ and $\mathcal{U}(u)$ in Eq. (64) we obtain (for $\epsilon_3 = 0$)

$$\Phi_1 = -\epsilon_2 \frac{u\mathfrak{A}_w^2}{2k_w}\mathcal{P}\left(\frac{w}{\mathfrak{A}_w}\right) - \epsilon_2 \frac{w\mathfrak{A}_u^2}{2k_u}\mathcal{P}\left(\frac{u}{\mathfrak{A}_u}\right) + \text{const}, \tag{68}$$

where $\mathcal{P}(y) = \arcsin(y) - y(1-y^2)^{1/2}$. Inserting the zero-order solutions given in Eq. (65) into Eq. (68) and inverting the hodograph transformation we can obtain explicit expressions for $u(x_+, x_-)$ and $w(x_+, x_-)$. However, as noted above for the corresponding perturbative solutions in Eqs. (27) and (25), these expressions include a term that exhibits a secular dependence on the $x_+$, $x_-$ coordinates. A procedure analogous to the one adopted in Eq. (28) can be used to remove this secular behavior as sketched in Appendix E.

### D. Lorentz invariant solutions

Equation (56) admits self-similar solution when the function $\Phi$ depends only on the variable $\xi = uw$ which is invariant under Lorentz boosts along $x$. These solutions are the hodograph counterpart of the solutions described by Eqs. (37), (40), and (41) in $x_+$, $x_-$ space. For the function $\Phi(\xi)$ we obtain

$$(1 - 4\epsilon_2\xi - 9\epsilon_3\xi^2)(\Phi' + \xi\Phi'') = -(2\epsilon_2\xi^2 - 6\epsilon_3\xi^3)\Phi'', \tag{69}$$

where $\Phi' = d\Phi/d\xi$. Introducing the function $U(\xi) = \Phi'$ Eq. (69) reduces to

$$U' + \frac{1 - 4\epsilon_2\xi - 9\epsilon_3\xi^2}{\xi(1 - 2\epsilon_2\xi - 3\epsilon_3\xi^2)}U = 0. \tag{70}$$

Integration of this equation yields

$$U(uw) = \frac{C}{uw(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2w^2)}. \tag{71}$$

For coordinates $x_- = wU$ and $x_+ = uU$ we have

$$x_- = \frac{C}{u(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2w^2)} \quad \text{and}$$
$$x_+ = \frac{C}{w(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2w^2)}, \tag{72}$$

which in the limit $\epsilon_2 = \epsilon_3 = 0$ coincide with Eq. (37). They can be rewritten as

$$x = \frac{2CB}{(E^2 - B^2)[1 + \epsilon_2(E^2 - B^2) - 3\epsilon_3(E^2 - B^2)^2/4]} \tag{73}$$

and

$$t = \frac{2CE}{(E^2 - B^2)[1 + \epsilon_2(E^2 - B^2) - 3\epsilon_3(E^2 - B^2)^2/4]} \tag{74}$$

The solution given by Eq. (71) describes two counterpropagating electromagnetic pulses with the electric and magnetic fields "cumulating" at the light cone $x^2 - t^2 = 0$ where the electric and magnetic fields formally tend to infinity. Note that the position where the cumulation occurs can be shifted by exploiting the translational invariance of the Lagrangian (3), i.e., by looking for solutions [see Eq. (61)] of the form $\Phi(\xi) + \delta_+ w + \delta_- u$. In the derivation of Eqs. (73) and (74) the renormalization procedure adopted for the corresponding solution in Sec. II F, which amounts to an amplitude dependent shift of the cumulation coordinates, has not been included.

For all these solutions the Poincaré invariant $\mathfrak{F} = F_{\mu\nu}F^{\mu\nu} = uw$ does not vanish for finite $x$ and $t$. In the case of solutions (73), (74) the dependence $\mathfrak{F}$ on $t$ and $x$ is given by

$$x^2 - t^2 = \frac{-4C^2}{\mathfrak{F}(1 + \epsilon_2 \mathfrak{F} - 3\epsilon_3 \mathfrak{F}^2/4)^2}. \qquad (75)$$

In the vicinity of the lines given by condition $x^2 - t^2 = 0$ in the $(x, t)$ plane the expression (75) cannot be used because here the electromagnetic field amplitude exceeds the critical QED field $E_S$.

The Lorentz invariant solutions derived above represent a special case of solutions obtained by using the hyperbolic coordinates in hodograph space $\xi = uw$ and $\varphi = (1/2)\ln(u/w)$. These solutions are briefly discussed in Appendix D.

### E. Standard form of the hodograph wave equation

The second order linear hyperbolic PDE given by Eq. (56) can be set in the standard form (see, e.g., Ref. [55])

$$\frac{\partial^2 \Phi}{\partial\zeta\partial\theta} + R(\Phi) = 0, \qquad (76)$$

by an appropriate redefinition of the independent variables $u$ and $w$. Here $R(\Phi)$ denotes terms linear in $\Phi$ containing up to first order derivatives. For the sake of simplicity in the following this transformation will be performed up to linear terms in $\epsilon_2$ and for $\epsilon_3 = 0$. We define the new independent variables

$$\zeta = u(1 - \epsilon_2 uw), \qquad \theta = w(1 - \epsilon_2 uw),$$
$$u = \zeta(1 + \epsilon_2 \zeta\theta), \qquad w = \theta(1 + \epsilon_2 \zeta\theta), \qquad (77)$$

and obtain (as stated above here and in the following only linear terms in $\epsilon_2$ are retained)

$$\frac{\partial^2 \Phi}{\partial\zeta\partial\theta} = 2\epsilon_2\left(\zeta\frac{\partial\Phi}{\partial\zeta} + \theta\frac{\partial\Phi}{\partial\theta}\right)(1 - 8\epsilon_2\zeta\theta)^{-1}$$
$$\sim 2\epsilon_2\left(\zeta\frac{\partial\Phi}{\partial\zeta} + \theta\frac{\partial\Phi}{\partial\theta}\right). \qquad (78)$$

Note that the field variables $\sqrt{2}\zeta = (E - B)[1 - \epsilon_2(B^2 - E^2)]$ and $\sqrt{2}\theta = -(E + B)[1 - \epsilon_2(B^2 - E^2)]$ are directly related to the perturbed light cone variables $X_+, X_-$ defined in Eq. (34) since

$$\theta = \frac{\partial a}{\partial X_+} \quad \text{and} \quad \zeta = \frac{\partial a}{\partial X_-}. \qquad (79)$$

Setting now $\Phi(\zeta, \theta) = \Phi_o(\zeta, \theta)(1 + 2\epsilon_2\zeta\theta)$ we obtain (to first order) the constant coefficient hyperbolic PDE

$$\frac{\partial^2 \Phi_o(\zeta, \theta)}{\partial\zeta\partial\theta} = 2\epsilon_2 \Phi_o(\zeta, \theta), \qquad (80)$$

which is isomorphic to the equation for linear transverse e.m. waves in a uniform plasma.

The solutions of Eq. (80) can be written in the general superposition form

$$\Phi_o(\zeta, \theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dk_\zeta dk_\theta \delta(k_\zeta k_\theta + 2\epsilon_2)$$
$$\times \tilde{\Phi}_o(k_\zeta, k_\theta)\exp\left[+i(k_\zeta\zeta + k_\theta\theta)\right] + \mathcal{CC}, \qquad (81)$$

where the condition $\delta(k_\zeta k_\theta + 2\epsilon_2)$ accounts for the "dispersion" in Eq. (80) and $\mathcal{CC}$ denotes complex conjugate. This dispersion in the hodograph equation can be traced back to the nonlinearity of the wave equation in $(x_+ - x_-)$ space.

#### *1. Conservation equation*

If we add the two equations that we derive by multiplying Eq. (80) by $\partial\Phi_o(\zeta, \theta)/\partial\zeta$ and by $\partial\Phi_o(\zeta, \theta)/\partial\theta$ respectively, we obtain the following conservation equation

$$\frac{\partial}{\partial\theta}\left[\frac{1}{2}\left(\frac{\partial\Phi_o}{\partial\zeta}\right)^2 - \epsilon_2\Phi_o^2\right] + \frac{\partial}{\partial\zeta}\left[\frac{1}{2}\left(\frac{\partial\Phi_o}{\partial\theta}\right)^2 - \epsilon_2\Phi_o^2\right] = 0, \qquad (82)$$

which is quadratic in the function $\Phi_0(\zeta, \theta)$, and is related to the Lorentz invariance of the Lagrangian $\mathcal{L}$, see remark below Eq. (61).

#### *2. An explicit inversion of the hodograph transformation*

Here we consider a special class of solutions of Eq. (80) that can be written as a superposition of "quasi-$\zeta$" modes ($k_\theta = -2\epsilon_2/k_\zeta$) and "quasi-$\theta$" modes ($k_\zeta = -2\epsilon_2/k_\theta$) as

$$\Phi(\zeta,\theta) = (1+2\epsilon_2\zeta\theta)[\Phi_{o,\zeta}(\zeta,\epsilon_2\theta) + \Phi_{o,\theta}(\theta,\epsilon_2\zeta)], \quad (83)$$

where $\Phi(\zeta,\theta) = \Phi_o(\zeta,\theta)(1+2\epsilon_2\zeta\theta)$ has been used with $\Phi_o(\zeta,\theta) = \Phi_{o\zeta}(\zeta,\epsilon_2\theta) + \Phi_{o\theta}(\theta,\epsilon_2\zeta)$ and

$$\Phi_{o\zeta}(\zeta,\epsilon_2\theta) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} dk_\zeta \tilde{\Phi}_{o\zeta}(k_\zeta)\exp[+i(k_\zeta\zeta - 2\epsilon_2\theta/k_\zeta)] + \mathcal{CC}, \quad (84)$$

$$\Phi_{o\theta}(\theta,\epsilon_2\zeta) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} dk_\theta \tilde{\Phi}_{o\theta}(k_\theta)\exp[+i(k_\theta\theta - 2\epsilon_2\zeta/k_\theta)] + \mathcal{CC}. \quad (85)$$

Here $\mathcal{CC}$ denotes complex conjugate. In order to be able to perform the hodograph transformation in an explicit analytical form we consider as an example the "standing wave"-type combination with amplitude $\mathcal{A}_\Phi$

$$\Phi(\zeta,\theta) = \mathcal{A}_\Phi(1+2\epsilon_2\zeta\theta)[\cos(k\zeta - 2\epsilon_2\theta/k) + \cos(k\theta - 2\epsilon_2\zeta/k)], \quad (86)$$

which in terms of $u$ and $w$ reads as

$$\Phi(u,w) = \mathcal{A}_\Phi(1+2\epsilon_2 uw)\{\cos[ku(1-\epsilon_2 uw) - 2\epsilon_2 w/k] + \cos[kw(1-\epsilon_2 uw) - 2\epsilon_2 u/k]\}. \quad (87)$$

Using Eq. (55), from Eq. (87) we obtain an explicit expression for $x_\pm = x_\pm(u,w)$ that can be inverted in a recursive form by keeping only leading order terms in $\epsilon_2$ in the amplitudes. Setting for the sake of algebraic simplicity $k=1$, we obtain

$$\begin{aligned} w = &-\arcsin\{x_+/\mathcal{A}_\Phi - 2\epsilon_2 u[\cos(u - \epsilon_2 w(u^2+2)) \\ &+ \cos[w - \epsilon_2 u(w^2+2)] \\ &+ \epsilon_2(u^2+2)\sin(u - \epsilon_2 w(u^2+2))\} \\ &- \epsilon_2(\arcsin x_-/\mathcal{A}_\Phi)[(\arcsin x_+/\mathcal{A}_\Phi)^2 + 2], \end{aligned} \quad (88)$$

$$\begin{aligned} u = &-\arcsin\{x_-/\mathcal{A}_\Phi - 2\epsilon_2 w[\cos(w - \epsilon_2 u(w^2+2)) \\ &+ \cos[u - \epsilon_2 w(u^2+2)] \\ &+ \epsilon_2(w^2+2)\sin(w - \epsilon_2 u(w^2+2))\} \\ &- \epsilon_2(\arcsin x_+/\mathcal{A}_\Phi)[(\arcsin x_-/\mathcal{A}_\Phi)^2 + 2], \end{aligned} \quad (89)$$

where the $u$, $w$ terms proportional to $\epsilon_2$ in the trigonometric terms on the right-hand side (r.h.s.) of Eqs. (87) and (89) have to be treated recursively and give rise to "harmonic-type" terms, as is characteristic of nonlinear inversions.

This will be particularly evident when considering a superposition of different solutions of Eq. (79), e.g., a discrete or a continuous superposition of solutions of the form given by Eq. (87) over a range of values of $k$. When inverted recursively, these solutions will make it possible to describe the effects of the vacuum nonlinearities on multiscale, electromagnetic counterpropagating waves.

## V. CONCLUSIONS AND DISCUSSIONS

In this article we have analyzed the main features of the interaction of counterpropagating electromagnetic fields in the quantum vacuum within the framework of the Euler-Heisenberg Lagrangian. We have restricted our analysis to the case of fields with the same transverse linear polarization in which case the invariant $\mathfrak{G} = F_{\mu\nu}\tilde{F}^{\mu\nu}$ vanishes (i.e., the term proportional to $\mathbf{E}\cdot\mathbf{B}$ drops from the Euler-Heisenberg Lagrangian).

The results described in this article have been obtained by adopting a combination of analytical methods that involve the direct search for particular solutions of the nonlinear Euler-Heisenberg wave equation in space-time light cone coordinates, the renormalization of perturbative solutions exhibiting a secular behavior and the use of the so called hodograph transformation. This transformation has been adopted in the literature in order to turn a system of quasilinear partial differential equations into a system of linear equations by interchanging the role of dependent and independent variables: see, e.g., Ref. [56] for the case of one-dimensional, compressional hydrodynamics or Ref. [57] for the case of nonlinear time evolution of the filamentation (Weibel) instability. In the case of Quantum Electrodynamics this transformation has been used in Ref. [9] for the study of the so called Born-Infeld equation [58]. When applying the hodograph transformation the Euler-Heisenberg wave equation turns out to be a linear hyperbolic equation to which standard solution methods can be applied. We have shown that when brought to standard form this equation is isomorphic (to leading order in the parameter $\epsilon_2$) to the wave equation of a linear electromagnetic wave in a homogeneous plasma with $2\epsilon_2$ playing the role of the square of the plasma frequency. This indicates that the dependence of the propagation velocity of the counterpropagating e.m. pulses in the $x$, $t$ coordinates on their amplitudes is turned into a standard dispersion phenomenon when the wave equation is expressed in terms of $E$, $B$ coordinates. While the hodograph solutions expressed in terms of $E$ and $B$ can be superimposed so as to construct new solutions, the inversion to $x$, $t$ coordinates is nonlinear: in other words the nonlinearity of the Euler-Heisenberg Lagrangian is shifted from the wane equation to the transformation itself. This latter may be algebraically involved and may require an iterative procedure. We have provided an explicit example of this inversion and shown that the inversion leads to the generation of "harmonic-type terms" as is characteristic of nonlinear inversions.

The relationship between the properties of the solutions of the Euler-Heisenberg wave equation obtained in the $x-t$ coordinates and of its hodograph transform in the $E-B$ coordinates has been discussed with special attention to the different form assumed in the two formulations by conservation laws and conserved quantities. These conservations arise from the translational invariance and from the Lorentz invariance of the Euler-Heisenberg Lagrangian. We have stressed in particular the effect of the Euler-Heisenberg correction to the e.m. Lagrangian on the energy momentum tensor and have shown that its trace does not vanish as would be the case in the linear limit.

We have also shown that, in accordance with previous results in the literature, the interaction of two counterpropagating pulses leads asymptotically only to a cumulative phase shift, a result that can be understood in terms of the energy and momentum conservation of massless particles in a head-on collision. On the contrary, during the interaction of two counterpropagating waves, the propagation velocity of each of them is reduced by a term that depends quadratically on the amplitude of the opposite propagating wave, as is consistent with the fact that the trace of the energy momentum tensor does not vanish.

The phase velocity of linear waves propagating in vacuum in the presence of large, steady and uniform electromagnetic fields (orthogonal to the direction of propagation along $x$) has been derived and shown to be smaller than the speed of light in vacuum by a term that, to leading order, depends on the square of the amplitudes of the steady electromagnetic fields. In the case where both the steady electric and magnetic fields do not vanish these phase velocities depend on whether the waves are propagating in the positive or in the negative $x$ direction.

Finally we observe that the determinant of the Hessian matrix consisting of the second order partial derivatives of $\Phi$ with respect to $u$ and $w$ that appear in Eq. (56) is negative as long as Eq. (56) remains hyperbolic. This property, which is automatically satisfied in the limit where we neglect the quantum vacuum terms, corresponds to solutions describing the propagation of waves. From Eq. (55) we see that the determinant of the Hessian coincides with the inverse of the Jacobian of the hodograph transformation from the $x^+, x^-$ coordinates to $u$ and $w$. Thus the use of the hodograph transformation (see, e.g., the treatment in Ref. [59] for the case of the nonlinear Schrödinger equation) will allow us to characterize (without any expansion in powers of the field amplitudes) the behavior of the electromagnetic fields in the neighborhood of singular curves (or points) in the $u$, $w$ plane where the Hessian determinant vanishes. On these curves the hodograph transformation breaks as $u$, $w$ cease to be single-valued functions of $x^+, x^-$, leading to a behavior that can be described within the framework of the so-called gradient catastrophe phenomena (see, e.g., [60]).


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## APPENDIX A: MOMENTUM VARIABLES OF THE EULER-HEISENBERG LAGRANGIAN

We define the field momenta $\Pi_u$, $\Pi_w$ in the standard way in terms of the Lagrangian $\mathcal{L}$ as

$$\Pi_u = \frac{\partial \mathcal{L}}{\partial(\partial a/\partial x_-)} = \frac{\partial \mathcal{L}}{\partial u}, \qquad \Pi_w = \frac{\partial \mathcal{L}}{\partial(\partial a/\partial x_+)} = \frac{\partial \mathcal{L}}{\partial w}, \tag{A1}$$

and find

$$\Pi_u = -\frac{w}{4\pi}(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2),$$
$$\Pi_w = -\frac{u}{4\pi}(1 - 2\epsilon_2 uw - 3\epsilon_3 u^2 w^2). \tag{A2}$$

The equations of motion (16) take the form

$$\partial_{x_+}\Pi_w + \partial_{x_-}\Pi_u = 0, \tag{A3}$$

which leads to Eq. (18) in the main text.

## APPENDIX B: EULER-HEISENBERG ENERGY MOMENTUM TENSOR IN $x$, $t$ COORDINATES

In light cone coordinates the energy momentum tensor, see Eq. (23), has the form

$$T = \frac{1}{4\pi}\begin{pmatrix} \epsilon_2 u^2 w^2 & -u^2(1-2\epsilon_2 uw) \\ -w^2(1-2\epsilon_2 uw) & \epsilon_2 u^2 w^2 \end{pmatrix}. \tag{B1}$$

In $x$, $t$ coordinate the corresponding mixed index tensor $\mathcal{T}^i_j, i, j = x, t$ is given by

$$\mathcal{T} = MTM \tag{B2}$$

where

$$M = M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{such that}$$
$$\begin{pmatrix} x_+ \\ x_- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}. \tag{B3}$$

This leads to

$$\mathcal{T} = \frac{1}{8\pi}\begin{pmatrix} -(u^2+w^2)(1-2\epsilon_2 uw)+2\epsilon_2 u^2w^2, & (u^2-w^2)(1-2\epsilon_2 uw), \\ -(u^2-w^2)(1-2\epsilon_2 uw), & (u^2+w^2)(1-2\epsilon_2 uw)+2\epsilon_2 u^2w^2 \end{pmatrix} \tag{B4}$$

and in terms of $E$, $B$ by

$$\mathcal{T} = \frac{1}{8\pi}\begin{pmatrix} -(E^2+B^2)+\epsilon_2(3B^4/2-E^4/2-B^2E^2), & -2EB+\epsilon_2(2EB^3-2E^3B), \\ 2EB-\epsilon_2(2EB^3-2E^3B) & (E^2+B^2)+\epsilon_2(3E^4/2-B^4/2-B^2E^2) \end{pmatrix}. \tag{B5}$$

Note that the trace of the energy momentum tensor ($\mathrm{Tr}(\mathcal{T}) = \mathrm{Tr}(T)$) does no longer vanish and is proportional to $\epsilon_2(E^2-B^2)^2$. Note in addition that, at a perfectly conducing boundary where $E=0$, both the energy density ($\mathcal{T}_t^t$) component and the radiation pressure $\mathcal{T}_x^x$ stress component are reduced with respect to the linear, non-interacting limit.

## APPENDIX C: THE HODOGRAPH TRANSFORMATION IN DIFFERENTIAL FORM AND EULER-HEISENBERG MOMENTA

Equations (17), (18), or equivalently Eqs. (17), (A3) can be written in the 2-form formalism as

$$\begin{aligned}(\partial_{x_+}u)dx_+\wedge dx_- - (\partial_{x_-}w)dx_+\wedge dx_- \\ = du\wedge dx_- + dw\wedge dx_+ = 0,\end{aligned} \tag{C1}$$

$$\begin{aligned}(\partial_{x_+}\Pi_w)dx_+\wedge dx_- + (\partial_{x_-}\Pi_u)dx_+\wedge dx_- \\ = d\Pi_w\wedge dx_- - d\Pi_w\wedge dx_+ = 0,\end{aligned} \tag{C2}$$

where the symbol $\wedge$ denotes exterior product [61]. Taking $u$ and $w$ as independent variables in Eq. (C1) (assuming as in Sec. III) that the Jacobian of the transformation is different from zero) we obtain

$$\begin{aligned}&(\partial_w x_-)du\wedge dw - (\partial_u x_+)du\wedge dw = 0, \\ &\to \quad \partial_w x_- = \partial_u x_+,\end{aligned} \tag{C3}$$

i.e., Eq. (53). Similarly, using $\Pi_u$, $\Pi_w$ as the independent variables in Eq. (C2) we obtain

$$\frac{\partial x_+}{\partial\Pi_w} + \frac{\partial x_-}{\partial\Pi_u} = 0, \tag{C4}$$

which leads to Eq. (54), after $\Pi_u$, $\Pi_w$ are expressed in terms of $u$, $w$ through Eq. (A2). Conversely, we can express $u$, $w$ in terms of $\Pi_u$, $\Pi_w$ and write the whole system of the hodograph equations in terms of the momenta $\Pi_u$, $\Pi_w$.

Note that the hodograph transformation procedure described above is also applicable to the more general case with vector potential $A_z = A(x,y,t)$. In this case however it would lead to nonlinear equations as can be easily seen, e.g., by appropriately reformulating Eq. (C1) as a 3-form ($dx\wedge dt \to dx\wedge dy\wedge dt$).

### 1. Conservations and Poisson brackets

We can rewrite the conservation equation (58) in the differential form

$$\begin{aligned}&(\partial_{x_+}\mathcal{A}_+)dx_+\wedge dx_- + (\partial_{x_-}\mathcal{A}_-)dx_+\wedge dx_- = 0 \\ &\to \quad d\mathcal{A}_+\wedge dx_- = d\mathcal{A}_-\wedge dx_+\end{aligned} \tag{C5}$$

and, imposing the hodograph transformation, we obtain

$$\begin{aligned}&[(\partial_u\mathcal{A}_+)(\partial_w x_-) - (\partial_u x_-)(\partial_w\mathcal{A}_+)]du\wedge dw \\ &\quad = [(\partial_u\mathcal{A}_-)(\partial_w x_+) - (\partial_u x_+)(\partial_w\mathcal{A}_-)]du\wedge dw \\ &\to \quad \{\mathcal{A}_+(u,w), x_-\}_{u,w} = \{\mathcal{A}_-(u,w), x_+\}_{u,w}.\end{aligned} \tag{C6}$$

## APPENDIX D: HYPERBOLIC COORDINATES

Instead of $x_+$ and $x_-$ we can use the hyperbolic coordinates

$$\begin{aligned}&\rho = x_+x_- = x^2 - t^2, \\ &\psi = (1/2)\ln(x_+/x_-) = \frac{1}{2}\ln\frac{1+t/x}{1-t/x} = \mathrm{arctan\,h}(t/x).\end{aligned} \tag{D1}$$

Under an infinitesimal (finite) Lorentz boost along $x$ [see Eq. (20)] we have

$$\begin{aligned}&\rho\to\rho, \qquad \psi\to\psi+\beta, \\ &\left(\psi\to\psi+\frac{1}{2}\ln\frac{1+\beta}{1-\beta} = \psi + \mathrm{arctan\,h}(\beta)\right).\end{aligned} \tag{D2}$$

### 1. Lagrangian in hyperbolic coordinates

Since the Euler-Heisenberg Lagrangian (12) is Lorentz invariant, when expressed in hyperbolic coordinates, it cannot depend explicitly on $\psi$. Starting from the Action $\mathcal{S}(a)$ in $x_+, x_-$ variables, bringing it to $\rho$, $\psi$ variables ad using the fact that the Jacobian of the transformation is equal to one, the new Lagrangian (with $\epsilon_3 = 0$) reads:

$$(-4\pi)\mathcal{L}_L(\rho,\psi) = \rho\left(\frac{\partial a}{\partial \rho}\right)^2 + \frac{1}{4\rho}\left(\frac{\partial a}{\partial \psi}\right)^2 - \epsilon_2\left[\rho\left(\frac{\partial a}{\partial \rho}\right)^2 + \frac{1}{4\rho}\left(\frac{\partial a}{\partial \psi}\right)^2\right]^2. \qquad (D3)$$

The self-similar solution Eq. (36) corresponds to $\partial a/\partial\psi = 0$ and can be derived directly from the Lagrangian $\mathcal{L}_L(\rho,\psi)$ in the convenient form given by Eq. (37). In the linear limit $\epsilon_2 = 0$ the Lagrangian $\mathcal{L}_L(\rho,\psi)$ can be expanded into "$\psi$-harmonics" and leads to power-law solutions. For $\epsilon_2 \neq 0$ these harmonics are coupled.

### 2. Hodograph equation in hyperbolic coordinates

In terms of the variables $\xi = uw$ and $\varphi = (1/2)\ln(u/w)$ Eq. (56) (with $\epsilon_3 = 0$) becomes

$$(1-4\epsilon_2\xi)\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Phi}{\partial\xi}\right) - \frac{1}{4\xi}\frac{\partial^2\Phi}{\partial\varphi^2}\right] + \epsilon_2\left(2\xi^2\frac{\partial^2\Phi}{\partial\xi^2} + \frac{1}{2}\frac{\partial^2\Phi}{\partial\varphi^2}\right) = 0. \qquad (D4)$$

Since Eq. (D4) is linear and its coefficients are independent of $\varphi$, its solutions can be decomposed into a two sided Poisson expansion, i.e., in $\cosh(\alpha\varphi)$ and $\sinh(\alpha\varphi)$ terms with $\alpha$ a real number. We obtain a family of ordinary differential equations that, with self-evident notation, can be written as

$$(1-4\epsilon_2\xi)\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Phi_\alpha}{\partial\xi}\right) - \frac{\alpha^2}{4\xi}\Phi_\alpha\right] + \epsilon_2\left(2\xi^2\frac{\partial^2\Phi_\alpha}{\partial\xi^2} + \frac{\alpha^2}{2}\Phi_\alpha\right) = 0. \qquad (D5)$$

In the linear limit ($\epsilon_2 = 0$) the solutions of Eq. (D5) are of the form $\Phi = C_1 w^\alpha + C_2 u^\alpha$ and, for positive integer values of $\alpha$, can be used as a polynomial basis in the noninteraction limit.

## APPENDIX E: RENORMALIZED HODOGRAPH SOLUTIONS FOR INTERACTING WAVES

In view of Eq. (28) we can rewrite Eq. (64) as

$$\mathcal{U}_{k_u}(u-\epsilon_2u^2w,\epsilon_2w) = \int_0^{u-\epsilon_2u^2w}\frac{du'/k_u}{1-2\epsilon_2u'w}\left[\arcsin\left(\frac{u'+\epsilon_2u'^2w}{A_u}\right) - \epsilon_2S_u\left(\arcsin\left(\frac{u'}{A_u}\right),\arcsin\left(\frac{w}{A_w}\right)\right)\right] + \text{const},$$

$$\mathcal{W}_{k_w}(w-\epsilon_2w^2u,\epsilon_2u) = \int_0^{w-\epsilon_2uw^2}\frac{dw'/k_w}{1-2\epsilon_2uw'}\left[\arcsin\left(\frac{w'+\epsilon_2w'^2u}{A_w}\right) - \epsilon_2S_w\left(\arcsin\left(\frac{u}{A_u}\right),\arcsin\left(\frac{w'}{A_w}\right)\right)\right] + \text{const} \qquad (E1)$$

Equation (E1) can be inverted (to first order in $\epsilon_2$) as

$$\begin{aligned} u(x_-,\epsilon_2x_+) &= A_u\sin(k_ux_- + \epsilon_2S_u(\arcsin(u/A_u),\arcsin(w/A_w)) \\ &= A_u\sin(k_ux_- + \epsilon_2S_u(k_ux_-,k_wx_+)), \\ w(x_+,\epsilon_2x_-) &= A_w\sin(k_wx_+ + \epsilon_2S_w(\arcsin(u/A_u),\arcsin(w/A_w)) \\ &= A_w\sin(k_wx_+ + \epsilon_2S_w(k_ux_-,k_wx_+)). \end{aligned} \qquad (E2)$$


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