Anyonic particle-vortex statistics and the nature of dense quark matter

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We show that $\mathbb{Z}_3$-valued particle-vortex braiding phases are present in high density quark matter. Certain mesonic and baryonic excitations, in the presence of a superfluid vortex, have orbital angular momentum quantized in units of $\hbar/3$. Such nonlocal topological features can distinguish phases whose realizations of global symmetries, as probed by local order parameters, are identical. If $\mathbb{Z}_3$ braiding phases and angular momentum fractionalization are absent in lower density hadronic matter, as is widely expected, then the quark matter and hadronic matter regimes of dense QCD must be separated by at least one phase transition.

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I. INTRODUCTION

The behavior of QCD as a function of baryon density, at vanishing temperature, is of fundamental importance to nuclear physics and astrophysics [1–4]. At low density, quarks and gluons have strong interactions and are bound into colorless hadrons, producing hadronic nuclear matter. At asymptotically high densities, one instead finds weakly-coupled quark matter [5]. Can the hadronic and quark matter regimes be smoothly connected, or are they necessarily separated by a phase transition?

We consider this question in the simplified, more symmetric setting of three-color, three-flavor QCD with degenerate quark masses and $SU(3)_V$ flavor symmetry.[1] The increase in symmetry gives some hope that questions of principle can be addressed in a sharper fashion. In this flavor-symmetric setting, there is a well-known conjecture of *quark-hadron continuity* due to Schäfer and Wilczek [9]. This conjecture is supported by a comparison of the expected pattern of low energy excitations and the realizations of conventional global symmetries at both high and low densities.

At asymptotically high densities, "color superconductivity" leads to a diquark "condensate" $\langle qq\rangle \neq 0$, which in turn induces nonzero gauge-invariant condensates with the schematic forms $\langle (qq)^3\rangle$ and $\langle \bar{q}\,\bar{q}\,qq\rangle$ [10]. The $\langle (qq)^3\rangle$ condensate signals spontaneous breaking of the $U(1)_B$ baryon number symmetry down to $\mathbb{Z}_2$, while the $\langle \bar{q}\,\bar{q}\,qq\rangle$ condensate signals, in the limit of massless quarks, spontaneous chiral symmetry breaking of the form $SU(3)_L\times SU(3)_R \to SU(3)_V$. The spontaneous breaking of $U(1)_B$ indicates that high density quark matter is a superfluid. At low densities, one expects an identical chiral symmetry breaking pattern in the massless limit, while $U(1)_B$ symmetry breaking is believed to arise (in the flavor symmetric theory) from the condensation of pairs of Λ hyperons with flavor content *uds*. The matching symmetry realization, along with apparently compatible patterns of low energy excitations, make it plausible that the quark and hadronic phases of QCD are smoothly connected, at least in the flavor symmetric limit [9].

However, there is no guarantee that distinct phases can always be distinguished by this Landau-Ginzburg style analysis based on local order parameters. Some transitions separating distinct phases can only be diagnosed by changes in behavior of topological observables such as the ground state degeneracy on large topologically nontrivial manifolds, or suitable nonlocal order parameters from which one infers, for example, particle-vortex braiding statistics [11–13].

Given this motivation, we examine topological ground state degeneracies and quark-vortex braiding statistics in asymptotically high-density quark matter, and compare results with the expected properties of hadronic nuclear matter. In high density quark matter, we find that quarks acquire nontrivial $\mathbb{Z}_3$ Aharonov-Bohm phases, arising from color holonomies, when encircling a superfluid vortex with minimal circulation. In terms of dressed gauge-invariant

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[1]When electromagnetic and weak interactions are included, and quarks have their physical masses, various phase transitions associated with, e.g., kaon condensation occur at high, but nonasymptotic, density [5–8]. Focusing on the flavor symmetric limit avoids these complications.

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quasiparticle excitations, this means that certain mesonic and baryonic excitations have orbital angular momentum quantized in units of $\hbar/3$ in the presence of a minimal superfluid vortex. These results can also be interpreted in terms of anyonic particle-vortex braiding statistics.

These topological features contrast sharply with the expected properties of superfluid hadronic matter, in which one expects conventional quantization of quasiparticle orbital angular momentum in units of $\hbar$. If the standard picture of the low density hadronic regime is correct, then the hadronic and quark matter regimes must be separated by at least one phase transition.

II. $U(1)$ SUPERCONDUCTORS

To set the stage for our QCD discussion we first review related aspects of BCS superconductors at zero temperature. (See, e.g., Ref. [14] for more detail.) Such systems can be modeled by an Abelian Higgs model,

$$\mathcal{L} = |D_\mu \phi|^2 + m^2|\phi|^2 + \frac{1}{2}\lambda|\phi|^4 - \frac{1}{4e^2}F_{\mu\nu}^2. \qquad (1)$$

The complex scalar field ϕ is assumed to have charge q under the $U(1)$ gauge symmetry, so $D_\mu\phi \equiv (\partial_\mu - iqA_\mu)\phi$, with $q = 2$ for real electron superconductors. Using sloppy perturbative language, ϕ gets a nonzero vacuum expectation value in the superconducting regime of large negative m^2, "breaking" the $U(1)$ gauge symmetry. But local gauge symmetries never truly break spontaneously [15]. Realizations of all conventional global symmetries remain unchanged as m^2 varies, and there are no gauge-invariant local operators which can serve as order parameters for superconductivity. Distinguishing the superconducting and normal phases requires looking beyond the conventional Landau-Ginzburg paradigm based on local order parameters.

The superconducting phase has $\mathbb{Z}_2$ topological order, arising from Higgsing of the $U(1)$ gauge symmetry down to $\mathbb{Z}_2$ [14]. This provides a sharp distinction from the normal phase. In this context, topological order [11–13] has two related consequences: a ground state degeneracy which depends on the topology of space, and nontrivial Aharonov-Bohm phases for transport of charged particles around magnetic vortex lines.

To examine the ground state degeneracy, one may compactify a single spatial direction with periodic boundary conditions (for all fields), and consider the theory on $\mathbb{R}^{1,2} \times S^1$. Ground state degeneracy arises from multiple minima of the holonomy effective potential $V_{\rm eff}(\Omega)$, where $\Omega \equiv e^{i\oint A}$ is the Wilson loop (or spatial holonomy) wrapping the compactified dimension. In the Higgs phase, the covariant gradient term of the Lagrangian (1) induces a (Meissner) mass for the photon. When the spatial holonomy $\Omega \neq 1$, this term also gives a tree level contribution to the holonomy potential,

$$V_{\rm eff}(\Omega) = \min_{k\in\mathbb{Z}} \frac{|v|^2}{L^2}(2\pi k - qa)^2 + \cdots, \qquad (2)$$

where $\Omega \equiv e^{ia}$, the ellipsis denotes higher order contributions (plus terms independent of Ω), $v^2 \equiv -m^2/\lambda$, and k is the winding number of the phase of the condensate around the S^1. Viewed as a function of $a = -i\ln\Omega$, gauge invariance requires the potential to be 2π periodic, but when $q > 1$ it actually has a finer periodicity of $2\pi/q$. For $q = 2$ there are two degenerate minima within the fundamental domain $[0, 2\pi)$, namely $a = 0$ and $a = \pi$, associated with $k = 0$ and $k = 1$, respectively. So $\langle\Omega\rangle = \pm 1$ and the ground state degeneracy is 2 on $\mathbb{R}^{1,2} \times S^1$. (On $\mathbb{R} \times T^3$, with charge q, the degeneracy is q^3.)

While one does not directly measure this ground state degeneracy experimentally, the $\Omega = -1$ minimum is related to the braiding statistics between charged particles and magnetic vortices [16]. The field configuration describing a superconducting vortex running along some straight path is, in cylindrical coordinates, given by

$$\phi(\theta, r) = f(r)e^{ik\theta}, \qquad A_\theta = ah(r)/r, \qquad (3)$$

where the radial functions $f(r)$ and $h(r)$ run monotonically from 0 at $r = 0$ to 1 at $r = \infty$, and $k \in \mathbb{Z}$. For the vortex to have finite energy per unit length, the covariant derivative of ϕ must vanish at large r, implying that $a = k/2$. So a minimal vortex carrying $k = 1$ units of magnetic flux has an azimuthal gauge field $A_\theta \sim 1/(2r)$ at large r, implying that the holonomy around a large circle surrounding the vortex is -1. More generally, if $\Omega[C]$ denotes the holonomy (Wilson loop) around some closed loop C, then

$$\langle\Omega[C]\rangle_{V[P]} = e^{i\pi\ell(C,P)}, \qquad (4)$$

where $\langle\cdots\rangle_{V[P]}$ denotes an expectation value in the presence of a vortex V running along some closed path P, and $\ell(C, P)$ is the intersection (linking) number of paths C and P, provided these paths are well separated.[2] This shows that particles of charge $q = 1$ have an Aharonov-Bohm phase of -1 when going around a vortex, demonstrating that particles and vortices have $\mathbb{Z}_2$ braiding statistics. To reiterate, the holonomy $\Omega[C]$ may be far from unity in the presence of vortices, or with nontrivial topology, even though electric and magnetic fields are completely screened in a superconductor [14,17].

[2]When fluctuations are taken into account, $|\langle\Omega\rangle|$ has perimeter law decay, and the left-hand side of expression (4) should really be the phase of the Wilson loop, or equivalently the ratio $\langle\Omega[C]\rangle_{V[P]}/\langle\Omega[C]\rangle$, where the denominator is the Wilson loop expectation value in the absence of the vortex.

III. HIGH DENSITY QCD

We now generalize this analysis to the case of QCD with $SU(3)$ flavor symmetry. Purely for simplicity of presentation, we assume that the quarks are massive, $m_q > 0$, so that the theory has a nonvanishing mass gap. (We comment on the chiral limit below.) We consider the limit of asymptotically large quark density, and vanishing temperature, so that the quark chemical potential μ (equal to 1/3 times the baryon chemical potential μ_B) is large compared to the strong scale Λ or the light quark mass m_q. This regime is weakly coupled at all length scales, thanks to Cooper pairing of quarks at the Fermi surface and the resulting color Meissner effect which suppresses long wavelength gauge field fluctuations. In gauge-variant language, the large μ, low temperature regime features a diquark condensate of the "color-flavor-locked" (CFL) form,

$$\langle q_a^i C q_b^j \rangle = K g(\mu)^{-1} \mu^2 \Delta \epsilon^{ijk} \epsilon_{abk} \equiv \Phi_{ab}^{ij}, \tag{5}$$

where K is a pure numerical factor, indices i, j denote color, a, b are flavor indices, and $\Delta \sim \mu g(\mu)^{-5} e^{-3\pi^2/(g(\mu)\sqrt{2})}$ is the pairing gap [5,18]. Equivalently, at long distances there are three emergent antifundamental Higgs fields, which may be viewed as forming a single 3×3 matrix valued scalar field,

$$(\phi)_m^l \equiv (4K)^{-1} g(\mu) \mu^{-2} \epsilon^{ijl} \epsilon_{abm} \Phi_{ab}^{ij}. \tag{6}$$

In the unitary gauge of Eq. (5), $\phi = \Delta \cdot \mathbb{1}$.

The determinant of ϕ is the physical, gauge invariant order parameter which was earlier written schematically as $\langle (qq)^3 \rangle$. Under the $U(1)_B$ baryon number symmetry, $\det \phi$ has charge 2. Since $\langle \det \phi \rangle \neq 0$, $U(1)_B$ symmetry is spontaneously broken down to $\mathbb{Z}_2$ and dense quark matter is a superfluid. Fluctuations in the condensate phase $\varphi \equiv -i \log \det \phi$ represent $U(1)_B$ Nambu-Goldstone bosons (NGBs), with associated the low energy effective action

$$S_{U(1)_B} = \int d^4x \frac{1}{2} f^2 [(\partial_t \varphi)^2 + v_s^2 (\nabla \varphi)^2]. \tag{7}$$

Here $f^2 \sim 6\mu^2/\pi^2$ and $v_s^2 \sim 1/3$ [19].

The action (7) is conventionally believed to be a valid long-distance effective action for dense QCD when $m_q > 0$. In the massless limit, $m_q \to 0$, there are additional Nambu-Goldstone bosons (pions, kaons, etc.) due to the spontaneous breaking of chiral symmetry [20]. However, at the level of two derivative terms in the effective action there is no coupling between phase fluctuations of the neutral superfluid condensate and these chiral symmetry NGBs. Consequently, such chiral symmetry NGBs play no role in any of the phenomena we discuss below, and may simply be ignored.

IV. COLOR HOLONOMIES

We would like to examine possible equilibrium states on topologically nontrivial spatial manifolds, and understand the related issue of particle-vortex braiding statistics. To this end, we follow the same procedure used above. We compactify one spatial dimension on a circle of circumference L, with periodic boundary conditions for all fields and L larger than all intrinsic length scales.[3] Fluctuations in the condensate (6), as well as the color Meissner effect, may be described by an effective Lagrangian,

$$\mathcal{L}_\phi = \kappa \mathrm{tr}[(D_t \phi)^\dagger (D_t \phi) + v_s^2 (D_i \phi)^\dagger (D_i \phi)] + V(\phi), \tag{8}$$

where $\kappa = O(\mu^2/\Delta^2)$, $D_\mu = \partial_\mu + iA_\mu$ is the color covariant derivative for an antifundamental, and the scalar potential $V(\phi)$ is minimized when $\phi/\Delta \in U(3)$ [20].

Let Ω denote the $SU(3)$ color holonomy around the S^1. The gradient term in the Lagrangian (8) generates a tree-level contribution to the holonomy effective potential $V_{\text{eff}}(\Omega)$.[4] One finds

$$V_{\text{eff}}(\Omega) = \frac{\kappa \Delta^2 v_s^2}{L^2} \min_{k \in \mathbb{Z}^3} \sum_{i=1}^{3} (2\pi k_i + \theta_i)^2 + \cdots, \tag{9}$$

where $\theta = (\theta_1, \theta_2, \theta_3)$ are the phases of the eigenvalues of Ω, with $\theta_3 \equiv -\theta_1 - \theta_2$, and $k \equiv (k_1, k_2, k_3)$ are the winding numbers of the eigenvalues of ϕ around the S^1.

Within the triangular fundamental domain given by $-2\theta_1 \le \theta_2 \le \theta_1$, $\theta_2 \le 2\pi - 2\theta_1$, and $0 \le \theta_1 \le 4\pi/3$, the holonomy potential has one global minimum and two degenerate local minima. The global minimum lies at $\theta = (0,0,0)$ and is associated with $k = (0,0,0)$. The two local minima are $\theta = (2\pi/3, 2\pi/3, -4\pi/3)$ and $\theta = (4\pi/3, -2\pi/3, -2\pi/3)$, and are associated with $k = (0,0,1)$ and $(-1,0,0)$, respectively. In other words, the global minimum is $\Omega = \mathbb{1}$, while the local minima lie at $\Omega = e^{\pm 2\pi i/3}\mathbb{1}$. The energy density at the local minima exceeds that at the global minimum by the amount

$$\Delta V_{\text{eff}} = \frac{4\pi^2 \kappa \Delta^2 v_s^2}{3L^2}. \tag{10}$$

Unlike the case of a BCS superconductor, high density QCD on $\mathbb{R}^3 \times S^1$ has a unique ground state, so the color superconducting state does not have conventional topological order. But local minima are present in which the

[3] On the resulting $\mathbb{R}^2 \times S^1$ spatial manifold, spontaneous breaking of continuous symmetries remains possible at zero temperature.

[4] Reference [21] examined one-loop contributions to the holonomy effective potential in dense QCD with flavor-twisted boundary conditions, but did not include this dominant tree-level contribution. Correcting this oversight eliminates the infinite sequence of alternating transitions discussed in Ref. [21].

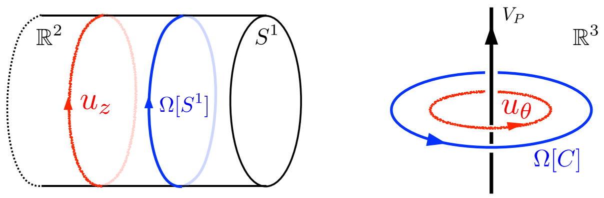

FIG. 1. Left: $SU(3)$ holonomy Ω wrapping the compactified circle on spatial manifold $\mathbb{R}^2 \times S^1$ in dense quark matter. Nontrivial $\mathbb{Z}_3$-valued holonomies are associated with nonzero superfluid flow around the S^1. Right: the related situation of a Wilson loop on curve C encircling a minimal superfluid vortex on path P, with linking number $\ell(C, P) = 1$.

holonomy is a nontrivial cube root of unity, with energy density (relative to the ground state) vanishing as $L \to \infty$, but only as an inverse power of L. This differs from generic metastable "false vacua" whose energy is linear in the spatial volume, and is also unlike the exponentially small (in L) energy differences typically present in gapped systems with degenerate infinite volume ground states.

To elucidate the physical interpretation of these nontrivial local minima, consider the superfluid flow velocity u_μ, given by the gradient of the phase of the gauge invariant condensate $\det \phi$ divided by twice the baryon chemical potential μ_B. Equivalently, $u_\mu = (2\mu_B)^{-1} \mathrm{tr} \phi^{-1} D_\mu \phi$. Evaluated in the homogeneous states described by the holonomy effective potential, one finds a superfluid velocity along the compactified direction given by

$$u_z = \frac{\pi}{\mu_B L}(k_1 + k_2 + k_3). \tag{11}$$

The $\Omega = \mathbb{1}$ global minimum has vanishing superfluid velocity, but the nontrivial local minima with $\Omega = e^{\pm 2\pi i/3}\mathbb{1}$ have $\sum_i k_i = \pm 1$, and hence have supercurrents flowing around the compact direction with the minimal nonzero quantized circulation. The same quantized circulation appears around superfluid vortices, and one may regard the circle-compactified theory as mimicking an annular region surrounding a superfluid vortex, as illustrated in Fig. 1.

V. VORTICES IN DENSE QUARK MATTER

Minimal circulation vortex configurations in high density quark matter were first examined in Ref. [22].[5] In the CFL regime, a suitable field ansatz for describing a minimal vortex on a path P running along the z axis is given by [22]:

$$\frac{\phi}{\Delta} = \mathrm{diag}[e^{i\theta} f(r), g(r), g(r)], \tag{12a}$$

$$A_\theta = \frac{h(r)}{r} \mathrm{diag}[-2a, a, a], \tag{12b}$$

with other gauge field components vanishing. The radial functions $f(r)$, $g(r)$, and $h(r)$ all approach 1 as $r \to \infty$, and f and h vanish at $r = 0$. The gauge invariant order parameter $\det(\phi/\Delta)$ approaches $e^{i\theta}$ far from the vortex core, showing that this ansatz describes a vortex with minimal $U(1)_B$ winding, or equivalently minimal superfluid circulation. Such vortices are necessarily present when superfluid quark matter rotates (above a critical frequency), as in neutron stars. The energy per unit length of any straight, infinitely long superfluid vortex is logarithmically IR divergent. Minimizing the long distance energy density of the configuration (12), proportional to $r^{-2}[(1-2a)^2 + 2a^2]$, gives $a = 1/3$.[6]

Now consider the $SU(3)$ holonomy Ω for some loop C in the presence of this minimal vortex running along the path P. Assume that the curves C and P are everywhere widely separated compared to the color magnetic penetration length. Then by direct contour integration of the gauge field (12b) one finds

$$\gamma \equiv \mathcal{N}^{-1} \langle \mathrm{tr}\Omega[C] \rangle_{V[P]} = \exp\left[\frac{2\pi i}{3} \ell(C, P)\right], \tag{13}$$

where $\ell(C, P) \in \mathbb{Z}$ is the linking number of the contours, and the normalization factor $\mathcal{N} = \langle \mathrm{tr}\Omega[C] \rangle$ is the expectation value without the vortex. This shows that fundamental representation quarks and vortices have $\mathbb{Z}_3$ braiding phases in high density quark matter.

VI. SCREENING AND FRACTIONALIZATION

How do the nontrivial color holonomies (13) affect physical quasiparticle excitations of dense QCD? How does this relate to color screening in a color superconductor? These issues were considered in Ref. [27] (see also Ref. [28]) which examined CFL vortices and discussed possible implications for quark-hadron continuity. This followed the interesting earlier work in Ref. [29] which relied on the gauge variant notion of color-magnetic flux to argue that vortices could not continue smoothly between the CFL and hadronic regimes. The authors of Ref. [27] correctly noted that gauge invariant aspects of "color-magnetic flux" must be encoded in color holonomies which encircle a vortex. However, Ref. [27] asserted that

[5] These vortices have variously been termed "semilocal" and "non-Abelian" [23–26]. The former terminology emphasizes that they are magnetic vortices as far as the $SU(3)$ gauge group is concerned but at the same time are superfluid vortices with logarithmically divergent energy per unit length. The latter terminology refers to the gauge-dependent notion of color-magnetic flux.

[6] Minimizing the complete energy determines the radial functions, which are monotonic and approach their asymptotic values exponentially rapidly on a scale set by the coherence length Δ^{-1} (for f and g) or the much shorter $O((g\mu)^{-1})$ color magnetic penetration length (for h) [22].

such holonomies must be trivial "because the condensate, as a color triplet, can completely screen the color charge of the probe quark." This is inconsistent with the above explicit calculation of the holonomy. Moreover, while static test quarks are screened in QCD, this only implies that the *magnitude* of a large Wilson loop will have perimeter-law behavior, $|\langle \mathrm{tr}\Omega[C]\rangle| \sim e^{-\delta m|C|}$, where $|C|$ denotes the perimeter and δm is a (renormalization scheme dependent) mass shift. But such screening does not constrain the *phase* of the expectation which, as we have seen, can be $\mathbb{Z}_3$-valued at high densities.

It is illuminating to recast this point in more physical terms and examine how nontrivial holonomies affect "screened" gauge invariant quasiparticle excitations. The holonomy evaluated above represents the phase, due to the color gauge field, acquired by a quark when encircling a minimal vortex in the direction of the superfluid flow. Such a quark, whose minimal excitation energy is the pairing gap Δ, can be dressed by the diquark condensate ϕ to produce a baryonic quasiparticle with the schematic structure $q\phi$. But the condensate is periodic, by construction, so this cannot affect the holonomy induced $\mathbb{Z}_3$ phase experienced by the quark. Equivalently, the condensate (12a) can be written as a product of $U(1)$ and $SU(3)$ factors; at large distance $\phi/\Delta \sim e^{i\theta/3} \times \mathrm{diag}[e^{2i\theta/3}, e^{-i\theta/3}, e^{-i\theta/3}]$. The angular variation of the $SU(3)$ factor cancels the color holonomy of the quark—but the $U(1)$ factor reinstates exactly the same overall phase change.

This $\mathbb{Z}_3$ phase implies that a $q\phi$ excitation has anyonic statistics with superfluid vortices. Just as in BCS superconductors [30], one can also interpret the holonomy induced phase $\alpha \equiv 2\pi/3$ as a shift in the allowed values of (orbital) angular momentum of this excitation in the presence of a vortex, $\alpha = 2\pi\Delta L_z/\hbar$, so $\Delta L_z = \hbar/3$. More precisely, for excitations far from the vortex core it is the azimuthal component of the "kinetic" or "covariant" angular momentum, $L_z \equiv \hat{z}\cdot\vec{r}\times(\vec{p}-\vec{A})$, which is fractionalized. This differs from the conserved angular momentum, or generator of rotations, which remains integer quantized. However, it is the kinetic angular momentum L_z which appears in the relation between angular velocity and angular momentum, $d\theta/dt = L_z/(mr_\perp^2)$, or equivalently in the rotational component of kinetic energy, $L_z^2/(2mr_\perp^2)$. In other words, a $q\phi$ excitation far from the vortex moves as if it is a free particle (in the absence of any gauge holonomy) with fractional angular momentum, $L_z/\hbar \in \mathbb{Z}+1/3$.

TABLE I. Shifts in orbital angular momentum, ΔL_z, of low energy excitations in the CFL phase in the presence of a minimal superfluid vortex. Here q denotes a quark excited above the Fermi surface, $\bar{q}$ denotes a corresponding hole, and ϕ is the diquark condensate. Each indicated combination can form a physical, gauge invariant excitation.

ΔL_z	0	$+\hbar/3$	$-\hbar/3$
Bosons	$q\bar{q}$	$\bar{q}\,\bar{q}\,\phi$	$qq\phi^*$
Fermions	qqq, $\bar{q}\,\bar{q}\,\bar{q}$	$q\phi$	$\bar{q}\phi^*$

Table I lists the analogous shifts in L_z for other possible quasiparticles in CFL color superconductors. Note that these considerations imply the existence of sharply distinct classes of both baryonic and mesonic excitations in the presence of a vortex.

VII. TOPOLOGY IN EFFECTIVE THEORY

We obtained the $\mathbb{Z}_3$ braiding phases (13) using the microscopic degrees of freedom of the QCD Lagrangian. However, the topological data in these braiding phases must also be encoded in any correct long distance description of the system. Since these braiding phases are insensitive to a bare quark mass which gaps out other NGBs (pions) associated with chiral symmetry breaking, this data must appear in the minimal effective field theory (EFT) describing the dynamics of $U(1)_B$ Nambu-Goldstone bosons and the response of various probes to superfluid fluctuations. The conventional effective action (7) cannot reproduce these braiding phases, and thus cannot be complete. What must be added to the EFT to fix this problem?

The correct EFT must reproduce the holonomy-vortex linking relation (13). When considering the theory on a nonsimply connected space (such as a toroidal compactification), it should also reproduce correct $\mathbb{Z}_3$-valued holonomies when there is superfluid flow around a noncontractible cycle. To construct such a theory, a natural starting point is a 4D topological quantum field theory (TQFT) known as BF theory [31–34]. This TQFT has the Euclidean action

$$S_{\mathrm{BF}} = \frac{ip}{2\pi}\int_{M_4} b_2 \wedge da, \tag{14}$$

where M_4 is a four-dimensional spacetime manifold, b_2 is a 2-form gauge field,[7] a is a 1-form gauge field, d denotes exterior differentiation, and $p \in \mathbb{Z}$. By itself, this action describes a $\mathbb{Z}_p$ discrete gauge theory [33,34]. We wish to relate holonomies of the gauge field a to the superfluid circulation. For holonomies encircling vortices, this may be accomplished by adding the term $\frac{1}{2\pi i}\int_{M_4} b_2 \wedge d^2\varphi$ to S_{BF}. The two-form $d^2\varphi$ is proportional to the superfluid vorticity and vanishes everywhere except at a vortex center where φ is ill defined. The resulting equation of motion, $pda = d^2\varphi$, connects the Abelian field strength da to the vorticity. By Stokes theorem, this is the same as equating the circulation $\oint_C d\varphi$ with p times the holonomy phase

[7] A p-form field is a totally antisymmetric rank-p tensor, and may be integrated over any dimension p manifold without needing a metric.

$\oint_C a$. For $p = 3$, this is precisely our linking number relation (13).

These considerations suggest that a proper low energy effective action for high-density QCD should be

$$S_{\rm eff} \stackrel{?}{=} \int_{M_4} \frac{f^2}{2} d\varphi \wedge \star d\varphi + \frac{i}{2\pi} \int_{M_4} b_2 \wedge (p da - d^2\varphi), \qquad (15)$$

where $\star$ denotes the Hodge dual and we chose units where $v_s = 1$. But this effective theory is not fully correct, as it cannot reproduce the correlation between holonomies and superfluid flow when spacetime is not simply connected, as in our earlier example of $\mathbb{R}^3 \times S^1$. We need an effective action that can connect integrals of a and $d\varphi$ around noncontractible curves. However, this is impossible with a local gauge-invariant 4D effective action.

This problem has a natural solution inspired by the Wess-Zumino-Witten term of chiral perturbation theory [35,36]. Let $M_5 = M_4 \times [0, 1]$ denote the 5D manifold which is the product of M_4 times an interval, and let $w \in [0, 1]$ be a coordinate along this interval. Regard the $w = 1$ boundary as the physical 4D spacetime and extend the superfluid condensate phase φ to a function $\tilde{\varphi}$ on M_5 (i.e., a mapping of M_5 into S^1) which coincides with φ at $w = 1$, is constant at $w = 0$, and is smooth and differentiable almost everywhere in the interior of M_5. When the phase φ winds around a physical vortex in M_4, the vortex worldsheet in M_4 will extend to a 3D world volume in M_5 on which $\tilde{\varphi}$ is ill defined (and around which $\tilde{\varphi}$ has nonzero winding). If φ has winding around some noncontractible cycle in M_4, then $\tilde{\varphi}$ will necessarily be ill defined on some 3D "vortex world volume" in the interior of M_5. (If M_4 is closed, this 3-surface is also closed.) We now replace the incomplete effective theory (15) with a 5D BF theory coupled to the vorticity of $\tilde{\varphi}$,

$$S_{\rm eff} = \int_{M_4} \frac{f^2}{2} d\varphi \wedge \star d\varphi + \frac{i}{2\pi} \int_{M_5} b_3 \wedge (p da - d^2\tilde{\varphi}). \qquad (16)$$

Here b_3 is a 3-form gauge field which is required to vanish at $w = 1$, while the gauge field a is required to vanish at $w = 0$.[8] The 5D equations of motion require that $p da = d^2\tilde{\varphi}$ and this, once again, implies that the circulation $\oint_C d\varphi$ around any closed curve C in M_5 coincide with p times the holonomy phase $\oint_C a$. And, importantly, this now includes noncontractible curves lying on the boundary at $w = 1$.

Consequently, the effective theory (16), involving a 5D topological term coupled to a 4D superfluid effective action, correctly reproduces the association between the $\mathbb{Z}_3$ holonomy and superfluid circulation in all geometries. The value of writing down Eq. (16) is simply to establish that it is possible to construct an effective action describing the dynamical $U(1)_B$ Goldstone boson field φ and some emergent color-singlet fields b and a, whose own dynamics are essentially trivial. The role of these emergent fields, which enter the new 5D term is simply to encode the nontrivial gauge-invariant holonomies of the color gauge fields which we found using a more microscopic description of the theory.

VIII. IMPLICATIONS

If the quark-hadron continuity conjecture of Ref. [9] is correct, then $\mathbb{Z}_3$ particle-vortex braiding phases and associated angular momentum fractionalization must persist as the density decreases all the way down to the onset of superfluidity in nuclear matter, at least sufficiently close to the $SU(3)$ flavor symmetric limit. This would imply that hadronic nuclear matter *cannot* be accurately described by a local effective theory involving colorless baryon and meson degrees of freedom, which is very hard to believe. Such a result would not be consistent with the standard picture that in low density nuclear matter, test quarks are screened by pair production of quark-antiquark pairs from the vacuum. The baryon-number violating condensate in nuclear matter cannot screen a test color charge, because this condensate has the quantum numbers of a dibaryon and is color-neutral. This should be contrasted with the situation in dense quark matter, where a diquark condensate can screen test quarks, leading to the explicit results described above concerning color holonomies that encircle superfluid vortices and associated fractionalization of orbital angular momentum.

The most plausible interpretation of our results is that quark-hadron continuity fails, with at least one phase transition separating high density quark matter, with its $\mathbb{Z}_3$ topological features, from lower density superfluid nuclear matter lacking these features. To understand why a discrete change in particle-vortex braiding statistics implies a thermodynamic phase transition, note that the ground state must have nonzero amplitudes for configurations in which vortex loops are present. Any such loop affects the quasiparticle spectrum in its vicinity, and hence affects fluctuation corrections of the ground state energy. Consequently, an abrupt change in particle-vortex braiding statistics should generate nonanalyticity in the ground state energy.

We emphasize that such a phase transition in dense nuclear matter does not conflict with the well-known Fradkin-Shenker results on Higgs-confinement complementarity [37], which demonstrate continuity between confinement and Higgs regimes in systems *without* a $U(1)$ global symmetry. In contrast, the presence of spontaneously broken $U(1)_B$ symmetry and the consequent

[8] The action (16) shifts by integer multiples of $2\pi i p$ under the gauge transformations $b \to b + d\lambda_{(2)}$ and $a \to a + d\lambda_{(0)}$, where $\lambda_{(2)}$ and $\lambda_{(0)}$ are 2-form and 0-form gauge functions constrained to vanish at $w = 1$ and at $w = 0$ respectively. Gauge invariance of the functional integral then implies that $p \in \mathbb{Z}$.

existence of superfluid vortex excitations plays a central role in our discussion of particle-vortex statistics.

Finally, we emphasize we have assumed coinciding quark masses and $SU(3)_V$ flavor symmetry throughout this paper. This flavor-symmetric limit was the setting for the quark-hadron continuity conjecture of Ref. [9]. Of course, the real world has distinct quark masses, plus electromagnetic and weak interactions. For asymptotically large μ, such flavor-breaking effects are negligible [5], but as μ decreases flavor symmetry breaking effects grow in importance. A detailed analysis of flavor-breaking effects is undoubtedly interesting, but is left to future work. Aside from the practical consideration of allowing a discussion of the physical point, an analysis of flavor symmetry breaking perturbations is also important to address theoretical points of principle. For example, as it stands, it is tempting to *define* the density-driven phase transition between nuclear matter and quark matter discussed above as a color confinement-deconfinement transition. But to make such a definition reasonable one would want to be sure that the transition persists for a range of flavor-symmetry breaking perturbations.

IX. OUTLOOK

The recognition that high density quark matter exhibits nontrivial topological features such as $\mathbb{Z}_3$ particle-vortex braiding statistics, angular momentum fractionalization, and an emergent $\mathbb{Z}_3$ discrete gauge field, shows that color superconductivity has novel physical signatures which are distinct from the more general phenomenon of ordinary superfluidity. This realization leads to numerous additional questions. What are observable signatures of angular momentum fractionalization around vortices? How do $\mathbb{Z}_3$ particle-vortex statistics affect quasiparticle dynamics and transport processes in rotating quark matter? If quark-hadron continuity fails, what is the nature of the phase transition(s) which separate these regimes? How do superfluid vortices in a rotating neutron star behave at such a phase interface [29]? Are there superfluids in condensed matter systems with similar particle-vortex statistics? We hope future work can shed light on these and related questions.

Each of the points in this chain of assertions is problematic. The authors of Ref. [38] begin with a construction of a purported effective theory which is based on a gauge-dependent analysis ignoring all gauge field components except those in a Cartan subgroup, and then treat individual components of the (untraced) holonomy as if they are physical observables. We fail to understand how this construction can make any sense. They then argue that their effective theory has an "emergent" $\mathbb{Z}_3$ two-form symmetry, and assert that "if QCD CFL is a superfluid phase with topological order there should be an emergent higher form symmetry, and it has to be spontaneously broken." We emphasize that at no point have we argued that high density QCD has topological order as conventionally defined (involving topology-dependent ground state degeneracy). In Sec. IV we explicitly discuss how the presence of equilibrium states with nonvanishing circulation and nontrivial holonomy around a close cycle is *not* topological order. Hence the entire supposition of Ref. [38] concerning genuine topological order and associated higher form symmetries is irrelevant.

Nevertheless, Ref. [38] then argues that their putative $\mathbb{Z}_3$ two-form symmetry cannot be spontaneously broken by first noting that a $\mathbb{Z}_3$ symmetry is a subgroup of an "emergent $U(1)$ symmetry" and then referencing the Mermin-Wagner theorem on the impossibility of spontaneously breaking continuous global symmetries in two or fewer dimensions. Once again, we fail to see the logic of this assertion because no $U(1)$ higher form symmetry of QCD was identified. Even when taken at face value, Ref. [38] only claims to identify such a symmetry approximately, as an emergent symmetry in the infrared, and only in the Higgs regime; no argument is given that the purported emergent symmetry is in fact a feature of full QCD.

Finally, it must be emphasized that nothing in Ref. [38] casts any doubt on, or even addresses, our explicit evaluation of nontrivial $\mathbb{Z}_3$ holonomies associated with superfluid circulation in high density QCD, the resulting physical effects discussed above, or the expectation that such topological features cannot be present in lower density hadronic matter.

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Note added.—After our work was posted on the arXiv, the paper [38] appeared, arguing that quark-hadron continuity is consistent with an unbroken "emergent two-form symmetry." We disagree with a number of assertions in this paper and its conclusions. However, nothing in Ref. [38] casts any doubt on, or even addresses, our explicit evaluation of nontrivial $\mathbb{Z}_3$ holonomies associated with superfluid circulation in high density QCD and the resulting physical effects discussed above.

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