Gauge invariant renormalizability of quantum gravity

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Using the Batalin-Vilkovisky technique and the background field method the proof of gauge invariant renormalizability is elaborated for a generic model of quantum gravity which is diffeomorphism invariant and has no other, potentially anomalous, symmetries. The gauge invariant renormalizability means that in all orders of loop expansion of the quantum effective action one can control deformations of the generators of gauge transformations which leave such an action invariant. In quantum gravity this means that one can maintain general covariance of the divergent part of effective action when the mean quantum field, ghosts, and antifields are switched off.

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I. INTRODUCTION

Renormalization is one of the main issues in quantum gravity. The traditional view on the difficulty of quantizing gravitational field is that the quantum general relativity is not renormalizable, while the renormalizable version of the theory includes fourth derivatives [1] and therefore it is not unitary. In the last decades this simple two-side story was getting more complicated, with the new models of superrenormalizable gravity, both polynomial [2] and nonpolynomial [3] (see also earlier papers [4,5]). Typically, these models intend to resolve the conflict between nonrenormalizability and nonunitarity by introducing more than four derivatives.

The main advantage of the nonpolynomial models is that the tree level propagator may have the unique physical pole corresponding to massless graviton. At the same time the dressed propagator has, typically, an infinite (countable) amount of the ghostlike states with complex poles [6] and hence the questions about physical contents and quantum consistency of such a theory remains open, especially taking into account the problems with reflection positivity [7] (see further discussion in [8]). It might happen that the construction of a consistent version of quantum gravity should not go through the S-matrix approach, since the flat limit and hence well-defined asymptotic states may not exist for the theories of gravity which are consistent even at the semiclassical level [9]. In this case the central question related to ghosts is the stability of the physically relevant classical solutions, and there are positive indications for the nonlocal models in this respect [10].

On the other hand, within the polynomial model one can prove the unitarity of the S-matrix within the Lee-Wick approach [11] to quantum gravity in four [12] and even higher dimensional space-times [13]. Furthermore, it is possible to make explicit one-loop calculations [14] which provide exact beta-functions in these theories due to the superrenormalizability of the theory. In the part of stability, the existing investigations concerned special backgrounds, namely cosmological [15,16] and black hole cases [17–19]. While the black hole results are not conclusive, the results for the cosmological backgrounds provide good intuitive understanding of the problem of stability in the gravity models with higher derivative ghosts.

Independent on the efforts in better understanding the role of ghosts and instabilities in both polynomial and nonpolynomial models, it would be useful to have a formal proof that these theories are renormalizable or superrenormalizable. The existing proofs concern only fourth derivative quantum gravity [1] (see also Refs. [20,21] as an application of a general approach [22]). In the present work we present the proof of a gauge invariant renormalizability in the general models of quantum gravity, which includes second derivative and higher derivative, polynomial and nonpolynomial models. The preliminary condition for the consideration which is given in the present paper is that

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there should be regularization which preserves the symmetries of the classical action. Thus our consideration cannot be directly applied to the models with conformal or chiral symmetries where one can expect to meet the corresponding anomalies. The consideration is based on the Batalin-Vilkovisky formalism, which enables one to analyse Becchi-Rouet-Stora-Tyutin (BRST) invariant renormalization of a wide class of gauge theories (including quantum gravity) without going into the details of a quantum gravity model, but using only the general structure of gauge algebra. The Batalin-Vilkovisky formalism use an algebraic approach to construct solutions of the master equation for different types of generating functionals of Green functions. In the present work we apply this formalism to establish the general structure of extended action and renormalized effective action for a quantum gravity model of a very general form within the background field formalism.

The paper is organized as follows. In Sec. II we formulate the Batalin-Vilkovisky formalism combined with the background field method in the case of quantum gravity. In Sec. III this formalism is applied to the formal proof of renormalizability in the model of quantum gravity of the general form. On the top of that we use the same formalism to briefly discuss the gauge fixing independence of the S-matrix of gravitational excitations in the theories of quantum gravity. Section IV discuss the renormalization of multiloop diagrams in the general type models of quantum gravity. The difference with the subsequent analysis of the power counting is that for the subdiagrams one has to keep the mean fields of the quantum metric, ghosts and auxiliary field, while for the power counting the mean fields can be omitted. Section V consists of the brief review of a power counting in quantum gravity, which enables one to classify the nonrenormalizable, renormalizable, and superrenormalizable models. Taking into account the contents of the previous sections, this classification is now based on a more solid background and we decided to include it here. Finally, in Sec. VI we draw our conclusions.

Condensed DeWitt's notations [23] are used in the paper. Right and left derivatives of a quantity f with respect to the variable φ are denoted as $\frac{\delta_r f}{\delta\varphi}$ and $\frac{\delta_l f}{\delta\varphi}$, correspondingly. The Grassmann parity and the ghost number of a quantity A are denoted by $\varepsilon(A)$ and $\mathrm{gh}(A)$, see, e.g., Eq. (23) in the last case. The condensed notation for the space-time integral in D dimensions, $\int dx = \int d^D x$ is used throughout the text.

II. QUANTUM GRAVITY IN THE BACKGROUND FIELD FORMALISM

Our starting point is an arbitrary action of a Riemann's metric, $S_0 = S_0(g)$, where $g = \{g_{\mu\nu}(x)\}$. The action is assumed invariant under the general coordinate transformations,

$$x'^\mu = f^\mu(x) \to x^\mu = x^\mu(x'),$$
$$g_{\mu\nu} \to g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}. \tag{1}$$

The standard examples of the theories of our interest are Einstein gravity with a cosmological constant term,

$$S_{\mathrm{EH}}(g) = -\frac{1}{\kappa^2}\int dx\sqrt{-g}(R + 2\Lambda) \tag{2}$$

and a general version of higher derivative gravity,

$$S(g) = S_{\mathrm{EH}}(g) + \int dx\sqrt{-g}\{R^{\mu\nu\alpha\beta}\Pi_1(\Box/M^2)R_{\mu\nu\alpha\beta} + R^{\mu\nu}\Pi_2(\Box/M^2)R_{\mu\nu} + R\Pi_3(\Box/M^2)R + \mathcal{O}(R^3_{\dots})\}, \tag{3}$$

where $\Pi_{1,2,3}$ are polynomial or nonpolynomial form factors and the last term represents nonquadratic in curvature terms. In quantum theory the action (3) may lead to the theory which is nonrenormalizable, renormalizable, or even superrenormalizable, depending on the choice of the functions $\Pi_{1,2,3}(x)$ and the nonquadratic terms.

The parameter M^2 in the form factors $\Pi_{1,2,3}(\Box/M^2)$ is a universal mass scale at which the quantum gravity effect becomes relevant. For instance, it can be the square of the Planck mass, but there may be other options, including multiple scale models, as analyzed in [24]. For the analysis presented below the unique necessary feature is that the action should be diffeomorphism invariant.

In the infinitesimal form the transformations (1) read

$$x'^\mu = x^\mu + \xi^\mu(x) \to x^\mu = x'^\mu - \xi^\mu(x'),$$
$$g_{\mu\nu} \to g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x), \tag{4}$$

where

$$\delta g_{\mu\nu}(x) = -\xi^\sigma(x)\partial_\sigma g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_\nu\xi^\sigma(x) - g_{\sigma\nu}(x)\partial_\mu\xi^\sigma(x). \tag{5}$$

The invariance of the action $S_0(g)$ under the transformations (5) can be expressed in the form of the Noether identity

$$\int dx\frac{\delta S_0(g)}{\delta g_{\mu\nu}(x)}\delta g_{\mu\nu}(x) = 0. \tag{6}$$

In what follows we will also need the transformation rule for vector fields $A_\mu(x)$ and $A^\mu(x)$,

$$\delta A_\mu(x) = -\xi^\sigma(x)\partial_\sigma A_\mu(x) - A_\sigma(x)\partial_\mu\xi^\sigma(x), \tag{7}$$

$$\delta A^\mu(x) = -\xi^\sigma(x)\partial_\sigma A^\mu(x) + A^\sigma(x)\partial_\sigma\xi^\mu(x). \tag{8}$$

Let us present the transformations (5) in the form

$$\delta g_{\mu\nu}(x) = \int dy R_{\mu\nu\sigma}(x,y;g)\xi^{\sigma}(y), \tag{9}$$

where

$$R_{\mu\nu\sigma}(x,y;g) = -\delta(x-y)\partial_{\sigma} g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_{\nu}\delta(x-y) - g_{\sigma\nu}(x)\partial_{\mu}\delta(x-y) \tag{10}$$

are the generators of gauge transformations of the metric tensor $g_{\mu\nu}$ with gauge parameters $\xi^{\sigma}(x)$. The algebra of gauge transformations is defined by the algebra of generators, which has the following form:

$$\int du \left[\frac{\delta R_{\mu\nu\sigma}(x,y;g)}{\delta g_{\alpha\beta}(u)} R_{\alpha\beta\gamma}(u,z;g) - \frac{\delta R_{\mu\nu\gamma}(x,z;g)}{\delta g_{\alpha\beta}(u)} R_{\alpha\beta\sigma}(u,y;g)\right] = -\int du R_{\mu\nu\lambda}(x,u;g) F^{\lambda}_{\sigma\gamma}(u,y,z), \tag{11}$$

where

$$F^{\lambda}_{\alpha\beta}(x,y,z) = \delta(x-y)\delta^{\lambda}_{\beta}\frac{\partial}{\partial x^{\alpha}}\delta(x-z) - \delta(x-z)\delta^{\lambda}_{\alpha}\frac{\partial}{\partial x^{\beta}}\delta(x-y), \tag{12}$$

$$F^{\lambda}_{\alpha\beta}(x,y,z) = -F^{\lambda}_{\beta\alpha}(x,z,y) \tag{13}$$

are structure functions of the gauge algebra which do not depend on the metric tensor $g_{\mu\nu}$. Therefore, independent on the form of the action, any theory of gravity looks like a gauge theory with closed gauge algebra and structure functions independent on the fields (metric tensor, in the case), i.e., similar to the Yang-Mills theory.

It proves useful to perform quantization of gravity on an external background, represented by a metric tensor $\bar{g}_{\mu\nu}(x)$. In the simplest case the Riemann space may be just the Minkowski space-time with the metric tensor $\eta_{\mu\nu} = \text{const}$. On the other hand, introducing an arbitrary background metric provides serious advantages, as we shall see in what follows. The standard reference on the background field formalism in quantum field theory is [25–27] (see also recent advances for the gauge theories in [28–32]).

Within the background field method the metric tensor $g_{\mu\nu}(x)$ is replaced by the sum

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad \text{such that} \qquad S_0(g) \to S_0(\bar{g}+h). \tag{14}$$

Here $h_{\mu\nu}(x)$ is called quantum metric and is regarded as a set of integration variables in the functional integrals for generating functionals of Green functions.

The action $S_0(\bar{g}+h)$ is a functional of two variables $\bar{g}$ and h and therefore it has additional symmetries because of extra degrees of freedom. Namely, it is invariant under the following transformations

$$\delta\bar{g}_{\mu\nu} = \epsilon_{\mu\nu} \quad \text{and} \quad \delta h_{\mu\nu} = -\epsilon_{\mu\nu} \tag{15}$$

with arbitrary symmetric tensor functions $\epsilon_{\nu\mu} = \epsilon_{\mu\nu} = \epsilon_{\mu\nu}(x)$. In particular, this means that there is an ambiguity is defining the gauge transformations for $\bar{g}$ and h. To fix this arbitrariness we require that the transformation of our interest has the right flat limit when $\bar{g}_{\mu\nu}(x)$ is traded for $\eta_{\mu\nu}$. Then the gauge transformation of the quantum metric fields $h_{\mu\nu}$ in the presence of external (fixed) background $\bar{g}$ should have the form

$$\delta h_{\mu\nu}(x) = \int dy R_{\mu\nu\sigma}(x,y;\bar{g}+h)\xi^{\sigma}, \tag{16}$$

while $\delta\bar{g}_{\mu\nu}(x) = 0$ and the action remains invariant, $\delta S_0(\bar{g}+h) = 0$.

Because of the similarity with the Yang-Mills field, the Faddeev-Popov quantization procedure is quite standard and the resulting action $S_{FP} = S_{FP}(\phi,\bar{g})$ has the form [33]

$$S_{FP} = S_0(\bar{g}+h) + S_{gh}(\phi,\bar{g}) + S_{gf}(\phi,\bar{g}). \tag{17}$$

Taking into account the presence of an external background metric $\bar{g}$, the ghost action has the form

$$S_{gh}(\phi,\bar{g}) = \int dx dy dz \sqrt{-\bar{g}(x)}\bar{C}^{\alpha}(x) H^{\beta\gamma}_{\alpha}(x,y;\bar{g},h) \times R_{\beta\gamma\sigma}(y,z;\bar{g}+h) C^{\sigma}(z), \tag{18}$$

with the notation

$$H^{\beta\gamma}_{\alpha}(x,y;\bar{g},h) = \frac{\delta\chi_{\alpha}(x;\bar{g},h)}{\delta h_{\beta\gamma}(y)}. \tag{19}$$

The $S_{gf}(\bar{g},h)$ is the gauge fixing action

$$S_{gf}(\phi,\bar{g}) = \int dx \sqrt{-\bar{g}(x)} B^{\alpha}(x)\chi_{\alpha}(x;\bar{g},h). \tag{20}$$

which corresponds to the singular gauge condition. For the nonsingular gauge condition the action has the form

$$S_{gf}(\phi,\bar{g}) = \int dx \sqrt{-\bar{g}(x)} \times \left[B^{\alpha}(x)\chi_{\alpha}(x;\bar{g},h) + \frac{1}{2}B^{\alpha}(x)\bar{g}_{\alpha\beta}(x)B^{\beta}(x)\right]. \tag{21}$$

In what follows we shall use the form (20), where $\chi_\alpha(x;\bar{g},h)$ are the gauge fixing functions, which are called to remove the degeneracy of the action $S_0(\bar{g}+h)$.

Let us introduce an important notation

$$\phi = \{\phi^i\} = \{h_{\mu\nu}, B^\alpha, C^\alpha, \bar{C}^\alpha\} \tag{22}$$

for the full set of quantum fields including quantum metric, Faddeev-Popov ghost, antighost and the Nakanishi-Lautrup auxiliary fields B^α. The Grassmann parity of these fields will be denoted as $\varepsilon(\phi^i) = \varepsilon_i$, such that for ghost and antighost $\varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = 1$, while for the auxiliary fields B^α and metric $\varepsilon(B^\alpha) = \varepsilon(h_{\mu\nu}) = 0$.

The conserved quantity called ghost number is defined for the same fields as

$$\mathrm{gh}(C^\alpha) = 1, \quad \mathrm{gh}(\bar{C}^\alpha) = -1 \quad \text{and} \quad \mathrm{gh}(B^\alpha) = \mathrm{gh}(h_{\mu\nu}) = 0. \tag{23}$$

For any admissible choice of gauge fixing functions $\chi_\alpha(x;\bar{g},h)$ action (17) is invariant under global supersymmetry (BRST symmetry) [34,35],[1]

$$\delta_B h_{\mu\nu}(x) = \int dy R_{\mu\nu\alpha}(x,y;\bar{g}+h) C^\alpha(y)\mu, \quad \delta_B B^\alpha(x) = 0,$$
$$\delta_B C^\alpha(x) = -C^\sigma(x)\partial_\sigma C^\alpha(x)\mu, \quad \delta_B \bar{C}^\alpha(x) = B^\alpha(x)\mu, \tag{24}$$

where μ is a constant Grassmann parameter. Let us present the BRST transformations (24) in the form

$$\delta_B \phi^i(x) = R^i(x;\phi,\bar{g})\mu, \quad \varepsilon(R^i(x;\phi,\bar{g})) = \varepsilon_i + 1, \tag{25}$$

where $R^i = \{R^{(h)}_{\mu\nu}, R^\alpha_{(B)}, R^\alpha_{(C)}, R^\alpha_{(\bar{C})}\}$ and

$$\begin{aligned} R^{(h)}_{\mu\nu}(x;\phi,\bar{g}) &= \int dy R_{\mu\nu\sigma}(x,y;\bar{g}+h) C^\sigma(y), \\ R^\alpha_{(B)}(x;\phi,\bar{g}) &= 0, \\ R^\alpha_{(C)}(x;\phi,\bar{g}) &= -C^\sigma(x)\partial_\sigma C^\alpha(x), \\ R^\alpha_{(\bar{C})}(x;\phi,\bar{g}) &= B^\alpha(x). \end{aligned} \tag{26}$$

Then the BRST invariance of the action S_{FP} reads

$$\int dx \frac{\delta_r S_{FP}}{\delta\phi^i(x)} R^i(x;\phi,\bar{g}) = 0. \tag{27}$$

The invariance property (27) can be expressed in a compact and useful form called Zinn-Justin equation, by introducing the set of additional variables $\phi^*_i(x)$. The new fields have Grassmann parities opposite to the corresponding fields $\phi^i(x)$, namely $\varepsilon(\phi^*_i) = \varepsilon_i + 1$.

The extended action $S = S(\phi, \phi^*, \bar{g})$ reads

$$S = S_{FP} + \int dx \phi^*_i(x) R^i(x;\phi,\bar{g}). \tag{28}$$

It easy to note that the new variables $\phi^*_i(x)$ serve as the sources to BRST generators (26). Then the relation (27) takes the standard form of the Zinn-Justin equation [38] for the action (28),

$$\int dx \frac{\delta_r S}{\delta\phi^i(x)} \frac{\delta_l S}{\delta\phi^*_i(x)} = 0, \tag{29}$$

One can note that using left and right derivatives in the last equation is relevant due to the nontrivial Grassmann parities of the involved quantities.

According to the terminology of Batalin-Vilkovisky formalism [39,40] the sources $\phi^*_i(x)$ are known as antifields. The fundamental notion in the Batalin-Vilkovisky formalism is the antibracket for two arbitrary functionals of fields and antifields, $F = F(\phi,\phi^*)$ and $G = G(\phi,\phi^*)$. The antibracket is defined as

$$(F,G) = \int dx \left[\frac{\delta_r F}{\delta\phi^i(x)} \frac{\delta_l G}{\delta\phi^*_i(x)} - \frac{\delta_r F}{\delta\phi^*_i(x)} \frac{\delta_l G}{\delta\phi^i(x)} \right], \tag{30}$$

which obeys the following properties:

(1) Grassmann parity relations

$$\varepsilon((F,G)) = \varepsilon(F) + \varepsilon(G) + 1 = \varepsilon((G,F)); \tag{31}$$

(2) Generalized antisymmetry

$$(F,G) = -(G,F)(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}; \tag{32}$$

(3) Leibniz rule

$$(F,GH) = (F,G)H + (F,H)G(-1)^{\varepsilon(G)\varepsilon(H)}; \tag{33}$$

(4) Generalized Jacobi identity

$$((F,G),H)(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \mathrm{cycle}(F,G,H) \equiv 0. \tag{34}$$

In terms of antibracket Eq. (29) can be written in a compact form,

$$(S,S) = 0, \tag{35}$$

which is the classical master equation of Batalin-Vilkovisky formalism [39,40]. This equation will be generalized to the quantum domain and extensively used

[1]The gravitational BRST transformations were introduced in [1,36,37].

to analyze renormalizability of quantum gravity in the next section.

Now we are in a position to formulate the quantum theory. The generating functional of Green functions is defined in the form of functional integral[2]

$$Z(J,\bar{g}) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{FP}(\phi,\bar{g}) + J\phi]\right\} = \exp\left\{\frac{i}{\hbar}W(J,\bar{g})\right\}, \tag{36}$$

where $W(J,\bar{g})$ is the generating functional of connected Green functions. In (36) the DeWitt notations are used, namely

$$J\phi = \int dx J_i(x)\phi^i(x), \quad \text{where} \quad J_i(x) = \{J^{\mu\nu}(x), J^{(B)}_\alpha(x), \bar{J}_\alpha(x), J_\alpha(x)\} \tag{37}$$

are external sources for the fields (22). The Grassmann parities and ghost numbers of these sources satisfy the relations

$$\varepsilon(J_i) = \varepsilon(\phi^i), \qquad \mathrm{gh}(J_i) = \mathrm{gh}(\phi^i). \tag{38}$$

Let us a detailed consideration of the generating functionals and their gauge dependence. As a first step, consider the vacuum functional $Z_\Psi(\bar{g})$, which corresponds to the choice of gauge fixing functional (27) in the presence of external fields $\bar{g}$,

$$Z_\Psi(\bar{g}) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_0(\bar{g}+h) + \Psi(\phi,\bar{g})\hat{R}(\phi,\bar{g})]\right\} = \exp\left\{\frac{i}{\hbar}W_\Psi(\bar{g})\right\}, \tag{39}$$

where we introduced the operator

$$\hat{R}(\phi,\bar{g}) = \int dx \frac{\delta_r}{\delta\phi^i(x)} R^i(x;\phi,\bar{g}) \tag{40}$$

and $\Psi(\phi,\bar{g})$ is the fermionic gauge fixing functional,

$$\Psi(\phi,\bar{g}) = \int dx \sqrt{-\bar{g}(x)}\bar{C}^\alpha \chi_\alpha(x;\bar{g},h). \tag{41}$$

Taking into account (40) and (41), the definition (39) becomes an expression

$$Z_\Psi(\bar{g}) = \int d\phi \exp\left\{\frac{i}{\hbar}S_{FP}(\phi,\bar{g})\right\}, \tag{42}$$

which is nothing but (36) without the source term in the exponential.

In order to take care about possible change of the gauge fixing, let $Z_{\Psi+\delta\Psi}$ be the modified vacuum functional corresponding to $\Psi(\phi,\bar{g}) + \delta\Psi(\phi,\bar{g})$, where $\delta\Psi(\phi,\bar{g})$ is an arbitrary infinitesimal functional with odd Grassmann parity. Besides from this requirement, $\delta\Psi(\phi,\bar{g})$ can be arbitrary, in particular it may be different from Eq. (41).

Taking into account (42), with the new term we get

$$Z_{\Psi+\delta\Psi}(\bar{g}) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{FP}(\phi,\bar{g}) + \delta\Psi(\phi,\bar{g})\hat{R}(\phi,\bar{g})]\right\}. \tag{43}$$

The next step is to make the change of variables ϕ^i in the form of BRST transformations (24) but with replacement of the constant parameter μ by a functional $\mu = \mu(\phi,\bar{g})$,

$$\phi^i(x) \to \phi'^i(x) = \phi^i(x) + R^i(x;\phi,\bar{g})\mu(\phi,\bar{g}) = \phi^i(x) + \Delta\phi^i(x). \tag{44}$$

In what follows we shall use short notations $R^i(x;\phi,\bar{g}) = R^i(x)$ and $\mu(\phi,\bar{g}) = \mu$. Due to the linearity of BRST transformations, action $S_{FP}(\phi,\bar{g})$ is invariant under (44) even for the nonconstant μ. It is easy to check that the Jacobian of transformations (44) reads [41][3]

$$J = J(\phi,\bar{g}) = \exp\left\{\int dx (-1)^{\varepsilon_i} M^i_i(x,x)\right\}, \tag{45}$$

where matrix $M^i_j(x,y)$ has the form

$$M^i_j(x,y) = \frac{\delta_r \Delta\phi^i(x)}{\delta\phi^j(y)} = (-1)^{\varepsilon_j+1}\frac{\delta_r\mu}{\delta\phi^j(y)}R^i(x) - (-1)^{\varepsilon_j(\varepsilon_i+1)}\frac{\delta_l R^i(x)}{\delta\phi^j(y)}\mu. \tag{46}$$

In Yang-Mills type theories due to antisymmetry properties of structure constants the following relation

$$\int dx (-1)^{\varepsilon_i}\frac{\delta_l R^i(x)}{\delta\phi^i(x)} = 0 \tag{47}$$

holds. Then from (45) and (46) it follows that

[2]Let us note that for exploring gauge invariance of renormalization we need to introduce a more general object $Z(J,\phi^*,\bar{g})$ which also depends on the set of antifields ϕ^*. This extended definition will be given below.

[3]Note that the Jacobian of the transformations (44) can be calculated exactly [42,43].

$$J = \exp\{-\mu(\phi,\bar{g})\hat{R}(\phi,\bar{g})\}. \tag{48}$$

Choosing the functional μ in the form

$$\mu = \frac{i}{\hbar}\delta\Psi(\phi,\bar{g}), \tag{49}$$

one can observe that the described change of variables in the functional integral completely compensates the modification in the expression (43) compared to the fiducial formula (42). Thus we arrive at the gauge independence of the vacuum functional

$$Z_\Psi(\bar{g}) = Z_{\Psi+\delta\Psi}(\bar{g}). \tag{50}$$

One can present this identity as vanishing variations of the vacuum functionals Z and W,

$$\delta_\Psi Z(\bar{g}) = 0 \Rightarrow \delta_\Psi W(\bar{g}) = 0. \tag{51}$$

Due to the invariance feature (50) we can omit the label Ψ in the definition of the generating functionals (36). Furthermore, it is known that due to the equivalence theorem [44] the invariance (50) implies that if the background metric $\bar{g}_{\mu\nu}$ admits asymptotic states (e.g., if it is a flat Minkowski metric), then the S-matrix in the theory of quantum gravity does not depend on the gauge fixing. It is remarkable that we can make this statement for an arbitrary model of quantum gravity (QG), even without requiring the locality of the classical action. One can say that if the theory admits the construction of the S-matrix, the last will be independent on the choice of the gauge fixing conditions. Let us note that this is true only within the conventional perturbative approach to quantum field theory, while the situation may be opposite in other approaches. For instance, the S-matrix is *not* invariant if it is constructed on the basis of the concept of average effective action related to functional renormalization group [45–48]. The corresponding proof for the Yang-Mills theory is based on the general result of Ref. [44] and can be found in Ref. [49]. We believe it can be directly generalized for the case of gravity. A similar situation takes place in the standard formulation of the Gribov-Zwanziger theory [50–52] when the corresponding effective action depends on the choice of gauge even on-shell [53,54]. This difficulty illustrates the situation which we meet when trying to go beyond the framework of perturbative field theory, that would be especially relevant in the case of quantum gravity.

The effective action $\Gamma(\Phi,\bar{g})$ is defined by means of Legendre transformation,

$$\Gamma(\Phi,\bar{g}) = W(J,\bar{g}) - J_i\Phi^i, \tag{52}$$

where $\Phi = \{\Phi^i\}$ are mean fields and J_i are the solutions of the equations

$$\frac{\delta W(J,\bar{g})}{\delta J_i} = \Phi^i \quad \text{and} \quad J_i\Phi^i = \int dx J_i(x)\Phi^i(x). \tag{53}$$

In terms of effective action the property (51) means the on-shell gauge fixing independence and reads

$$\delta_\Psi\Gamma(\Phi,\bar{g})\Big|_{\frac{\delta\Gamma(\Phi,\bar{g})}{\delta\Phi}=0} = 0, \tag{54}$$

i.e., the effective action evaluated on its extremal does not depend on gauge.

Until now we did not assume that the background metric may transform under the general coordinate transformation. This was a necessary approach, as it was explained after the definition of the splitting (14) of the metric into background and quantum parts. However, since effective action is defined, one can perform the coordinate transformation for the background metric $\bar{g}_{\mu\nu}$ together with the corresponding transformation for the quantum metric. It is important that this transformation does not lead either to the change of the form of the Faddeev-Popov action (17) or to the change of the transformation rules for the auxiliary and ghost fields.

Thus, consider a variation of the background metric under general coordinates transformations of external metric tensor $\bar{g}_{\mu\nu}$, treating it as a symmetric tensor, hence

$$\delta^{(c)}_\omega \bar{g}_{\mu\nu} = R_{\mu\nu\sigma}(\bar{g})\omega^\sigma. \tag{55}$$

The symbol (c) indicates that the transformation concerns the background metric, i.e., in the sector of classical fields.

In the quantum fields sector $h_{\mu\nu}$ the form of the transformations is fixed by the requirement of invariance of the action,

$$\begin{aligned}\delta^{(q)}_\omega h_{\mu\nu} &= R_{\mu\nu\sigma}(h)\omega^\sigma \\ &= -\omega^\sigma\partial_\sigma h_{\mu\nu} - h_{\mu\sigma}\partial_\nu\omega^\sigma - h_{\sigma\nu}\partial_\mu\omega^\sigma,\end{aligned} \tag{56}$$

where the symbol (q) indicates the gauge transformations in the sector of quantum fields. Then we have

$$\delta_\omega S_0(\bar{g}+h) = 0, \qquad \delta_\omega = (\delta^{(c)}_\omega + \delta^{(q)}_\omega). \tag{57}$$

With these definitions, for the variation of $Z(\bar{g})$ we have

$$\begin{aligned}\delta^{(c)}_\omega Z(\bar{g}) = \frac{i}{\hbar}\int d\phi[\delta^{(c)}_\omega S_0(\bar{g}+h) + \delta^{(c)}_\omega S_{gh}(\phi,\bar{g}) \\ + \delta^{(c)}_\omega S_{gf}(\phi,\bar{g})]\exp\left\{\frac{i}{\hbar}S_{FP}(\phi,\bar{g})\right\}.\end{aligned} \tag{58}$$

Let us stress that here we consider the transformations of $\bar{g}$ only, that is why the $\delta^{(q)}$ does not enter into the last expression.

Using a change of variables in the functional integral (58) one can try to arrive at the relation $\delta^{(c)}_\omega Z(\bar{g}) = 0$ to prove invariance of $Z(\bar{g})$ under the transformations (55).

In the analysis of the gauge fixing action $S_{gf}(\phi, \bar{g})$ we can use that this action depends only on the three variables $h_{\mu\nu}$, B^α and $\bar{g}_{\mu\nu}$. Also, for the two of them, $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, the transformation law is already defined in (55) and (56). Thus, we need to define the transformation for the remaining field B^α. This unknown transformation rule $\delta^{(q)}_\omega B^\alpha$ should be chosen in such a way that it compensates the variation of $S_{gf}(\phi, \bar{g})$ caused by the transformations of $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$. Therefore, we have

$$\delta_\omega S_{gf} = \int dx \sqrt{-\bar{g}}[(\delta^{(q)}_\omega B^\alpha + \omega^\sigma \partial_\sigma B^\alpha)\chi_\alpha(\bar{g}, h) + B^\alpha \omega^\sigma \partial_\sigma \chi_\alpha(\bar{g}, h) + B^\alpha \delta_\omega \chi_\alpha(\bar{g}, h)]. \tag{59}$$

The gauge fixing functions χ_α are not independent, since they are constructed from the metric, which is transformed as a tensor, according to Eq. (55). Thus the variation of the gauge fixing functions χ_α has the form (7) for the vector fields,

$$\delta_\omega \chi_\alpha = -\omega^\sigma \partial_\sigma \chi_\alpha - \chi_\sigma \partial_\alpha \omega^\sigma. \tag{60}$$

The transformation of the auxiliary field B can be chosen by the covariance arguments, following the rule (8). This gives

$$\delta^{(q)}_\omega B^\alpha = -\omega^\sigma \partial_\sigma B^\alpha + B^\sigma \partial_\sigma \omega^\alpha \tag{61}$$

and provides the desired relation

$$\delta_\omega S_{gf} = 0. \tag{62}$$

In the same way one can analyze the variation of the ghost action and find its invariance,

$$\delta_\omega S_{gh} = 0, \tag{63}$$

for the following transformation laws for the ghost fields $\bar{C}^\alpha$ and C^α:

$$\begin{aligned} \delta^{(q)}_\omega \bar{C}^\alpha(x) &= -\omega^\sigma(x)\partial_\sigma \bar{C}^\alpha(x) + \bar{C}^\rho \partial_\rho \omega^\alpha(x), \\ \delta^{(q)}_\omega C^\alpha(x) &= -\omega^\sigma(x)\partial_\sigma C^\alpha(x) + C^\rho \partial_\rho \omega^\alpha(x). \end{aligned} \tag{64}$$

All in all, we conclude that the Faddeev-Popov action S_{FP} is invariant

$$\delta_\omega S_{FP} = 0 \tag{65}$$

under the new version of gauge transformations, which is based on the background transformations of all fields ϕ and $\bar{g}$ including (55), (56), (61), and (64).

As a consequence of (65), vacuum functional possesses gauge invariance too,

$$\delta_\omega Z(\bar{g}) = \delta^{(c)}_\omega Z(\bar{g}) = 0. \tag{66}$$

The same statement is automatically valid for the background effective action, that is the effective action with switched off mean fields Φ^i.

As we shall see in what follows, one can use Eq. (66) to prove the gauge invariance of an important object $\Gamma(\bar{g}) = \Gamma(\Phi = 0, \bar{g})$, that means

$$\delta^{(c)}_\omega \Gamma(\bar{g}) = 0. \tag{67}$$

Indeed, this relation is one of the main targets of our work. It shows that when the mean quantum fields $\Phi = \{h, C, \bar{C}, B\}$ are switched off (later on we shall see how this should be done), the remaining effective action of the background metric is covariant.

It is useful to start by exploring the gauge invariance property of generating functionals of our interest off-shell. To this end it is useful to present the background transformations (55), (56), (61), and (64) in the form

$$\delta^{(c)}_\omega \bar{g}_{\mu\nu} = R_{\mu\nu\sigma}(\bar{g})\omega^\sigma, \qquad \delta^{(q)}_\omega \phi^i = \mathcal{R}^i_\sigma(\phi)\omega^\sigma, \tag{68}$$

where the generators $\mathcal{R}^i_\sigma(\phi)$ are linear in the quantum fields ϕ and do not depend on the background metric $\bar{g}$. The general form of the transformation of an arbitrary functional [let it be $\Gamma = \Gamma(\phi, \bar{g})$] can be written in the form

$$\delta_\omega \Gamma = \delta^{(c)}_\omega \Gamma + \frac{\delta_r \Gamma}{\delta \phi^i} \mathcal{R}^i_\sigma(\phi)\omega^\sigma. \tag{69}$$

Consider the variation of the generating functional $Z(J, \bar{g})$ (36), under the gauge transformations of the background metric

$$\delta^{(c)}_\omega Z(J,\bar{g}) = \frac{i}{\hbar}\int d\phi \delta^{(c)}_\omega S_{FP}(\phi,\bar{g}) \exp\left\{\frac{i}{\hbar}[S_{FP}(\phi,\bar{g}) + J\phi]\right\}. \tag{70}$$

Using the background transformations in the sector of quantum fields ϕ and taking into account that for the linear change of variables the Jacobian of this transformation is independent on the fields, we arrive at the relation

$$\frac{i}{\hbar}\int d\phi \{\delta^{(q)}_\omega S_{FP}(\phi, \bar{g}) + J\delta^{(q)}_\omega \phi\} \exp\left\{\frac{i}{\hbar}[S_{FP}(\phi, \bar{g}) + J\phi]\right\} = 0. \tag{71}$$

On the other hand, from (65) and (71) it follows that

$$\delta^{(c)}_\omega Z(J,\bar{g}) = \frac{i}{\hbar}\int d\phi J_j \mathcal{R}^j_\sigma(\phi)\omega^\sigma \exp\left\{\frac{i}{\hbar}[S_{FP}(\phi,\bar{g})+J\phi]\right\}, \tag{72}$$

or

$$\delta^{(c)}_\omega Z(J,\bar{g}) = \frac{i}{\hbar}J_j\mathcal{R}^j_\sigma\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)Z(J,\bar{g})\omega^\sigma. \tag{73}$$

In terms of the generating functional of connected Green functions, $W = W(J,\bar{g}) = -i\hbar \ln Z(J,\bar{g})$, the relation (73) reads

$$\delta^{(c)}_\omega W(J,\bar{g}) = J_j\mathcal{R}^j_\sigma\left(\frac{\delta W}{\delta J}\right)\omega^\sigma, \tag{74}$$

where we used linearity of generators $\mathcal{R}^i_\sigma(\phi)$ with respect to ϕ.

Once again, consider the generating functional of vertex functions (effective action),

$$\Gamma = \Gamma(\Phi,\bar{g}) = W(J,\bar{g}) - J\Phi, \tag{75}$$

where

$$\Phi^j = \frac{\delta_l W}{\delta J_j}, \qquad \frac{\delta_r\Gamma}{\delta\Phi^j} = -J_j \quad \text{and} \quad \delta W = \delta\Gamma \tag{76}$$

under the variation of external metric and the mean fields (68). In terms of Γ the relation (74) becomes

$$\delta^{(c)}_\omega\Gamma(\Phi,\bar{g}) = -\frac{\delta_r\Gamma}{\delta\Phi^j}\mathcal{R}^j_\sigma(\Phi)\omega^\sigma, \tag{77}$$

or, using the identity (69), simply

$$\delta_\omega\Gamma(\Phi,\bar{g}) = 0 \tag{78}$$

if the variations of all variables (68) are taken into account.

It is important that the relations (77) and (78) serve as a proof of the fundamental property (67). In order to see this, one has to note that the generators of quantum fields (56), (64), and (61) have linear dependence of these fields. As a result one meets the following limit for the generators $\mathcal{R}^i_\sigma(\Phi)$ when the mean fields are switched off:

$$\lim_{\Phi\to 0}\mathcal{R}^j_\sigma(\Phi) = 0, \tag{79}$$

Thus the effective action Γ is invariant under nondeformed background transformations and repeats the invariance property of the Faddeev-Popov action S_{FP}.

Let us come back to formulating the instruments required for the proof of renormalizability. In the renormalization program based on Batalin-Vilkovisky formalism the extended action $S = S(\phi,\phi^*,\bar{g})$ (28) and corresponding extended generating functionals of Green functions $Z = Z(J,\phi^*,\bar{g})$, and of connected Green functions $W = W(J,\phi^*,\bar{g})$,

$$Z(J,\phi^*,\bar{g}) = \int d\phi \exp\left\{\frac{i}{\hbar}[S(\phi,\phi^*,\bar{g})+J\phi]\right\} = \exp\left\{\frac{i}{\hbar}W(J,\phi^*,\bar{g})\right\}, \tag{80}$$

play the role of precursor for the full effective action, which satisfies the quantum version of Eq. (35).

Due to the invariance of S_{FP} under background fields transformations, the variation of S takes the special form

$$\delta_\omega S(\phi,\phi^*,\bar{g}) = \phi^*_i\delta_\omega R^i(\phi,\bar{g}), \tag{81}$$

that shows that the action is gauge invariant on the hypersurface $\phi^*_i = 0$. The variations $\delta_\omega R^i(\phi,\bar{g})$ are quadratic in the sector of fields $h_{\mu\nu}$ and C^α and linear in the sector of field $\bar{C}^\alpha$. Using the condensed DeWitt's notation one can write the variations of the generators $\delta_\omega R^i(\phi,\bar{g})$ in the following compact form:

$$\begin{aligned}
\delta_\omega R^{(h)}_{\mu\nu}(\phi,\bar{g}) &= -\omega^\sigma\partial_\sigma R_{\mu\nu\lambda}(\bar{g}+h)C^\lambda - \partial_\mu\omega^\sigma R_{\sigma\nu\lambda}(\bar{g}+h)C^\lambda \\
&\quad - \partial_\nu\omega^\sigma R_{\mu\sigma\lambda}(\bar{g}+h)C^\lambda, \\
\delta_\omega R^\alpha_{(B)}(\phi,\bar{g}) &= 0, \\
\delta_\omega R^\alpha_{(C)}(\phi,\bar{g}) &= \omega^\sigma\partial_\sigma(C^\lambda\partial_\lambda C^\alpha) - C^\lambda\partial_\lambda C^\sigma\partial_\sigma\omega^\alpha, \\
\delta_\omega R^\alpha_{(\bar{C})}(\phi,\bar{g}) &= -\omega^\sigma\partial_\sigma B^\alpha + B^\sigma\partial_\sigma\omega^\alpha.
\end{aligned} \tag{82}$$

Let us now consider the variation of the extended generating functional $Z(J,\phi^*,\bar{g})$ (80) under the gauge transformations of external metric $\bar{g}$,

$$\delta^{(c)}_\omega Z(J,\phi^*,\bar{g}) = \frac{i}{\hbar}\int d\phi(\delta^{(c)}_\omega S_{FP}(\phi,\bar{g}) + \phi^*_i\delta^{(c)}_\omega R^i(\phi,\bar{g})) \times \exp\left\{\frac{i}{\hbar}[S(\phi,\phi^*\bar{g})+J\phi]\right\}. \tag{83}$$

Making the change of variables ϕ^i according to (56), (61), and (64) in the functional integral and taking into account the triviality of the corresponding Jacobian, we arrive at the relation

$$\frac{i}{\hbar}\int d\phi\{\delta^{(q)}_\omega S_{FP}(\phi,\bar{g}) + \phi^*_i\delta^{(q)}_\omega R^i(\phi,\bar{g}) + J_i\delta^{(q)}_\omega\phi^i\} \times \exp\left\{\frac{i}{\hbar}[S(\phi,\phi^*\bar{g})+J\phi]\right\} = 0. \tag{84}$$

Combining Eqs. (83) and (84) and using the gauge invariance of S_{FP} (65) we obtain

$$\delta_\omega^{(c)} Z(J,\phi^*,\bar{g}) = \frac{i}{\hbar}\int d\phi\{\phi_i^*\delta_\omega R^i(\phi,\bar{g}) + J_i\mathcal{R}^i_\sigma(\phi)\omega^\sigma\} \times \exp\left\{\frac{i}{\hbar}[S(\phi,\phi^*\bar{g}) + J\phi]\right\}, \qquad (85)$$

or, equivalently,

$$\delta_\omega^{(c)} Z(J,\phi^*,\bar{g}) = \frac{i}{\hbar}\phi_i^*\delta_\omega R^i\left(\frac{\hbar}{i}\frac{\delta}{\delta J},\bar{g}\right)Z(J,\phi^*,\bar{g}) + \frac{i}{\hbar}J_i\mathcal{R}^i_\sigma\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)Z(J,\phi^*,\bar{g})\omega^\sigma. \qquad (86)$$

In terms of the generating functional of connected Green functions $W = W(J,\phi^*,\bar{g})$ the relation (86) reads

$$\delta_\omega^{(c)} W(J,\phi^*,\bar{g}) = \phi_i^*\delta_\omega R^i\left(\frac{\delta W}{\delta J} + \frac{\hbar}{i}\frac{\delta}{\delta J},\bar{g}\right)\mathbf{1} + J_i\mathcal{R}^i_\sigma\left(\frac{\delta W}{\delta J}\right)\omega^\sigma, \qquad (87)$$

where the symbol $\mathbf{1}$ means that the operator acts on the numerical unit, $\mathbf{1} = 1$. In the case of functional derivative one has $\frac{\delta}{\delta\phi}\mathbf{1} = 0$, but since in many cases the expressions are nonlinear, this is a useful notation.

The extended generating functional of vertex functions (extended effective action) is defined in a standard way through the Legendre transformation of $W = W(J,\phi^*,\bar{g})$ introduced in Eq. (80),

$$\Gamma(\Phi,\phi^*,\bar{g}) = W(J,\phi^*,\bar{g}) - J\Phi, \qquad \Phi^j = \frac{\delta_l W}{\delta J_j}, \qquad \frac{\delta_r\Gamma}{\delta\Phi^j} = -J_j. \qquad (88)$$

As usual,

$$(\Gamma'')_{ij}\times(W'')^{jk} = \frac{\delta_r}{\delta J_k}\left(\frac{\delta_l W}{\delta J_i}\right)\times\frac{\delta_l}{\delta\Phi^i}\left(\frac{\delta_r\Gamma}{\delta\Phi^j}\right) = -\delta^k_j, \qquad (89)$$

where we introduced a compact notation for the second variational derivatives of Γ and W.

It proves useful to introduce the following notations:

$$\delta_\omega\bar{R}^i(\Phi,\phi^*,\bar{g}) = \delta_\omega R^i(\hat{\Phi},\bar{g})\mathbf{1}, \qquad \hat{\Phi}^j = \Phi^j + i\hbar(\Gamma''^{-1})^{jk}\frac{\delta_l}{\delta\Phi^k}, \qquad (90)$$

where the symbol $(\Gamma''^{-1})^{jk}$ denotes the matrix inverse to the matrix of second derivatives of the functional Γ defined in (89),

$$(\Gamma''^{-1})^{ik}(\Gamma'')_{kj} = \delta^i_j. \qquad (91)$$

Using these notations, in terms of extended effective action the equation (87) rewrites as

$$\delta_\omega^{(c)}\Gamma(\Phi,\phi^*,\bar{g}) = -\frac{\delta_r\Gamma}{\delta\Phi^i}\mathcal{R}^i_\sigma(\Phi)\omega^\sigma + \phi_i^*\delta_\omega\bar{R}^i(\Phi,\phi^*,\bar{g}), \qquad (92)$$

or, using the relation (69), in the form

$$\delta_\omega\Gamma(\Phi,\phi^*,\bar{g}) = \phi_i^*\delta_\omega\bar{R}^i(\Phi,\phi^*,\bar{g}). \qquad (93)$$

At this point we can draw a general conclusion from our consideration of quantum gravity theories in the background field formalism. At the nonrenormalized level any covariant quantum gravity theory has the following general property: the extended quantum action $S = S(\phi,\phi^*,\bar{g})$ satisfies the classical master (Zinn-Justin) equation of the Batalin-Vilkovisky formalism [39,40], as we already anticipated in Eq. (35). And, moreover, the extended effective action $\Gamma = \Gamma(\Phi,\phi^*,\bar{g})$ also satisfies the classical master equation,

$$(\Gamma,\Gamma) = 0. \qquad (94)$$

The functionals $S = S(\phi,\phi^*,\bar{g})$ and $\Gamma = \Gamma(\Phi,\phi^*,\bar{g})$ are invariant under the background gauge transformations

$$\delta_\omega S|_{\phi^*=0} = 0, \qquad \delta_\omega\Gamma|_{\phi^*=0} = 0, \qquad (95)$$

on the hypersurface $\phi^* = 0$ and, more general, satisfy the relations (81) and (93).

III. GAUGE-INVARIANT RENORMALIZABILITY

Up to now we were considering the nonrenormalized generating functionals of Green functions. The next step is to prove the gauge invariant renormalizability, that is the property of renormalized generating functionals. In the framework of Batalin-Vilkovisky formalism one can prove the BRST invariant renormalizability which means the preservation of basic equations (35) for the extended action $S = S(\phi,\phi^*,\bar{g})$ and an identical equation (94) for the extended effective action $\Gamma = \Gamma(\Phi,\Phi^*,\bar{g})$ after renormalization, that means

$$(S_R,S_R) = 0 \quad \text{and} \quad (\Gamma_R,\Gamma_R) = 0. \qquad (96)$$

Let us remember that the "classical" actions S and S_R are nothing but zero-order approximations of the loop expansions in the parameter $\hbar$ of the effective actions Γ and Γ_R. In this sense Eq. (35) is the zero order approximation of Eq. (94) and what we have to do now is to extend these two equations to the renormalized quantities S_R and Γ_R. Our strategy will be to make this extension order by order in the loop expansion parameter $\hbar$. Then we will prove that the renormalized actions S_R and Γ_R obey the gauge invariance property.

A. BRST invariant renormalization

As a first step, consider the one-loop approximation for $\Gamma = \Gamma(\Phi,\Phi^*,\bar{g})$. For the uniformity of notations we use $\Phi^* = \phi^*$ for the antifields in what follows. The effective action can be presented in the form

$$\Gamma = \Gamma^{(1)} + \mathcal{O}(\hbar^2) = S + \hbar[\Gamma^{(1)}_{\rm div} + \Gamma^{(1)}_{\rm fin}] + \mathcal{O}(\hbar^2), \qquad (97)$$

where $S = S(\Phi, \Phi^*, \bar{g})$ and $\Gamma^{(1)}_{\rm div}$ and $\Gamma^{(1)}_{\rm fin}$ denote the divergent and finite parts of the one-loop approximation for Γ.

In the local models of quantum gravity the locality of the divergent part of effective action is guaranteed by the Weinberg's theorem [55] (see also [56] for an alternative proof). Furthermore, even if the starting action is nonlocal, the UV divergences may be described by local functionals, just because the high energy domain always corresponds to the short-distance limit. And in the case of UV divergences the energies are infinitely high, hence the distances should be infinitely short, that does not leave space to the nonlocalities. As it was argued in Refs. [3,57,58], the UV divergent part of effective action for a wide class of models of quantum gravity is local, including the ones with a nonlocal classical action. Thus we assume that $\Gamma^{(1)}_{\rm div}$ is a local functional. Since it determines the form of the counterterms of the one-loop renormalized action

$$S_{1R} = S - \hbar\Gamma^{(1)}_{\rm div}, \qquad (98)$$

the last is also a local functional. Furthermore, from the expansion of the divergent parts of Eqs. (94) and (97) up to the first order in $\hbar$ follows that $\Gamma^{(1)}_{\rm div}$ and $\Gamma^{(1)}_{\rm fin}$ satisfy the equation

$$\begin{aligned} 0 = (\Gamma, \Gamma) &= (S, S) + 2\hbar(S, \Gamma^{(1)}_{\rm div}) + 2\hbar(S, \Gamma^{(1)}_{\rm fin}) + O(\hbar^2) \\ &= 2\hbar(S, \Gamma^{(1)}_{\rm div}) + 2\hbar(S, \Gamma^{(1)}_{\rm fin}) + O(\hbar^2). \end{aligned} \qquad (99)$$

In the first order in $\hbar$ we have a vanishing sum of the two terms, one of them is infinite and hence it has to vanish independent on another one. Therefore

$$(S, \Gamma^{(1)}_{\rm div}) = 0. \qquad (100)$$

Let us consider

$$(S_{1R}, S_{1R}) = (S, S) - 2\hbar(S, \Gamma^{(1)}_{\rm div}) + \hbar^2(\Gamma^{(1)}_{\rm div}, \Gamma^{(1)}_{\rm div}). \qquad (101)$$

Taking into account (35) and (100), we find the relation

$$(S_{1R}, S_{1R}) = \hbar^2 E_2, \qquad (102)$$

where E_2 is an unknown functional. Thus we have shown that S_{1R} satisfies the classical master equation (35) up to the terms of order $\hbar^2$,

$$E_2 = \frac{1}{2}(\Gamma^{(1)}_{\rm div}, \Gamma^{(1)}_{\rm div}). \qquad (103)$$

The one-loop effective action Γ_{1R} can be constructed by adding a local counterterm to the $\mathcal{O}(\hbar)$ part of Eq. (97). As usual, the counterterm has the divergent part which cancel the divergence of $\Gamma^{(1)}_{\rm div}$, and the remaining contribution is finite and typically depends on the renormalization parameter μ. This contribution is not only finite, but also satisfies the same symmetries as the initial action S. Thus, the sum of (97) and the counterterm, that is Γ_{1R}, also satisfies the same symmetries. Since we are not interested in the dependence on μ in this work, we shall simply use (98) and assume that Γ_{1R} is constructed by following the procedure of quantization described above, with S replaced by S_{1R}.

Being constructed in this way, the functional Γ_{1R} is finite in the one-loop approximation and satisfies the equation

$$(\Gamma_{1R}, \Gamma_{1R}) = \hbar^2 E_2 + O(\hbar^3). \qquad (104)$$

Now we are in a position to make the second step. Consider the one-loop renormalized effective action in the form which takes into account the $\mathcal{O}(\hbar^2)$-terms,

$$\Gamma_{1R} = S + \hbar\Gamma^{(1)}_{\rm fin} + \hbar^2(\Gamma^{(2)}_{1,\rm div} + \Gamma^{(2)}_{1,\rm fin}) + O(\hbar^3). \qquad (105)$$

Here $\Gamma^{(2)}_{1,\rm div}$ and $\Gamma^{(2)}_{1,\rm fin}$ are divergent and finite $\mathcal{O}(\hbar^2)$ parts of the two-loop effective action constructed on the basis of S_{1R} instead of S. The divergent part $\Gamma^{(2)}_{1,\rm div}$ of the two-loop approximation for Γ_{1R} determines the two-loop renormalization for S_{2R} according to

$$S_{2R} = S_{1R} - \hbar^2\Gamma^{(2)}_{1,\rm div} \qquad (106)$$

and satisfies the equation

$$(S, \Gamma^{(2)}_{1,\rm div}) = E_2.$$

As a third step consider

$$(S_{2R}, S_{2R}) = \hbar^3 E_3 + O(\hbar^4). \qquad (107)$$

We have found that S_{2R} satisfies the master equations up to the terms $\hbar^3 E_3$, where

$$E_3 = \frac{1}{2}(\Gamma^{(1)}_{\rm div}, \Gamma^{(2)}_{1,\rm div}), \qquad (108)$$

The effective action Γ_{2R} is generated by replacing S_{2R} into functional integral instead of S. Therefore, Γ_{2R} is automatically finite in the two-loop approximation,

$$\Gamma_{2R} = S + \hbar\Gamma^{(1)}_{\rm fin} + \hbar^2\Gamma^{(2)}_{1,\rm fin} + \hbar^3(\Gamma^{(3)}_{2,\rm div} + \Gamma^{(3)}_{2,\rm fin}) + O(\hbar^4)$$

and satisfies the equation

$$(\Gamma_{2R}, \Gamma_{2R}) = \hbar^3 E_3 + O(\hbar^4). \tag{109}$$

By applying the induction method we find that the totally renormalized action S_R is given by the expression

$$S_R = S - \sum_{n=1}^{\infty} \hbar^n \Gamma^{(n)}_{n-1,\mathrm{div}}. \tag{110}$$

We assume that $\Gamma^{(n)}_{n-1,\mathrm{div}}$ and $\Gamma^{(n)}_{n-1,\mathrm{fin}}$ are the divergent and finite parts of the n-loop approximation for the effective action, which is already finite in $(n-1)$-loop approximation, since it is constructed on the basis of the action $S_{(n-1)R}$.

The action (110) is a local functional and satisfies the classical master equations exactly,

$$(S_R, S_R) = 0. \tag{111}$$

It means the preservation of the BRST symmetry of renormalized action S_R that corresponds exactly to the BRST cohomology on local functionals with ghost number 0 [59,60].

The renormalized effective action Γ_R is finite in each order of the loop expansion in the powers of $\hbar$,

$$\Gamma_R = S + \sum_{n=1}^{\infty} \hbar^n \Gamma^{(n)}_{n-1,\mathrm{fin}}, \tag{112}$$

and satisfies the analog of Slavnov-Taylor identities [61–63] in Yang-Mills theory (see also [64] for the pedagogical introduction),

Thus the renormalized action S_R and the effective action Γ_R satisfy the classical master equation and the Ward (or Slavnov-Taylor) identity, respectively.

B. Gauge invariance of renormalized background effective action

As far as our main target is the symmetries of the renormalized effective action, the next stage of our consideration will be to generalize the transformation relations (81) and (93) for the renormalized functionals S_R and Γ_R. In the one-loop approximation from (93) follows that

$$\delta_\omega \Gamma(\Phi, \Phi^*, \bar{g}) = \Phi^*_i \delta_\omega R^i(\Phi, \bar{g}) + \hbar \Phi^*_i \delta_\omega \bar{R}^{i(1)}_{\mathrm{div}}(\Phi, \Phi^*, \bar{g}) + \hbar \Phi^*_i \delta_\omega \bar{R}^{i(1)}_{\mathrm{fin}}(\Phi, \Phi^*, \bar{g}) + O(\hbar^2), \tag{113}$$

where the condensed notations (90) were used. In the last expression $\delta_\omega \bar{R}^{i(1)}_{\mathrm{div}}(\Phi, \Phi^*, \bar{g})$ and $\delta_\omega \bar{R}^{i(1)}_{\mathrm{fin}}(\Phi, \Phi^*, \bar{g})$ are divergent and finite parts of the one-loop approximation for the gauge transformations $\delta_\omega \bar{R}^i(\Phi, \Phi^*, \bar{g})$, correspondingly.

On the other hand, from (97) we have

$$\delta_\omega \Gamma(\Phi, \Phi^*, \bar{g}) = \delta_\omega S(\Phi, \Phi^*, \bar{g}) + \hbar \delta_\omega \Gamma^{(1)}_{\mathrm{div}} + \hbar \delta_\omega \Gamma^{(1)}_{\mathrm{fin}} + O(\hbar^2). \tag{114}$$

The comparison of the relations (113) and (114) tells us that

$$\delta_\omega \Gamma^{(1)}_{\mathrm{div}} = \Phi^*_i \delta_\omega \bar{R}^{i(1)}_{\mathrm{div}}(\Phi, \Phi^*, \bar{g}), \tag{115}$$

$$\delta_\omega \Gamma^{(1)}_{\mathrm{fin}} = \Phi^*_i \delta_\omega \bar{R}^{i(1)}_{\mathrm{fin}}(\Phi, \Phi^*, \bar{g}). \tag{116}$$

From Eq. (115) and the definition (98) follows that the one-loop renormalized action $S_{1R} = S_{1R}(\Phi, \Phi^*, \bar{g})$ transforms according to

$$\delta_\omega S_{1R} = \Phi^*_i \delta_\omega R^{i(1)}_R, \tag{117}$$

where

$$R^{i(1)}_R = R^{i(1)}_R(\Phi, \Phi^*, \bar{g}) = \delta_\omega R^i(\Phi, \bar{g}) - \hbar \delta_\omega \bar{R}^{i(1)}_{\mathrm{div}}(\Phi, \Phi^*, \bar{g}). \tag{118}$$

The last relations mean that the action S_{1R} is invariant under the background gauge transformations with one-loop deformed gauge generators $R^{i(1)}_R$ (118) on the hypersurface $\Phi^* = 0$. Furthermore, due to Eq. (117) the functional Γ_{1R} obeys the transformation rule

$$\delta_\omega \Gamma_{1R} = \Phi^*_i \delta_\omega R^i + \hbar \Phi^*_i \delta_\omega \bar{R}^{i(1)}_{\mathrm{fin}} + \hbar^2 (\Phi^*_i \delta_\omega \bar{R}^{i(2)}_{1,\mathrm{div}} + \Phi^*_i \delta_\omega \bar{R}^{i(2)}_{1,\mathrm{fin}}) + O(\hbar^3), \tag{119}$$

where $\delta_\omega \bar{R}^{i(2)}_{1,\mathrm{div}} = \delta_\omega \bar{R}^{i(2)}_{1,\mathrm{div}}(\Phi, \Phi^*, \bar{g})$ and $\delta_\omega \bar{R}^{i(2)}_{1,\mathrm{fin}} = \delta_\omega \bar{R}^{i(2)}_{1,\mathrm{fin}}(\Phi, \Phi^*, \bar{g})$ are related to $\Gamma^{(2)}_{1,\mathrm{div}}$ and $\Gamma^{(2)}_{1,\mathrm{fin}}$ (105) as

$$\delta_\omega \Gamma^{(2)}_{1,\mathrm{div}} = \Phi^*_i \delta_\omega \bar{R}^{i(2)}_{1,\mathrm{div}}, \qquad \delta_\omega \Gamma^{(2)}_{1,\mathrm{fin}} = \Phi^*_i \delta_\omega \bar{R}^{i(2)}_{1,\mathrm{fin}}. \tag{120}$$

Therefore the functional Γ_{1R} is finite in one-loop approximation and is invariant under the background gauge transformations up to the second order in $\hbar$ on the hypersurface $\Phi^* = 0$.

Applying the induction method one can show that the renormalized functionals S_R and Γ_R satisfy the properties[4]

$$\delta_\omega S_R = \Phi^*_i \delta_\omega R^i_R, \qquad \delta_\omega \Gamma_R = \Phi^*_i \delta_\omega \bar{R}^i_R, \tag{121}$$

where

[4]We note that these statements are very close to the results concerning preservation of global symmetries of initial classical action at quantum level when the effective action of theory under consideration is invariant under deformed global transformations of all fields [65].

$$\delta_\omega R^i_R = \delta_\omega R^i - \hbar\delta_\omega \bar{R}^{i(1)}_{\rm div} - \hbar^2\delta_\omega \bar{R}^{i(2)}_{1,{\rm div}} - \cdots, \qquad (122)$$

$$\delta_\omega \bar{R}^i_R = \delta_\omega R^i + \hbar\delta_\omega \bar{R}^{i(1)}_{\rm fin} + \hbar^2\delta_\omega \bar{R}^{i(2)}_{1,{\rm fin}} + \cdots. \qquad (123)$$

It is important that $\delta_\omega \bar{R}^i_R$ defined in (123) are finite.

The last observation is that, in case of local theories the quantities $\delta_\omega R^i_R$ (122) are local due to the Weinberg's theorem [55], while in the nonlocal models of quantum gravity there are also strong arguments in favor of locality of divergences [3,58], including the transformations δ_ω.

The important consequence of the results (121) is that we can state that renormalized functionals $S_R(\Phi,\bar{g}) = S_R(\Phi,\Phi^*=0,\bar{g})$ and $\Gamma_R(\Phi,\bar{g}) = \Gamma_R(\Phi,\Phi^*=0,\bar{g})$ satisfy the same equations

$$\delta_\omega S_R(\Phi,\bar{g}) = 0, \qquad \delta_\omega \Gamma_R(\Phi,\bar{g}) = 0, \qquad (124)$$

as nonrenormalized functionals $S(\Phi,\bar{g}) = S_{FP}(\Phi,\bar{g})$ and $\Gamma(\Phi,\bar{g})$ in (65) and (78), respectively. Then from (124) we deduce the invariance for renormalized background functionals $S_R(\bar{g}) = S(\Phi=0,\bar{g})$ and $\Gamma(\bar{g}) = \Gamma(\Phi=0,\bar{g})$ under general coordinate transformations of external background metric $\bar{g}$,

$$\delta^{(c)}_\omega S_R(\bar{g}) = 0, \qquad \delta^{(c)}_\omega \Gamma_R(\bar{g}) = 0. \qquad (125)$$

These properties repeat exactly the invariance of initial action $S_0(\bar{g})$ and $\Gamma(\bar{g})$ in (66).

C. Comparison with the proof based on cohomology

In order to understand better the relevance of the results described above, let us present a short historical review of the subject. The first proof of the gauge invariant renormalizability in quantum gravity was given by Stelle in the famous 1977 paper [1]. The considerations in this paper concerned only the renormalizable model of quantum gravity. However, most of the analysis is quite general and can be applied to many covariant models of QG, not only to the general four-derivative gravity. After that there were many important publications devoted to the invariant renormalizability in gauge theories of a general form, including gravity. One can say that the progress in understanding renormalizability of quantum gravity was performed in relatively small steps after [1], that does not mean at all that the progress in this area was irrelevant.

The most significant achievement in this respect was the demonstration of BRST invariant renormalizability in the theories which may be not renormalizable by power counting. In particular, in 1982 it was formulated the first proof for the general gauge theories [66], based on the Batalin-Vilkovisky formalism [39,40]. The approach in this paper assumed the regularization procedure respecting the gauge invariance of initial classical action and zero volume divergences, $\delta(0)=0$. Within the Batalin-Vilkovisky formalism one can prove that the full gauge fixed action $S=S(\phi,\phi^*)$ satisfies the classical master equation $(S,S)=0$, generalizing the Zinn-Justin equation [38]. The next step is to show that the generating functional of vertex functions (effective action), $\Gamma=\Gamma(\Phi,\Phi^*)$, constructed on the basis of $S=S(\phi,\phi^*)$, satisfies the Ward identity being the same master equation, $(\Gamma,\Gamma)=0$. Applying the minimal substraction scheme one can prove that both local functional of renormalized action S_R and renormalized effective action Γ_R satisfy the corresponding master equations, $(S_R,S_R)=0$ and $(\Gamma_R,\Gamma_R)=0$. The proof is valid for any boundary condition related to an initial gauge invariant action and for arbitrary choice of gauge fixing functions. Furthermore, the renormalization procedure of [66] can be described in terms of anticanonical transformations (for recent developments, see [67,68]) which are defined as transformations preserving the antibracket (we use terminology of the standard review [69]).

An alternative, albeit very close, approach to prove the BRST invariant renormalization of general gauge theories [70], is based on the use of cohomologies of nilpotent BRST operator, $\hat{s}$, associated with adjoint operation of the antibracket of the action S with an arbitrary functional F, $\hat{s}F=(S,F)$ [59,60]. The detailed description of this approach can be found in [69] and in the chapter 17.3 of the Weinberg's book [64]. Let us note that this approach does not directly cover the useful formalism of background field formalism, apparently for this reason the use of the linear background field gauges is discussed in the next chapter of [64].

Indeed, the background field formalism [25–27] represents a powerful approach to study quantum properties of gauge theories, allowing us to keep the gauge invariance, or general covariance in the quantum gravity case, at all stages of quantum calculations. From the viewpoint of the quantization of gauge systems this method corresponds to the special choice of a boundary condition and to the special choice of gauge fixing functions. However, since the background field method requires the presence of an "external" field in the course of the Lagrangian quantization, this formalism should be considered as a very special case which requires special care. Indeed, this special case attracted a great deal of attention recently, see, e.g., the papers [28–32]. We believe that the consistent treatment of this method in quantum gravity that we presented in the previous subsections, will contribute to a better general understanding of the formalism.

In the present work, we mainly follow the approach of [66] (and subsequent [20,21] for the quantum gravity case) but, for the first time, we consider the BRST renormalizability in the background field method from the very beginning. As a result, we prove that both renormalized action S_R and effective action Γ_R satisfy the original gauge symmetry (125), when antifields, ghosts, and the mean quantum metric are switched off.

The main result concerning renormalizability is essentially the same as the one in the original work of Stelle [1] and in all subsequent publications mentioned above. However, it is easy to see that the treatment of renormalization in the previous subsections is different from the approach in the works based on cohomology. Starting from the second loop, we have the terms such as the right-hand side of Eq. (102), which violate the form of the master equations (96). This fact represents a difficulty for the approach of [70], while in our case it is solved automatically. The solution of this problem in [70] implies the modification of the one-loop divergence by introducing into it the term $\mathcal{O}(\hbar^2)$. The procedure can be continued to higher than the second loops, and at the end the full perturbative expansion satisfies the equation for cohomology or, in our notations, the master equations (96). We leave it to the reader to compare the two approached. The additional benefits of our method is the proof of the invariance (124) for the renormalized effective action Γ_R, which is a finite nonlocal object (even if the boundary condition corresponds to a local covariant action). All in all, we believe that the present work represents one more relevant step forward in the consistent description of gauge invariant renormalization of quantum gravity theories.

IV. OBSERVATION ABOUT MULTILOOP RENORMALIZATION

In order to apply the results derived in the previous section to the analysis of renormalization, one cannot go directly to the power counting for the renormalized effective action (125). The reason is that the power counting provides information only about the last integral of the multi-loop diagram. In the last integration we can really switch off not only the antifields, but also the mean fields of quantum metric, ghosts, and the auxiliary field B. On the other hand, in the internal integrals of subdiagrams, one has to hold all the mean fields, while the antifields can be switched off. Thus, before classifying the theories of quantum gravity according to their renormalization properties, it is useful to formulate the procedure of invariant renormalization of multi-loop diagrams.

The most general object is the renormalized background effective action $\Gamma_R = \Gamma_R(\Phi,\Phi^*,\bar{g})$ (112) can be found as a solution to the following functional derivative equation

$$\Gamma_R(\Phi,\Phi^*,\bar{g}) = S_R(\Phi,\Phi^*,\bar{g}) - i\hbar\ln\int D\phi\exp\left\{\frac{i}{\hbar}\left[S_R(\Phi+\phi,\Phi^*,\bar{g}) - S_R(\Phi,\Phi^*,\bar{g}) - \frac{\delta\Gamma_R(\Phi,\Phi^*,\bar{g})}{\delta\Phi}\phi\right]\right\}. \quad (126)$$

Switching off all the antifields, this boils down to the equation for the reduced effective action functional $\bar{\Gamma}_R(\Phi,\bar{g}) = \Gamma_R(\Phi,\Phi^*=0,\bar{g})$, satisfying the equation

$$\bar{\Gamma}_R(\Phi,\bar{g}) = \bar{S}_R(\Phi,\bar{g}) - i\hbar\ln\int D\phi\exp\left\{\frac{i}{\hbar}\left[\bar{S}_R(\Phi+\phi,\bar{g}) - \bar{S}_R(\Phi,\bar{g}) - \frac{\delta\bar{\Gamma}_R(\Phi,\bar{g})}{\delta\Phi}\phi\right]\right\}, \quad (127)$$

where $\bar{S}_R(\Phi,\bar{g}) = S_R(\Phi,\Phi^*=0,\bar{g})$. This is exactly the object, which is sufficient to deal with to consider the renormalization of the subdiagrams. The two important observations are as follows. First, both Eqs. (126) and (127) are closed expressions for effective actions with respect to the corresponding fields. For (126) this statement is trivial, since all fields are included. For (127) this means that the right-hand side is written in terms of the background metric and the mean fields, without invoking antifields. Second, the effective action $\Gamma_R(\Phi,\bar{g})$ obeys the symmetries such as BRST and the combined background transformation δ_ω.

As a result, we can guarantee that the renormalization of p-loop diagrams occurs in a completely covariant way. Up to the last integration the divergences are removed such that we get the $p-1$ order of the loop expansion of (127), and in the last (surface) integral one can switch off all the means fields, arriving at the functional

$$\Gamma_R(\bar{g}) = \bar{\Gamma}_R(\Phi=0,\bar{g}). \quad (128)$$

This functional satisfies the equation

$$\Gamma_R(\bar{g}) = S_R(\bar{g}) - i\hbar\ln\int D\phi\exp\left\{\frac{i}{\hbar}\left[\bar{S}_R(\phi,\bar{g}) - S_R(\bar{g}) - \frac{\delta\bar{\Gamma}_R(\Phi,\bar{g})}{\delta\Phi}\bigg|_{\Phi=0}\phi\right]\right\}, \quad (129)$$

where $S_R(\bar{g}) = \bar{S}_R(\Phi=0,\bar{g})$. It is easy to see that the main object of the background field formalism (BFM) in quantum gravity, namely the effective action (129), is not a closed expression, in the sense explained above. At the same time, we have proved in the previous sections that it is a covariant functional. Together with the locality of divergences, this result enables one to evaluate the power counting, as it is done in the next section.

V. POWER COUNTING AND CLASSIFICATION OF QUANTUM GRAVITY MODELS

Equations (121) show that with the antifields switched off, for $\Phi^*=0$, the renormalized action S_R and effective action Γ_R are both gauge invariant quantities. In particular, this means that if we restrict the attention by the standard nonextended generating functional of the Green functions, without introducing sources for the ghosts C, $\bar{C}$ and the auxiliary field B, the effective action will be metric-dependent and generally covariant functional. This statement concerns both divergences and the finite part of renormalized effective action.

As far as we are interested in renormalizability of the theory, our main focus should be on the structure of divergences. In this case one can use the power counting arguments to classify the theories of quantum gravity to the nonrenormalizable, renormalizable, and superrenormalizable models. The power counting in quantum gravity is especially simple, because the metric field is dimensionless. As a result, the dimension of a Feynman diagram is divided between the internal momenta defining divergences and the external momenta, or the number of metric derivatives in the counterterms.

It is clear that the simple structure of power counting in *higher derivative* quantum gravity, as described above, requires that the following two conditions are fulfilled: (i) The propagator of the gravitational field should be homogeneous in the powers of momenta. This means, in particular, that the free equations for different modes of the gravitational field (tensor, vector, and scalar) are of the same order in derivatives after the gauge fixing is implemented through the Faddeev-Popov procedure. (ii) The propagator of gauge ghosts must have the same powers of momenta as all modes of the gravitational field.

In order to ensure these two conditions are fulfilled, the standard Faddeev-Popov procedure needs to be modified. Instead of the conventional gauge fixing term (singular or not) and the usual ghost action, there must be a modified term of the form

$$S_{gf} = \int d^4x\sqrt{-g}\chi^\alpha Y_{\alpha\beta}\chi^\beta, \tag{130}$$

$$S_{gh} = \int d^4x\sqrt{-g}\bar{C}^\alpha Y_{\alpha\beta}M^\beta_\lambda C^\lambda, \tag{131}$$

where, according to (18) and (19),

$$M^\beta_\lambda = H^{\rho\sigma}_\beta(x,y;\bar{g},h)R_{\rho\sigma\lambda}(y,z;\bar{g}+h). \tag{132}$$

The choice of the weight operator $Y_{\alpha\beta}$ should be done in such a way that the total amount of derivatives in the expressions (130) and (131) are the same as in the action of the model of quantum gravity under consideration. For instance, in the quantum gravity based on general relativity $Y_{\alpha\beta} = \theta g_{\alpha\beta}$, where θ is a constant gauge fixing parameter. In case of the fourth order gravity one has to take [1,71,72]

$$Y_{\alpha\beta} = \theta_1\delta_{\alpha\beta}\Box + \theta_2\nabla_\alpha\nabla_\beta + \theta_3 R_{\alpha\beta} + \theta_4\delta_{\alpha\beta}R, \tag{133}$$

where $\theta_{1,2,3,4}$ are gauge fixing constants. In the case of six-derivative superrenormalizable gravity model [2] $\theta_{1,2,3,4}$ should be linear functions of d'Alembertian operator $\Box$, plus the possible linear in curvature tensor terms, for the eight-derivative quantum gravity the parameters $\theta_{1,2,3,4}$ become quadratic functions of $\Box$, etc.

An important question is how to incorporate the modified gauge fixing and ghost actions (130) and (131) into the proof of gauge invariant renormalizability which we developed in the previous Sec. III.

The simplest possibility in this direction is as follows. The effective action in the superrenormalizable quantum gravity theories with more than four derivatives does not depend on the gauge fixing [2]. This fact can be explained by covariance, power counting and by the fact that the gauge fixing dependencies vanish on-shell. At higher loops the on-shell condition involves not only classical equations of motion, but also the loop corrections. However, the classical part is included and it has more than four derivatives. On the other hand, quantum corrections in these models may have at most four derivatives in the polynomial part, such that the gauge dependence is ruled out. Thus the scheme based on the weight function (133) with $\theta_{1,2,3,4}$ being at least linear functions of a $\Box$, does not affect the loop corrections, regardless it is critically important for correctly evaluating the power counting in these theories. This argument looks convincing and its output is eventually correct, but it is indeed based on a logical loophole. We have the proof of covariance based on the conventional gauge fixing, leading to a nonhomogeneous propagator. At the same time the power counting that is another element of the presented argument, is essentially based on the homogeneity of the propagator (see below, and also in [2,6] and [6]). Hence we really need to modify the standard Faddeev-Popov procedure in this case and see whether something has to be changed in the proof given in the previous section.

Consider $\chi_\alpha = \chi_\alpha(x;\bar{g},h)$ being a standard gauge fixing functions used in previous sections. We can introduce the set of two differential operators, Y^β_α and $Y_{1\alpha\beta}$. These weight operators must have the structure of tensor fields of types (1,1) and (0,2), respectively, and cannot depend on the quantum metric $h_{\mu\nu}$,

$$Y^\beta_\alpha(x,y) = Y^\beta_\alpha(x,y;\bar{g},\bar{\Box}) \quad \text{and} \quad Y_{1\alpha\beta}(x,y) = Y_{1\alpha\beta}(x,y;\bar{g},\bar{\Box}). \tag{134}$$

The next step is to modify the gauge fixing functions χ_α, by the following rule:

$$\chi^{\text{mod}}_\alpha(x;\bar{g},h,B) = \int dy\bigg[Y^\beta_\alpha(x,y;\bar{g},\bar{\Box})\chi_\beta(y;\bar{g},h) + \frac{1}{2}Y_{1\alpha\beta}(x,y;\bar{g},\bar{\Box})B^\beta(y)\bigg] \tag{135}$$

and construct the corresponding gauge fixing functional,

$$\Psi^{\rm mod}(\phi,\bar{g}) = \int dx\sqrt{-\bar{g}}\bar{C}^\alpha(x)\chi^{\rm mod}_\alpha(x;\bar{g},h,B). \quad (136)$$

According to what we previously learned, the transformation law of $\chi^{\rm mod}_\alpha$ coincides with the transformation rule of tensor fields of type (0,1). Then the modified Faddeev-Popov action is constructed in the standard manner, using the generator of BRST transformations, $\hat{R}(\phi,\bar{g})$,

$$S^{\rm mod}_{FP}(\phi,\bar{g}) = S_0(\bar{g}+h) + \Psi^{\rm mod}(\phi,\bar{g})\hat{R}(\phi,\bar{g}). \quad (137)$$

The explicit form of the second term in the right-hand side of (137) is

$$\begin{aligned}
&\Psi^{\rm mod}(\phi,\bar{g})\hat{R}(\phi,\bar{g})\\
&\quad = \int dxdydzdu\sqrt{-\bar{g}(x)}\bar{C}^\alpha(x)Y^\beta_\alpha(x,u;\bar{g},\bar{\Box})\\
&\qquad \times H^{\gamma\sigma}_\beta(u,y;\bar{g},h)R_{\gamma\sigma\rho}(y,z;\bar{g}+h)C^\rho(z)\\
&\qquad + \int dxdy\sqrt{-\bar{g}(x)}\Big[B^\alpha(x)Y^\beta_\alpha(x,y;\bar{g},\bar{\Box})\chi_\beta(y;\bar{g},h)\\
&\qquad + \frac{1}{2}B^\alpha(x)Y_{1\alpha\beta}(x,y;\bar{g},\bar{\Box})B^\beta(y)\Big].
\end{aligned}$$

It is easy to see that the first term in the right-hand side of the last formula is exactly of the desired form (131) with (132), if the weight operator is properly defined. The key observation is that, since the transformation rules for the terms in the Faddeev-Popov action depend only on the type of the tensor fields, all the main statements of the previous sections remain valid for the new choice of the gauge fixing functions (135).

Consider a special choice of the operator $Y_{1\alpha\beta}$,

$$\begin{aligned}
&Y_{1\alpha\beta}(x,y) = \bar{g}_{\alpha\gamma}(x)(Y^{-1})^\gamma_\beta(x,y), \quad \text{where}\\
&\int dz Y^\gamma_\alpha(x,z)(Y^{-1})^\beta_\gamma(z,y) = \delta^\beta_\alpha\delta(x-y),\\
&Y^\beta_\alpha(x,y) = Y^\beta_\alpha(x;\bar{g},\bar{\Box})\delta(x-y).
\end{aligned}$$

Since the problem of the homogeneity in the ghost sector is already resolved by Eq. (138), we need to deal only with the propagator of the quantum metric $h_{\mu\nu}$. Integrating over the fields B^α in the functional integral defines the generating functional of Green functions it terms of $\bar{C}^\alpha$, C^α and $h_{\mu\nu}$. As a results we obtain the functional determinant that is equal to

$$[{\rm Det}\, Y^\beta_\alpha(x,y)]^{1/2}, \quad (138)$$

and does not depend on the variables (quantum fields) of integration. Let us note that the factor (138) is well-known in both fourth derivative quantum gravity [71,72] and superrenormalizable models [2,14], but we got it a new way here.

After all, we need the following modifications:

$$\begin{aligned}
\Psi^{\rm mod}(\phi,\bar{g})\hat{R}(\phi,\bar{g}) + \int dx\sqrt{-\bar{g}(x)}J^{(B)}_\alpha(x)B^\alpha(x) \to &\int dxdydz\sqrt{-\bar{g}(x)}\bar{C}^\alpha(x)Y^\beta_\alpha(x;\bar{g},\bar{\Box})H^{\gamma\sigma}_\beta(x,y;\bar{g},h)R_{\gamma\sigma\rho}(y,z;\bar{g}+h)C^\rho(z)\\
&-\frac{1}{2}\int dx\sqrt{-\bar{g}(x)}\chi^\alpha(x;\bar{g},h)Y^\beta_\alpha(x;\bar{g},\bar{\Box})\chi_\beta(x;\bar{g},h)\\
&-\frac{1}{2}\int dx\sqrt{-\bar{g}(x)}J^{(B)\alpha}(x)Y^\beta_\alpha(x;\bar{g},\bar{\Box})J^{(B)}_\beta(x)\\
&-\int dx\sqrt{-\bar{g}(x)}J^{(B)}_\alpha(x)\chi^\alpha(x;\bar{g},h), \quad (139)
\end{aligned}$$

where the notations

$$\begin{aligned}
\chi^\alpha(x;\bar{g},h) &= \bar{g}^{\alpha\beta}(x)\chi_\beta(x;\bar{g},h),\\
J^{(B)\alpha}(x) &= \bar{g}^{\alpha\beta}(x)J^{(B)}_\beta(x) \quad (140)
\end{aligned}$$

are used. It is easy to see that the second term in the expression (139) is exactly what is needed for the homogeneity condition (130). At the same time the terms with the source of the auxiliary field $B_\alpha(x)$ remains and this opens the possibility to define the corresponding mean field in a standard way.

As far as the problem of homogeneity and introduction of (130) and (131) has been solved, we are in a position to review the power counting and classify the models of quantum gravity. For this sake, consider the Feynman diagrams with n vertices, $l_{\rm int}$ internal lines, and p loops. It is easy to verify that these three quantities satisfy the topological relation

$$l_{\rm int} = p + n - 1. \quad (141)$$

Another relation links the superficial degree of divergence D of the diagram and the total number of momenta external lines of the diagram d with the power of momenta in the inverse propagator of internal line r_l and the number of vertices K_ν with ν momenta. The formula of our interest is [1]

$$D + d = \sum_{l_{\rm int}} (4 - r_l) - 4n + 4 + \sum_\nu K_\nu. \tag{142}$$

As the first example, let us see how these two formulas work for the quantum gravity based on general relativity. In the theory without cosmological constant we have $r_l = 2$ and $K_2 = n$. Replacing these numbers into (142) and using (141) we arrive at

$$D + d = (4 - 2) l_{\rm int} - 4n + 4 - 2n = 4 + 2p. \tag{143}$$

For the logarithmic divergences $D = 0$ and we discover that the dimension of covariant counterterms grows with the number of loops as $d = 4 + 2p$. The theory is obviously nonrenormalizable. In the presence of cosmological constant the quantity d becomes smaller $d = 4 + 2p - 2K_0$ with each vertex without derivatives, and the loss of dimension is compensated by the powers of the cosmological constant. The results of the previous section and locality of divergences enable one to use the quantity of d to write down all possible counterterms in any loop order p. For $p = 1$ there are $\mathcal{O}(R^2_{\dots})$ and $\Box R$ type divergences [73], for $p = 2$ we meet $\mathcal{O}(R^3_{\dots})$ [74,75], etc.

The next example is the fourth derivative quantum gravity [1]. In this case one can modify the definition of ghost action in such a way that $r_l = 4$ for both metric and ghost propagators. Also, there are vertices with four K_4, two K_2 and zero K_0 derivatives. Combining (142) and (141) it is easy to get

$$D + d = 4 - 2K_2 - 4K_0. \tag{144}$$

The results of the previous section (for this theory the renormalizability was originally demonstrated in [1]) show that the divergences are covariant. Since they are also local, this means that if we include all terms of dimension four into the classical action,

$$S_{\rm 4DQG} = S_{\rm EH} - \int d^4x \sqrt{-g} \left\{ \frac{1}{2\lambda} C^2 + \frac{1}{\rho} E_4 + \tau \Box R + \frac{\omega}{3\lambda} R^2 \right\}, \tag{145}$$

then the divergences will repeat the form of the classical action. Thus, such a theory is multiplicatively renormalizable. In Eq. (145) we used the standard (in quantum gravity) basis for the four derivative terms, with C^2 being the square of the Weyl tensor

$$C^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3} R^2 \tag{146}$$

and E_4 is the integrand of the Gauss-Bonnet topological invariant,

$$E_4 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2. \tag{147}$$

The next example of our interest is the model (3) with functions $\Pi_1(x)$, $\Pi_2(x)$, and $\Pi_3(x)$ being polynomials of the same order $k \geq 1$ [2],

$$\Pi_{1,2,3}(x) = a_0^{1,2,3} x^k + a_1^{1,2,3} x^{k-1} + \cdots + a_{k-1}^{1,2,3} x + a_k^{1,2,3}. \tag{148}$$

The terms with $\Pi_{1,2,3}(x)$ have at most $2k + 4$ derivatives of the metric. The terms $+\mathcal{O}(R^3_{\dots})$ should satisfy the same restriction on the number of derivatives. Then we have $r_l = 2k + 4$ and the maximal number of derivatives in the vertices is also $\nu = 2k + 4$. If we are interested in the diagrams with the strongest divergences, $K_{4k+4} = n$. Once again, combining (142) and (141) for the maximally divergent diagrams it is easy to arrive at the result

$$D + d = 4 + 2k(1 - p). \tag{149}$$

This formula shows that for the logarithmic divergences at the one-loop order $p = 1$ and we have $d = 4$. Taking the covariance and locality arguments into account, the one-loop divergences repeat the form of the four-derivative action (145). Thus, the theory (3) can be renormalizable only if the coefficients $a_k^{1,2,3}$ in Eq. (148) are all nonzero, and the Einstein-Hilbert action with the cosmological constant is also included.

In case of $k \geq 3$ Eq. (149) tells us that there are no divergences beyond the first loop. For $k = 2$ we have only the cosmological constant divergences at two loop order. Finally, in the case of $k = 1$ there are cosmological constant-type divergences at three loop order and linear in R divergences at two loops. Obviously, the theory is superrenormalizable. Let us stress that in this case we have locality guaranteed due to the Weinber's theorem and covariance holds since we proved it in the previous section.

Finally, let us consider an example of the nonlocal gravity. The main proposal of this kind of model is to avoid the presence of higher derivative massive ghost in the spectrum of tree-level theory while keeping the theory renormalizable [3–5]. The general analysis of how the freedom from ghosts can be achieved can be found in [3,57,58] and we will not repeat this part, since our purpose here is the study of renormalization. It is sufficient for us to give an example of the theory which satisfies the ghost-free condition. The typical Euclidean space propagator in such a theory has the form

$$G(p) \propto \frac{1}{p^2} \exp\{-p^2/M^2\}. \tag{150}$$

Since gravity action is always nonpolynomial, this structure of propagator means that the vertices have the UV behavior which is at least proportional to

$$V(p) \propto p^2 \exp\{p^2/M^2\}. \tag{151}$$

The proof of the gauge-invariant renormalizability which we achieved in Sec. III is based only on the hypothesis of diffeomorphism invariance of the classical action. Therefore it is perfectly well applicable to the nonlocal models. Thus, the question of whether these theories are renormalizable depend only on power counting and locality of divergences. The power counting in this case represents a serious problem, because the expression (142) boils down to the indefinite difference of the $\infty - \infty$ type. However, there is a solution [6], which is based on the topological relation (141). It is clear from Eqs. (150) and (151) that the diagrams with $l_{\rm int} > n$ will be convergent, while those with $l_{\rm int} < n$ will be strongly (to say the least) divergent. Thus the logarithmic divergences will be the maximal ones only if $l_{\rm int} = n$, that gives $p = 1$. This means that all diagrams beyond one-loop order are finite (except one-loop subdiagrams, as usual). Furthermore, in the one-loop case all exponentials cancel out and the diagram has divergences which are of the same order as in the quantum GR. Taking covariance of divergences into account, this means that the one-loop divergences are of the four-derivative type (145).

There are two consequences of the power counting which we have described. The first is that the exponential nonlocal model has the power counting which is exactly the same as the polynomial model (3), (148) with $k \geq 3$. In other words, such a theory is superrenormalizable by power counting. However, the theory which is free from ghosts and has one-loop divergences cannot be even renormalizable, because all the coefficients of four-derivative terms should be precisely fine-tuned to provide the structure of the propagator (150) required for absence of ghosts. The problem can be alleviated by introducing a specially fine tuned $\mathcal{O}(R^3_{\dots})$ terms called "killers" [58] (see also earlier discussion in [2] for the polynomial models). These terms can make the theory finite, but still do not guarantee the ghostfree structure in the dressed propagator [6]. All in all, the nonlocal ghostfree models meet the problem of absolutely precise fine-tuning, which cannot be maintained upon (even finite) renormalization, even if the theory is superrenormalizable. Together with the problem is physical unitarity [7] this situation makes nonlocal theories less prospective, but of course they still remain very interesting models to study.

Finally, we note that in the polynomial models (3), (148) there are no problems with locality of divergent parts of effective action, and hence the proof of gauge invariant renormalizability can be used to give solid background to the power counting arguments.

VI. CONCLUSIONS

We described in detail the general proof of that the diffeomorphism invariance can be maintained in quantum gravity theories. The main advantages of the approach of the present paper is related to the explicit form of variation of extended effective action under the gauge transformations of all fields appearing in the background field formalism. The derived form of these variations can be applied to an arbitrary gravity theory which respects diffeomorphism invariance. The variation has a very special form, providing an exact invariance of the effective action when the antifields (sources for the BRST generators) are switched off.

After switching off the mean field of quantum metric, Faddeev-Popov ghosts, auxiliary field and antifields, the divergent part of effective action possess general covariance, and this important property holds in all orders of the perturbative loop expansion. This statement is proved correct for generic models of quantum gravity, including the ones with higher derivatives and even with certain (phenomenologically interesting) models with nonlocalities. Starting from covariance and using power counting and locality of the counterterms one can easily classify the models of quantum gravity into nonrenormalizable, renormalizable, and superrenormalizable versions.

On the other hand, we have extended the usual statement concerning the gauge invariance of the background effective action up to the gauge invariance of effective action depending on the mean quantum fields. Furthermore, we extended all mentioned results from the nonrenormalizable effective action to renormalized one. The gauge invariance of renormalized extended effective under the renormalized finite gauge transformations has been proved on the hypersurface of switched off antifields. An important consequence of the last result is the gauge invariance of renormalized background effective action under deformed gauge transformations of background metric for any covariant quantum gravity theory.

One of the possible applications of the new developments of the present work is that our treatment of background field method can be extended to the case of nonlinear gauges, which was never done [76]. This is an interesting problem to solve, because in the recent years there were several publications of different authors on the gauge and parametrization dependence in quantum gravity (see, e.g., [77,78] and further references therein). From the background field method side, the nonlinear change of parametrization may transform the linear gauge into nonlinear. Thus, it would be interesting to include the nonlinear gauges into this consideration. From this perspective, our work can be seen as a preparation for a solid field theoretical analysis of this issue.

It is tempting to extend the results achieved in this work to the nonperturbative domain. Unfortunately this cannot be done for the standard versions of average effective action, since the last does not admit the consistent on-shell limit in the case of gauge fields. In this respect the most promising is the new version of functional renormalization group which is based on the composite fields, as introduced in [49] for the Yang-Mills fields. However, for this end one has to extend this new scheme to quantum gravity and, most difficult, to learn how it can be used for making practical calculations. As a reward we can hope to get a consistent nonperturbative treatment of not only vector gauge fields, but also gravity.

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