Holographic entropy cone for five regions

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Even though little is known about the quantum entropy cone for $N \geq 4$ subsystems, holographic techniques allow one to get a handle on the subspace of entropy vectors corresponding to states with gravity duals. For static spacetimes and N boundary subsystems, this space is a convex polyhedral cone known as the holographic entropy cone $\mathcal{C}_N$ for N regions. While an explicit description of $\mathcal{C}_N$ was accomplished for all $N \leq 4$ in the initial study, the information given about larger N was only partial already for $\mathcal{C}_5$. This paper provides a complete construction of $\mathcal{C}_5$ by exhibiting graph models for every extreme ray orbit generating the cone defined by all proven holographic entropy inequalities for $N = 5$. The question of whether there exist additional inequalities for five parties is thus settled with a negative answer. The conjecture that $\mathcal{C}_5$ coincides with the analogous cone for dynamical spacetimes is also supported by demonstrating that the information quantities defining its facets are primitive.

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I. INTRODUCTION

The most broadly studied limit of the AdS/CFT correspondence conjectures a holographic duality between certain strongly coupled gauge theories and classical gravity [1,2]. More explicitly, for a pair of gauge-gravity dual theories, the AdS/CFT dictionary poses a specific spacetime geometry as the gravitational counterpart of a given state in the Hilbert space of the quantum theory. On grounds of such a duality, it is of interest to determine which quantum states are dual to classical bulk geometries. A remarkable finding from the study of holographic entanglement is that, regardless of the theory, quantum states with particular patterns of correlations do not admit smooth geometric duals [3,4].

At the heart of this result lies the Ryu-Takayanagi (RT) proposal, which states that for static bulk geometries the entanglement entropy S_A of a spatial region A of the boundary conformal field theory is given by [5–7]

$$S_A = \min_{\mathcal{A}} \frac{\text{area}\mathcal{A}}{4G_N}, \tag{1}$$

where the minimization is performed over all bulk codimension-2 surfaces homologous to A and such that $\partial A = \partial\mathcal{A}$. The Hubeny-Rangamani-Takayanagi (HRT) prescription gives the covariant generalization of RT that applies to arbitrary dynamical spacetimes [7,8]. That the RT formula should reproduce the results of the von Neumann entropy for arbitrary partitions of a quantum state establishes a necessary condition for the existence of a smooth bulk dual. The discovery that there exist valid holographic entropy inequalities which are not true in quantum theory means that this necessary condition is not met by arbitrary quantum states. In particular, this is the case for the inequality known as monogamy of mutual information (MMI) [3],

$$I_2(\mathrm{A}:\mathrm{BC}) \geq I_2(\mathrm{A}:\mathrm{B}) + I_2(\mathrm{A}:\mathrm{C}), \tag{2}$$

defined here in terms of the mutual information $I_2(\mathrm{A}:\mathrm{B}) = S_\mathrm{A} + S_\mathrm{B} - S_\mathrm{AB}$, and where A, B and C stand for three disjoint regions. This inequality has been proven true holographically for arbitrary dynamical spacetimes [3,9], yet is easily violated quantum mechanically (e.g., by the Greenberger-Horne-Zeilinger state).

It follows that characterizing entanglement properties via entropy inequalities provides a powerful criterion to determine whether a quantum state can possibly be geometric (i.e., whether it can be holographically dual to a classical geometry). The formalization of this idea was carried out in Ref. [10], which introduced what is known as the holographic entropy cone to parametrize the space of allowed entropies for geometric states. The purpose of this work is to continue the systematic study and enumeration of holographic entropy inequalities. The picture for four and fewer regions was completed in Ref. [10], where partial results were also found for five regions. Building on this previous work, the complete construction of the holographic entropy cone for five regions is produced here. A direct corollary is

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that no additional entropy inequalities are needed for the completion of the five-party cone.

II. FRAMEWORK AND APPROACH

Let Σ be a spacelike slice of the spacetime manifold of a quantum field theory in a state which admits a holographic description in terms of a smooth bulk geometry. Consider $N \in \mathbb{Z}^+$ arbitrary nonempty codimension-1 disjoint subsets $X_i \subset \Sigma$, where $i \in [N] \equiv \{1, \ldots, N\}$. Any such X_i will be referred to as a monochromatic region of color i. One also defines the set of polychromatic indices $\wp_N$ as the power set of $[N]$ with the empty set removed. The latter has cardinality $D = 2^N - 1$ and its elements $I \in \wp_N$ are used to label polychromatic regions $X_I \equiv \bigcup_{i \in I} X_i$. Denoting the entanglement entropy of each region X_I by S_I, one may construct a D-tuple $\vec{S} \equiv \{S_I | I \in \wp_N\}$. Canonically ordering its entries by increasing cardinality of I and then lexicographically, $\vec{S} \in \mathbb{R}^D$ defines an entropy vector.

Every entropy S_I of a collection of regions can be computed holographically to leading order in the central charge of the boundary theory using the HRT prescription [8]. For static bulk geometries for which this construct reduces to the RT formula (1) [5], the space $\mathcal{C}_N \subset \mathbb{R}^D$ of all physically realizable holographic entropy vectors $\vec{S} \in \mathbb{R}^D$ is known as the holographic entropy cone $\mathcal{C}_N$ for N regions. It was shown in Ref. [10] that this space is indeed a convex cone which is closed, rational and polyhedral. The Farkas-Minkowski-Weyl theorem recasts polyhedrality into the existence of two dual representations of such convex cones [11]:

(1) Facet representation: $\mathcal{C}_N$ can be constructed as the intersection of a finite number of half-spaces specified by entropy inequalities of the form $\vec{S} \cdot \vec{Q}_j \geq 0$, where $\vec{Q}_j \in \mathbb{R}^D$. The minimal collection $\{\vec{Q}_j \in \mathbb{R}^D\}$ of such vectors is unique and geometrically defines the support hyperplanes or facets of the cone.

(2) Extreme ray representation: $\mathcal{C}_N$ can be finitely generated as the conical hull of a set of vectors. The minimal collection of such vectors is unique and consists of the extreme rays $\{\vec{e}_k \in \mathcal{C}_N\}$ of the cone, i.e., the vectors in $\mathcal{C}_N$ which cannot be conically spanned by other vectors in $\mathcal{C}_N$.

Importantly, since $\mathcal{C}_N$ is closed, for every extreme ray $\vec{e}_k \in \mathcal{C}_N$ there exists a bulk geometry and a choice of boundary regions such that their corresponding entropy vector $\vec{S} \propto \vec{e}_k$ [10]. Also, by virtue of being rational, the facet vectors and extreme rays of $\mathcal{C}_N$ can be written with integer coordinates. In particular, by the non-negativity of entanglement entropy, every $\vec{e}_k$ has non-negative integer entries, as will be seen.

Constructing the holographic entropy cone $\mathcal{C}_N$ for N regions amounts to finding a representation of it. A collection of proven entropy inequalities for N parties does not necessarily provide a complete representation of $\mathcal{C}_N$. More specifically, supposing that such a collection of inequalities represents a cone $\tilde{\mathcal{C}}_N$, that they are true entropy inequalities only guarantees that $\mathcal{C}_N \subseteq \tilde{\mathcal{C}}_N$. Proving that the facets of the two cones in fact coincide is better done in the dual description in terms of extreme rays. In particular, if for every extreme ray of $\tilde{\mathcal{C}}_N$ one is able to find a geometry whose entropy vector lies on it, then convexity immediately implies that $\mathcal{C}_N = \tilde{\mathcal{C}}_N$.

This strategy was implemented in Ref. [10] to construct the holographic entropy cones for $N \leq 4$. For $N = 5$, the authors successfully found and proved by contraction five new entropy inequalities, but left as an open question whether this set was complete. A thorough understanding of $\mathcal{C}_5$ has thus been lacking. In this work, the complete representation of the holographic entropy cone $\mathcal{C}_5$ for five regions is provided by explicit construction of its extreme rays. One of the outcomes is that there are no new holographic inequalities for five parties, so that the facets of $\mathcal{C}_5$ are precisely certain upliftings of known inequalities for $N \leq 3$ and the five new ones proven in Ref. [10].

The construction of the extreme rays of $\mathcal{C}_5$ is given here in terms of graph models as introduced in Ref. [10]. The key theorem behind this combinatorial approach is that $\vec{S} \in \mathcal{C}_N$ if and only if there exists a graph model that realizes $\vec{S}$. In other words, the holographic entropy cone and the analogously defined graph-model entropy cone are identical. A graph model for N parties is an undirected graph (V, E) with V vertices and E edges, where a subset $\partial V \subseteq V$ is colored by a map $c: \partial V \to [N]$. As the nomenclature suggests, a vertex colored by i stands as the graph representative of the monochromatic region X_i in the boundary theory. The elements of ∂V are thus called boundary vertices, while those in the complement $V \backslash \partial V$ are called bulk vertices. Edges are assigned non-negative edge capacities by a weight map $E \to \mathbb{R}_{\geq 0}$. Then, the entropy S_I of a polychromatic subset of boundary vertices $\partial V_I \equiv c^{-1}[I] \subset \partial V$ is given by the maximum flow between multisources V_I and multitargets $\partial V \backslash \partial V_I$ which respects the edge capacities. By the max-flow min-cut theorem, this is equivalent to the prescription that defines S_I as the total weight in the minimum cut which disconnects source from sink. Physically, the latter is equivalent to the RT prescription, while the former corresponds to the bit-thread formulation of entanglement [12].

III. THE HOLOGRAPHIC ENTROPY CONE FOR FIVE REGIONS

The action of the symmetric group S_N which relabels the regions X_i clearly leaves $\mathcal{C}_N$ invariant. This symmetry extends to an S_{N+1} symmetry which implements the exchange of any X_i with the purifier $O \equiv \Sigma \backslash \bigcup_{i \in [N]} X_i$. Henceforth, statements about symmetries refer to the

TABLE I. Representatives for each of the eight inequality orbits of the holographic entropy cone $\mathcal{C}_5$ for five regions. Respectively, their orbit lengths are 15, 20, 45, 72, 10, 60, 60 and 90, thus defining 372 facets for $\mathcal{C}_5$ in a 31-dimensional entropy space.

1. $S_{\mathrm{A}} + S_{\mathrm{B}} \geq S_{\mathrm{AB}}$
2. $S_{\mathrm{AB}} + S_{\mathrm{AC}} + S_{\mathrm{BC}} \geq S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}} + S_{\mathrm{ABC}}$
3. $S_{\mathrm{ABC}} + S_{\mathrm{ADE}} + S_{\mathrm{BCDE}} \geq S_{\mathrm{A}} + S_{\mathrm{BC}} + S_{\mathrm{DE}} + S_{\mathrm{ABCDE}}$
4. $S_{\mathrm{ABC}} + S_{\mathrm{ABD}} + S_{\mathrm{ACE}} + S_{\mathrm{BDE}} + S_{\mathrm{CDE}} \geq S_{\mathrm{AB}} + S_{\mathrm{AC}} + S_{\mathrm{BD}} + S_{\mathrm{CE}} + S_{\mathrm{DE}} + S_{\mathrm{ABCDE}}$
5. $S_{\mathrm{ABC}} + S_{\mathrm{ABD}} + S_{\mathrm{ABE}} + S_{\mathrm{ACD}} + S_{\mathrm{ACE}} + S_{\mathrm{ADE}} + S_{\mathrm{BCE}} + S_{\mathrm{BDE}} + S_{\mathrm{CDE}} \geq S_{\mathrm{AB}} + S_{\mathrm{AC}} + S_{\mathrm{AD}} + S_{\mathrm{BE}} + S_{\mathrm{CE}} + S_{\mathrm{DE}} + S_{\mathrm{BCD}} + S_{\mathrm{ABCE}} + S_{\mathrm{ABDE}} + S_{\mathrm{ACDE}}$
6. $3S_{\mathrm{ABC}} + 3S_{\mathrm{ABD}} + S_{\mathrm{ABE}} + S_{\mathrm{ACD}} + 3S_{\mathrm{ACE}} + S_{\mathrm{ADE}} + S_{\mathrm{BCD}} + S_{\mathrm{BCE}} + S_{\mathrm{BDE}} + S_{\mathrm{CDE}} \geq 2S_{\mathrm{AB}} + 2S_{\mathrm{AC}} + S_{\mathrm{AD}} + S_{\mathrm{AE}} + S_{\mathrm{BC}} + 2S_{\mathrm{BD}} + 2S_{\mathrm{CE}} + S_{\mathrm{DE}} + 2S_{\mathrm{ABCD}} + 2S_{\mathrm{ABCE}} + S_{\mathrm{ABDE}} + S_{\mathrm{ACDE}}$
7. $2S_{\mathrm{ABC}} + S_{\mathrm{ABD}} + S_{\mathrm{ABE}} + S_{\mathrm{ACD}} + S_{\mathrm{ADE}} + S_{\mathrm{BCE}} + S_{\mathrm{BDE}} \geq S_{\mathrm{AB}} + S_{\mathrm{AC}} + S_{\mathrm{AD}} + S_{\mathrm{BC}} + S_{\mathrm{BE}} + S_{\mathrm{DE}} + S_{\mathrm{ABCD}} + S_{\mathrm{ABCE}} + S_{\mathrm{ABDE}}$
8. $S_{\mathrm{AD}} + S_{\mathrm{BC}} + S_{\mathrm{ABE}} + S_{\mathrm{ACE}} + S_{\mathrm{ADE}} + S_{\mathrm{BDE}} + S_{\mathrm{CDE}} \geq S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}} + S_{\mathrm{D}} + S_{\mathrm{AE}} + S_{\mathrm{DE}} + S_{\mathrm{BCE}} + S_{\mathrm{ABDE}} + S_{\mathrm{ACD}E}$

TABLE II. Representatives for each of the eight proto-entropic configurations which generate the information quantities associated to each respective inequality orbit as a primitive of the holographic entropy cone $\mathcal{C}_5$ for five regions. [a] Here, $\bar{k} \equiv [5]\backslash\{k\}$ for $k \in [5]$ and $\bar{I} \equiv [5]\backslash I$ for $I \subset [5]$. The notation for building blocks is adapted from Ref. [15]: $\mathscr{C}^\circ[I]$ denotes the canonical building block with a connected surface computing the entropy of I, whereas $\mathscr{C}^*[I(J)]$ and $\mathscr{C}^{\circledast}[I(J)]$ refer to the noncanonical building blocks constructed in Sec. 6 of Ref. [15] with and without a connected surface for J, respectively (see Figs. 5(a), 5(c) and 5(d) in Ref. [15] for respective examples of $\mathscr{C}^\circ$, $\mathscr{C}^*$ and $\mathscr{C}^{\circledast}$).

1. $\bigsqcup_{I\in\wp_N\backslash\{\mathrm{AB}\}} \mathscr{C}^\circ[I]$
2. $\bigsqcup_{I\in\wp_N\backslash\{\mathrm{ABC}\}} \mathscr{C}^\circ[I]$
3. $\bigsqcup_{k\in\{\mathrm{B,C,D,E}\}} \mathscr{C}_5^{\circledast}[k(\bar{k})] \sqcup \mathscr{C}_4^*[\mathrm{B}(\overline{\mathrm{BE}})] \sqcup \mathscr{C}_4^*[\mathrm{B}(\overline{\mathrm{BD}})]$
4. $\bigsqcup_{k\in[5]} \mathscr{C}_5^{\circledast}[k(\bar{k})] \sqcup \mathscr{C}_4^*[\mathrm{A}(\overline{\mathrm{AE}})] \sqcup \mathscr{C}_4^*[\mathrm{A}(\overline{\mathrm{AD}})] \sqcup \mathscr{C}_4^*[\mathrm{B}(\overline{\mathrm{BC}})]$
5. $\bigsqcup_{k\in\{\mathrm{A,B,C,D}\}} \mathscr{C}_5^{\circledast}[k(\bar{k})] \sqcup \mathscr{C}_4^*[\mathrm{B}(\overline{\mathrm{BD}})]$
6. $\bigsqcup_{k\in[5]} \mathscr{C}_5^{\circledast}[k(\bar{k})] \sqcup \mathscr{C}_4^*[\mathrm{B}(\overline{\mathrm{BE}})] \sqcup \mathscr{C}_4^*[\mathrm{C}(\overline{\mathrm{CD}})]$
7. $\bigsqcup_{k\in[5]} \mathscr{C}_5^{\circledast}[k(\bar{k})] \sqcup \mathscr{C}_4^*[\mathrm{A}(\overline{\mathrm{AE}})]$
8. $\mathscr{C}_5^{\circledast}[E(\bar{E})] \sqcup \mathscr{C}_4^*[\mathrm{B}(\overline{\mathrm{BC}})] \sqcup \mathscr{C}_4^*[\mathrm{C}(\overline{\mathrm{BC}})]$

[a] For inequalities 3–8 in Table I, the necessary canonical building blocks required to reach rank $D-1$ can be straightforwardly obtained by completing the span of the orthogonal complement of the associated information quantity and are thus omitted for clarity.

extended symmetry group S_{N+1}. The following subsections detail the description of $\mathcal{C}_5$ in its two representations.

A. Facets

The starting point of the strategy described above is a set of true inequalities for $N = 5$ which is to be proven complete. This set consists of 372 inequalities, which reduce to just eight when quotiented by symmetry. Table I shows a representative inequality for each symmetry orbit [13]. The first three are upliftings of well-known inequalities for $N \leq 3$, whereas the last five are new to $N = 5$. Inequality 1 is the trivial uplifting of subadditivity, whose orbit includes instances of the Araki-Lieb inequality too. Inequalities 2 and 3 are two different upliftings of MMI, which can be more compactly written in terms of the tripartite information as $I_3(\mathrm{A}:\mathrm{B}:\mathrm{C}) \leq 0$ and $I_3(\mathrm{A}:\mathrm{BC}:\mathrm{DE}) \leq 0$, respectively. Inequality 4 is the five-region instance of an infinite family of cyclic entropy inequalities [10,14]. Like inequality 4, the remaining four were proven by contraction for the RT case in Ref. [10]. This set of inequalities defines a cone in entropy space which will be shown to be precisely the holographic entropy cone $\mathcal{C}_5$ for five regions in the next section. A natural question, however, is how to arrive at these inequalities, in particular the last five, in the first place. As for now, only Ref. [15] succeeded in algebraically deriving these as

TABLE III. Representatives for each of the 19 extreme ray orbits of the holographic entropy cone $\mathcal{C}_5$ for five regions.

1. (1 0 0 0 0; 1 1 1 1 0 0 0 0 0 0; 1 1 1 1 1 1 0 0 0 0; 1 1 1 1 0; 1)
2. (1 1 1 0 0; 2 2 1 1 2 1 1 1 1 0; 1 2 2 2 2 1 2 2 1 1; 1 1 2 2 2; 1)
3. (1 1 1 1 0; 2 2 2 1 2 2 1 2 1 1; 3 3 2 3 2 2 3 2 2 2; 2 3 3 3 3; 2)
4. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 3 3 3 3 3 3 3 3 3 3; 2 2 2 2 2; 1)
5. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 3 3 3 3 3 3 3 3 3 3; 4 4 4 4 4; 3)
6. (1 1 1 1 2; 2 2 2 3 2 2 3 2 3 3; 3 3 4 3 4 4 3 4 4 4; 4 3 3 3 3; 2)
7. (1 1 1 2 2; 2 2 3 3 2 3 3 3 3 4; 3 4 4 4 4 5 4 4 5 5; 5 5 4 4 4; 3)
8. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 3 3 3 3 3 3 3 3 3 1; 2 2 2 2 2; 1)
9. (1 1 1 1 2; 2 2 2 32 2 3 2 3 3; 3 3 4 3 4 4 3 4 4 2; 4 3 3 3 3; 2)
10. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 2 3 3 3 3 3 2 3 3 2; 2 2 2 2 2; 1)
11. (1 1 2 2 2; 2 3 3 3 3 3 3 4 4 4; 4 4 4 5 5 3 5 3 5 4; 4 4 4 3 3; 2)
12. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 3 3 2 3 3 2 3 2 3 2; 2 2 2 2 2; 1)
13. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 32 3 3 3 3 3 2 3 2; 2 2 2 2 2; 1)
14. (2 2 2 2 3; 4 4 4 5 4 4 5 4 5 5; 6 4 7 6 7 7 6 5 7 5; 6 5 5 5 5; 3)
15. (3 3 3 3 3; 6 6 6 6 6 6 6 6 6 6; 7 7 5 9 7 7 9 9 9 9; 6 6 6 6 6; 3)
16. (1 1 1 1 1; 2 2 2 2 2 2 2 2 2 2; 3 3 2 2 3 3 2 2 3 3; 2 2 2 2 2; 1)
17. (2 2 2 2 3; 4 4 4 5 4 4 5 4 5 5; 4 6 5 6 7 5 6 7 7 7; 6 5 5 5 5; 3)
18. (3 3 3 3 3; 6 6 6 6 6 6 6 6 6 6; 5 9 7 9 9 7 7 9 9 7; 6 6 6 6 6; 3)
19. (3 3 3 3 3; 6 6 6 6 6 6 6 6 6 6; 7 9 5 7 9 7 9 9 9 7; 6 6 6 6 6; 3)

candidate inequalities using the formalism of the holographic entropy arrangement [15,16].

It is worth remarking that, as defined, $\mathcal{C}_N$ is the space of holographic entropy vectors for states with time-reflection symmetry to which the RT prescription applies. In principle, lifting this restriction to the fully covariant HRT case could allow for a larger space of entropy vectors, the HRT holographic entropy cone $\mathcal{C}_N^{\mathrm{HRT}} \supseteq \mathcal{C}_N$. While the original RT-based proof of strong subadditivity [17] was extended to dynamical setups and proofs of MMI [3,9], it has been argued that the same methods may not be generalizable to nonstatic proofs for the five-party inequalities [18]. Alternative bit-thread-based proofs of MMI [19,20] may lend themselves to generalizations to larger-N inequalities and covariance, but this is yet to be explored. The validity of inequalities 4–8 for dynamical spacetimes thus remains an open question which has only been verified in specific setups [21–23]. However, a suggestive indication that $\mathcal{C}_N^{\mathrm{HRT}}$ is no larger than $\mathcal{C}_N$ is precisely the algebraic derivation of these from the holographic entropy arrangement, which is defined for arbitrary spacetimes. More importantly, all facets of $\mathcal{C}_5$ can be shown to be primitive quantities as defined in Refs. [15,16], thus corresponding to phase transitions of entangling surfaces for arbitrary geometric states. Explicitly, using the proto-entropic formalism and notation for building blocks established in Ref. [15],

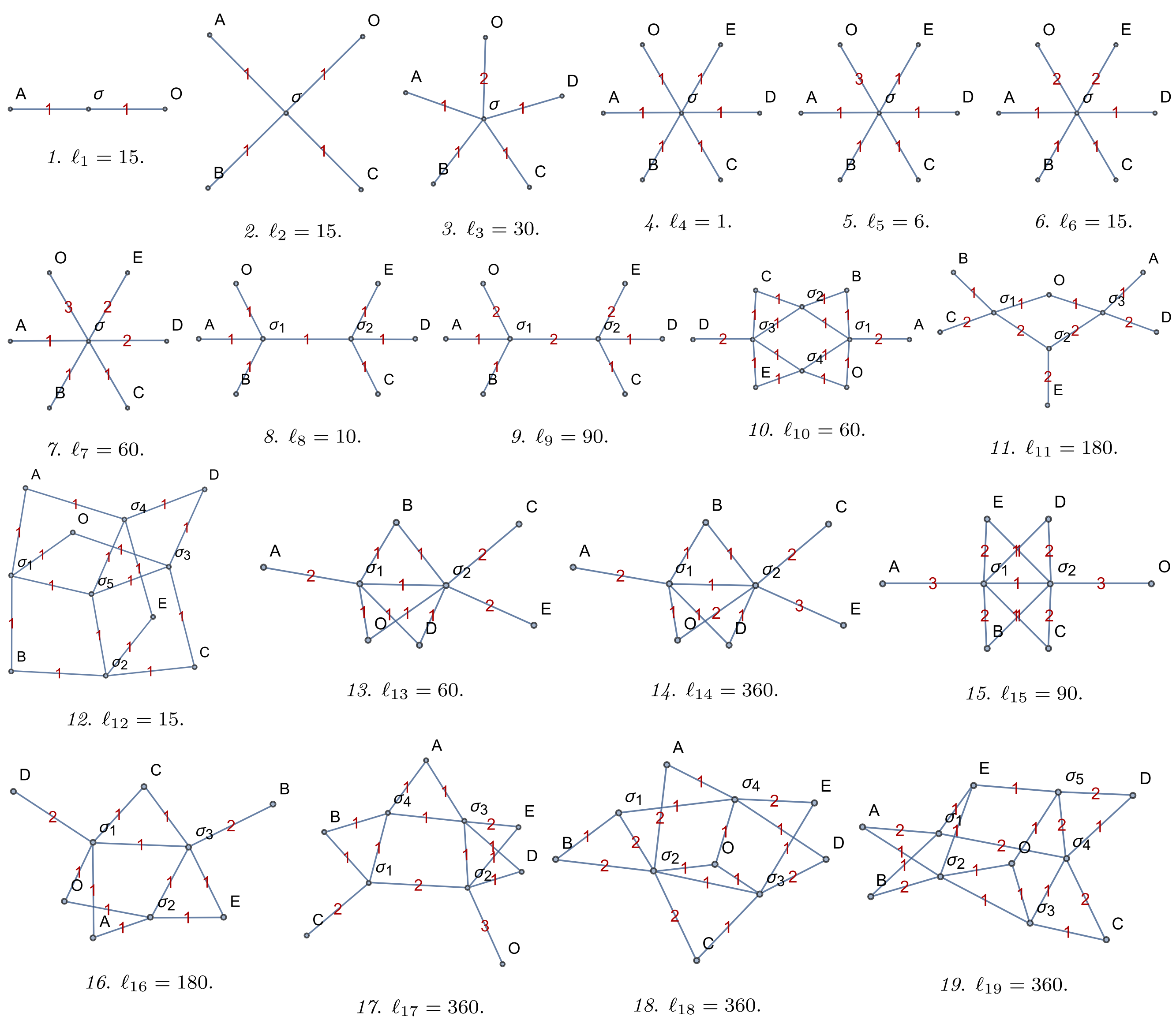

FIG. 1. Graph models realizing the ray representatives in Table III, corresponding to each of the 19 extreme ray orbits of the holographic entropy cone $\mathcal{C}_5$ for five regions. Graphs are numbered according to the extreme ray they generate, and captioned by the length ℓ_k of their orbit. Boundary vertices are labeled by their monochromatic index, bulk vertices by σ_n, with $n \in \mathbb{Z}^+$ enumerating them, and edges are labeled by their capacity. Boundary vertices of pure regions are omitted.

Table II provides a set of configurations which suffice to generate the information quantities associated to inequalities 1–8 as primitive, respectively. It is remarkable that, besides canonical building blocks, only the nonadjoining configurations $\mathscr{C}_4^*$ and $\mathscr{C}_5^{\circledast}$ are needed to generate all facets of the polyhedron for $N = 5$ up to symmetries (see Table II for notation). Note also the necessity of considering non-simply connected boundary topologies with enveloping, for otherwise the I_n theorem would preclude the construction of these quantities as primitive [16]. The configurations in Table II strongly support the conjecture in Ref. [15] that the holographic entropy cone and polyhedron are indeed the same object.

B. Extreme rays

The cone specified above by its 372 facets admits a dual description in terms of 2267 extreme rays. The latter can be grouped into 19 distinct symmetry orbits, such that one may focus on a single representative ray per orbit. Table III shows one such choice of representatives [24], while Fig. 1 provides every graph model needed to construct the holographic entropy cone $\mathcal{C}_5$ for five regions [25]. The first seven rays continue the pattern of being realizable by star graphs, which prove sufficient to generate all extreme rays for $N \leq 4$. However, the other 12 exhibit much richer structure, both in terms of nonplanarity and reduced symmetry.

IV. CONCLUSION

The holographic entropy cone $\mathcal{C}_N$ is now known for all $N \leq 5$. Besides the infinite family of cyclic inequalities, an understanding of the general N case remains elusive. Early explorations of $N = 6$ reveal that $\mathcal{C}_6$ consists of at least 19 valid (i.e., proven by contraction), linearly independent orbits of holographic entropy inequalities. The following is an example of one such six-party inequality [26]: $S_{\mathrm{AB}} + S_{\mathrm{ABC}} + S_{\mathrm{ACD}} + S_{\mathrm{ADE}} + S_{\mathrm{BCD}} + S_{\mathrm{BDE}} + S_{\mathrm{CDE}} + S_{\mathrm{CDF}} + S_{\mathrm{DEF}} + S_{\mathrm{ABCE}} \geq S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{AC}} + S_{\mathrm{BC}} + S_{\mathrm{CD}} + 2S_{\mathrm{DE}} + S_{\mathrm{DF}} + S_{\mathrm{ABE}} + S_{\mathrm{ABCD}} + S_{\mathrm{CDEF}} + S_{\mathrm{ABCDE}}$.

Any constructive approach to exploring $\mathcal{C}_N$ for larger N must overcome the difficulty of dealing with an entropy space of $2^N - 1$ dimensions. Already the dual description problem, for which no efficient algorithm is known, can only be feasibly solved up to symmetry [27]. Moreover, most aspects of the problem suffer a combinatorial explosion which is doubly exponential in N and any hope to proceed constructively must be accompanied by a strategy to tame the combinatorics. In particular, it is indispensable to turn the tables regarding the large degree of redundancy in the structure of $\mathcal{C}_N$ and use its symmetry to one's advantage. Nevertheless, it would ultimately be desirable to understand $\mathcal{C}_N$ for arbitrary N. This will most likely require reducing the problem to an algebraic question rather than a combinatorial one, potentially along the lines of the formalism in Ref. [15,16].

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[13] For the explicit construction of $N = 5$ objects, monochromatic indices are assigned alphabetic values rather than positive integers, and polychromatic indices are written by juxtaposition of letters rather than tuples.

[14] In the literature, this inequality is often quoted with manifest cyclicity of the subsystem labels. For consistency with the notation used for the other inequalities, symbols and their labels are ordered lexicographically here.

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[23] R. J. Caginalp, The holographic entropy cone in AdS-Vaidya spacetimes, arXiv:1905.00544.

[24] Semicolons separate entropy entries corresponding to polychromatic indices of different cardinality. The ordering is given by: (A, B, C, D, E; AB, AC, AD, AE, BC, BD, BE, CD, CE, DE; ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE; ABCD, ABCE, ABDE, ACDE, BCDE; ABCDE).

[25] In producing these graphs, the choice has been to fix the number of boundary vertices to $5 + 1$, such that the coloring map is bijective for the five regions and their purifier O. An explicit construction of an associated wormhole geometry can be accomplished by operations that bring the graph to a suitable form without changing its entropies, as explained in Ref. [10].

[26] This inequality has been proven using the contraction-map method introduced in Ref. [10]. It can also be generated as a primitive quantity using the building blocks from [15], thus demonstrating that it is a facet of $\mathcal{C}_6$. The full construction of $\mathcal{C}_6$ is work in progress.

[27] D. Bremner, M. D. Sikiric, and A. Schuermann, Polyhedral representation conversion up to symmetries, CRM Proc. Lect. Notes **48**, 45 (2007).