Dual 2 + 1D loop quantum gravity on the edge

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In a recent paper, we introduced a new discretization scheme for gravity in $2+1$ dimensions. Starting from the continuum theory, this new scheme allowed us to rigorously obtain the discrete phase space of loop gravity, coupled to particlelike "edge mode" degrees of freedom. In this work, we expand on that result by considering the most general choice of integration during the discretization process. We obtain a family of polarizations of the discrete phase space. In particular, one member of this family corresponds to the usual loop gravity phase space, while another corresponds to a new polarization, dual to the usual one in several ways. We study its properties, including the relevant constraints and the symmetries they generate. Furthermore, we motivate a relation between the dual polarization and teleparallel gravity.

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I. INTRODUCTION

The theory of general relativity famously describes gravity as a result of the curvature of spacetime itself. Furthermore, the geometry of spacetime is assumed to be torsionless by employing the Levi-Civita connection, which is torsionless by definition. While this is the most popular formulation, there exists an alternative but mathematically equivalent formulation called teleparallel gravity [1–3], differing from general relativity only by a boundary term. In this formulation, one instead uses the Weitzenböck connection, which is flat by definition. The gravitational degrees of freedom (d.o.f.) are then encoded in the torsion of the spacetime geometry.

Loop quantum gravity [4] is a popular approach towards the formulation of a consistent and physically relevant theory of quantum gravity. In the canonical version of the theory [5], one starts by rewriting general relativity in the Hamiltonian formulation and quantizing using the familiar Dirac procedure [6]. One finds a fully constrained system, that is, the Hamiltonian is simply a sum of constraints.

In $2+1$ spacetime dimensions, where gravity is topological [7], there are two such constraints:

(i) the Gauss (or torsion) constraint, which imposes zero torsion everywhere, and

(ii) the curvature (or flatness) constraint, which imposes zero curvature everywhere.

In the classical theory, it does not matter which constraint is imposed first. However, in the quantum theory, it does matter, since the Hilbert space is defined in terms of representations of the symmetries generated by the constraints. The first constraint that we impose is used to define the kinematics of the theory, while the second constraint will encode the dynamics. Thus, it seems natural to identify general relativity with the quantization in which the Gauss constraint is imposed first, and teleparallel gravity with that in which the curvature constraint is imposed first.

Indeed, in loop quantum gravity, which is a quantization of general relativity, the Gauss constraint is imposed first. This is done by selecting, as the basis for the kinematical Hilbert space, the spin network basis [8] of rotation-invariant states. Then, the curvature constraint is imposed at the dynamical level in order to obtain the Hilbert space of physical states.

In [9], an alternative choice was suggested where the order of constraints is reversed. The curvature constraint is imposed first by employing the group network basis of translation-invariant states, and the Gauss constraint is the one which encodes the dynamics. This dual loop quantum gravity quantization is the quantum counterpart of teleparallel gravity, and could be used to study the dual vacua proposed in [10,11].

In this paper, we will only deal with the classical theory. We will explore a family of discretizations which includes, in particular, three cases of interest:

(i) The loop gravity phase space, which is the classical version of the spin network basis [12]. This case was studied in detail in our paper [13] and is related to $2+1$D general relativity. We will provide a more rigorous derivation of some results, in particular the discrete curvature constraint, and additional subtle details which were missing in our initial treatment. The phase space obtained in this case contains the phase space of spin networks, plus curvature and torsion excitations corresponding to edge modes which do not cancel.

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(ii) Dual loop gravity, which is the classical version of the group network basis. This case was first studied in [14] in the simple case where there are no curvature or torsion excitations. It is intuitively related to teleparallel gravity. Here, we will study this case carefully, incorporating the edge modes as was done in [13] for the loop gravity case. We will rigorously derive the discrete constraints and the symmetry transformations they generate. The resulting phase space will contain the phase space of group networks, plus the same curvature and torsion obtained in the previous case.

(iii) A mixed phase space, containing both loop gravity and its dual, which is intuitively related to Chern-Simons theory [15], as we will motivate below. In this case our formalism should be related to existing results [16–22].

Crucial to our formalism is the separation of discretization into two steps. This procedure was first utilized, in the $3+1$-dimensional case, in [23,24], but without considering any curvature and torsion. The steps are as follows:

(1) Subdivision, or decomposition into subsystems. More precisely, we define a cellular decomposition[1] on our 2-dimensional spatial manifold. This structure has a dual structure, which as we will see, will be the spin network graph.

(2) Truncation, or coarse-graining of the subsystems. In this step, we assume that there is arbitrary curvature and torsion inside each loop of the spin network. We then "compress" the information about the geometry into a single point, or vertex, inside the loop. Since the only way to probe the geometry is by looking at the holonomies and fluxes on the loops of the spin network, the observables before and after this truncation are the same.

The edge modes, mentioned earlier, are the final piece of our formalism. When discretizing gauge theories, and gravity in particular, a major problem is preserving gauge invariance despite the discreteness of the resulting theory. The presence of boundaries can be shown to introduce new d.o.f., called edge modes [25–27],[2] which may be used to *dress* observables and make them gauge invariant. These edge modes are associated to new boundary symmetries, which transform them and control the gluing map between subsystems.

As we will see below, the edge modes at the boundaries of the cells in our cellular decomposition will mostly cancel with the edge modes on the boundaries of the adjacent cells. However, there will also be edge modes at the vertices of the cells, which will not have anything to cancel with. These d.o.f. will survive the discretization process, and introduce a particlelike phase space [32,33] for the curvature and torsion, which we then interpret as mass and spin respectively.

One might expect that the geometry will be encoded in the constraints alone, by imposing that a loop of holonomies sees the curvature inside it and a loop of fluxes sees the torsion inside it. As we will see, while the constraints do indeed encode the geometry, the presence of the edge modes enforces the inclusion of the curvature and torsion themselves as additional phase space variables.

A. Basic definitions and notation

Consider a group $G \ltimes \mathfrak{g}^* \cong T^*G$, which is[3] a generalization of the Euclidean or Poincaré group. One possible option is

$$\mathrm{ISU}(2) \cong \mathrm{SU}(2) \ltimes \mathbb{R}^3, \tag{1}$$

but we will keep it general. The algebra for this group is given by

$$[\mathbf{P}_i, \mathbf{P}_j] = 0, \qquad [\mathbf{J}_i, \mathbf{J}_j] = f_{ij}{}^k \mathbf{J}_k, \qquad [\mathbf{J}_i, \mathbf{P}_j] = f_{ij}{}^k \mathbf{P}_k, \tag{2}$$

where $f_{ij}{}^k$ are the structure constants.[4] The algebra indices i, j, k go from 1 to $\dim \mathfrak{g}$, which is e.g., 3 for $\mathfrak{su}(2)$. The generators $\mathbf{J}_i$ are the rotation generators, and they correspond to a non-Abelian group G, while the generators $\mathbf{P}_i$ are the translation generators, and they correspond to an Abelian normal subgroup $\mathfrak{g}^*$.

Notation-wise, all Lie algebra elements and Lie-algebra-valued forms will be written in bold font to distinguish them from Lie group element or Lie-group-valued forms. Furthermore, we will use calligraphic font for $G \ltimes \mathfrak{g}^*$ or $\mathfrak{g} \oplus \mathfrak{g}^*$-valued forms (which will rarely be of interest) and Roman font for G, $\mathfrak{g}$ or $\mathfrak{g}^*$-valued forms.

Given any two Lie-algebra-valued forms $\mathbf{A}$, $\mathbf{B}$ of degrees $\deg \mathbf{A}$ and $\deg \mathbf{B}$ respectively, we define the graded commutator:

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{A} \wedge \mathbf{B} - (-1)^{\deg \mathbf{A} \deg \mathbf{B}} \mathbf{B} \wedge \mathbf{A}. \tag{3}$$

We also define a dot (inner) product, also known as the Killing form, on the generators as follows:

$$\mathbf{J}_i \cdot \mathbf{P}_j = \delta_{ij}, \qquad \mathbf{J}_i \cdot \mathbf{J}_j = \mathbf{P}_i \cdot \mathbf{P}_j = 0. \tag{4}$$

Given two Lie-algebra-valued forms, the dot product is defined to include a wedge product. Thus, if $\mathbf{A} \equiv A^i \mathbf{J}_i$ is a

[1]The cells in this decomposition can take any shape.

[2]See also [28] for a more intuitive discussion and [29–31] for the case of $3+1$-dimensional gravity.

[3]The notation T^*G signifies the cotangent bundle of G.

[4]They satisfy antisymmetry $f_{ij}{}^k = -f_{ij}{}^k$ and the Jacobi identity $f_{[ij}{}^l f_{k]l}{}^m = 0$. For $\mathfrak{su}(2)$ we have $f_{ij}{}^k = \epsilon_{ij}{}^k$ where $\epsilon_{ij}{}^k$ is the Levi-Civita symbol.

pure rotation and $\mathbf{B} \equiv B^i \mathbf{P}_i$ is a pure translation, which will usually be the case,[5] we have

$$\mathbf{A} \cdot \mathbf{B} \equiv A^i \wedge B_i. \tag{6}$$

Finally, in addition to the exterior derivative d and the interior product ι on spacetime, we introduce a variational exterior derivative δ and a variational interior product I on field space. These operators act analogously to d and ι, and in particular they are nilpotent, e.g., $\delta^2 = 0$, and satisfy the graded Leibniz rule. Degrees of differential forms are counted with respect to spacetime and field space separately; for example, if f is a 0-form then $\mathrm{d}\delta f$ is a 1-form on spacetime, due to d, and independently also a 1-form on field space, due to δ. The dot product defined above also includes an implicit wedge product with respect to field-space forms, such that e.g., $\delta \mathbf{f} \cdot \delta \mathbf{g} = -\delta \mathbf{g} \cdot \delta \mathbf{f}$ if $\mathbf{f}$ and $\mathbf{g}$ are 0-forms.

B. The Chern-Simons action and 2 + 1D gravity

Let M be a $2+1$-dimensional spacetime manifold and let Σ be a 2-dimensional spatial manifold such that $M = \Sigma \times \mathbb{R}$ where $\mathbb{R}$ represents time. Let us also define the Chern-Simons connection 1-form $\mathcal{A}$, valued in $\mathfrak{g} \oplus \mathfrak{g}^*$:

$$\mathcal{A} \equiv \mathbf{A} + \mathbf{E} \equiv A^i \mathbf{J}_i + E^i \mathbf{P}_i, \tag{7}$$

where $\mathbf{A} \equiv A^i \mathbf{J}_i$ is the $\mathfrak{g}$-valued connection 1-form and $\mathbf{E} \equiv E^i \mathbf{P}_i$ is the $\mathfrak{g}^*$-valued frame field 1-form. The $\mathfrak{g} \oplus \mathfrak{g}^*$-valued curvature 2-form $\mathcal{F}$ is then defined as

$$\mathcal{F} \equiv \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}], \tag{8}$$

and it may be split into

$$\mathcal{F} \equiv \mathbf{F} + \mathbf{T} \equiv F^i \mathbf{J}_i + T^i \mathbf{P}_i, \tag{9}$$

where $\mathbf{F} \equiv F^i \mathbf{J}_i$ is the $\mathfrak{g}$-valued curvature 2-form and $\mathbf{T} \equiv T^i \mathbf{P}_i$ is the $\mathfrak{g}^*$-valued torsion 2-form, and they are defined in terms of $\mathbf{A}$ and $\mathbf{E}$ as

$$\mathbf{F} \equiv \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}], \qquad \mathbf{T} \equiv \mathrm{d}_{\mathbf{A}} \mathbf{E} \equiv \mathrm{d}\mathbf{E} + [\mathbf{A}, \mathbf{E}], \tag{10}$$

where $\mathrm{d}_{\mathbf{A}} \equiv \mathrm{d} + [\mathbf{A}, \cdot]$ is the covariant exterior derivative.

In our notation, the Chern-Simons action is given by

$$S[\mathcal{A}] = \frac{1}{2} \int_M \mathcal{A} \cdot \left(\mathrm{d}\mathcal{A} + \frac{1}{3}[\mathcal{A}, \mathcal{A}] \right), \tag{11}$$

and its variation is

$$\delta S[\mathcal{A}] = \int_M \left(\mathcal{F} \cdot \delta\mathcal{A} - \frac{1}{2}\mathrm{d}(\mathcal{A} \cdot \delta\mathcal{A}) \right). \tag{12}$$

From this we can read the equation of motion

$$\mathcal{F} = 0, \tag{13}$$

and, from the boundary term, the symplectic potential

$$\Theta[\mathcal{A}] \equiv -\frac{1}{2} \int_\Sigma \mathcal{A} \cdot \delta\mathcal{A}, \tag{14}$$

which gives us the symplectic form

$$\Omega[\mathcal{A}] \equiv \delta\Theta[\mathcal{A}] = -\frac{1}{2} \int_\Sigma \delta\mathcal{A} \cdot \delta\mathcal{A}. \tag{15}$$

Furthermore, we can write the action[6] in terms of $\mathbf{A}$ and $\mathbf{E}$:

$$S[\mathbf{A}, \mathbf{E}] = \int_M \left(\mathbf{E} \cdot \mathbf{F} - \frac{1}{2}\mathrm{d}(\mathbf{A} \cdot \mathbf{E}) \right). \tag{17}$$

This is the action for $2+1$D gravity, with an additional boundary term (which is usually disregarded by assuming M has no boundary). Using the identity $\delta\mathbf{F} = \mathrm{d}_{\mathbf{A}}\delta\mathbf{A}$, we find the variation of the action is

$$\delta S[\mathbf{A}, \mathbf{E}] = \int_M \left(\mathbf{F} \cdot \delta\mathbf{E} + \mathbf{T} \cdot \delta\mathbf{A} - \frac{1}{2}\mathrm{d}(\mathbf{E} \cdot \delta\mathbf{A} + \mathbf{A} \cdot \delta\mathbf{E}) \right), \tag{18}$$

and thus we see that the equations of motion are

$$\mathbf{F} = 0, \qquad \mathbf{T} = 0, \tag{19}$$

and the symplectic potential is

$$\Theta[\mathbf{A}, \mathbf{E}] \equiv -\frac{1}{2} \int_\Sigma (\mathbf{E} \cdot \delta\mathbf{A} + \mathbf{A} \cdot \delta\mathbf{E}). \tag{20}$$

Of course, (19) and (20) may be easily derived from (13) and (14).

[5]In the general case, which will only be relevant for our discussion of Chern-Simons theory in the next subsection, for $\mathfrak{g} \oplus \mathfrak{g}^*$-valued forms $\mathcal{A} \equiv \mathcal{A}_J^i \mathbf{J}_i + \mathcal{A}_P^i \mathbf{P}_i$ and $\mathcal{B} \equiv \mathcal{B}_J^i \mathbf{J}_i + \mathcal{B}_P^i \mathbf{P}_i$ we have

$$\mathcal{A} \cdot \mathcal{B} = \delta_{ij} (\mathcal{A}_J^i \wedge \mathcal{B}_P^j + \mathcal{A}_P^i \wedge \mathcal{B}_J^j). \tag{5}$$

[6]Here we use the following identities, derived from the properties of the dot product (4) and the graded commutator:

$$\mathbf{A} \cdot \mathrm{d}\mathbf{A} = \mathbf{E} \cdot \mathrm{d}\mathbf{E} = [\mathbf{E}, \mathbf{E}] = \mathbf{A} \cdot [\mathbf{A}, \mathbf{A}] = \mathbf{E} \cdot [\mathbf{A}, \mathbf{E}] = 0. \tag{16}$$

C. Phase space polarizations and teleparallel gravity

The symplectic potential (20) results in the symplectic form

$$\Omega \equiv \delta\Theta = -\int_\Sigma \delta\mathbf{E}\cdot\delta\mathbf{A}. \tag{21}$$

In fact, one may obtain the same symplectic form using a family of potentials of the form

$$\Theta_\lambda = -\int_\Sigma ((1-\lambda)\mathbf{E}\cdot\delta\mathbf{A} + \lambda\mathbf{A}\cdot\delta\mathbf{E}), \tag{22}$$

where the parameter $\lambda \in [0,1]$ determines the polarization of the phase space. This potential may be obtained from a family of actions of the form

$$S_\lambda = \int_M (\mathbf{E}\cdot\mathbf{F} - \lambda \mathrm{d}(\mathbf{A}\cdot\mathbf{E})), \tag{23}$$

where the difference lies only in the boundary term and thus does not affect the physics. Hence the choice of polarization does not matter in the continuum, but it will be very important in the discrete theory, as we will see below.

The equations of motion for any action of the form (23) (or constraints, in the Hamiltonian formulation) are, as we have seen, as follows:

(i) the torsion (or Gauss) constraint $\mathbf{T}=0$, and
(ii) the curvature constraint $\mathbf{F}=0$.

Now, recall that general relativity is formulated using the Levi-Civita connection, which is torsionless by definition. Thus, the torsion constraint $\mathbf{T}=0$ can really be seen as defining the connection $\mathbf{A}$ to be torsionless, and thus selecting the theory to be general relativity. In this case, $\mathbf{F}=0$ is the true equation of motion, describing the dynamics of the theory.

In the teleparallel formulation of gravity we instead use the Weitzenböck connection, which is defined to be flat but not necessarily torsionless. In this formulation, we interpret the curvature constraint $\mathbf{F}=0$ as defining the connection $\mathbf{A}$ to be flat, while $\mathbf{T}=0$ is the true equation of motion.

There are three cases of interest when considering the choice of the parameter λ. The case $\lambda=0$ is the one most suitable for $2+1$D general relativity:

$$S_{\lambda=0} = \int_M \mathbf{E}\cdot\mathbf{F}, \qquad \Theta_{\lambda=0} = -\int_\Sigma \mathbf{E}\cdot\delta\mathbf{A}, \tag{24}$$

since it indeed produces the familiar action for $2+1$D gravity. The case $\lambda=1/2$ is one most suitable for $2+1$D Chern-Simons theory:

$$S_{\lambda=\frac{1}{2}} = \int_M \left(\mathbf{E}\cdot\mathbf{F} - \frac{1}{2}\mathrm{d}(\mathbf{A}\cdot\mathbf{E})\right),$$
$$\Theta_{\lambda=\frac{1}{2}} = -\frac{1}{2}\int_\Sigma (\mathbf{E}\cdot\delta\mathbf{A} + \mathbf{A}\cdot\delta\mathbf{E}), \tag{25}$$

since it corresponds to the Chern-Simons action (17). Finally, the case $\lambda=1$ is one most suitable for $2+1$D teleparallel gravity:

$$S_{\lambda=1} = \int_M (\mathbf{E}\cdot\mathbf{F} - \mathrm{d}(\mathbf{A}\cdot\mathbf{E})), \qquad \Theta_{\lambda=1} = -\int_\Sigma \mathbf{A}\cdot\delta\mathbf{E}, \tag{26}$$

as explained in [34].

Further details about the different polarizations may be found in [14]. However, the discretization procedure in that paper did not take into account possible curvature and torsion d.o.f. In the rest of this paper, we will include these d.o.f. in the discussion by generalizing our results in [13] to include all possible polarizations of the phase space.

II. THE DISCRETE GEOMETRY

A. The cellular decomposition and its dual

We embed a cellular decomposition Δ and a dual cellular decomposition Δ^* in our 2-dimensional spatial manifold Σ. These structures consist of the following elements, where each element of Δ is uniquely dual to an element of Δ^*:

Δ		Δ^*
0-cells (vertices) v	dual to	2-cells (faces) f_v
1-cells (edges) e	dual to	1-cells (links) e^*
2-cells (cells) c	dual to	0-cells (nodes) c^*

The 1-skeleton graph $\Gamma \subset \Delta$ is the set of all vertices and edges of Δ. Its dual is the spin network graph $\Gamma^* \subset \Delta^*$, the set of all nodes and links of Δ^*. Both graphs are oriented, and we write $e=(vv')$ to indicate that the edge e starts at the vertex v and ends at v', and $e^*=(cc')^*$ to indicate that the link e^* starts at the node c^* and ends at c'^*. Furthermore, since edges are where two cells intersect, we write $e=(cc') \equiv \partial c \cap \partial c'$ to denote that the edge e is the intersection of the boundaries ∂c and $\partial c'$ of the cells c and c' respectively. If the link e^* is dual to the edge e, then we have that $e=(cc')$ and $e^*=(cc')^*$; therefore the notation is consistent. This construction is illustrated in Fig. 1 (taken from [13]).

For the purpose of doing calculations, it will prove useful to introduce disks D_v around each vertex v. The disks have a radius R, small enough that the entire disk D_v is inside the face f_v for every v. We also define punctured disks v^*, which are obtained from the full disks D_v by removing the vertex v, which is at the center, and a cut C_v, connecting v to an arbitrary point v_0 on the boundary ∂D_v. Thus

$$v^* \equiv D_v \backslash (\{v\} \cup C_v). \tag{27}$$

The punctured disks are equipped with a cylindrical coordinate system (r_v, ϕ_v) such that $r_v \in (0,R)$ and $\phi_v \in (\alpha_v - \frac{1}{2}, \alpha_v + \frac{1}{2})$; note that ϕ_v is scaled by 2π, so it

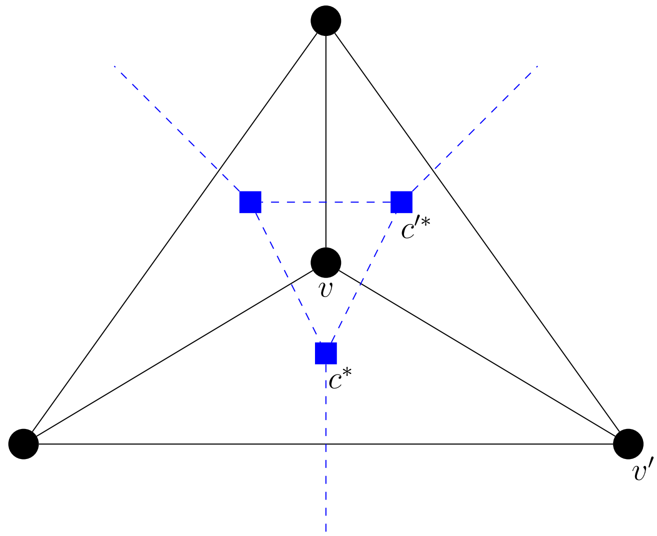

FIG. 1. A simple piece of the cellular decomposition Δ, in black, and its dual spin network Γ^*, in blue. The vertices v of the 1-skeleton $\Gamma \subset \Delta$ are shown as black circles, while the nodes c^* of Γ^* are shown as blue squares. The edges $e \in \Gamma$ are shown as black solid lines, while the links $e^* \in \Gamma^*$ are shown as blue dashed lines. In particular, two nodes c^* and c'^*, connected by a link $e^* = (cc')^*$, are labeled, as well as two vertices v and v', connected by an edge $e = (vv') = (cc') = c \cap c'$, which is dual to the link e^*. There is one face in the illustration, f_v, which is the triangle enclosed by the three blue links at the center.

has a period of 1, for notational brevity. The boundary of the punctured disk is such that

$$\partial v^* = \partial_0 v^* \cup C_v \cup \partial_R v^*, \qquad (28)$$

where $\partial_0 v^*$ is the inner boundary at $r_v = 0$, C_v is the cut at $\phi_v = \alpha_v - \frac{1}{2}$, and $\partial_R v^*$ is the outer boundary at $r_v = R$, and the point where the cut meets the outer boundary is $v_0 \equiv (R, \alpha_v - \frac{1}{2})$. Note that $\partial_R v^* = \partial D_v$. The punctured disk is illustrated in Fig. 2 (taken from [13]).

The outer boundary $\partial_R v^*$ of each disk is composed of arcs (vc_i) such that

$$\partial_R v^* = \bigcup_{i=1}^{N_v} (vc_i), \qquad (29)$$

where N_v is the number of cells around v and the cells are enumerated $c_1, \ldots, c_{N_v}$. Similarly, the boundary ∂c of the cell c is composed of edges (cc_i) and arcs (cv_i) such that

$$\partial c = \bigcup_{i=1}^{N_c} ((cc_i) \cup (cv_i)), \qquad (30)$$

where N_c is the number of cells adjacent to c or, equivalently, the number of vertices around c. We will use these decompositions during the discretization process.

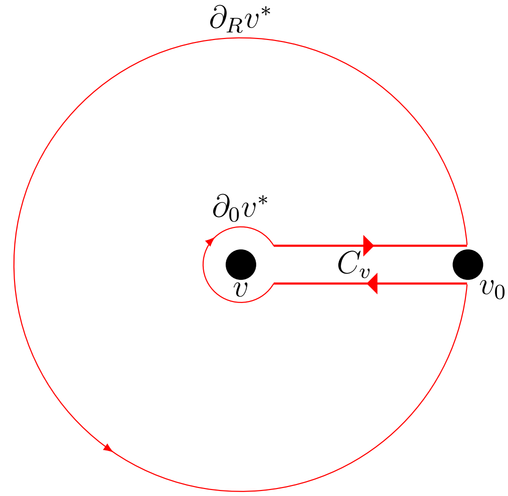

FIG. 2. The punctured disk v^*. The figure shows the vertex v, cut C_v, inner boundary $\partial_0 v^*$, outer boundary $\partial_R v^*$, and reference point v_0.

B. Truncating the geometry to the vertices

1. Motivation

Before the equations of motion (i.e., the curvature and torsion constraints $\mathbf{F} = \mathbf{T} = 0$) are applied, the geometry on Σ can have arbitrary curvature and torsion. We would like to capture the "essence" of the curvature and torsion and encode them on codimension 2 defects.

For this purpose, we can imagine looking at every possible loop on the spin network graph Γ^* and taking a holonomy in $G \ltimes \mathfrak{g}^*$ around it. This holonomy will have a part valued in $\mathfrak{g}$, which will encode the curvature, and a part valued in $\mathfrak{g}^*$, which will encode the torsion.

A loop of the spin network is the boundary ∂f_v of a face f_v. Since the face is dual to a vertex v, the natural place to encode the geometry would be at the vertex. Thus, we will place the defects at the vertices, and give them the appropriate values in $\mathfrak{g} \oplus \mathfrak{g}^*$ obtained by the holonomies.

The disks D_v defined above are in a 1-to-1 correspondence with the faces f_v. In fact, we can imagine deforming the disks such that they cover the faces, and their boundaries ∂D_v are exactly the loops ∂f_v. Thus, we may perform calculations on the disks instead on the faces.

This intuitive and qualitative motivation will be made precise in the following subsections.

2. The Chern-Simons connection on the disks

We define the Chern-Simons[7] connection on the punctured disk v^* as follows:

[7]Recall that we use calligraphic font to denote forms valued in the double $G \ltimes \mathfrak{g}^*$, and bold calligraphic font for forms valued in its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$.

$$\mathcal{A}|_{v^*} \equiv \overset{\circ}{\mathcal{H}}_v^{-1} \mathrm{d}\overset{\circ}{\mathcal{H}}_v \equiv \mathcal{H}_v^{-1}\mathrm{d}\mathcal{H}_v + \mathcal{H}_v^{-1}\boldsymbol{\mathcal{M}}_v\mathcal{H}_v\mathrm{d}\phi_v, \quad (31)$$

where

(i) $\overset{\circ}{\mathcal{H}}_v$ is a nonperiodic $G \ltimes \mathfrak{g}^*$-valued 0-form defined as $\overset{\circ}{\mathcal{H}}_v \equiv \mathrm{e}^{\boldsymbol{\mathcal{M}}_v\phi_v}\mathcal{H}_v$,

(ii) $\mathcal{H}_v$ is a periodic[8] $\mathfrak{g} \oplus \mathfrak{g}^*$-valued 0-form,

(iii) $\boldsymbol{\mathcal{M}}_v$ is a constant element of the Cartan subalgebra $\mathfrak{h} \oplus \mathfrak{h}^*$ of $\mathfrak{g} \oplus \mathfrak{g}^*$.

Note that this connection is related by a gauge transformation of the form $\boldsymbol{\mathcal{A}}_0 \mapsto \mathcal{H}_v^{-1}\mathrm{d}\mathcal{H}_v + \mathcal{H}_v^{-1}\boldsymbol{\mathcal{A}}_0\mathcal{H}_v$ to a connection $\boldsymbol{\mathcal{A}}_0$ defined as follows:

$$\boldsymbol{\mathcal{A}}_0 \equiv \boldsymbol{\mathcal{M}}_v\mathrm{d}\phi_v. \quad (32)$$

The connection $\boldsymbol{\mathcal{A}}_0$ satisfies $[\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{A}}_0] = 0$, so its curvature is $\boldsymbol{\mathcal{F}}_0 \equiv \mathrm{d}\boldsymbol{\mathcal{A}}_0$. This curvature vanishes everywhere on the punctured disk (which excludes the point v), since $\mathrm{d}^2\phi_v = 0$. However, at the origin of our coordinate system, i.e., the vertex v, ϕ_v is not well defined, so we cannot guarantee that $\boldsymbol{\mathcal{F}}_0$ vanishes at v itself.

In fact, we can show that it must not vanish there. If we integrate the curvature on the full disk D_v using Stokes' theorem, we get

$$\int_{D_v} \boldsymbol{\mathcal{F}}_0 = \oint_{\partial D_v} \boldsymbol{\mathcal{A}}_0 = \boldsymbol{\mathcal{M}}_v \oint_{\partial D_v} \mathrm{d}\phi_v = \boldsymbol{\mathcal{M}}_v, \quad (33)$$

where $\oint_{\partial D_v} \mathrm{d}\phi_v = 1$ since we are using coordinates scaled by 2π, and we used the fact that $\boldsymbol{\mathcal{M}}_v$ is constant. We conclude that, since $\boldsymbol{\mathcal{F}}_0$ vanishes everywhere on v^*, and yet it integrates to a finite value at D_v, the curvature $\boldsymbol{\mathcal{F}}_0$ must take the form of a Dirac delta function centered at v:

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{M}}_v\delta(v), \quad (34)$$

where $\delta(v)$ is a distributional 2-form such that for any 0-form f,

$$\int_\Sigma f\delta(v) \equiv f(v). \quad (35)$$

The final step is to gauge-transform back from $\boldsymbol{\mathcal{A}}_0$ to the initial connection $\boldsymbol{\mathcal{A}}$ defined in (31). The curvature transforms in the usual way, $\boldsymbol{\mathcal{F}}_0 \mapsto \mathcal{H}_v^{-1}\boldsymbol{\mathcal{F}}_0\mathcal{H}_v \equiv \boldsymbol{\mathcal{F}}$, so we get

$$\boldsymbol{\mathcal{F}}|_{D_v} = \mathcal{H}_v^{-1}\boldsymbol{\mathcal{M}}_v\mathcal{H}_v\delta(v) \equiv \boldsymbol{\mathcal{P}}_v\delta(v), \quad (36)$$

where we defined

$$\boldsymbol{\mathcal{P}}_v \equiv \mathcal{H}_v^{-1}\boldsymbol{\mathcal{M}}_v\mathcal{H}_v. \quad (37)$$

Note again that, while $\boldsymbol{\mathcal{F}}|_{D_v}$ (on the full disk) does not vanish, $\boldsymbol{\mathcal{F}}|_{v^*}$ (on the punctured disk) does vanish.

3. The connection and frame field on the disks

Now that we have defined the Chern-Simons connection 1-form $\boldsymbol{\mathcal{A}}$ and found its curvature $\boldsymbol{\mathcal{F}}$ on the disks, we split $\boldsymbol{\mathcal{A}}$ into a $\mathfrak{g}$-valued connection 1-form $\mathbf{A}$ and a $\mathfrak{g}^*$-valued frame field 1-form $\mathbf{E}$ as defined in (7). Similarly, we split $\boldsymbol{\mathcal{F}}$ into a $\mathfrak{g}$-valued curvature 2-form $\mathbf{F}$ and a $\mathfrak{g}^*$-valued torsion 2-form $\mathbf{T}$ as defined in (9).

From (7) we get

$$\mathbf{A}|_{v^*} = \overset{\circ}{h}_v^{-1}\mathrm{d}\overset{\circ}{h}_v, \qquad \mathbf{E}|_{v^*} = \overset{\circ}{h}_v^{-1}\mathrm{d}\overset{\circ}{\mathbf{x}}_v\overset{\circ}{h}_v, \quad (38)$$

where

(i) $\overset{\circ}{h}_v$ is a nonperiodic G-valued 0-form and $\overset{\circ}{\mathbf{x}}_v$ is a nonperiodic $\mathfrak{g}^*$-valued 0-form such that[9]

$$\overset{\circ}{h}_v \equiv \mathrm{e}^{\mathbf{M}_v\phi_v}h_v, \qquad \overset{\circ}{\mathbf{x}}_v \equiv \mathrm{e}^{\mathbf{M}_v\phi_v}(\mathbf{x}_v + \mathbf{S}_v\phi_v)\mathrm{e}^{-\mathbf{M}_v\phi_v}, \quad (39)$$

(ii) h_v is a periodic G-valued 0-form,

(iii) $\mathbf{x}_v$ is a periodic $\mathfrak{g}^*$-valued 0-form,

(iv) $\mathbf{M}_v$ is a constant element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, such that $\mathbf{M}_v \equiv M_v\mathbf{J}_1$ where $\mathbf{J}_1$ is the Cartan generator,

(v) $\mathbf{S}_v$ is a constant element of the Cartan subalgebra $\mathfrak{h}^*$ of $\mathfrak{g}^*$, such that $\mathbf{S}_v \equiv S_v\mathbf{P}_1$ where $\mathbf{P}_1$ is the Cartan generator, and

(vi) by construction $[\mathbf{M}_v, \mathbf{S}_v] = 0$.

The full expressions for $\mathbf{A}$ and $\mathbf{E}$ on v^* in terms of h_v and $\mathbf{x}_v$ are as follows:

$$\begin{aligned}\mathbf{A}|_{v^*} &= h_v^{-1}\mathrm{d}h_v + h_v^{-1}\mathbf{M}_v h_v\mathrm{d}\phi_v,\\ \mathbf{E}|_{v^*} &= h_v^{-1}\mathrm{d}\mathbf{x}_v h_v + h_v^{-1}(\mathbf{S}_v + [\mathbf{M}_v, \mathbf{x}_v])h_v\mathrm{d}\phi_v. \end{aligned} \quad (40)$$

Furthermore, from (9) we get

$$\mathbf{F}|_{D_v} = \mathbf{p}_v\delta(v), \qquad \mathbf{T}|_{D_v} = \mathbf{j}_v\delta(v), \quad (41)$$

where $\mathbf{p}_v$, $\mathbf{j}_v$ represent the momentum and angular momentum respectively:

[8]By "periodic" we mean that, under $\phi \mapsto \phi + 1$, the nonperiodic variable $\overset{\circ}{\mathcal{H}}_v$ gets an additional factor of $\mathrm{e}^{\boldsymbol{\mathcal{M}}_v}$ due to the term $\mathrm{e}^{\boldsymbol{\mathcal{M}}_v\phi_v}$, while the periodic variable $\mathcal{H}_v$ is invariant. (Recall that we are scaling ϕ by 2π, so the period is 1 and not 2π.)

[9]This notation differs from the one we used in [13]. For the periodic variables, we used h and $\mathbf{y}$ in [13]. Here, we still use h, but instead of $\mathbf{y}$ we use $\mathbf{x}$ due to this variable's relation to the flux $\mathbf{X}$, as shown below. For the nonperiodic variables, we used u and $\mathbf{w}$ in [13]. Here we use $\overset{\circ}{h}$ and $\overset{\circ}{\mathbf{x}}$ in order to avoid introducing additional letters, which might be confusing. The circle above the letter conveys that it involves the angular variable ϕ and is thus nonperiodic.

$$\mathbf{p}_v \equiv h_v^{-1}\mathbf{M}_v h_v, \qquad \mathbf{j}_v \equiv h_v^{-1}(\mathbf{S}_v + [\mathbf{M}_v, \mathbf{x}_v])h_v. \tag{42}$$

In terms of $\mathbf{p}_v$ and $\mathbf{j}_v$, we may write $\mathbf{A}$ and $\mathbf{E}$ on the disk as follows:

$$\mathbf{A}|_{v^*} = h_v^{-1}\mathrm{d}h_v + \mathbf{p}_v \mathrm{d}\phi_v, \qquad \mathbf{E}|_{v^*} = h_v^{-1}\mathrm{d}\mathbf{x}_v h_v + \mathbf{j}_v \mathrm{d}\phi_v. \tag{43}$$

It is clear that the first term in each definition is flat and torsionless, while the second term (involving $\mathbf{p}_v$ and $\mathbf{j}_v$ respectively) is the one which contributes to the curvature and torsion at v. Since the punctured disk v^* does not include v itself, the curvature and torsion vanish everywhere on it:

$$\mathbf{F}|_{v^*} = 0, \qquad \mathbf{T}|_{v^*} = 0. \tag{44}$$

As before, while $\mathbf{F}$ and $\mathbf{T}$ do not vanish on the full disk D_v, they do vanish on v^*. We call this type of geometry a piecewise flat and torsionless geometry.[10] Given a particular spin network Γ^*, and assuming that information about the curvature and torsion may only be obtained by taking holonomies along the loops of this spin network, the piecewise flat and torsionless geometry carries, at least intuitively, the exact same information as the arbitrary geometry we had before.

4. The connection and frame field on the cells

Now that we have defined $\mathbf{A}$ and $\mathbf{E}$ on the punctured disks v^*, defining them on the cells c is a piece of cake. The geometry inside the cells is flat and torsionless everywhere, not distributional. Thus, the expressions for $\mathbf{A}$ and $\mathbf{E}$ on c are analogous to the first term in each of the expressions in (40), which is the flat and torsionless term:

$$\mathbf{A}|_c = h_c^{-1}\mathrm{d}h_c, \qquad \mathbf{E}|_c = h_c^{-1}\mathrm{d}\mathbf{x}_c h_c, \tag{45}$$

where h_c is a G-valued 0-form and $\mathbf{x}_c$ is a $\mathfrak{g}^*$-valued 0-form. Of course, by construction, the curvature and torsion associated to this connection and frame field vanish everywhere on the cell:

$$\mathbf{F}|_c = 0, \qquad \mathbf{T}|_c = 0. \tag{46}$$

C. Dressed holonomies and edge modes

Consider the definition $\mathbf{A}|_c = h_c^{-1}\mathrm{d}h_c$ for $\mathbf{A}$ in terms of h_c. Note that $\mathbf{A}$ is invariant under the left action transformation $h_c \mapsto g_c h_c$ for some constant $g_c \in G$. Thus, inverting the definition $\mathbf{A}|_c = h_c^{-1}\mathrm{d}h_c$ to find h_c in terms of $\mathbf{A}$, we get

$$h_c(x) = h_c(c^*)\overrightarrow{\exp}\int_{c^*}^x \mathbf{A}, \tag{47}$$

where $\overrightarrow{\exp}$ is a path-ordered exponential, and $h_c(c^*)$ is a new d.o.f. which does not exist in $\mathbf{A}$. The notation suggests that it is the holonomy "from c^* to itself," but it is in general not the identity. The notation $h_c(c^*)$ is just a placeholder for the edge mode which "dresses" the holonomy.

For the "undressed" holonomy—which is simply the path-ordered exponential from the node c^* to some point x—we thus have

$$\overrightarrow{\exp}\int_{c^*}^x \mathbf{A} = h_c^{-1}(c^*)h_c(x). \tag{48}$$

Similarly, the definition $\mathbf{A}|_{v^*} = h_v^{-1}\mathrm{d}h_v + h_v^{-1}\mathbf{M}_v h_v \mathrm{d}\phi_v$ is invariant under $h_v \mapsto g_v h_v$, but only if g_v is in H, the Cartan subgroup of G, since it must commute with $\mathbf{M}_v$. Inverting the relation $\mathbf{A}|_{v^*} = \mathring{h}_v^{-1}\mathrm{d}\mathring{h}_v$, we get

$$\mathring{h}_v(x) = h_v(v)\overrightarrow{\exp}\int_v^x \mathbf{A}, \tag{49}$$

where again the edge mode $h_v(v)$ is a new d.o.f. The undressed holonomy is then

$$\overrightarrow{\exp}\int_v^x \mathbf{A} = h_v^{-1}(v)\mathring{h}_v(x) = h_v^{-1}(v)\mathrm{e}^{\mathbf{M}_v\phi_v(x)}h_v(x). \tag{50}$$

From (48) and (50), we may construct general path-ordered exponentials from some point x to another point y by breaking the path from x to y such that it passes through an intermediate point. If that point is the node c^*, then we get

$$\begin{aligned}\overrightarrow{\exp}\int_x^y \mathbf{A} &= \left(\overrightarrow{\exp}\int_x^{c^*}\mathbf{A}\right)\left(\overrightarrow{\exp}\int_{c^*}^y \mathbf{A}\right)\\ &= (h_c^{-1}(x)h_c(c^*))(h_c^{-1}(c^*)h_c(y))\\ &= h_c^{-1}(x)h_c(y),\end{aligned} \tag{51}$$

and if it's the vertex v, we similarly get

$$\begin{aligned}\overrightarrow{\exp}\int_x^y \mathbf{A} &= \left(\overrightarrow{\exp}\int_x^{v}\mathbf{A}\right)\left(\overrightarrow{\exp}\int_{v}^y \mathbf{A}\right)\\ &= h_v^{-1}(x)\mathrm{e}^{\mathbf{M}_v(\phi_v(y)-\phi_v(x))}h_v(y).\end{aligned} \tag{52}$$

Furthermore, we may use the continuity relations (101) and (102) (to be discussed later) to obtain a relation between the path-ordered integrals and the holonomies $h_{cc'}$ and h_{cv}. If $y \in (cc')$ then we can write

$$\overrightarrow{\exp}\int_x^y \mathbf{A} = h_c^{-1}(x)h_{cc'}h_{c'}(y), \tag{53}$$

and if $y \in (cv)$ then we can write

[10] The question of whether the geometry we have defined here has a notion of a "continuum limit", e.g., by shrinking the loops to points such that the discrete defects at the vertices become continuous curvature and torsion, is left for future work.

$$\overrightarrow{\exp}\int_x^y \mathbf{A} = h_c^{-1}(x)h_{cv}\mathring{h}_v(y) = h_c^{-1}(x)h_{cv}\mathrm{e}^{\mathbf{M}_v\phi_v(y)}h_v(y). \tag{54}$$

Note that, in particular,

$$\overrightarrow{\exp}\int_{c^*}^{c'^*} \mathbf{A} = h_c^{-1}(c^*)h_{cc'}h_{c'}(c'^*). \tag{55}$$

A similar discussion applies to the translational holonomies $\mathbf{x}_c$ and $\mathbf{x}_v$, and one finds two new d.o.f., $\mathbf{x}_c(c^*)$ and $\mathbf{x}_v(v)$.

III. DISCRETIZING THE SYMPLECTIC POTENTIAL

A. The choice of polarization

Recall that there is a family of symplectic potential given by (22)

$$\Theta_\lambda = -\int_\Sigma ((1-\lambda)\mathbf{E}\cdot\delta\mathbf{A} + \lambda\mathbf{A}\cdot\delta\mathbf{E}). \tag{56}$$

We would like to replace $\mathbf{A}$ and $\mathbf{E}$ by their discretized expressions given by (45) and (38). Before we do this for each cell and disk individually, let us consider a toy model where we simply take $\mathbf{A} = h^{-1}\mathrm{d}h$ and $\mathbf{E} = h^{-1}\mathrm{d}\mathbf{x}h$ for some G-valued 0-form h and $\mathfrak{g}^*$-valued 0-form $\mathbf{x}$ over the entire manifold Σ. We begin by calculating the variations of these expressions, obtaining

$$\delta\mathbf{A} = \delta(h^{-1}\mathrm{d}h) = h^{-1}(\mathrm{d}\mathbf{\Delta}h)h, \tag{57}$$

$$\delta\mathbf{E} = \delta(h^{-1}\mathrm{d}\mathbf{x}h) = h^{-1}(\mathrm{d}\delta\mathbf{x} + [\mathrm{d}\mathbf{x},\mathbf{\Delta}h])h, \tag{58}$$

where we have defined the notation $\mathbf{\Delta}h \equiv \delta h h^{-1}$ for the Maurer-Cartan form on field space. Thus, we have

$$\Theta_\lambda = -\int_\Sigma ((1-\lambda)\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h + \lambda\mathrm{d}hh^{-1}\cdot(\mathrm{d}\delta\mathbf{x} + [\mathrm{d}\mathbf{x},\mathbf{\Delta}h])), \tag{59}$$

where we used the cyclicity of the dot product to cancel some group elements. Now, the first term is very simple; in fact, it is clearly an exact 2-form, and thus may be easily integrated. However, the second term is complicated, and it is unclear if it can be integrated. Nevertheless, we know that every choice of λ leads to the same symplectic form:

$$\Omega = \delta\Theta_\lambda = -\int_\Sigma \delta\mathbf{E}\cdot\delta\mathbf{A} = -\int_\Sigma (\mathrm{d}\delta\mathbf{x} + [\mathrm{d}\mathbf{x},\mathbf{\Delta}h])\cdot\mathrm{d}\mathbf{\Delta}h. \tag{60}$$

Furthermore, we have seen from (23) that the difference between different polarizations amounts to the addition of a boundary term and is equivalent to an integration by parts. Thus, we employ the following trick. First we take $\lambda = 0$ in Θ_λ, so that it becomes the 2 + 1D gravity polarization:

$$\Theta = -\int_\Sigma \mathbf{E}\cdot\delta\mathbf{A}. \tag{61}$$

Then, in the discretization process, we obtain

$$\Theta = -\int_\Sigma \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h. \tag{62}$$

The integrand in an exact 2-form, and thus may be integrated in two equivalent ways:

$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h = \mathrm{d}(\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h) = -\mathrm{d}(\mathrm{d}\mathbf{x}\cdot\mathbf{\Delta}h). \tag{63}$$

Note that the 1-forms $\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h$ and $\mathrm{d}\mathbf{x}\cdot\mathbf{\Delta}h$ differ only by a boundary term of the form $\mathrm{d}(\mathbf{x}\cdot\mathbf{\Delta}h)$, and they may be obtained from each other with integration by parts, just as for the different polarizations. In fact, we may write

$$\mathbf{E}\cdot\delta\mathbf{A} = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h = \lambda\mathrm{d}(\mathbf{x}\cdot\mathrm{d}\mathbf{\Delta}h) - (1-\lambda)\mathrm{d}(\mathrm{d}\mathbf{x}\cdot\mathbf{\Delta}h). \tag{64}$$

We claim that, even though technically both options are equivalent discretizations of the $\lambda = 0$ polarization in (56), there is in fact reason to believe that the choice of λ in (56) corresponds to the same choice of λ in (64). We will motivate this by showing that the choice $\lambda = 0$ corresponds to the usual loop gravity polarization, which is associated with usual general relativity, while the choice $\lambda = 1$ corresponds to a dual polarization which, as we will see, is associated with teleparallel gravity.

B. Decomposing the spatial manifold

As we have seen, the spatial manifold Σ is decomposed into cells c and disks v^*. The whole manifold Σ may be recovered by taking the union of the cells with the closures of the disks (recall that the vertices v are not in v^*, they are on their boundaries):

$$\Sigma = \left(\bigcup_c c\right) \cup \left(\bigcup_v v^* \cup \partial v^*\right). \tag{65}$$

Here, we are assuming that the cells and punctured disks are disjoint; the disks "eat into" the cells. We can thus split Θ into contributions from each cell c and punctured disk v^*:

$$\Theta = \sum_c \Theta_c + \sum_v \Theta_{v^*}, \tag{66}$$

where

$$\Theta_c = -\int_c \mathbf{E}\cdot\delta\mathbf{A}, \qquad \Theta_{v^*} = -\int_{v^*} \mathbf{E}\cdot\delta\mathbf{A}. \tag{67}$$

Given the discretizations (45) and (38), we replace h, $\mathbf{x}$ in (64) with h_c, $\mathbf{x}_c$ or $\mathring{h}_v, \mathring{\mathbf{x}}_v$ respectively, and then integrate using Stokes's theorem to obtain:

$$\Theta_c = \int_{\partial c} ((1-\lambda)\mathrm{d}\mathbf{x}_c \cdot \mathbf{\Delta} h_c - \lambda \mathbf{x}_c \cdot \mathrm{d}\mathbf{\Delta} h_c), \quad (68)$$

$$\Theta_{v^*} = \int_{\partial v^*} ((1-\lambda)\mathrm{d}\mathring{\mathbf{x}}_v \cdot \mathbf{\Delta}\mathring{h}_v - \lambda \mathring{\mathbf{x}}_v \cdot \mathrm{d}\mathbf{\Delta}\mathring{h}_v). \quad (69)$$

In the next few subsections, we will manipulate these expressions so that they can be integrated once again to obtain truly discrete symplectic potentials.

C. The vertex and cut contributions

The boundary ∂v^* splits into three contributions: one from the inner boundary $\partial_0 v^*$ (which is the vertex v), one from the cut C_v, and one from the outer boundary $\partial_R v^*$. Thus we have

$$\Theta_{v^*} = -\Theta_{\partial_0 v^*} - \Theta_{C_v} + \Theta_{\partial_R v^*}, \quad (70)$$

where the minus sign comes from the fact that orientation of the outer boundary is opposite to that of the inner boundary. Here we will discuss the first two terms, while the contribution from the outer boundary $\partial_R v^*$ will be calculated in Sec. III E.

Writing the terms in the integrand explicitly in terms of $\mathbf{x}_v$, h_v using (39), and making use of the identities

$$\mathrm{d}\mathring{\mathbf{x}}_v = \mathrm{e}^{\mathbf{M}_v\phi_v}(\mathrm{d}\mathbf{x}_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{x}_v])\mathrm{d}\phi_v)\mathrm{e}^{-\mathbf{M}_v\phi_v}, \quad (71)$$

$$\mathbf{\Delta}\mathring{h}_v = \mathrm{e}^{\mathbf{M}_v\phi_v}(\delta\mathbf{M}_v\phi_v + \mathbf{\Delta} h_v)\mathrm{e}^{-\mathbf{M}_v\phi_v}, \quad (72)$$

$$\mathrm{d}\mathbf{\Delta}\mathring{h}_v = \mathrm{e}^{\mathbf{M}_v\phi_v}(\mathrm{d}\mathbf{\Delta} h_v + (\delta\mathbf{M}_v + [\mathbf{M}_v, \mathbf{\Delta} h_v])\mathrm{d}\phi_v)\mathrm{e}^{-\mathbf{M}_v\phi_v}, \quad (73)$$

we get

$$\mathrm{d}\mathring{\mathbf{x}}_v \cdot \mathbf{\Delta}\mathring{h}_v = (\mathrm{d}\mathbf{x}_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{x}_v])\mathrm{d}\phi_v) \cdot (\delta\mathbf{M}_v\phi_v + \mathbf{\Delta} h_v), \quad (74)$$

$$\mathring{\mathbf{x}}_v \cdot \mathrm{d}\mathbf{\Delta}\mathring{h}_v = (\mathbf{x}_v + \mathbf{S}_v\phi_v) \cdot (\mathrm{d}\mathbf{\Delta} h_v + (\delta\mathbf{M}_v + [\mathbf{M}_v, \mathbf{\Delta} h_v])\mathrm{d}\phi_v). \quad (75)$$

The integral on the inner boundary $\partial_0 v^*$ is easily calculated, since $\mathbf{x}_v$ and h_v obtain the constant values $\mathbf{x}_v(v)$ and $h_v(v)$ on the inner boundary. Hence $\mathrm{d}\mathbf{x}_v(v) = \mathrm{d}\mathbf{\Delta} h_v(v) = 0$, and these expressions simplify to[11]

$$\mathrm{d}\mathring{\mathbf{x}}_v \cdot \mathbf{\Delta}\mathring{h}_v|_{\partial_0 v^*} = (\phi_v\mathbf{S}_v \cdot \delta\mathbf{M}_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{x}_v(v)]) \cdot \mathbf{\Delta} h_v(v))\mathrm{d}\phi_v, \quad (76)$$

$$\mathring{\mathbf{x}}_v \cdot \mathrm{d}\mathbf{\Delta}\mathring{h}_v|_{\partial_0 v^*} = (\phi_v\mathbf{S}_v \cdot \delta\mathbf{M}_v + \mathbf{x}_v(v) \cdot (\delta\mathbf{M}_v + [\mathbf{M}_v, \mathbf{\Delta} h_v(v)]))\mathrm{d}\phi_v. \quad (77)$$

To evaluate the contribution from the inner boundary, we integrate from $\phi_v = \alpha_v - 1/2$ to $\phi_v = \alpha_v + 1/2$. Then since

$$\int_{\alpha_v - 1/2}^{\alpha_v + 1/2} \mathrm{d}\phi_v = 1, \qquad \int_{\alpha_v - 1/2}^{\alpha_v + 1/2} \phi_v \mathrm{d}\phi_v = \alpha_v, \quad (78)$$

we get

$$\begin{aligned}\Theta_{\partial_0 v^*} = &(1-2\lambda)\alpha_v\mathbf{S}_v \cdot \delta\mathbf{M}_v \\ &+ (1-\lambda)(\mathbf{S}_v + [\mathbf{M}_v, \mathbf{x}_v(v)]) \cdot \mathbf{\Delta} h_v(v) \\ &- \lambda\mathbf{x}_v(v) \cdot (\delta\mathbf{M}_v + [\mathbf{M}_v, \mathbf{\Delta} h_v(v)]),\end{aligned} \quad (79)$$

which may be simplified to

$$\begin{aligned}\Theta_{\partial_0 v^*} = &(1-2\lambda)\alpha_v\mathbf{S}_v \cdot \delta\mathbf{M}_v + (1-\lambda)\mathbf{S}_v \cdot \mathbf{\Delta} h_v(v) \\ &- \lambda\mathbf{x}_v(v) \cdot \delta\mathbf{M}_v + [\mathbf{M}_v, \mathbf{x}_v(v)] \cdot \mathbf{\Delta} h_v(v).\end{aligned} \quad (80)$$

Next, we have the cut C_v. Since $\mathrm{d}\phi_v = 0$ on the cut, we have a significant simplification:

$$\mathrm{d}\mathring{\mathbf{x}}_v \cdot \mathbf{\Delta}\mathring{h}_v|_{C_v} = \mathrm{d}\mathbf{x}_v \cdot (\delta\mathbf{M}_v\phi_v + \mathbf{\Delta} h_v), \quad (81)$$

$$\mathring{\mathbf{x}}_v \cdot \mathrm{d}\mathbf{\Delta}\mathring{h}_v|_{C_v} = (\mathbf{x}_v + \mathbf{S}_v\phi_v) \cdot \mathrm{d}\mathbf{\Delta} h_v. \quad (82)$$

In fact, the cut has two sides: one at $\phi_v = \alpha_v - 1/2$ and another at $\phi_v = \alpha_v + 1/2$, with opposite orientation. Let us label them C_v^- and C_v^+ respectively. Any term that does not depend explicitly on ϕ_v will vanish when we take the difference between both sides of the cut, since they only differ by the value of ϕ_v. Thus only the terms $\mathrm{d}\mathbf{x}_v \cdot \delta\mathbf{M}_v\phi_v$ and $\mathbf{S}_v \cdot \mathrm{d}\mathbf{\Delta} h_v\phi_v$ survive. The relevant contribution from each side of the cut is therefore

$$\begin{aligned}\Theta_{C_v^\pm} &= \int_{r=0}^{R} ((1-\lambda)\mathrm{d}\mathbf{x}_v \cdot \delta\mathbf{M}_v\phi_v - \lambda\mathbf{S}_v \cdot \mathrm{d}\mathbf{\Delta} h_v\phi_v)\Big|_{\phi_v = \alpha_v \pm 1/2} \\ &= \left(\alpha_v \pm \frac{1}{2}\right)\left((1-\lambda)\delta\mathbf{M}_v \cdot \int_{r=0}^{R} \mathrm{d}\mathbf{x}_v - \lambda\mathbf{S}_v \cdot \int_{r=0}^{R} \mathrm{d}\Delta h_v\right) \\ &= \left(\alpha_v \pm \frac{1}{2}\right)((1-\lambda)\delta\mathbf{M}_v \cdot (\mathbf{x}_v(v_0) - \mathbf{x}_v(v)) \\ &\quad - \lambda\mathbf{S}_v \cdot (\mathbf{\Delta} h_v(v_0) - \mathbf{\Delta} h_v(v))),\end{aligned}$$

[11]Here we used the identity $[\mathbf{A}, \mathbf{B}] \cdot \mathbf{C} = \mathbf{A} \cdot [\mathbf{B}, \mathbf{C}]$ to get $[\mathbf{M}_v, \mathbf{x}_v] \cdot \delta\mathbf{M}_v = \mathbf{x}_v \cdot [\delta\mathbf{M}_v, \mathbf{M}_v] = 0$ and $\mathbf{S}_v \cdot [\mathbf{M}_v, \mathbf{\Delta} h_v] = \mathbf{\Delta} h_v \cdot [\mathbf{S}_v, \mathbf{M}_v] = 0$.

where the point at $r = 0$ is the vertex v, and the point at $r = R$ and $\phi_v = \alpha_v \pm 1/2$ is labeled v_0. Taking the difference between both sides of the cut, we thus get the total contribution:

$$\begin{aligned}\Theta_{C_v} &= \Theta_{C_v^+} - \Theta_{C_v^-} \\ &= \left(\left(\alpha_v + \frac{1}{2}\right) - \left(\alpha_v - \frac{1}{2}\right)\right)((1-\lambda)(\mathbf{x}_v(v_0) - \mathbf{x}_v(v)) \cdot \delta\mathbf{M}_v \\ &\quad - \lambda\mathbf{S}_v \cdot (\mathbf{\Delta}h_v(v_0) - \mathbf{\Delta}h_v(v))) \\ &= (1-\lambda)(\mathbf{x}_v(v_0) - \mathbf{x}_v(v)) \cdot \delta\mathbf{M}_v \\ &\quad - \lambda\mathbf{S}_v \cdot (\mathbf{\Delta}h_v(v_0) - \mathbf{\Delta}h_v(v)).\end{aligned}$$

Adding up the contributions from the inner boundary and the cut, we obtain the vertex symplectic potential $\Theta_v \equiv -(\Theta_{\partial_0 v^*} + \Theta_{C_v})$:

$$\Theta_v = -(1-2\lambda)\alpha_v \mathbf{S}_v \cdot \delta\mathbf{M}_v - \mathbf{S}_v \cdot (\mathbf{\Delta}h_v(v) - \lambda\Delta h_v(v_0)) \tag{83}$$

$$\begin{aligned}&+ (\mathbf{x}_v(v) - (1-\lambda)\mathbf{x}_v(v_0)) \cdot \delta\mathbf{M}_v \\ &- [\mathbf{M}_v, \mathbf{x}_v(v)] \cdot \mathbf{\Delta}h_v(v).\end{aligned} \tag{84}$$

D. The "particle" potential

Let $\mathbf{x}_v^{\|}(v_0)$ be the component of $\mathbf{x}_v(v_0)$ parallel to $\mathbf{S}_v$:

$$\mathbf{x}_v(v_0) \equiv \mathbf{x}_v^{\|}(v_0) + \mathbf{x}_v^{\perp}(v_0), \qquad \mathbf{x}_v^{\|}(v_0) \equiv (\mathbf{x}_v(v_0) \cdot \mathbf{J}_1)\mathbf{P}_1, \tag{85}$$

where $\mathbf{J}_1$ and $\mathbf{P}_1$ are the Cartan generator of rotations and translations respectively, and we remind the reader that the dot product is defined in (4) as $\mathbf{J}_i \cdot \mathbf{P}_j = \delta_{ij}$ and $\mathbf{J}_i \cdot \mathbf{J}_j = \mathbf{P}_i \cdot \mathbf{P}_j = 0$. Similarly, let $\mathbf{\Delta}^{\|}h_v(v_0)$ be the component of $\mathbf{\Delta}h_v(v_0)$ parallel to $\mathbf{M}_v$:

$$\begin{aligned}\Delta h_v(v_0) &\equiv \mathbf{\Delta}^{\|}h_v(v_0) + \mathbf{\Delta}^{\perp}h_v(v_0), \\ \mathbf{\Delta}^{\|}h_v(v_0) &\equiv (\mathbf{\Delta}h_v(v_0) \cdot \mathbf{P}_1)\mathbf{J}_1.\end{aligned} \tag{86}$$

Let us now define a $\mathfrak{g}$-valued 0-form $\mathbf{\Delta}H_v$, which is a 1-form on field space (i.e., a variation[12]):

$$\Delta H_v \equiv \mathbf{\Delta}h_v(v) - \lambda\Delta^{\|}h_v(v_0), \tag{87}$$

[12]Despite the suggestive notation, in principle ΔH_v need not be of the form $\delta H_v H_v^{-1}$ for some G-valued 0-form H_v. It can instead be of the form $\delta\mathbf{h}_v$ for some $\mathfrak{g}$-valued 0-form $\mathbf{h}_v$. Its precise form is left implicit, and we merely assume that there is a solution for either H_v or $\mathbf{h}_v$ in terms of $h_v(v)$ and $h_v(v_0)$.

and a $\mathfrak{g}^*$-valued 0-form $\mathbf{X}_v$ called the vertex flux:

$$\mathbf{X}_v \equiv \mathbf{x}_v(v) - (1-\lambda)\mathbf{x}_v^{\|}(v_0) - (1-2\lambda)\alpha_v \mathbf{S}_v. \tag{88}$$

Then since $\mathbf{S}_v \cdot \mathbf{\Delta}h_v(v_0) = \mathbf{S}_v \cdot \mathbf{\Delta}^{\|}h_v(v_0)$ we have

$$\mathbf{S}_v \cdot (\mathbf{\Delta}h_v(v) - \lambda\Delta h_v(v_0)) = \mathbf{S}_v \cdot \mathbf{\Delta}H_v, \tag{89}$$

and since $\mathbf{x}_v(v_0) \cdot \delta\mathbf{M}_v = \mathbf{x}_v^{\|}(v_0) \cdot \delta\mathbf{M}_v$ we have

$$(\mathbf{x}_v(v) - (1-\lambda)\mathbf{x}_v(v_0) - (1-2\lambda)\alpha_v\mathbf{S}_v) \cdot \delta\mathbf{M}_v = \mathbf{X}_v \cdot \delta\mathbf{M}_v. \tag{90}$$

Furthermore, since $[\mathbf{M}_v, \mathbf{x}_v^{\|}(v_0)] = [\mathbf{M}_v, \mathbf{S}_v] = 0$ and $[\mathbf{M}_v, \mathbf{X}_v] \cdot \mathbf{\Delta}^{\|}h_v(v_0) = 0$ we have

$$[\mathbf{M}_v, \mathbf{x}_v(v)] \cdot \mathbf{\Delta}h_v(v) = [\mathbf{M}_v, \mathbf{X}_v] \cdot \mathbf{\Delta}H_v. \tag{91}$$

Therefore (83) becomes

$$\Theta_v = \mathbf{X}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v]) \cdot \mathbf{\Delta}H_v. \tag{92}$$

This potential resembles that of a point particle with mass $\mathbf{M}_v$ and spin $\mathbf{S}_v$. Note that the free parameter λ has been absorbed into $\mathbf{X}_v$ and ΔH_v, so this potential is obtained independently of the value of λ and thus the choice of polarization.

E. The edge and arc contributions

To summarize our progress so far, we now have

$$\Theta = \sum_c \Theta_c + \sum_v \Theta_{\partial_R v^*} + \sum_v \Theta_v, \tag{93}$$

where

$$\Theta_c = \int_{\partial c} ((1-\lambda)\mathrm{d}\mathbf{x}_c \cdot \mathbf{\Delta}h_c - \lambda\mathbf{x}_c \cdot \mathrm{d}\mathbf{\Delta}h_c), \tag{94}$$

$$\Theta_{\partial_R v^*} = \int_{\partial_R v^*} ((1-\lambda)\mathrm{d}\mathring{\mathbf{x}}_v \cdot \mathbf{\Delta}\mathring{h}_v - \lambda\mathring{\mathbf{x}}_v \cdot \mathrm{d}\mathbf{\Delta}\mathring{h}_v), \tag{95}$$

and Θ_v is given by (92). In order to simplify $\Theta_{\partial_R v^*}$, we recall from Sec. II A that the boundary ∂c of the cell c is composed of edges (cc_i) and arcs (cv_i) such that

$$\partial c = \bigcup_{i=1}^{N_c} ((cc_i) \cup (cv_i)), \tag{96}$$

while the outer boundary $\partial_R v^*$ of the disk v^* is composed of arcs (vc_i) such that

$$\partial_R v^* = \bigcup_{i=1}^{N_v} (vc_i), \tag{97}$$

where N_v is the number of cells around v. Importantly, in terms of orientation, $(cc') = (c'c)^{-1}$ and $(cv) = (vc)^{-1}$. We thus see that each edge (cc') is integrated over exactly twice, once from the integral over ∂c and once from the integral over $\partial c'$ with opposite orientation, and similarly each arc (cv) is integrated over twice, once from ∂c and once from $\partial_R v^*$ with opposite orientation. Hence we may rearrange the sums and integrals as follows:

$$\Theta = \sum_{(cc')} \Theta_{cc'} + \sum_{(vc)} \Theta_{vc} + \sum_{v} \Theta_v, \tag{98}$$

where

$$\Theta_{cc'} \equiv \int_{(cc')} ((1-\lambda)(\mathrm{d}\mathbf{x}_c \cdot \mathbf{\Delta} h_c - \mathrm{d}\mathbf{x}_{c'} \cdot \mathbf{\Delta} h_{c'}) - \lambda(\mathbf{x}_c \cdot \mathrm{d}\mathbf{\Delta} h_c - \mathbf{x}_{c'} \cdot \mathrm{d}\mathbf{\Delta} h_{c'})), \tag{99}$$

$$\Theta_{vc} \equiv \int_{(vc)} ((1-\lambda)(\mathrm{d}\mathring{\mathbf{x}}_v \cdot \mathbf{\Delta} \mathring{h}_v - \mathrm{d}\mathbf{x}_c \cdot \mathbf{\Delta} h_c) - \lambda(\mathring{\mathbf{x}}_v \cdot \mathrm{d}\mathbf{\Delta} \mathring{h}_v - \mathbf{x}_c \cdot \mathrm{d}\mathbf{\Delta} h_c)). \tag{100}$$

Next, we note that the connection $\mathbf{A}$ and frame field $\mathbf{E}$ are defined using different variables on each cell and disk, but overall they must be continuous on the entire spatial manifold Σ. This implies that the variables from each cell and disk, when evaluated on the edges and arcs, must be related via continuity relations, which are, for the edges (cc'),

$$h_{c'} = h_{c'c} h_c, \quad \mathbf{x}_{c'} = h_{c'c}(\mathbf{x}_c - \mathbf{x}_c^{c'}) h_{cc'}, \quad \text{on}(cc'), \tag{101}$$

and for the arcs (vc)

$$h_c = h_{cv} \mathring{h}_v, \quad \mathbf{x}_c = h_{cv}(\mathring{\mathbf{x}}_v - \mathbf{x}_v^c) h_{vc}, \quad \text{on}(vc), \tag{102}$$

where $h_{cc'}$, h_{cv}, $\mathbf{x}_c^{c'}$ and $\mathbf{x}_c^v$ are all constant and satisfy

$$h_{cc'} = h_{c'c}^{-1}, \qquad h_{vc} = h_{cv}^{-1}, \\ \mathbf{x}_c^{c'} = -h_{cc'} \mathbf{x}_{c'}^c h_{c'c}, \qquad \mathbf{x}_c^v = -h_{cv} \mathbf{x}_v^c h_{vc}. \tag{103}$$

By plugging these relations into $\Theta_{cc'}$ and Θ_{vc} and simplifying, using the identities

$$\Delta h_{c'} = h_{c'c}(\mathbf{\Delta} h_c - \mathbf{\Delta} h_c^{c'}) h_{cc'}, \\ \mathbf{\Delta} h_c = h_{cv}(\mathbf{\Delta} \mathring{h}_v - \mathbf{\Delta} h_v^c) h_{vc}, \tag{104}$$

where $\mathbf{\Delta} h_c^{c'} \equiv \delta h_{cc'} h_{c'c}$ and $\Delta h_v^c \equiv \delta h_{vc} h_{cv}$, we find

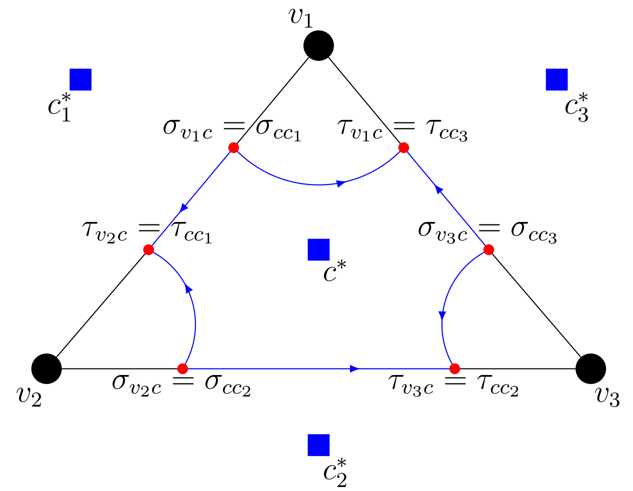

FIG. 3. The intersection points (red circles) of truncated edges and arcs along the oriented boundary ∂c (blue arrows).

$$\Theta_{cc'} = (1-\lambda)\mathbf{\Delta} h_c^{c'} \cdot \int_{(cc')} \mathrm{d}\mathbf{x}_c - \lambda \mathbf{x}_c^{c'} \cdot \int_{(cc')} \mathrm{d}\mathbf{\Delta} h_c, \tag{105}$$

$$\Theta_{vc} = (1-\lambda)\mathbf{\Delta} h_v^c \cdot \int_{(vc)} \mathrm{d}\mathring{\mathbf{x}}_v - \lambda \mathbf{x}_v^c \cdot \int_{(vc)} \mathrm{d}\mathbf{\Delta} \mathring{h}_v. \tag{106}$$

F. Holonomies and fluxes

Let us label the source and target points of the edge (cc') as $\sigma_{cc'}$ and $\tau_{cc'}$ respectively, and the source and target points of the arc (vc) as σ_{vc} and τ_{vc} respectively, where σ stands for "source" and τ for "target":

$$(cc') \equiv (\sigma_{cc'} \tau_{cc'}), \qquad (vc) \equiv (\sigma_{vc} \tau_{vc}). \tag{107}$$

This labeling is illustrated in Fig. 3 (taken from [13]). We now define holonomies and fluxes on the edges and their dual links, and on the arcs and their dual line segments.

1. Holonomies on the links and segments

The rotational[13] holonomy $h_{cc'}$ comes from the continuity relations (101). Its role is relating the variables $h_c, \mathbf{x}_c$ on the cell c to the variables $h_{c'}, \mathbf{x}_{c'}$ on the cell c'. Now, in the relation $h_c(x) = h_{cc'} h_{c'}(x)$, the holonomy on the left-hand side is from the node c^* to a point x on the edge (cc'). Therefore, the holonomy on the right-hand side should also take us from c^* to x. Since $h_{c'}(x)$ is the holonomy from c'^* to x, we see that $h_{cc'}$ must take us from c^* to c'^*. In other

[13]Recall that we are dealing with a generalized Euclidean or Poincaré group $G \ltimes \mathfrak{g}^*$ where G represents rotations and $\mathfrak{g}^*$ represents translations (or generalizations thereof). $h_{cc'}$ is valued in G and is thus a rotational holonomy, while $\mathbf{x}_c^{c'}$ is valued in $\mathfrak{g}^*$ and is thus a translational holonomy.

words, the holonomy $h_{cc'}$ is exactly the holonomy from c^* to c'^*, along[14] the link $(cc')^*$.

Thus we define[15] holonomies along the links $(cc')^*$:

$$H_{cc'} \equiv h_{cc'}, \qquad \mathbf{\Delta} H_c^{c'} \equiv \delta H_{cc'} H_{c'c}. \tag{108}$$

Similarly, the holonomy h_{vc} comes from the continuity relations (102), and it takes us from the vertex v to the node c^*. We define $(vc)^*$ to be the line segment connecting v to c^*; it is dual to the arc (vc) and its inverse is $(cv)^*$. We then define holonomies along the segments $(vc)^*$:

$$H_{vc} \equiv h_{vc}, \qquad \mathbf{\Delta} H_v^c \equiv \delta H_{vc} H_{cv}. \tag{109}$$

The inverse holonomies follow immediately from the relations $h_{cc'}^{-1} = h_{c'c}$ and $h_{vc}^{-1} = h_{cv}$:

$$H_{cc'}^{-1} = H_{c'c}, \qquad H_{vc}^{-1} = H_{cv}. \tag{110}$$

2. Fluxes on the edges and arcs

From the integral in the first term of (105), we are inspired to define fluxes along the edges (cc'):

$$\tilde{\mathbf{X}}_c^{c'} \equiv \int_{(cc')} \mathrm{d}\mathbf{x}_c = \mathbf{x}_c(\tau_{cc'}) - \mathbf{x}_c(\sigma_{cc'}). \tag{111}$$

The tilde specifies that the flux $\tilde{\mathbf{X}}_c^{c'}$ is on the edge (cc') dual to the link $(cc')^*$; the flux $\mathbf{X}_c^{c'}$, to be defined below, is on the link, and similarly we will define $\tilde{H}_{cc'}$ to be the holonomy on the edge, while $H_{cc'}$ is the holonomy on the link.

The flux $\tilde{\mathbf{X}}_c^{c'}$ is a composition of two translational holonomies. The holonomy $-\mathbf{x}_c(\sigma_{cc'})$ takes us from the point $\sigma_{cc'}$ to the node c^*, and then the holonomy $\mathbf{x}_c(\tau_{cc'})$ takes us from c^* to $\tau_{cc'}$. Hence, the composition of these holonomies is a translational holonomy from $\sigma_{cc'}$ to $\tau_{cc'}$, that is, along[16] the edge (cc'), as claimed.

To find the inverse flux we use $(cc') = (c'c)^{-1}$, $\sigma_{cc'} = \tau_{c'c}$ and (101):

$$\begin{aligned}\tilde{\mathbf{X}}_{c'}^{c} &\equiv \int_{(c'c)} \mathrm{d}\mathbf{x}_{c'} = \mathbf{x}_{c'}(\tau_{c'c}) - \mathbf{x}_{c'}(\sigma_{c'c}) \\ &= h_{c'c}(\mathbf{x}_c(\sigma_{cc'}) - \mathbf{x}_c(\tau_{cc'}))h_{cc'} = -H_{c'c}\tilde{\mathbf{X}}_c^{c'} H_{cc'}. \end{aligned} \tag{112}$$

Similarly, from the first integral in (106) we are inspired to define fluxes along the arcs (vc):

$$\tilde{\mathbf{X}}_v^c \equiv \int_{(vc)} \mathrm{d}\mathring{\mathbf{x}}_v = \mathring{\mathbf{x}}_v(\tau_{vc}) - \mathring{\mathbf{x}}_v(\sigma_{vc}). \tag{113}$$

Note that this time, the two translational holonomies are composed at v. As for the inverse, we define $\tilde{\mathbf{X}}_c^v$ as follows and use (102) to find a relation with $\tilde{\mathbf{X}}_v^c$, taking into account the fact that $(cv) = (vc)^{-1}$ and $\sigma_{cv} = \tau_{vc}$:

$$\begin{aligned}\tilde{\mathbf{X}}_c^v &\equiv \int_{(cv)} \mathrm{d}\mathbf{x}_c = \mathbf{x}_c(\tau_{cv}) - \mathbf{x}_c(\sigma_{cv}) \\ &= h_{cv}(\mathring{\mathbf{x}}_v(\sigma_{vc}) - \mathring{\mathbf{x}}_v(\tau_{vc}))h_{vc} = -H_{cv}\tilde{\mathbf{X}}_v^c H_{vc}. \end{aligned} \tag{114}$$

In conclusion, we have the relations

$$\tilde{\mathbf{X}}_{c'}^{c} = -H_{c'c}\tilde{\mathbf{X}}_c^{c'} H_{cc'}, \qquad \tilde{\mathbf{X}}_c^v = -H_{cv}\tilde{\mathbf{X}}_v^c H_{vc}. \tag{115}$$

3. Holonomies on the edges and arcs

The holonomies and fluxes defined thus far will be used in the $\lambda = 0$ polarization. In the $\lambda = 1$ (dual) polarization, let us define holonomies along the edges (cc') and holonomies along the arcs (vc):

$$\tilde{H}_{cc'} \equiv h_c^{-1}(\sigma_{cc'})h_c(\tau_{cc'}), \qquad \mathbf{\Delta}\tilde{H}_c^{c'} \equiv \delta\tilde{H}_{cc'}\tilde{H}_{c'c}, \tag{116}$$

$$\tilde{H}_{vc} \equiv \mathring{h}_v^{-1}(\sigma_{vc})\mathring{h}_v(\tau_{vc}), \qquad \mathbf{\Delta}\tilde{H}_v^c \equiv \delta\tilde{H}_{vc}\tilde{H}_{cv}. \tag{117}$$

As with $\tilde{\mathbf{X}}_c^{c'}$, the holonomy $\tilde{H}_{cc'}$ starts from $\sigma_{cc'}$, goes to c^* via $h_c^{-1}(\sigma_{cc'})$, and then goes to $\tau_{cc'}$ via $h_c(\tau_{cc'})$. Therefore it is indeed a holonomy along the edge (cc'). Similarly, the holonomy $\tilde{H}_{vc}$ starts from σ_{vc}, goes to v via $\mathring{h}_v^{-1}(\sigma_{vc})$, and then goes to τ_{vc} via $\mathring{h}_v(\tau_{vc})$. Therefore it is indeed a holonomy along the arc (vc).

The difference compared to $\tilde{\mathbf{X}}_c^{c'}$ is that in $\tilde{H}_{cc'}$ we have rotational instead of translational holonomies, and the composition of holonomies is (non-Abelian) multiplication instead of addition. As before, the tilde specifies that the holonomy is on the edges or arcs and not the dual links or segments.

The variations of these holonomies are

$$\begin{aligned}\mathbf{\Delta}\tilde{H}_c^{c'} &= h_c^{-1}(\sigma_{cc'})(\mathbf{\Delta}h_c(\tau_{cc'}) - \mathbf{\Delta}h_c(\sigma_{cc'}))h_c(\sigma_{cc'}) \\ &= h_c^{-1}(\sigma_{cc'})\left(\int_{(cc')}\mathrm{d}\mathbf{\Delta}h_c\right)h_c(\sigma_{cc'}), \end{aligned} \tag{118}$$

$$\begin{aligned}\mathbf{\Delta}\tilde{H}_v^c &= \mathring{h}_v^{-1}(\sigma_{vc})(\mathbf{\Delta}\mathring{h}_v(\tau_{vc}) - \mathbf{\Delta}\mathring{h}_v(\sigma_{vc}))\mathring{h}_v(\sigma_{vc}) \\ &= \mathring{h}_v^{-1}(\sigma_{vc})\left(\int_{(vc)}\mathrm{d}\mathbf{\Delta}\mathring{h}_v\right)\mathring{h}_v(\sigma_{vc}). \end{aligned} \tag{119}$$

Thus, we see that they relate to the integrals in the second terms of (105) and (106).

[14]Since the geometry is flat, the actual path taken does not matter, only that it starts at c^* and ends at c'^*. We may therefore assume without loss of generality that the path taken by $h_{cc'}$ is, in fact, along the link $(cc')^*$.

[15]The change from lowercase h to uppercase H is only symbolic here, but it will become more meaningful when we define other holonomies and fluxes below.

[16]Again, since the geometry is flat, the path passing through the node c^* is equivalent to the path going along the edge (cc').

Since $(cc') = (c'c)^{-1}$, it is obvious that $\tilde{H}_{cc'}^{-1} = \tilde{H}_{c'c}$. Furthermore, by combining (117) with (102) we may obtain an expression for $\tilde{H}_{vc}$ in terms of h_c:

$$\tilde{H}_{vc} = h_c^{-1}(\sigma_{vc}) h_c(\tau_{vc}). \tag{120}$$

If we now define

$$\tilde{H}_{cv} \equiv h_c^{-1}(\sigma_{cv}) h_c(\tau_{cv}), \tag{121}$$

then using the relations $\sigma_{cv} = \tau_{vc}$ and $\tau_{cv} = \sigma_{vc}$, which come from the fact that $(vc) = (cv)^{-1}$, it is easy to see that $\tilde{H}_{vc}^{-1} = \tilde{H}_{cv}$. In conclusion, the inverses of these holonomies satisfy the relationships

$$\tilde{H}_{cc'}^{-1} = \tilde{H}_{c'c}, \qquad \tilde{H}_{vc}^{-1} = \tilde{H}_{cv}. \tag{122}$$

4. Fluxes on the links and segments

Just as we defined the holonomies on the links and segments from the variables $h_{cc'}$ and h_{vc}, which were used in the continuity relations (101) and (102), we can similarly define the fluxes on the links and segments from the variables $\mathbf{x}_c^{c'}$ and $\mathbf{x}_v^c$. These will, again, be used in the dual polarization.

Let us define fluxes along the links $(cc')^*$ and segments $(vc)^*$:

$$\mathbf{X}_c^{c'} \equiv h_c^{-1}(\sigma_{cc'}) \mathbf{x}_c^{c'} h_c(\sigma_{cc'}), \qquad \mathbf{X}_v^c \equiv \overset{\circ}{h}{}_v^{-1}(\sigma_{vc}) \mathbf{x}_v^c \overset{\circ}{h}_v(\sigma_{vc}). \tag{123}$$

The factors of $h_c(\sigma_{cc'})$ and $\overset{\circ}{h}_v(\sigma_{vc})$ are needed because they appear alongside the integrals in the variations (118) and (119). Thus, if we want the second terms in (105) and (106) to look like we want them to, we must include these extra factors in the definition of the fluxes. The fluxes are still translational holonomies between two cells (in the case of $\mathbf{x}_c^{c'}$) or a cell and a disk (in the case of $\mathbf{x}_v^c$), but they contain an extra rotation at the starting point.

The inverse link flux $\mathbf{X}_{c'}^c$ follows from (101), (103) and $\sigma_{cc'} = \tau_{c'c}$, while the inverse segment flux $\mathbf{X}_c^v \equiv h_c^{-1}(\sigma_{cv}) \mathbf{x}_c^v h_c(\sigma_{cv})$ follows from (102), (103) and $\sigma_{cv} = \tau_{vc}$:

$$\mathbf{X}_{c'}^c = -\tilde{H}_{cc'}^{-1} \mathbf{X}_c^{c'} \tilde{H}_{cc'}, \qquad \mathbf{X}_c^v = -\tilde{H}_{vc}^{-1} \mathbf{X}_v^c \tilde{H}_{vc}. \tag{124}$$

5. The symplectic potential in terms of the holonomies and fluxes

With the holonomies and fluxes defined above, we find that we can write the symplectic potential on the edges and arcs, (105) and (106), as

$$\Theta_{cc'} = (1-\lambda) \tilde{\mathbf{X}}_c^{c'} \cdot \mathbf{\Delta} H_c^{c'} - \lambda \mathbf{X}_c^{c'} \cdot \mathbf{\Delta} \tilde{H}_c^{c'}, \tag{125}$$

$$\Theta_{vc} = (1-\lambda) \tilde{\mathbf{X}}_v^c \cdot \mathbf{\Delta} H_v^c - \lambda \mathbf{X}_v^c \cdot \mathbf{\Delta} \tilde{H}_v^c. \tag{126}$$

The full symplectic potential becomes

$$\begin{aligned} \Theta = & \sum_{(cc')} ((1-\lambda) \tilde{\mathbf{X}}_c^{c'} \cdot \mathbf{\Delta} H_c^{c'} - \lambda \mathbf{X}_c^{c'} \cdot \mathbf{\Delta} \tilde{H}_c^{c'}) \\ & + \sum_{(vc)} ((1-\lambda) \tilde{\mathbf{X}}_v^c \cdot \mathbf{\Delta} H_v^c - \lambda \mathbf{X}_v^c \cdot \mathbf{\Delta} \tilde{H}_v^c) \\ & + \sum_v (\mathbf{X}_v \cdot \delta \mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v]) \cdot \mathbf{\Delta} H_v). \end{aligned}$$

Notice how the holonomies and fluxes are always dual to each other: one with tilde (on the edges/arcs) and one without tilde (on the links/segments). For the $\lambda = 0$ polarization, the holonomies are on the links $(cc')^*$ and segments $(vc)^*$ and the fluxes are on their dual edges (cc') and arcs (vc). This is the polarization considered in [13], and corresponds to the usual loop gravity picture. For the $\lambda = 1$ (dual) polarization, we have the opposite case: the fluxes are on the links $(cc')^*$ and segments $(vc)^*$ and the holonomies are on their dual edges (cc') and arcs (vc). For any other choice of λ, we have a combination of both polarizations.

The phase space corresponding to $\mathbf{X} \cdot \mathbf{\Delta} H$ for some flux $\mathbf{X}$ and holonomy H is called the holonomy-flux phase space, and it is the classical phase space of the spin networks which appear in loop quantum gravity.

IV. THE GAUSS AND CURVATURE CONSTRAINTS

We have seen that, in the continuum, the constraints are $\mathbf{F} = \mathbf{T} = 0$. Let us see how they translate to constraints on the discrete phase space. There will be two types of constraints: the curvature constraints which corresponds to $\mathbf{F} = 0$, and the Gauss constraints which correspond to $\mathbf{T} = 0$. The constraints will be localized in three different types of places: on the cells, on the disks, and on the faces. After deriving all of the constraints and showing that they are identically satisfied in our construction, we will summarize and interpret them. The reader who is not interested in the details of the calculation may wish to skip to Sec. IV D.

A. Derivation of the constraints on the cells

1. The Gauss constraint on the cells

The cell Gauss constraint $\mathbf{G}_c$ will impose the torsionlessness condition $\mathbf{T} \equiv \mathrm{d}_{\mathbf{A}} \mathbf{E} = 0$ inside the cells:

$$\begin{aligned} 0 = \mathbf{G}_c \equiv & \int_c h_c (\mathrm{d}_{\mathbf{A}} \mathbf{E}) h_c^{-1} = \int_c \mathrm{d}(h_c \mathbf{E} h_c^{-1}) \\ = & \int_{\partial c} h_c \mathbf{E} h_c^{-1} = \int_{\partial c} \mathrm{d}\mathbf{x}_c. \end{aligned} \tag{127}$$

As we have seen, ∂c is composed of edges (cc_i) and arcs (cv_i) such that

$$\partial c = \bigcup_{i=1}^{N_c} ((cc_i) \cup (cv_i)). \qquad (128)$$

Therefore we can split the integral as follows:

$$\mathbf{G}_c = \sum_{c' \ni c} \int_{(cc')} \mathrm{d}\mathbf{x}_c + \sum_{v \ni c} \int_{(cv)} \mathrm{d}\mathbf{x}_c, \qquad (129)$$

where $c' \ni c$ means "all cells c' adjacent to c" and $v \ni c$ means "all vertices v adjacent to c."

Using the fluxes defined in (111) and (114), we get[17]

$$\mathbf{G}_c = \sum_{c' \ni c} \tilde{\mathbf{X}}_c^{c'} + \sum_{v \ni c} \tilde{\mathbf{X}}_c^{v} = 0. \qquad (130)$$

This constraint is satisfied identically in our construction. Indeed, from (111) and (114) we have

$$\tilde{\mathbf{X}}_c^{c'} = \mathbf{x}_c(\tau_{cc'}) - \mathbf{x}_c(\sigma_{cc'}), \qquad \tilde{\mathbf{X}}_c^{v} = \mathbf{x}_c(\tau_{cv}) - \mathbf{x}_c(\sigma_{cv}). \qquad (131)$$

Since $\tau_{cc_i} = \sigma_{cv_i}$ and $\tau_{cv_i} = \sigma_{cc_{i+1}}$ (the end of an edge is the beginning of an arc and the end of an arc is the beginning of an edge), and $\tau_{cv_{N_c}} = \sigma_{cc_1}$ (the end of the last arc is the beginning of the first edge), it is easy to see that the sum $\sum_{c' \ni c} \tilde{\mathbf{X}}_c^{c'} + \sum_{v \ni c} \tilde{\mathbf{X}}_c^{v}$ evaluates to zero.

2. The curvature constraint on the cells

The cell curvature constraint F_c will impose that $\mathbf{F} \equiv \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}] = 0$ inside the cells. An equivalent condition is that the holonomy around the cell evaluates to the identity:

$$1 = F_c \equiv \overrightarrow{\exp} \int_{\partial c} \mathbf{A}. \qquad (132)$$

Since $\partial c = \bigcup_{i=1}^{N_c} ((cc_i) \cup (cv_i))$, we may decompose this as a product of path-ordered exponentials over edges and arcs:

$$F_c = \prod_{i=1}^{N_c} \left(\overrightarrow{\exp} \int_{(cc_i)} \mathbf{A} \right) \left(\overrightarrow{\exp} \int_{(cv_i)} \mathbf{A} \right). \qquad (133)$$

Furthermore, since the geometry is flat, we may deform the paths so that instead of going along the edges and arcs, it passes through the node c^*. From (48) we have that

$$\overrightarrow{\exp} \int_{c^*}^{x} \mathbf{A} = h_c^{-1}(c^*) h_c(x), \qquad (134)$$

so

$$\begin{aligned} \overrightarrow{\exp} \int_{(cc_i)} \mathbf{A} &= \overrightarrow{\exp} \int_{\sigma_{cc_i}}^{\tau_{cc_i}} \mathbf{A} \\ &= \left(\overrightarrow{\exp} \int_{\sigma_{cc_i}}^{c^*} \mathbf{A} \right) \left(\overrightarrow{\exp} \int_{c^*}^{\tau_{cc_i}} \mathbf{A} \right) \\ &= h_c^{-1}(\sigma_{cc_i}) h_c(\tau_{cc_i}) = \tilde{H}_{cc_i}, \end{aligned} \qquad (135)$$

where we used the definition (116) of the holonomy on the edge. Note that the contribution from $h_c(c^*)$ cancels. Similarly, we find

$$\overrightarrow{\exp} \int_{(cv_i)} \mathbf{A} = h_c^{-1}(\sigma_{cv_i}) h_c(\tau_{cv_i}) = \tilde{H}_{cv_i}, \qquad (136)$$

where we used (121). Hence we obtain

$$F_c = \prod_{i=1}^{N_c} \tilde{H}_{cc_i} \tilde{H}_{cv_i} = 1. \qquad (137)$$

This is the curvature constraint on the cells. It is easy to show that it is satisfied identically in our construction. Indeed, using again the relations $\tau_{cc_i} = \sigma_{cv_i}$, $\tau_{cv_i} = \sigma_{cc_{i+1}}$ and $\tau_{cv_{N_c}} = \sigma_{cc_1}$, we immediately see that

$$\begin{aligned} \prod_{i=1}^{N_c} \tilde{H}_{cc_i} \tilde{H}_{cv_i} &= \prod_{i=1}^{N_c} (h_c^{-1}(\sigma_{cc_i}) h_c(\tau_{cc_i}))(h_c^{-1}(\sigma_{cv_i}) h_c(\tau_{cv_i})) \\ &= 1, \end{aligned} \qquad (138)$$

as desired.

B. Derivation of the constraints on the disks

Since we have places the curvature and torsion excitations inside the disks, the constraints on the disks must involve these excitations—namely, $\mathbf{M}_v$ and $\mathbf{S}_v$. We will now see that this is indeed the case.

1. The Gauss constraint on the disks

The disk Gauss constraint $\mathbf{G}_v$ will impose the torsionlessness condition $\mathbf{T} \equiv \mathrm{d}_{\mathbf{A}} \mathbf{E} = 0$ inside the *punctured*[18] disks:

[17]Note that in [13] we used a different convention for $\mathbf{X}_c^v$. This resulted in a relative minus sign between the two terms, which does not appear in this paper.

[18]As we have seen, we only have $\mathbf{T} = 0$ inside the punctured disk v^*; at the vertex v itself there is torsion, but v is not part of v^*. Instead, it is on its (inner) boundary. As can be seen from Fig. 2, the path we take here, as given by (140), does not enclose the vertex, and therefore the interior of the path is indeed torsionless.

$$0 = \mathbf{G}_v \equiv \int_{v^*} \mathring{h}_v (\mathrm{d}_{\mathbf{A}} \mathbf{E}) \mathring{h}_v^{-1} = \int_{v^*} \mathrm{d}(\mathring{h}_v \mathbf{E} \mathring{h}_v^{-1})$$
$$= \int_{\partial v^*} \mathring{h}_v \mathbf{E} \mathring{h}_v^{-1} = \int_{\partial v^*} \mathrm{d}\mathring{\mathbf{x}}_v. \qquad (139)$$

The boundary ∂v^* is composed of the inner boundary $\partial_0 v^*$, the outer boundary $\partial_R v^*$, and the cut C_v:

$$\partial v^* = \partial_0 v^* \cup \partial_R v^* \cup C_v. \qquad (140)$$

Hence

$$\mathbf{G}_v = \int_{\partial_R v^*} \mathrm{d}\mathring{\mathbf{x}}_v - \int_{\partial_0 v^*} \mathrm{d}\mathring{\mathbf{x}}_v - \int_{C_v} \mathrm{d}\mathring{\mathbf{x}}_v, \qquad (141)$$

where the minus signs represent the relative orientations of each piece. On the inner boundary $\partial_0 v^*$, we use the fact that $\mathbf{x}_v$ takes the constant value $\mathbf{x}_v(v)$ to obtain

$$\int_{\partial_0 v^*} \mathrm{d}\mathring{\mathbf{x}}_v = \mathrm{e}^{\mathbf{M}_v \phi_v} (\mathbf{x}_v(v) + \mathbf{S}_v \phi_v) \mathrm{e}^{-\mathbf{M}_v \phi_v} \Big|_{\phi_v = \alpha_v - \frac{1}{2}}^{\alpha_v + \frac{1}{2}}$$
$$= \mathbf{S}_v + \mathrm{e}^{\mathbf{M}_v (\alpha_v - \frac{1}{2})} (\mathrm{e}^{\mathbf{M}_v} \mathbf{x}_v(v) \mathrm{e}^{-\mathbf{M}_v} - \mathbf{x}_v(v)) \mathrm{e}^{-\mathbf{M}_v (\alpha_v - \frac{1}{2})}.$$

The outer boundary $\partial_R v^*$ splits into arcs, and we use the definition (113) of the flux:

$$\int_{\partial_R v^*} \mathrm{d}\mathring{\mathbf{x}}_v = \sum_{c \in v} \int_{(vc)} \mathrm{d}\mathring{\mathbf{x}}_v = \sum_{c \in v} \tilde{\mathbf{X}}_v^c. \qquad (142)$$

On the cut C_v, we have contributions from both sides, one at $\phi_v = \alpha_v - \frac{1}{2}$ and another at $\phi_v = \alpha_v + \frac{1}{2}$ with opposite orientation. Since $\mathrm{d}\phi_v = 0$ on the cut, we have

$$\mathrm{d}\mathring{\mathbf{x}}_v|_{C_v} = \mathrm{e}^{\mathbf{M}_v \phi_v} \mathrm{d}\mathbf{x}_v \mathrm{e}^{-\mathbf{M}_v \phi_v}, \qquad (143)$$

and thus

$$\int_{C_v} \mathrm{d}\mathring{\mathbf{x}}_v = \int_{r=0}^{R} \left(\mathrm{e}^{\mathbf{M}_v \phi_v} \mathrm{d}\mathbf{x}_v \mathrm{e}^{-\mathbf{M}_v \phi_v} \Big|_{\phi_v = \alpha_v - \frac{1}{2}}^{\phi_v = \alpha_v + \frac{1}{2}} \right)$$
$$= \mathrm{e}^{\mathbf{M}_v (\alpha_v - \frac{1}{2})} (\mathrm{e}^{\mathbf{M}_v} (\mathbf{x}_v(v_0) - \mathbf{x}_v(v)) \mathrm{e}^{-\mathbf{M}_v}$$
$$- (\mathbf{x}_v(v_0) - \mathbf{x}_v(v))) \mathrm{e}^{-\mathbf{M}_v (\alpha_v - \frac{1}{2})},$$

since $\mathbf{x}_v$ has the value $\mathbf{x}_v(v_0)$ at $r = R$ and $\mathbf{x}_v(v)$ at $r = 0$ on the cut.

Adding up the integrals, we find that the Gauss constraint on the disk is

$$\mathbf{G}_v = \sum_{c \in v} \tilde{\mathbf{X}}_v^c - \mathbf{S}_v$$
$$- \mathrm{e}^{\mathbf{M}_v (\alpha_v - \frac{1}{2})} (\mathrm{e}^{\mathbf{M}_v} \mathbf{x}_v(v_0) \mathrm{e}^{-\mathbf{M}_v} - \mathbf{x}_v(v_0)) \mathrm{e}^{-\mathbf{M}_v (\alpha_v - \frac{1}{2})} = 0. \qquad (144)$$

In fact, since this constraint is used as a generator of symmetries (as we will see below), it automatically comes dotted with a Cartan element $\boldsymbol{\beta}_v$, which commutes with $\mathrm{e}^{\mathbf{M}_v}$. Therefore, the last term may be ignored, and the constraint simplifies to

$$\boldsymbol{\beta}_v \cdot \mathbf{G}_v = \boldsymbol{\beta}_v \cdot \left(\sum_{c \in v} \tilde{\mathbf{X}}_v^c - \mathbf{S}_v \right) = 0. \qquad (145)$$

Thus it may also be written

$$\sum_{c \in v} \tilde{\mathbf{X}}_v^c = \mathbf{S}_v. \qquad (146)$$

To see that this constraint is satisfied identically in our construction, let us combine (113) with (39) to obtain

$$\tilde{\mathbf{X}}_v^c = \mathbf{S}_v (\tau_{vc} - \sigma_{vc}) + \mathrm{e}^{\mathbf{M}_v \tau_{vc}} \mathbf{x}_v(\tau_{vc}) \mathrm{e}^{-\mathbf{M}_v \tau_{vc}}$$
$$- \mathrm{e}^{\mathbf{M}_v \sigma_{vc}} \mathbf{x}_v(\sigma_{vc}) \mathrm{e}^{-\mathbf{M}_v \sigma_{vc}}, \qquad (147)$$

where we used a slight abuse of notation by using σ_{vc} and τ_{vc} to denote the corresponding angles, $\sigma_{vc} \equiv \phi_v(\sigma_{vc})$ and $\tau_{vc} \equiv \phi_v(\tau_{vc})$. Let us now sum over the fluxes for each arc. Since $\tau_{vc_i} = \sigma_{vc_{i+1}}$ (each arc ends where the next one starts) and $\tau_{vc_{N_v}} = \sigma_{vc_1} + 1$ (the last arc ends a full circle after the first arc began[19]), we get

$$\sum_{i=1}^{N_v} \tilde{\mathbf{X}}_v^{c_i} = \mathbf{S}_v + \mathrm{e}^{\mathbf{M}_v \sigma_{vc_1}} (\mathrm{e}^{\mathbf{M}_v} \mathbf{x}_v(\sigma_{vc_1}) \mathrm{e}^{-\mathbf{M}_v} - \mathbf{x}_v(\sigma_{vc_1})) \mathrm{e}^{-\mathbf{M}_v \sigma_{vc_1}}. \qquad (148)$$

Choosing without loss of generality the point v_0 to be at the beginning of the first edge, $v_0 = \sigma_{vc_1}$, and recalling that this point corresponds to the angle $\phi_v = \alpha_v - \frac{1}{2}$, we indeed obtain precisely the constraint (144).

2. The curvature constraint on the disks

The disk curvature constraint F_v will impose that $\mathbf{F} \equiv \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}] = 0$ inside the punctured disks.[20] An equivalent condition is that the holonomy around the punctured disk evaluates to the identity:

[19] Recall that we are using scaled angles such that a full circle corresponds to 1 instead of 2π.

[20] Again, we only have $\mathbf{F} = 0$ inside the punctured disk v^*; at the vertex v itself, there is curvature. However, the path of integration does not enclose the vertex, and therefore the interior of the path is indeed flat.

$$1 = F_v \equiv \overrightarrow{\exp} \int_{\partial v^*} \mathbf{A} = \overrightarrow{\exp}\left(\int_{C_v^-} \mathbf{A}\right)\overrightarrow{\exp}\left(\int_{\partial_R v^*} \mathbf{A}\right) \times \overrightarrow{\exp}\left(\int_{C_v^+} \mathbf{A}\right)\overrightarrow{\exp}\left(\int_{\partial_0 v^*} \mathbf{A}\right). \quad (149)$$

Let us describe the path of integration step by step, referring to Fig. 2:

(i) We start at v, at the polar coordinates $r_v = 0$ and $\phi_v = \alpha_v - 1/2$.
(ii) We take the path C_v^- along the cut at $\phi_v = \alpha_v - 1/2$ from $r_v = 0$ to $r_v = R$.
(iii) We go around the outer boundary $\partial_R v^*$ of the disk at $r_v = R$ from $\phi_v = \alpha_v - 1/2$ to $\phi_v = \alpha_v + 1/2$.
(iv) We take the path C_v^+ along the cut at $\phi_v = \alpha_v + 1/2$ from $r_v = R$ to $r_v = 0$.
(v) Finally, we go around the inner boundary $\partial_0 v^*$ of the disk at $r_v = 0$ from $\phi_v = \alpha_v + 1/2$ to $\phi_v = \alpha_v - 1/2$, back to our starting point.

Let us evaluate each term individually. On C_v^- and C_v^+ we have[21] from (50)

$$\overrightarrow{\exp}\left(\int_{C_v^-} \mathbf{A}\right) = \overrightarrow{\exp} \int_v^{v_0} \mathbf{A} = h_v^{-1}(v) h_v(v_0), \quad (150)$$

$$\overrightarrow{\exp}\left(\int_{C_v^+} \mathbf{A}\right) = \overrightarrow{\exp} \int_{v_0}^{v} \mathbf{A} = h_v^{-1}(v_0) h_v(v). \quad (151)$$

On the inner boundary we have, again using (50),

$$\overrightarrow{\exp} \int_{\partial_0 v^*} \mathbf{A} = \overrightarrow{\exp} \int_{v(\phi_v = \alpha_v + 1/2)}^{v(\phi_v = \alpha_v - 1/2)} \mathbf{A} = h_v^{-1}(v) \mathrm{e}^{-\mathbf{M}_v} h_v(v), \quad (152)$$

since h_v is periodic. The minus sign comes from the fact that we are going from a larger angle to a smaller angle. Finally, on the outer boundary we have, splitting into arcs and then using (136) and $(vc) = (cv)^{-1}$,

$$\overrightarrow{\exp} \int_{\partial_R v^*} \mathbf{A} = \prod_{c \in v} \left(\overrightarrow{\exp} \int_{(vc)} \mathbf{A}\right) = \prod_{c \in v} \tilde{H}_{vc}. \quad (153)$$

In conclusion, the curvature constraint on the disks is

$$F_v = h_v^{-1}(v) h_v(v_0) \left(\prod_{c \in v} \tilde{H}_{vc}\right) h_v^{-1}(v_0) \mathrm{e}^{-\mathbf{M}_v} h_v(v) = 1. \quad (154)$$

In fact, we can multiply both sides by $h_v^{-1}(v_0) h_v(v)$ from the left and $h_v^{-1}(v) h_v(v_0)$ and obtain, after redefining F_v,

$$F_v \equiv \left(\prod_{c \in v} \tilde{H}_{vc}\right) h_v^{-1}(v_0) \mathrm{e}^{-\mathbf{M}_v} h_v(v_0) = 1. \quad (155)$$

This may be written more suggestively as

$$\prod_{c \in v} \tilde{H}_{vc} = h_v^{-1}(v_0) \mathrm{e}^{\mathbf{M}_v} h_v(v_0). \quad (156)$$

Let us now show that this constraint is satisfied identically in our construction. From (117) we have

$$\tilde{H}_{vc} \equiv \mathring{h}_v^{-1}(\sigma_{vc}) \mathring{h}_v(\tau_{vc}), \quad (157)$$

and using the definition $\mathring{h}_v \equiv \mathrm{e}^{\mathbf{M}_v \phi_v} h_v$ from (39) we get

$$\tilde{H}_{vc} = h_v^{-1}(\sigma_{vc}) \mathrm{e}^{\mathbf{M}_v(\phi_v(\tau_{vc}) - \phi_v(\sigma_{vc}))} h_v(\tau_{vc}). \quad (158)$$

Now, consider the product

$$\prod_{c \in v} \tilde{H}_{vc} = \prod_{i=1}^{N_v} h_v^{-1}(\sigma_{vc_i}) \mathrm{e}^{\mathbf{M}_v(\phi_v(\tau_{vc_i}) - \phi_v(\sigma_{vc_i}))} h_v(\tau_{vc_i}). \quad (159)$$

This is a telescoping product; the term $h_v(\tau_{vc_i})$ always cancels the term $h_v^{-1}(\sigma_{vc_{i+1}})$ in the next factor in the product. After the cancellations take place, we are left only with $h_v^{-1}(\sigma_{vc_1})$, the product of exponents

$$\prod_{i=1}^{N_v} \mathrm{e}^{\mathbf{M}_v(\phi_v(\tau_{vc_i}) - \phi_v(\sigma_{vc_i}))} = \mathrm{e}^{\mathbf{M}_v}, \quad (160)$$

where we used the fact that the angles sum to 1, and $h_v(\tau_{vc_{N_v}}) = h_v(\sigma_{vc_1})$. In conclusion,

$$\prod_{c \in v} \tilde{H}_{vc} = h_v^{-1}(\sigma_{vc_1}) \mathrm{e}^{\mathbf{M}_v} h_v(\sigma_{vc_1}). \quad (161)$$

If we then choose, without loss of generality, the point v_0 (which defines the cut C_v) to be at σ_{vc_1} (where c_1 is an arbitrarily chosen cell), we get

$$\prod_{c \in v} \tilde{H}_{vc} = h_v^{-1}(v_0) \mathrm{e}^{\mathbf{M}_v} h_v(v_0), \quad (162)$$

and we see that the constraint is indeed identically satisfied.

[21]Note that the angle $\phi_v(x)$ in the term $\mathrm{e}^{\mathbf{M}_v \phi_v(x)}$ in (50) refers to the difference in angles between the starting point and the final point; therefore, it vanishes in this case since the path along the cut is purely radial.

C. Derivation of the constraints on the faces

We have seen that the Gauss constraints, as we have defined them, involve the fluxes on the edges and arcs. Since these fluxes are not part of the phase space for $\lambda = 1$, these constraints cannot be imposed in that case. Similarly, the curvature constraints involve the holonomies on the edges and arcs and therefore will not work for the case $\lambda = 0$. This is a result of formulating both constraints on the cells and disks, which then requires us to use the holonomies and fluxes on the edges and arcs which are on their boundaries.

Alternatively, instead of demanding that the torsion and curvature vanish on the cells and disks, we may demand that they vanish on the faces f_v created by the spin network links. Since the (closures of the) faces cover the entire spatial manifold Σ, this is entirely equivalent.

This alternative form is obtained by deforming (or expanding) the disks such that they coincide with the faces. The inner boundary $\partial_0 v^* \to \partial_0 f_v$ is still the vertex v. The outer boundary $\partial_R v^* \to \partial_R f_v$ now consists of links $(c_i c_{i+1})^*$, where $i \in \{1, \ldots, N_v\}$ and $c_{N_v+1} \equiv c_1$. The point v_0 on the outer boundary can now be identified, without loss of generality, with the node c_1^*. Thus, the cut $C_v \to C_{f_v}$ now extends from v to c_1^*.

Since the spatial manifold Σ is now composed solely of the union of the closures of the faces, and not cells and disks, we only need one type of Gauss constraint and one type of curvature constraint. Let us derive them now.

1. The Gauss constraint on the faces

The face Gauss constraint $\mathbf{G}_{f_v}$ will impose the torsionlessness condition $\mathbf{T} \equiv \mathrm{d}_{\mathbf{A}} \mathbf{E} = 0$ inside the faces:

$$0 = \mathbf{G}_{f_v} \equiv \int_{f_v} \overset{\circ}{h}_v (\mathrm{d}_{\mathbf{A}} \mathbf{E}) \overset{\circ}{h}_v^{-1} = \int_{f_v} \mathrm{d}\left(\overset{\circ}{h}_v \mathbf{E} \overset{\circ}{h}_v^{-1}\right) = \int_{\partial f_v} \overset{\circ}{h}_v \mathbf{E} \overset{\circ}{h}_v^{-1} = \int_{\partial f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v. \quad (163)$$

The boundary ∂f_v is composed of the inner boundary $\partial_0 f_v$, the outer boundary $\partial_R f_v$, and the cut C_{f_v}:

$$\mathbf{G}_{f_v} = \int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v - \int_{\partial_0 f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v - \int_{C_{f_v}} \mathrm{d}\overset{\circ}{\mathbf{x}}_v, \quad (164)$$

where the minus signs represent the relative orientations of each piece. On the inner boundary $\partial_0 f_v$, we use the fact that $\mathbf{x}_v$ takes the constant value $\mathbf{x}_v(v)$ to obtain as for $\partial_0 v^*$ above:

$$\int_{\partial_0 f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \mathbf{S}_v + e^{\mathbf{M}_v(\alpha_v - \frac{1}{2})} (\mathrm{e}^{\mathbf{M}_v} \mathbf{x}_v(v) \mathrm{e}^{-\mathbf{M}_v} - \mathbf{x}_v(v)) \times \mathrm{e}^{-\mathbf{M}_v(\alpha_v - \frac{1}{2})}. \quad (165)$$

On the cut C_v, we have as before

$$\int_{C_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \mathrm{e}^{\mathbf{M}_v(\alpha_v - \frac{1}{2})} (\mathrm{e}^{\mathbf{M}_v} (\mathbf{x}_v(v_0) - \mathbf{x}_v(v)) \mathrm{e}^{-\mathbf{M}_v} - (\mathbf{x}_v(v_0) - \mathbf{x}_v(v))) \mathrm{e}^{-\mathbf{M}_v(\alpha_v - \frac{1}{2})}. \quad (166)$$

The outer boundary $\partial_R f_v$ splits into links:

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} \int_{c_i^*}^{c_{i+1}^*} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} (\overset{\circ}{\mathbf{x}}_v(c_{i+1}^*) - \overset{\circ}{\mathbf{x}}_v(c_i^*)). \quad (167)$$

Now, (102) can be inverted[22] to get

$$\overset{\circ}{\mathbf{x}}_v = h_{vc} \mathbf{x}_c h_{cv} + \mathbf{x}_v^c. \quad (168)$$

Plugging into (167), we get

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} (h_{vc_{i+1}} \mathbf{x}_{c_{i+1}} (c_{i+1}^*) h_{c_{i+1} v} - h_{vc_i} \mathbf{x}_{c_i} (c_i^*) h_{c_i v} + \mathbf{x}_v^{c_{i+1}} - \mathbf{x}_v^{c_i}). \quad (169)$$

In fact, we can get rid of the first two terms, since the sum is telescoping: each term of the from $h_{vc_i} \mathbf{x}_{c_i}(c_i^*) h_{c_i v}$ for $i = j$ is canceled[23] by a term of the form $h_{vc_{i+1}} \mathbf{x}_{c_{i+1}} (c_{i+1}^*) h_{c_{i+1} v}$ for $i = j - 1$. Thus we get

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} (\mathbf{x}_v^{c_{i+1}} - \mathbf{x}_v^{c_i}). \quad (170)$$

Next, we note that from (101) we have

$$h_{cc'} = h_c h_{c'}^{-1}, \qquad \mathbf{x}_c^{c'} = \mathbf{x}_c - h_{cc'} \mathbf{x}_{c'} h_{c'c}, \quad (171)$$

and if we plug in (102) for h_c, $h_{c'}$, $\mathbf{x}_c$ and $\mathbf{x}_{c'}$ we get

$$h_{cc'} = h_{cv} h_{vc'}, \quad (172)$$

$$\mathbf{x}_c^{c'} = h_{cv} (\overset{\circ}{\mathbf{x}}_v - \mathbf{x}_v^c) h_{vc} - h_{cc'} h_{c'v} (\overset{\circ}{\mathbf{x}}_v - \mathbf{x}_v^{c'}) h_{vc'} h_{c'c}. \quad (173)$$

From (172) we see that $h_{cc'} h_{c'v} = h_{cv}$. Plugging this into (173), we get the simplified expression

$$\mathbf{x}_c^{c'} = h_{cv} (\mathbf{x}_v^{c'} - \mathbf{x}_v^c) h_{vc}. \quad (174)$$

[22]Note that (102) is only valid on the arc (vc), which is the boundary between c and v^*. However, since we have expanded the disks, the arcs now coincide with the links, with every arc (vc) intersecting the two links connected to the node c^*. Thus the equation is still valid at c^* itself.

[23]Of course, $\mathbf{x}_v^{c_{i+1}}$ and $\mathbf{x}_v^{c_i}$ also cancel each other, but we choose to leave them.

Therefore, we may rewrite (170) as

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} h_{vc_i} \mathbf{x}_{c_i}^{c_{i+1}} h_{c_i v}. \tag{175}$$

Finally, we recall from (123) the definition of the fluxes on the links:

$$\mathbf{X}_c^{c'} \equiv h_c^{-1}(\sigma_{cc'}) \mathbf{x}_c^{c'} h_c(\sigma_{cc'}) = h_c^{-1}(v_0) \mathbf{x}_c^{c'} h_c(v_0). \tag{176}$$

In the second equality we use the fact that, since we have deformed the disks, the source point $\sigma_{cc'}$ of the edge (cc') lies on the spin network itself, and we can further deform the edge such that $\sigma_{cc'} = v_0$. Plugging into (175), we obtain

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} h_{vc_i} h_{c_i}(v_0) \mathbf{X}_{c_i}^{c_{i+1}} h_{c_i}^{-1}(v_0) h_{c_i v}. \tag{177}$$

Finally, from (102) we have $h_{vc} h_c = \overset{\circ}{h}_v$, and we get

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \overset{\circ}{h}_v(v_0) \left(\sum_{i=1}^{N_v} \mathbf{X}_{c_i}^{c_{i+1}} \right) \overset{\circ}{h}{}_v^{-1}(v_0). \tag{178}$$

Adding up the integrals in (164), we obtain the Gauss constraint on the faces:

$$\mathbf{G}_{f_v} = \overset{\circ}{h}_v(v_0) \left(\sum_{i=1}^{N_v} \mathbf{X}_{c_i}^{c_{i+1}} \right) \overset{\circ}{h}{}_v^{-1}(v_0) - \mathbf{S}_v - \mathrm{e}^{\mathbf{M}_v(\alpha_v - \frac{1}{2})} (\mathrm{e}^{\mathbf{M}_v} \mathbf{x}_v(v_0) \mathrm{e}^{-\mathbf{M}_v} - \mathbf{x}_v(v_0)) \mathrm{e}^{-\mathbf{M}_v(\alpha_v - \frac{1}{2})} = 0. \tag{179}$$

Just like the Gauss constraint on the disks, this can be simplified by noting that the constraint comes dotted with an element $\boldsymbol{\beta}_{f_v}$ of the Cartan subalgebra, which commutes with $\mathbf{M}_v$:

$$\boldsymbol{\beta}_{f_v} \cdot \mathbf{G}_{f_v} = \boldsymbol{\beta}_{f_v} \cdot \left(h_v(v_0) \left(\sum_{i=1}^{N_v} \mathbf{X}_{c_i}^{c_{i+1}} \right) h_v^{-1}(v_0) - \mathbf{S}_v \right) = 0, \tag{180}$$

where we used the fact that $\overset{\circ}{h}_v = \mathrm{e}^{\mathbf{M}_v \phi_v} h_v$ and the $\mathrm{e}^{\mathbf{M}_v \phi_v}$ part commutes with $\boldsymbol{\beta}_{f_v}$. Thus, Gauss constraint on the faces may be rewritten in a simplified way:

$$\mathbf{G}_{f_v} \equiv \sum_{i=1}^{N_v} \mathbf{X}_{c_i}^{c_{i+1}} - h_v^{-1}(v_0) \mathbf{S}_v h_v(v_0) = 0. \tag{181}$$

Let us now show that this constraint is satisfied identically. We have from the definition of $\overset{\circ}{\mathbf{x}}_v$:

$$\begin{aligned} \int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v &= \sum_{i=1}^{N_v} \int_{c_i^*}^{c_{i+1}^*} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \sum_{i=1}^{N_v} (\overset{\circ}{\mathbf{x}}_v(c_{i+1}^*) - \overset{\circ}{\mathbf{x}}_v(c_i^*)) \\ &= \sum_{i=1}^{N_v} (\mathrm{e}^{\mathbf{M}_v \phi_v(c_{i+1}^*)} \mathbf{x}_v(c_{i+1}^*) \mathrm{e}^{-\mathbf{M}_v \phi_v(c_{i+1}^*)} \\ &\quad - \mathrm{e}^{\mathbf{M}_v \phi_v(c_i^*)} \mathbf{x}_v(c_i^*) \mathrm{e}^{-\mathbf{M}_v \phi_v(c_i^*)} \\ &\quad + \mathbf{S}_v(\phi_v(c_{i+1}^*) - \phi_v(c_i^*))). \end{aligned}$$

The sum is telescoping, and every term cancels the previous one. However, in the term with $i = N_v$, we have

$$\phi_v(c_{N_v+1}^*) = \phi_v(c_1^*) + 1, \tag{182}$$

since ϕ_v, unlike $\mathbf{x}_v$, is not periodic. Therefore, the first and last terms do not cancel each other. If we furthermore choose $v_0 \equiv c_1^*$, we get

$$\int_{\partial_R f_v} \mathrm{d}\overset{\circ}{\mathbf{x}}_v = \mathbf{S}_v + \mathrm{e}^{\mathbf{M}_v \phi_v(v_0)} (\mathrm{e}^{\mathbf{M}_v} \mathbf{x}_v(v_0) \mathrm{e}^{-\mathbf{M}_v} - \mathbf{x}_v(v_0)) \times \mathrm{e}^{-\mathbf{M}_v \phi_v(v_0)}. \tag{183}$$

Then, using (178) we immediately obtain (179), as desired.

2. *The curvature constraint on the faces*

The face curvature constraint F_{f_v} will impose that $\mathbf{F} \equiv \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}] = 0$ inside the faces. As before, an equivalent condition is that the holonomy around the face evaluates to the identity:

$$\begin{aligned} 1 = F_{f_v} &\equiv \overrightarrow{\exp} \int_{\partial f_v} \mathbf{A} \\ &= \overrightarrow{\exp} \left(\int_{C_v^-} \mathbf{A} \right) \overrightarrow{\exp} \left(\int_{\partial_R f_v} \mathbf{A} \right) \\ &\quad \times \overrightarrow{\exp} \left(\int_{C_v^+} \mathbf{A} \right) \overrightarrow{\exp} \left(\int_{\partial_0 f_v} \mathbf{A} \right). \end{aligned} \tag{184}$$

On C_v^- and C_v^+ we have as before

$$\overrightarrow{\exp} \left(\int_{C_v^-} \mathbf{A} \right) = \overrightarrow{\exp} \int_v^{v_0} \mathbf{A} = h_v^{-1}(v) h_v(v_0), \tag{185}$$

$$\overrightarrow{\exp} \left(\int_{C_v^+} \mathbf{A} \right) = \overrightarrow{\exp} \int_{v_0}^{v} \mathbf{A} = h_v^{-1}(v_0) h_v(v). \tag{186}$$

On the inner boundary we have

$$\overrightarrow{\exp} \int_{\partial_0 f_v} \mathbf{A} = \overrightarrow{\exp} \int_{v(\phi_v = \alpha_v + 1/2)}^{v(\phi_v = \alpha_v - 1/2)} \mathbf{A} = h_v^{-1}(v) \mathrm{e}^{-\mathbf{M}_v} h_v(v). \tag{187}$$

Finally, we decompose the outer boundary (which is now a loop on the spin network) into links:

$$\overrightarrow{\exp}\int_{\partial_R f_v}\mathbf{A}=\prod_{i=1}^{N_v}\left(\overrightarrow{\exp}\int_{c_i^*}^{c_{i+1}^*}\mathbf{A}\right). \tag{188}$$

From (55) we know that

$$\overrightarrow{\exp}\int_{c^*}^{c'^*}\mathbf{A}=h_c^{-1}(c^*)h_{cc'}h_{c'}(c'^*), \tag{189}$$

and therefore

$$\begin{aligned}\overrightarrow{\exp}\int_{\partial_R f_v}\mathbf{A}&=\prod_{i=1}^{N_v}h_{c_i}^{-1}(c_i^*)h_{c_ic_{i+1}}h_{c_{i+1}}(c_{i+1}^*)\\&=h_{c_1}^{-1}(v_0)\left(\prod_{i=1}^{N_v}h_{c_ic_{i+1}}\right)h_{c_1}(v_0),\end{aligned} \tag{190}$$

where we used the choice $v_0\equiv c_1^*$ and the fact that the product is telescoping, that is, each term $h_{c_{i+1}}(c_{i+1}^*)$ cancels the term $h_{c_{i+1}}^{-1}(c_{i+1}^*)$ which follows it, except the first and last terms, which have nothing to cancel with.

Joining the integrals, we get

$$\begin{aligned}&h_v^{-1}(v)h_v(v_0)h_{c_1}^{-1}(v_0)\left(\prod_{i=1}^{N_v}h_{c_ic_{i+1}}\right)h_{c_1}(v_0)h_v^{-1}(v_0)\\&\quad\times \mathrm{e}^{-\mathbf{M}_v}h_v(v)=1.\end{aligned} \tag{191}$$

From (102) we find that

$$h_{c_1}(v_0)h_v^{-1}(v_0)=h_{c_1v}, \tag{192}$$

and thus

$$h_v^{-1}(v)h_{vc_1}\left(\prod_{i=1}^{N_v}h_{c_ic_{i+1}}\right)h_{c_1v}\mathrm{e}^{-\mathbf{M}_v}h_v(v)=1. \tag{193}$$

For the last step, since we have the identity on the right-hand side, we may cycle the group elements and rewrite the constraint as follows:

$$F_{f_v}\equiv\left(\prod_{i=1}^{N_v}h_{c_ic_{i+1}}\right)h_{c_1v}\mathrm{e}^{-\mathbf{M}_v}h_{vc_1}=1. \tag{194}$$

Switching to the notation of (108) and (109), we rewrite this as

$$F_{f_v}\equiv\left(\prod_{i=1}^{N_v}H_{c_ic_{i+1}}\right)H_{c_1v}\mathrm{e}^{-\mathbf{M}_v}H_{vc_1}=1. \tag{195}$$

An even nicer form of this constraint is

$$\prod_{i=1}^{N_v}H_{c_ic_{i+1}}=H_{c_1v}\mathrm{e}^{\mathbf{M}_v}H_{vc_1}. \tag{196}$$

In other words, the loop of holonomies on the left-hand side would be the identity if there is no curvature, that is, $\mathbf{M}_v=0$.

To show that this constraint is satisfied identically, we use (52) with $x=c^*$ and $y=c'^*$:

$$\overrightarrow{\exp}\int_{c^*}^{c'^*}\mathbf{A}=h_v^{-1}(c^*)\mathrm{e}^{\mathbf{M}_v(\phi_v(c'^*)-\phi_v(c^*))}h_v(c'^*). \tag{197}$$

Comparing with (55), we see that

$$h_c^{-1}(c^*)h_{cc'}h_{c'}(c'^*)=h_v^{-1}(c^*)\mathrm{e}^{\mathbf{M}_v(\phi_v(c'^*)-\phi_v(c^*))}h_v(c'^*), \tag{198}$$

and therefore

$$h_{cc'}=h_{cv}\mathrm{e}^{\mathbf{M}_v(\phi_v(c'^*)-\phi_v(c^*))}h_{vc'}. \tag{199}$$

We now use this to rewrite the left-hand side of (196) as follows:

$$\prod_{i=1}^{N_v}h_{c_ic_{i+1}}=\prod_{i=1}^{N_v}h_{c_iv}\mathrm{e}^{\mathbf{M}_v(\phi_v(c_{i+1}^*)-\phi_v(c_i^*))}h_{vc_{i+1}}. \tag{200}$$

Again, we have a telescoping product, and after canceling terms we are left with

$$\prod_{i=1}^{N_v}h_{c_ic_{i+1}}=h_{c_1v}\mathrm{e}^{\mathbf{M}_v}h_{vc_1}, \tag{201}$$

which is exactly (196) after using (108) and (109).

D. Summary and interpretation

In conclusion, we have obtained[24] Gauss constraints $\mathbf{G}_c,\mathbf{G}_v,\mathbf{G}_{f_v}$ and curvature constraints F_c, F_v, F_{f_v} for each cell c, disk v^* and face f_v:

$$\mathbf{G}_c\equiv\sum_{i=1}^{N_c}(\tilde{\mathbf{X}}_c^{c_i}+\tilde{\mathbf{X}}_c^{v_i})=0, \tag{202}$$

[24] One might wonder about the appearance of $h_v(v_0)$ in (204) and (206), since the true phase space variable is H_v, defined implicitly in (87) as a function of $h_v(v)$ and $h_v(v_0)$. It is possible that there is an expression for these two constraints in terms of H_v instead of $h_v(v_0)$, but since we only have an *implicit* definition for H_v in terms of its variation ΔH_v, it is unclear how to obtain it. For now, we simply assume that both H_v and $h_v(v_0)$ are phase space variables. See also footnote 12.

$$\mathbf{G}_v \equiv \sum_{i=1}^{N_v} \tilde{\mathbf{X}}_v^{c_i} - \mathbf{S}_v = 0, \tag{203}$$

$$\mathbf{G}_{f_v} \equiv \sum_{i=1}^{N_v} \mathbf{X}_{c_i}^{c_{i+1}} - h_v^{-1}(v_0)\mathbf{S}_v h_v(v_0) = 0, \tag{204}$$

$$F_c \equiv \prod_{i=1}^{N_c} \tilde{H}_{cc_i}\tilde{H}_{cv_i} = 1, \tag{205}$$

$$F_v \equiv \left(\prod_{i=1}^{N_v} \tilde{H}_{vc_i}\right) h_v^{-1}(v_0)\mathrm{e}^{-\mathbf{M}_v} h_v(v_0) = 1, \tag{206}$$

$$F_{f_v} \equiv \left(\prod_{i=1}^{N_v} H_{c_i c_{i+1}}\right) H_{c_1 v}\mathrm{e}^{-\mathbf{M}_v} H_{vc_1} = 1. \tag{207}$$

The Gauss constraint on the cell c can also be written as

$$\sum_{c'\ni c} \tilde{\mathbf{X}}_c^{c'} = -\sum_{v\ni c} \tilde{\mathbf{X}}_c^{v}. \tag{208}$$

It tells us that the sum of fluxes along the edges and arcs surrounding c is zero, as expected given that the interior of c is flat. Alternatively, we may say that the sum of fluxes along the edges is prevented from summing to zero by the presence of the fluxes on the arcs.

The Gauss constraint on the punctured disk v^* can also be written as

$$\sum_{c\in v} \tilde{\mathbf{X}}_v^{c} = \mathbf{S}_v. \tag{209}$$

It tells us that the sum of fluxes on the arcs of the disk is prevented from summing to zero due to the torsion at the vertex v, as encoded in the parameter $\mathbf{S}_v$. Note that if $\mathbf{S}_v = 0$, that is, there is no torsion at v, then the constraint becomes simply $\sum_{c\in v} \tilde{\mathbf{X}}_v^c = 0$.

Importantly, notice that the sum $\sum_{v\ni c} \tilde{\mathbf{X}}_c^v$ on the right-hand side of (208) is over all the fluxes on the arcs surrounding a particular cell c, while the sum $\sum_{c\in v} \tilde{\mathbf{X}}_v^c$ on the left-hand side of (209) is over all the fluxes on the arcs surrounding a particular disk v^*. While the sums look alike at first sight, they are completely different and one cannot be exchanged for the other.

The Gauss constraint on the face f_v can also be written as

$$\sum_{i=1}^{N_v} \mathbf{X}_{c_i}^{c_{i+1}} = h_v^{-1}(v_0)\mathbf{S}_v h_v(v_0). \tag{210}$$

It tells us that the sum of fluxes on the link forming the boundary of the face is prevented from summing to zero due to the torsion at the vertex v, as encoded in the parameter $\mathbf{S}_v$.

The curvature constraint on the cell c is

$$\prod_{i=1}^{N_c} \tilde{H}_{cc_i}\tilde{H}_{cv_i} = 1. \tag{211}$$

It is analogous to the cell Gauss constraint, and imposes that the product of holonomies along the boundary of the cell is the identity.

The curvature constraint on the punctured disk v^* can also be written as

$$\prod_{c\in v} \tilde{H}_{vc} = h_v^{-1}(v_0)\mathrm{e}^{\mathbf{M}_v} h_v(v_0). \tag{212}$$

On the left-hand side, we have a loop of holonomies around the vertex v. If $\mathbf{M}_v = 0$, that is, there is no curvature at v, then the right-hand side becomes the identity, as we would expect. Otherwise, it is a quantity which depends on the curvature. The curvature constraint on the disks is thus analogous to the Gauss constraint on the disks, with torsion replaced by curvature.

Finally, the curvature constraint on the face f_v can also be written as

$$\prod_{i=1}^{N_v} H_{c_i c_{i+1}} = H_{c_1 v}\mathrm{e}^{\mathbf{M}_v} H_{vc_1}. \tag{213}$$

It has the same meaning as the one on the disks, except that the loop of holonomies around the vertex v is now composed of links instead of arcs.

V. THE CONSTRAINTS AS GENERATORS OF SYMMETRIES

A. The discrete symplectic form

The discrete symplectic potential we have found is

$$\begin{aligned}\Theta = &\sum_{(cc')}((1-\lambda)\tilde{\mathbf{X}}_c^{c'} \cdot \mathbf{\Delta} H_c^{c'} - \lambda \mathbf{X}_c^{c'} \cdot \mathbf{\Delta}\tilde{H}_c^{c'}) \\ &+ \sum_{(vc)}((1-\lambda)\tilde{\mathbf{X}}_v^{c} \cdot \mathbf{\Delta} H_v^{c} - \lambda \mathbf{X}_v^{c} \cdot \mathbf{\Delta}\tilde{H}_v^{c}) \\ &+ \sum_{v}(\mathbf{X}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v]) \cdot \mathbf{\Delta} H_v).\end{aligned}$$

In the second line, we can use (114), that is, $\tilde{\mathbf{X}}_c^v = -H_{cv}\tilde{\mathbf{X}}_v^c H_{vc}$, to write

$$\tilde{\mathbf{X}}_v^c \cdot \mathbf{\Delta} H_v^c = (-H_{vc}\tilde{\mathbf{X}}_c^v H_{cv}) \cdot (\delta H_{vc} H_{cv}) = \tilde{\mathbf{X}}_c^v \cdot \mathbf{\Delta} H_c^v. \tag{214}$$

Thus, the labels c and v may be freely exchanged. Using the identity $\delta\Delta H = \frac{1}{2}[\mathbf{\Delta} H, \mathbf{\Delta} H]$, we find that the corresponding symplectic form $\Omega \equiv \delta\Theta$ is

$$\Omega = \sum_{(cc')}\left((1-\lambda)\left(\delta\tilde{\mathbf{X}}_c^{c'}\cdot\mathbf{\Delta}H_c^{c'}+\frac{1}{2}\tilde{\mathbf{X}}_c^{c'}\cdot[\mathbf{\Delta}H_c^{c'},\mathbf{\Delta}H_c^{c'}]\right)-\lambda\left(\delta\mathbf{X}_c^{c'}\cdot\mathbf{\Delta}\tilde{H}_c^{c'}+\frac{1}{2}\mathbf{X}_c^{c'}\cdot[\mathbf{\Delta}\tilde{H}_c^{c'},\mathbf{\Delta}\tilde{H}_c^{c'}]\right)\right)$$
$$+\sum_{(vc)}\left((1-\lambda)\left(\delta\tilde{\mathbf{X}}_v^{c}\cdot\mathbf{\Delta}H_v^{c}+\frac{1}{2}\tilde{\mathbf{X}}_v^{c}\cdot[\mathbf{\Delta}H_v^{c},\mathbf{\Delta}H_v^{c}]\right)-\lambda\left(\delta\mathbf{X}_v^{c}\cdot\mathbf{\Delta}\tilde{H}_v^{c}+\frac{1}{2}\mathbf{X}_v^{c}\cdot[\mathbf{\Delta}\tilde{H}_v^{c},\mathbf{\Delta}\tilde{H}_v^{c}]\right)\right)$$
$$+\sum_{v}\left(\delta\mathbf{X}_v\cdot\delta\mathbf{M}_v-(\delta\mathbf{S}_v+[\delta\mathbf{M}_v,\mathbf{X}_v]+[\mathbf{M}_v,\delta\mathbf{X}_v])\cdot\mathbf{\Delta}H_v-\frac{1}{2}(\mathbf{S}_v+[\mathbf{M}_v,\mathbf{X}_v])\cdot[\mathbf{\Delta}H_v,\mathbf{\Delta}H_v]\right).$$

We now look for transformations[25] with parameters $g_c \equiv e^{\boldsymbol{\beta}_c}$, $g_v \equiv e^{\boldsymbol{\beta}_v}$, $\mathbf{z}_c$ and $\mathbf{z}_v$ such that

$$I_{\boldsymbol{\beta}_c}\Omega \propto -\boldsymbol{\beta}_c\cdot\delta\mathbf{G}_c, \qquad I_{\boldsymbol{\beta}_v}\Omega \propto -\boldsymbol{\beta}_v\cdot\delta\mathbf{G}_v, \tag{215}$$

$$I_{\mathbf{z}_c}\Omega \propto -\mathbf{z}_c\cdot\mathbf{\Delta}F_c, \qquad I_{\mathbf{z}_v}\Omega \propto -\mathbf{z}_v\cdot\mathbf{\Delta}F_v. \tag{216}$$

We will see that the proportionality coefficients will be λ-dependent.

B. The Gauss constraints as generators of rotations

1. The Gauss constraint on the cells

Let us consider the rotation transformation with parameter $\boldsymbol{\beta}_c$ defined by

$$\mathcal{L}_{\boldsymbol{\beta}_c}H_{cc'} = \boldsymbol{\beta}_c H_{cc'}, \qquad \mathcal{L}_{\boldsymbol{\beta}_c}H_{cv} = \boldsymbol{\beta}_c H_{cv},$$
$$\mathcal{L}_{\boldsymbol{\beta}_c}\tilde{\mathbf{X}}_c^{c'} = [\boldsymbol{\beta}_c,\tilde{\mathbf{X}}_c^{c'}], \qquad \mathcal{L}_{\boldsymbol{\beta}_c}\tilde{\mathbf{X}}_c^{v} = [\boldsymbol{\beta}_c,\tilde{\mathbf{X}}_c^{v}], \tag{217}$$

such that any other variables (in particular, those unrelated to the particular c of choice) are unaffected. Applying it to Ω and using the identity $I_{\boldsymbol{\beta}_c}\Delta H_c^{c'} = I_{\boldsymbol{\beta}_c}\Delta H_c^{v} = \boldsymbol{\beta}_c$, we get

$$I_{\boldsymbol{\beta}_c}\Omega = \sum_{c'\ni c}(1-\lambda)([\boldsymbol{\beta}_c,\tilde{\mathbf{X}}_c^{c'}]\cdot\mathbf{\Delta}H_c^{c'}$$
$$-\delta\tilde{\mathbf{X}}_c^{c'}\cdot\boldsymbol{\beta}_c+\tilde{\mathbf{X}}_c^{c'}\cdot[\boldsymbol{\beta}_c,\mathbf{\Delta}H_c^{c'}])$$
$$+\sum_{v\ni c}(1-\lambda)([\boldsymbol{\beta}_c,\tilde{\mathbf{X}}_c^{v}]\cdot\mathbf{\Delta}H_c^{v}$$
$$-\delta\tilde{\mathbf{X}}_c^{v}\cdot\boldsymbol{\beta}_c+\tilde{\mathbf{X}}_c^{v}\cdot[\boldsymbol{\beta}_c,\mathbf{\Delta}H_c^{v}]).$$

However, the first and last triple products in each line cancel each other, and we are left with

$$I_{\boldsymbol{\beta}_c}\Omega = -(1-\lambda)\boldsymbol{\beta}_c\cdot\left(\sum_{c'\ni c}\delta\tilde{\mathbf{X}}_c^{c'}+\sum_{v\ni c}\delta\tilde{\mathbf{X}}_c^{v}\right)$$
$$= -(1-\lambda)\boldsymbol{\beta}_c\cdot\delta\mathbf{G}_c.$$

Hence this transformation is generated by the cell Gauss constraint $\mathbf{G}_c$, given by (202), as long as $\lambda \neq 1$.

2. The Gauss constraint on the disks

Next we consider the rotation transformation with parameter $\boldsymbol{\beta}_v$ defined by

$$\mathcal{L}_{\boldsymbol{\beta}_v}H_{vc} = \boldsymbol{\beta}_v H_{vc}, \qquad \mathcal{L}_{\boldsymbol{\beta}_v}\tilde{\mathbf{X}}_v^{c} = [\boldsymbol{\beta}_v,\tilde{\mathbf{X}}_v^{c}],$$
$$\mathcal{L}_{\boldsymbol{\beta}_v}H_v = (1-\lambda)\boldsymbol{\beta}_v H_v, \qquad \mathcal{L}_{\boldsymbol{\beta}_v}\mathbf{X}_v = (1-\lambda)[\boldsymbol{\beta}_v,\mathbf{X}_v], \tag{218}$$

such that any other variables (in particular, those unrelated to the particular v of choice) are unaffected. Importantly, we choose the 0-form $\boldsymbol{\beta}_v$ to be valued in the Cartan subalgebra, so it commutes with $\mathbf{M}_v$ and $\mathbf{S}_v$. Applying the transformation to Ω and using the identities $I_{\boldsymbol{\beta}_v}\Delta H_v^c = \boldsymbol{\beta}_v$ and $I_{\boldsymbol{\beta}_v}\Delta H_v = (1-\lambda)\boldsymbol{\beta}_v$, we get

$$I_{\boldsymbol{\beta}_v}\Omega = (1-\lambda)\sum_{c\in v}([\boldsymbol{\beta}_v,\tilde{\mathbf{X}}_v^{c}]\cdot\mathbf{\Delta}H_v^c-\delta\tilde{\mathbf{X}}_v^{c}\cdot\boldsymbol{\beta}_v+\tilde{\mathbf{X}}_v^{c}\cdot[\boldsymbol{\beta}_v,\mathbf{\Delta}H_v^c])$$
$$+(1-\lambda)([\boldsymbol{\beta}_v,\mathbf{X}_v]\cdot\delta\mathbf{M}_v-[\mathbf{M}_v,[\boldsymbol{\beta}_v,\mathbf{X}_v]]\cdot\mathbf{\Delta}H_v)$$
$$+(1-\lambda)((\delta\mathbf{S}_v+[\delta\mathbf{M}_v,\mathbf{X}_v]+[\mathbf{M}_v,\delta\mathbf{X}_v])\cdot\boldsymbol{\beta}_v$$
$$-(\mathbf{S}_v+[\mathbf{M}_v,\mathbf{X}_v])\cdot[\boldsymbol{\beta}_v,\mathbf{\Delta}H_v]).$$

Isolating $\boldsymbol{\beta}_v$ and using the fact that it commutes with $\mathbf{M}_v$ and $\mathbf{S}_v$, we see that most terms cancel,[26] and we get

$$I_{\boldsymbol{\beta}_v}\Omega = -(1-\lambda)\boldsymbol{\beta}_v\cdot\left(\sum_{c\in v}\delta\tilde{\mathbf{X}}_v^{c}-\delta\mathbf{S}_v\right) = -(1-\lambda)\boldsymbol{\beta}_v\cdot\mathbf{G}_v. \tag{220}$$

Hence this transformation is generated by the disk Gauss constraint $\mathbf{G}_v$, given by (203), as long as $\lambda \neq 1$.

3. The Gauss constraint on the faces

Lastly, we consider the rotation transformation with parameter $\boldsymbol{\beta}_{f_v}$ defined by

[25]The transformations will be given by the action of the Lie derivative $\mathcal{L}_{\mathbf{a}} \equiv I_{\mathbf{a}}\delta + \delta I_{\mathbf{a}}$ where $I_{\mathbf{a}}$ is the variational interior product with respect to $\mathbf{a}$. In the literature the notation $\delta_{\mathbf{a}}$ is often used instead, but we avoid it in order to prevent confusion with the variational exterior derivative δ.

[26]In this calculation, we make use of the Jacobi identity:

$$[\boldsymbol{\beta}_v,[\mathbf{M}_v,\mathbf{X}_v]]+[\mathbf{M}_v,[\mathbf{X}_v,\boldsymbol{\beta}_v]] = -[\mathbf{X}_v,[\boldsymbol{\beta}_v,\mathbf{M}_v]] = 0. \tag{219}$$

$$\mathcal{L}_{\beta_{f_v}}\tilde{H}_{cc'} = -\boldsymbol{\beta}_{f_v}\tilde{H}_{cc'}, \qquad \mathcal{L}_{\beta_{f_v}}\mathbf{X}_c^{c'} = -[\boldsymbol{\beta}_{f_v}, \mathbf{X}_c^{c'}],$$
$$\mathcal{L}_{\beta_{f_v}}H_v = \lambda\bar{\boldsymbol{\beta}}_{f_v}H_v, \qquad \mathcal{L}_{\beta_{f_v}}\mathbf{X}_v = \lambda[\bar{\boldsymbol{\beta}}_{f_v}, \mathbf{X}_v], \tag{221}$$

such that any other variables (in particular, those unrelated to the particular v of choice) are unaffected, and such that

$$\boldsymbol{\beta}_{f_v} \equiv h_v^{-1}(v_0)\bar{\boldsymbol{\beta}}_{f_v}h_v(v_0), \tag{222}$$

where $\bar{\boldsymbol{\beta}}_{f_v}$ is valued in the Cartan subalgebra. Applying the transformation to Ω, we get after a calculation analogous to the one we did for the disks,

$$I_{\beta_{f_v}}\Omega = -\lambda\left(\boldsymbol{\beta}_{f_v}\cdot\sum_{c'\in c}\delta\mathbf{X}_c^{c'} - \bar{\boldsymbol{\beta}}_{f_v}\cdot\delta\mathbf{S}_v\right)$$
$$= -\lambda\boldsymbol{\beta}_{f_v}\cdot\left(\sum_{c'\in c}\delta\mathbf{X}_c^{c'} - h_v^{-1}(v_0)\delta\mathbf{S}_v h_v(v_0)\right).$$

The variation of the Gauss constraint (204) is

$$\delta\mathbf{G}_{f_v} = \sum_{i=1}^{N_v}\delta\mathbf{X}_{c_i}^{c_{i+1}} - h_v^{-1}(v_0)(\delta\mathbf{S}_v + [\mathbf{S}_v, \boldsymbol{\Delta}h_v(v_0)])h_v(v_0), \tag{223}$$

but since $\bar{\boldsymbol{\beta}}_{f_v}$ is in the Cartan we have $\bar{\boldsymbol{\beta}}_{f_v}\cdot[\mathbf{S}_v, \boldsymbol{\Delta}h_v(v_0)] = 0$, so this simplifies to

$$\boldsymbol{\beta}_{f_v}\cdot\delta\mathbf{G}_{f_v} = \boldsymbol{\beta}_{f_v}\cdot\left(\sum_{i=1}^{N_v}\delta\mathbf{X}_{c_i}^{c_{i+1}} - h_v^{-1}(v_0)\delta\mathbf{S}_v h_v(v_0)\right). \tag{224}$$

Thus, in conclusion,

$$I_{\beta_{f_v}}\Omega = -\lambda\boldsymbol{\beta}_{f_v}\cdot\delta\mathbf{G}_{f_v}, \tag{225}$$

and this transformation is generated by the face Gauss constraint $\mathbf{G}_v$, given by (204), as long as $\lambda \neq 0$.

C. The curvature constraints as generators of translations

1. The curvature constraint on the cells

For the curvature constraint on the cells, we would like to find a translation transformation with parameter $\mathbf{z}_c$ such that

$$I_{\mathbf{z}_c}\Omega = -\mathbf{z}_c\cdot\boldsymbol{\Delta}F_c. \tag{226}$$

First, we should calculate $\boldsymbol{\Delta}F_c$. Recall that

$$F_c \equiv \prod_{i=1}^{N_c}\tilde{H}_{cc_i}\tilde{H}_{cv_i} = 1. \tag{227}$$

To simplify the calculation, let us define $K_i \equiv \tilde{H}_{cc_i}\tilde{H}_{cv_i}$ such that we may write

$$F_c = \prod_{i=1}^{N}K_i = K_1\cdots K_N, \tag{228}$$

where we omit the subscript c on N_c for brevity. Then

$$\begin{aligned}\delta F_c &= \delta K_1K_2\cdots K_N + K_1\delta K_2K_3\cdots K_N + \cdots\\ &\quad + K_1\cdots K_{N-2}\delta K_{N-1}K_N + K_1\cdots K_{N-1}\delta K_N\\ &= \boldsymbol{\Delta}K_1K_1K_2\cdots K_N + K_1\Delta K_2K_2K_3\cdots K_N + \cdots\\ &\quad + K_1\cdots K_{N-2}\boldsymbol{\Delta}K_{N-1}K_{N-1}K_N + K_1\cdots K_{N-1}\boldsymbol{\Delta}K_NK_N,\end{aligned}$$

where $\boldsymbol{\Delta}K_i \equiv \delta K_iK_i^{-1}$. Hence

$$\begin{aligned}\boldsymbol{\Delta}F_c &\equiv \delta F_cF_c^{-1}\\ &= \boldsymbol{\Delta}K_1 + K_1\Delta K_2K_1^{-1} + \cdots\\ &\quad + (K_1\cdots K_{N-2})\boldsymbol{\Delta}K_{N-1}(K_1\cdots K_{N-2})^{-1}\\ &\quad + (K_1\cdots K_{N-1})\boldsymbol{\Delta}K_N(K_1\cdots K_{N-1})^{-1}\\ &\equiv \sum_{i=1}^{N}(K_1\cdots K_{i-1})\boldsymbol{\Delta}K_i(K_1\cdots K_{i-1})^{-1},\end{aligned}$$

where $K_1\cdots K_{i-1} \equiv 1$ for $i = 1$. For conciseness, we may define χ_i such that $\chi_1 \equiv 1$ and, for $i > 1$,

$$\chi_i \equiv K_1\cdots K_{i-1} = \tilde{H}_{cc_1}\tilde{H}_{cv_1}\cdots\tilde{H}_{cc_{i-1}}\tilde{H}_{cv_{i-1}}, \tag{229}$$

and write

$$\boldsymbol{\Delta}F_c = \sum_{i=1}^{N}\chi_i\Delta K_i\chi_i^{-1}. \tag{230}$$

Plugging in $K_i \equiv \tilde{H}_{cc_i}\tilde{H}_{cv_i}$ back, and using the identity

$$\Delta K_i = \boldsymbol{\Delta}\tilde{H}_c^{c_i} + \tilde{H}_{cc_i}\Delta\tilde{H}_c^{v_i}\tilde{H}_{c_ic} \tag{231}$$

we get

$$\Delta F_c = \sum_{i=1}^{N}\chi_i(\boldsymbol{\Delta}\tilde{H}_c^{c_i} + \tilde{H}_{cc_i}\Delta\tilde{H}_c^{v_i}\tilde{H}_{c_ic})\chi_i^{-1}. \tag{232}$$

Now, if we transform only the dual fluxes $\mathbf{X}_c^{c'}$ and $\mathbf{X}_c^{v}$ (for a particular c), then we get

$$I_{\mathbf{z}_c}\Omega = -\lambda\sum_{i=1}^{N_c}(\mathcal{L}_{\mathbf{z}_c}\mathbf{X}_c^{c_i}\cdot\boldsymbol{\Delta}\tilde{H}_c^{c_i} + \mathcal{L}_{\mathbf{z}_c}\mathbf{X}_c^{v_i}\cdot\boldsymbol{\Delta}\tilde{H}_c^{v_i}). \tag{233}$$

Comparing with (232), we see that if we take

$$\mathcal{L}_{\mathbf{z}_c}\mathbf{X}_c^{c_i}=\chi_i^{-1}\mathbf{z}_c\chi_i, \quad \mathcal{L}_{\mathbf{z}_c}\mathbf{X}_c^{v_i}=\tilde{H}_{c_ic}\chi_i^{-1}\mathbf{z}_c\chi_i\tilde{H}_{cc_i}, \tag{234}$$

we will obtain

$$I_{\mathbf{z}_c}\Omega = -\lambda\mathbf{z}_c\cdot\mathbf{\Delta}F_c, \tag{235}$$

as required. Hence this transformation is generated by the cell curvature constraint F_c, given by (205), as long as $\lambda\neq 0$.

2. The curvature constraint on the disks

As in the cell case, we would like to find a translation transformation with parameter $\mathbf{z}_v$ such that

$$I_{\mathbf{z}_v}\Omega = -\mathbf{z}_v\cdot\mathbf{\Delta}F_v, \tag{236}$$

where

$$F_v \equiv \left(\prod_{i=1}^{N_v}\tilde{H}_{vc_i}\right)h_v^{-1}(v_0)\mathrm{e}^{-\mathbf{M}_v}h_v(v_0)=1. \tag{237}$$

First, we should calculate $\mathbf{\Delta}F_v$. Let us define, omitting the subscript v on N_v for brevity,

$$K_i\equiv\tilde{H}_{vc_i}, \qquad i\in\{1,\ldots,N\}, \tag{238}$$

$$K_{N+1}\equiv h_v^{-1}(v_0)\mathrm{e}^{-\mathbf{M}_v}h_v(v_0), \tag{239}$$

and

$$\chi_1\equiv 1, \qquad \chi_i\equiv K_1\cdots K_{i-1}. \tag{240}$$

Then we may calculate similarly to the previous subsection

$$F_v=\prod_{i=1}^{N+1}K_i \Rightarrow \mathbf{\Delta}F_v=\sum_{i=1}^{N+1}\chi_i\Delta K_i\chi_i^{-1}. \tag{241}$$

Note that for $i=N+1$ we have

$$\chi_{N+1}\equiv K_1\cdots K_N = F_vK_{N+1}^{-1}=F_vh_v^{-1}(v_0)\mathrm{e}^{\mathbf{M}_v}h_v(v_0), \tag{242}$$

and since we are imposing $F_v=1$, we get simply

$$\chi_{N+1}=h_v^{-1}(v_0)\mathrm{e}^{\mathbf{M}_v}h_v(v_0). \tag{243}$$

Furthermore, using the fact that

$$\mathbf{\Delta}K_{N+1}=h_v^{-1}(v_0)(\mathrm{e}^{-\mathbf{M}_v}\Delta h_v(v_0)\mathrm{e}^{\mathbf{M}_v}-\mathbf{\Delta}h_v(v_0)-\delta\mathbf{M}_v)\times h_v(v_0), \tag{244}$$

we see that

$$\chi_{N+1}\mathbf{\Delta}K_{N+1}\chi_{N+1}^{-1}=h_v^{-1}(v_0)(\mathbf{\Delta}h_v(v_0)-\mathrm{e}^{\mathbf{M}_v}\Delta h_v(v_0)\mathrm{e}^{-\mathbf{M}_v}-\delta\mathbf{M}_v)h_v(v_0). \tag{245}$$

Therefore, we finally obtain the result

$$\mathbf{\Delta}F_v=\sum_{i=1}^{N_v}\chi_i\Delta\tilde{H}_v^{c_i}\chi_i^{-1}+h_v^{-1}(v_0)\times(\mathbf{\Delta}h_v(v_0)-\mathrm{e}^{\mathbf{M}_v}\Delta h_v(v_0)\mathrm{e}^{-\mathbf{M}_v}-\delta\mathbf{M}_v)h_v(v_0). \tag{246}$$

Now, let us take

$$\mathbf{z}_v\equiv h_v^{-1}(v_0)\bar{\mathbf{z}}_vh_v(v_0), \tag{247}$$

where $\bar{\mathbf{z}}_v$ is a 0-form valued in the Cartan subalgebra, and calculate $\mathbf{z}_v\cdot\mathbf{\Delta}F_v$. We find that, since $[\bar{\mathbf{z}}_v,\mathbf{M}_v]=0$, the terms $\mathbf{\Delta}h_v(v_0)-\mathrm{e}^{\mathbf{M}_v}\Delta h_v(v_0)\mathrm{e}^{-\mathbf{M}_v}$ cancel out and we are left with

$$\mathbf{z}_v\cdot\mathbf{\Delta}F_v=\mathbf{z}_v\cdot\left(\sum_{i=1}^{N_v}\chi_i\Delta\tilde{H}_v^{c_i}\chi_i^{-1}-h_v^{-1}(v_0)\delta\mathbf{M}_vh_v(v_0)\right). \tag{248}$$

We may now derive the appropriate transformation. If we transform only the segment flux $\mathbf{X}_v^c$ and the vertex flux $\mathbf{X}_v$ (for a particular v), then we get

$$I_{\mathbf{z}_v}\Omega=-\lambda\sum_{i=1}^{N_v}\mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v^{c_i}\cdot\mathbf{\Delta}\tilde{H}_v^{c_i}+\mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v\cdot(\delta\mathbf{M}_v+[\mathbf{M}_v,\mathbf{\Delta}H_v]). \tag{249}$$

Comparing with (248), we see that if we take

$$\mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v^{c_i}=\chi_i^{-1}\mathbf{z}_v\chi_i,\mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v=\lambda\bar{\mathbf{z}}_v, \tag{250}$$

we will obtain, since $\bar{\mathbf{z}}_v\cdot[\mathbf{M}_v,\mathbf{\Delta}H_v]=0$,

$$I_{\mathbf{z}_v}\Omega=-\lambda\mathbf{z}_v\cdot\left(\sum_{i=1}^{N_v}\chi_i\Delta\tilde{H}_v^{c_i}\chi_i^{-1}-h_v^{-1}(v_0)\delta\mathbf{M}_vh_v(v_0)\right) = -\lambda\mathbf{z}_v\cdot\mathbf{\Delta}F_v, \tag{251}$$

as required. Hence this transformation is generated by the disk curvature constraint F_v, given by (206), as long as $\lambda\neq 0$.

3. The curvature constraint on the faces

We would now like to find a translation transformation with parameter $\mathbf{z}_{f_v}$ such that

$$I_{\mathbf{z}_{f_v}}\Omega = -\mathbf{z}_{f_v} \cdot \mathbf{\Delta} F_{f_v}, \tag{252}$$

where

$$F_{f_v} \equiv \left(\prod_{i=1}^{N_v} H_{c_i c_{i+1}}\right) H_{c_1 v} \mathrm{e}^{-\mathbf{M}_v} H_{v c_1} = 1. \tag{253}$$

As before, to calculate $\mathbf{\Delta} F_{f_v}$ we define, omitting the subscript v on N_v for brevity,

$$K_i \equiv H_{c_i c_{i+1}}, \qquad i \in \{1, \dots, N\}, \tag{254}$$

$$K_{N+1} \equiv H_{c_1 v} \mathrm{e}^{-\mathbf{M}_v} H_{v c_1}, \tag{255}$$

$$\chi_1 \equiv 1, \qquad \chi_i \equiv K_1 \cdots K_{i-1}. \tag{256}$$

Then a similar calculation to the previous section gives

$$\begin{aligned} \mathbf{\Delta} F_{f_v} = & \sum_{i=1}^{N_v} \chi_i \Delta H_{c_i}^{c_{i+1}} \chi_i^{-1} \\ & + H_{c_1 v} (\mathbf{\Delta} H_v^{c_1} - \mathrm{e}^{\mathbf{M}_v} \Delta H_v^{c_1} \mathrm{e}^{\mathbf{M}_v} - \delta \mathbf{M}_v) H_{v c_1}, \end{aligned} \tag{257}$$

and if we take

$$\mathbf{z}_{f_v} \equiv H_{c_1 v} \bar{\mathbf{z}}_{f_v} H_{v c_1}, \tag{258}$$

where $\bar{\mathbf{z}}_{f_v}$ is a 0-form valued in the Cartan subalgebra, we get

$$\mathbf{z}_{f_v} \cdot \mathbf{\Delta} F_{f_v} = \mathbf{z}_{f_v} \cdot \left(\sum_{i=1}^{N_v} \chi_i \Delta H_{c_i}^{c_{i+1}} \chi_i^{-1} - H_{c_1 v} \delta \mathbf{M}_v H_{v c_1}\right). \tag{259}$$

We may now derive the appropriate transformation. If we transform only the edge flux $\tilde{\mathbf{X}}_c^{c'}$ and the vertex flux $\mathbf{X}_v$ (for a particular v), then we get

$$\begin{aligned} I_{\mathbf{z}_{f_v}}\Omega = & (1-\lambda) \sum_{i=1}^{N_v} \mathcal{L}_{\mathbf{z}_{f_v}} \tilde{\mathbf{X}}_{c_i}^{c_{i+1}} \cdot \mathbf{\Delta} H_c^{c'} \\ & + \mathcal{L}_{\mathbf{z}_{f_v}} \mathbf{X}_v \cdot (\delta \mathbf{M}_v + [\mathbf{M}_v, \mathbf{\Delta} H_v]). \end{aligned} \tag{260}$$

Comparing with (259), we see that if we take

$$\begin{aligned} \mathcal{L}_{\mathbf{z}_{f_v}} \tilde{\mathbf{X}}_{c_i}^{c_{i+1}} = & -\chi_i^{-1} \mathbf{z}_{f_v} \chi_i, \mathcal{L}_{\mathbf{z}_v} \mathbf{X}_v \\ & = (1-\lambda) H_{v c_1} \mathbf{z}_{f_v} H_{c_1 v}, \end{aligned} \tag{261}$$

we will obtain

$$\begin{aligned} I_{\mathbf{z}_{f_v}}\Omega = & -(1-\lambda) \mathbf{z}_{f_v} \cdot \left(\sum_{i=1}^{N_v} \chi_i \Delta H_{c_i}^{c_{i+1}} \chi_i^{-1} - H_{c_1 v} \delta \mathbf{M}_v H_{v c_1}\right) \\ = & -(1-\lambda) \mathbf{z}_{f_v} \cdot \mathbf{\Delta} F_{f_v}, \end{aligned} \tag{262}$$

as required. Hence this transformation is generated by the face curvature constraint F_v, given by (206), as long as $\lambda \neq 0$.

D. Conclusions

We have found that the Gauss constraints $\mathbf{G}_c, \mathbf{G}_v, \mathbf{G}_{f_v}$ and curvature constraints F_c, F_v, F_{f_v} for each cell c, disk v^* and face f_v, given by (202)–(207), generate transformations with rotation parameters $\boldsymbol{\beta}_c, \boldsymbol{\beta}_v, \boldsymbol{\beta}_{f_v}$ and translations parameters $\mathbf{z}_c, \mathbf{z}_v, \mathbf{z}_{f_v}$ as follows:

$$\begin{aligned} & I_{\boldsymbol{\beta}_c}\Omega = -(1-\lambda) \boldsymbol{\beta}_c \cdot \delta \mathbf{G}_c, \qquad I_{\boldsymbol{\beta}_v}\Omega = -(1-\lambda) \boldsymbol{\beta}_v \cdot \delta \mathbf{G}_v, \\ & I_{\boldsymbol{\beta}_{f_v}}\Omega = -\lambda \boldsymbol{\beta}_{f_v} \cdot \delta \mathbf{G}_{f_v}, \end{aligned} \tag{263}$$

$$\begin{aligned} & I_{\mathbf{z}_c}\Omega = -\lambda \mathbf{z}_c \cdot \mathbf{\Delta} F_c, \qquad I_{\mathbf{z}_v}\Omega = -\lambda \mathbf{z}_v \cdot \mathbf{\Delta} F_v, \\ & I_{\mathbf{z}_{f_v}}\Omega = -(1-\lambda) \mathbf{z}_{f_v} \cdot \mathbf{\Delta} F_{f_v}. \end{aligned} \tag{264}$$

The Gauss constraint on the cell c generates rotations of the holonomies on the links $(cc')^*$ and segments $(cv)^*$ connected to the node c^* and the fluxes on the edges (cc') and arcs (cv) surrounding c:

$$\begin{aligned} & \mathcal{L}_{\boldsymbol{\beta}_c} H_{cc'} = \boldsymbol{\beta}_c H_{cc'}, \qquad \mathcal{L}_{\boldsymbol{\beta}_c} H_{cv} = \boldsymbol{\beta}_c H_{cv}, \\ & \mathcal{L}_{\boldsymbol{\beta}_c} \tilde{\mathbf{X}}_c^{c'} = [\boldsymbol{\beta}_c, \tilde{\mathbf{X}}_c^{c'}], \qquad \mathcal{L}_{\boldsymbol{\beta}_c} \tilde{\mathbf{X}}_c^{v} = [\boldsymbol{\beta}_c, \tilde{\mathbf{X}}_c^{v}], \end{aligned} \tag{265}$$

where $\boldsymbol{\beta}_c$ is a $\mathfrak{g}^*$-valued 0-form.

The Gauss constraint on the disk v^* generates rotations of the holonomies on the segments $(vc)^*$ connected to the vertex v and the fluxes on the arcs (vc) surrounding v^*, as well as the holonomy and flux on the vertex v itself:

$$\begin{aligned} & \mathcal{L}_{\boldsymbol{\beta}_v} H_{vc} = \boldsymbol{\beta}_v H_{vc}, \qquad \mathcal{L}_{\boldsymbol{\beta}_v} \tilde{\mathbf{X}}_v^{c} = [\boldsymbol{\beta}_v, \tilde{\mathbf{X}}_v^{c}], \\ & \mathcal{L}_{\boldsymbol{\beta}_v} H_v = (1-\lambda) \boldsymbol{\beta}_v H_v, \qquad \mathcal{L}_{\boldsymbol{\beta}_v} \mathbf{X}_v = (1-\lambda) [\boldsymbol{\beta}_v, \mathbf{X}_v], \end{aligned} \tag{266}$$

where $\boldsymbol{\beta}_v$ is a 0-form valued in the Cartan subalgebra $\mathfrak{h}^*$ of $\mathfrak{g}^*$.

The Gauss constraint on the face f_v generates rotations of the fluxes on the links $(cc')^*$ surrounding f_v and the holonomies on their dual edges (cc'), as well as the holonomy and flux on the vertex v itself:

$$\begin{aligned} & \mathcal{L}_{\boldsymbol{\beta}_{f_v}} \tilde{H}_{cc'} = -\boldsymbol{\beta}_{f_v} \tilde{H}_{cc'}, \qquad \mathcal{L}_{\boldsymbol{\beta}_{f_v}} \mathbf{X}_c^{c'} = -[\boldsymbol{\beta}_{f_v}, \mathbf{X}_c^{c'}], \\ & \mathcal{L}_{\boldsymbol{\beta}_{f_v}} H_v = \lambda \bar{\boldsymbol{\beta}}_{f_v} H_v, \qquad \mathcal{L}_{\boldsymbol{\beta}_{f_v}} \mathbf{X}_v = \lambda [\bar{\boldsymbol{\beta}}_{f_v}, \mathbf{X}_v], \end{aligned} \tag{267}$$

where $\bar{\boldsymbol{\beta}}_{f_v}$ is a 0-form valued in the Cartan subalgebra $\mathfrak{h}^*$ of $\mathfrak{g}^*$ and $\boldsymbol{\beta}_{f_v} \equiv h_v^{-1}(v_0) \bar{\boldsymbol{\beta}}_{f_v} h_v(v_0)$.

The curvature constraint on the cell c generates translations[27] of the fluxes on the links $(cc')^*$ and segments $(cv)^*$ connected to the node c^*:

$$\mathcal{L}_{\mathbf{z}_c}\mathbf{X}_c^{c_i} = \chi_i^{-1}\mathbf{z}_c\chi_i, \quad \mathcal{L}_{\mathbf{z}_c}\mathbf{X}_c^{v_i} = \tilde{H}_{c_i c}\chi_i^{-1}\mathbf{z}_c\chi_i\tilde{H}_{cc_i}, \tag{268}$$

where

$$\chi_1 \equiv 1, \qquad \chi_i = \tilde{H}_{cc_1}\tilde{H}_{cv_1}\cdots\tilde{H}_{cc_{i-1}}\tilde{H}_{cv_{i-1}}, \tag{269}$$

and $\mathbf{z}_c$ is a $\mathfrak{g}$-valued 0-form.

The curvature constraint on the disk v^* generates translations of the fluxes on the segments $(vc)^*$ connected to the vertex v, as well as the flux on the vertex v itself:

$$\mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v^{c_i} = \chi_i^{-1}\mathbf{z}_v\chi_i, \qquad \mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v = \lambda\bar{\mathbf{z}}_v, \tag{270}$$

where

$$\chi_1 \equiv 1, \qquad \chi_i \equiv \tilde{H}_{vc_i}\cdots\tilde{H}_{vc_{i-1}}, \tag{271}$$

$\bar{\mathbf{z}}_v$ is a 0-form valued in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and $\mathbf{z}_v \equiv h_v^{-1}(v_0)\bar{\mathbf{z}}_v h_v(v_0)$.

The curvature constraint on the face f_v generates translations of the fluxes on the edges (cc') dual to the links surrounding the face f_v, as well as the flux on the vertex v itself:

$$\mathcal{L}_{\mathbf{z}_{f_v}}\tilde{\mathbf{X}}_{c_i}^{c_{i+1}} = -\chi_i^{-1}\mathbf{z}_{f_v}\chi_i, \qquad \mathcal{L}_{\mathbf{z}_v}\mathbf{X}_v = (1-\lambda)H_{vc_1}\mathbf{z}_{f_v}H_{c_1 v}, \tag{272}$$

where

$$\chi_1 \equiv 1, \qquad \chi_i \equiv H_{c_1 c_2}\cdots H_{c_{i-1}c_i}, \tag{273}$$

and $\mathbf{z}_{f_v}$ is a 0-form valued in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Importantly, in the case $\lambda = 0$, the usual loop gravity polarization, the curvature constraints on the cells and disks do not generate any transformations since $I_{\mathbf{z}_c}\Omega = I_{\mathbf{z}_v}\Omega = 0$. Similarly, for the case $\lambda = 1$, the dual polarization, the Gauss constraints on the cells and disks do not generate any transformations since $I_{\beta_c}\Omega = I_{\beta_v}\Omega = 0$. Of course, the reason for this is that, as we noted earlier, these constraints are formulated in the first place in terms of holonomies and fluxes which only exist in a particular polarization. Thus for $\lambda = 0$ we must instead use the curvature constraint on the faces,[28] and for $\lambda = 1$ we must instead use the Gauss constraint on the faces.

In the hybrid polarization with $\lambda = 1/2$, all of the discrete variables exist: there are holonomies and fluxes on both the links/edges and the arcs/segments. Therefore, in this polarization all six types of constraints may be consistently formulated using the available variables, and all of them generate transformations.

VI. SUMMARY AND OUTLOOK

In this paper, we generalized the work of [13] to include the most general possible discretization. We discovered a family of polarizations of the discrete phase space, given by different values of the parameter λ. Of these, the three cases of interest are $\lambda = 0$, $\lambda = 1$ and $\lambda = 1/2$.

In the $\lambda = 0$ case, which is the one we discussed in [13], the holonomies are on the links (and segments) and the fluxes are on their corresponding edges (and arcs), as in the familiar case of loop gravity. The Gauss constraints on the cells and disks generate rotations for all of the discrete variables, while the curvature constraints on the faces generate translations only for the fluxes on the edges and vertices.

In the $\lambda = 1$ case, the positions of the holonomies and fluxes are reversed. The holonomies are on the edges (and arcs) and the fluxes are on their corresponding links (and segments). The curvature constraints on the cells and disks generate translations for all of the fluxes, while the Gauss constraints on the faces generate rotations only for the fluxes on the links, holonomies on the edges, and fluxes and holonomies on the vertices.

Finally, in the $\lambda = 1/2$ case, we have the variables for both polarizations simultaneously. All six types of constraints exist, and each of them generates its associated transformations.

Intuitively, we may now conclude that the $\lambda = 0$ polarization corresponds to usual $2+1$D general relativity, while $\lambda = 1$ (the dual polarization) corresponds to teleparallel gravity. This intuition is motivated by the fact that, as we have seen, in the $\lambda = 1$ polarization the holonomies and fluxes switch places, and thus the curvature and torsion (and their respective constraints) also switch places.

Since $2+1$D general relativity has curvature but zero torsion, and teleparallel gravity has torsion but zero curvature, it makes sense to claim that these polarizations are related. Indeed, this is why we used the same parameter λ in both (22) and (56). Since the choice $\lambda = 1/2$ in (22) corresponds to Chern-Simons theory, we may further claim that the $\lambda = 1/2$ polarization in the discrete case is a discretization of Chern-Simons theory. Thus,

(i) the polarization $\lambda = 0$ corresponds to $2+1$D general relativity,
(ii) the polarization $\lambda = 1/2$ corresponds to Chern-Simons theory, and
(iii) the polarization $\lambda = 1$ corresponds to teleparallel gravity.

A discussion of quantization in different polarizations is provided in [9]. There, it is shown that in the $\lambda = 0$ case, the Gauss constraint is imposed at the kinematical level while the curvature constraint encodes the dynamics. In the

[27] Note that the curvature constraints do not transform any holonomies, since the holonomies are unaffected by translations.

[28] Which is indeed what we did in [13].

$\lambda = 1$, the roles of the constraints are reversed. This again motivates a relation between the $\lambda = 1$ case and teleparallel gravity. The relation of the $\lambda = 1/2$ case to Chern-Simons theory is motivated in [14]. We leave a more in-depth discussion and analysis of the relations between the $\lambda = 1$ case and teleparallel gravity, and between the $\lambda = 1/2$ case and Chern-Simons theory, to future work.

Following our exhaustive study of discretization of $2 + 1$D gravity, it is our goal to adapt this discretization scheme to the physically relevant case of $3 + 1$D gravity. While in the $2 + 1$D case there is only one place where an integration may be performed in two different ways, in the $3 + 1$D case there are two such integrations, since we have one more dimension. We expect to find both $3 + 1$D general relativity and $3 + 1$D teleparallel gravity as different polarizations of the discrete phase space. The discretization in $3 + 1$D dimensions will be presented in an upcoming paper [35].

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