Derrick's theorem in curved spacetime

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We extend Derrick's theorem to the case of a generic irrotational curved spacetime adopting a strategy similar to the original proof. We show that a static relativistic star made of real scalar fields is never possible regardless of the geometrical properties of the (static) spacetimes. The generalized theorem offers a tool that can be used to check the stability of localized solutions of a number of types of scalar field models as well as of compact objects of theories of gravity with a nonminimally coupled scalar degree of freedom.

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I. INTRODUCTION

Derrick's theorem [1] constitutes one of the most important results on localized solutions of the Klein-Gordon in Minkowski spacetime. The theorem was developed originally as an attempt to build a model for nonpointlike elementary particles [2,3] based on the now well-known concept of a "quasiparticle." Wheeler was the first to suggest the idea of an electromagnetic quasiparticle which he called a *geon*. In spite of the fact that Wheeler's geons do not really exist, other models were proposed (and are still studied) in which geons are composed of other fields in various settings. There are even (time dependent) formulations of this idea which are based on gravitational waves [4].

It is clear that, in the exploration of the idea that fundamental particles could be some form of geons, a crucial problem is to infer the stability of the geon itself. Derrick's theorem deals specifically with the stability of geons made of scalar fields. In particular, Derrick found that, in flat spacetimes, the Klein-Gordon equation cannot have a static solution with finite energy [5].

In relativistic astrophysics, Derrick's theorem has profound consequences: its proof implies that no stable boson star can be constructed with real scalar fields, and therefore that the existence of these objects requires more complex fields. Indeed the term *boson stars* nowadays is largely used to refer to complex scalar field stars, which are also called *Q balls* [6].

The consequences of Derrick's result span many different field of physics, from low energy phenomena to QCD, to nonlinear phenomena, to pure mathematics (see, e.g., the list of papers citing Ref. [1]). This is due to the fact that Derrick's results is related to a very general property of a class of differential equations called "Euclidean scalar field equations" to which the static Klein-Gordon equation belongs. In particular, Derrick's theorem is a direct consequence of the so-called Pohozaev identity [7,8]. This identity is akin to the well-known virial theorem as it relates the kinetic and potential energy of a localized scalar field configuration.

The original Derrick's theorem is limited to the case of flat spacetime, and since its publication a number of works have been published considering particular cases, metrics, or matter fields (see, e.g., Refs. [9–15]). However, no general proof of this theorem in curved spacetime and backreaction has been given. The purpose of this work is to provide such a generalization. The proof is based on the use of the $1+1+2$ covariant approach [16–18]. With this tool, we will be able to extend Derrick's results to the case of a curved spacetime. We will also discuss the consequences of such results on compact objects in some types of modifications of general relativity.

The paper is organized as follows: In Sec. II, we will describe briefly the $1+1+2$ formalism which will be the main tool of our proof. In Sec. III, we will use the $1+1+2$ formalism to prove Derrick's theorem in flat spacetime. In Sec. IV, we will extend this theorem to curved spacetimes. In Sec. V, we will analyze the effect of backreaction on the results of Sec IV. Section VI explores the effect that scalar field coupling might have on the proof of the extended Derrick theorem. Section VII is dedicated to the application of the generalized Pohozaev identity to different relevant models of scalar fields. Section VIII concerns the application of Derrick's theorem to nonminimally coupled theories of gravity. Finally Sec. IX is dedicated to the conclusions.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper and Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative, and a comma corresponds to partial differentiation. We use the $-,+,+,+$ signature, and the Riemann tensor is defined by

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$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \quad (1)$$

where the $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e., they are symmetric in the lower indices) defined by

$$\Gamma^a_{bd} = \frac{1}{2} g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}), \quad (2)$$

and g_{ab} is the metric tensor. The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{acbd}. \quad (3)$$

Finally, round brackets around indices of a given tensor represent symmetrization of these indices, whereas square brackets represent antisymmetization:

$$X_{(ab)} = \frac{1}{2}(X_{ab} + X_{ba}),$$
$$X_{[ab]} = \frac{1}{2}(X_{ab} - X_{ba}). \quad (4)$$

II. SOME ELEMENTS OF THE 1 + 1 + 2 COVARIANT FORMALISM

In what follows, we will make use of the $1+1+2$ covariant formalism [16–18] to construct a proof of Derrick's theorem in curved spacetime and in the context of modified gravity.

In the $1+1+2$ formalism, a generic spacetime is foliated in two surfaces, which we will call Υ, by the definition of a timelike and a spacelike congruence represented by the vectors u_a and e_a, respectively. The metric tensor can then be decomposed as

$$g_{ab} = -u_a u_b + e_a e_b + N_{ab}, \quad (5)$$

where N_{ab} is, at the same time, a projector operator and the metric of Υ. It will be useful also to define a three surface W with metric $h_{ab} = e_a e_b + N_{ab}$.

In line with the above decomposition, we can define three differential operators: a dot ($\dot{}$) represents the projection of the covariant derivative along u_a, e.g.,

$$\dot{X}^{a..b}{}_{c..d} = u^e \nabla_e X^{a..b}{}_{c..d}; \quad (6)$$

a hat ($\hat{}$) denotes the projection of the covariant derivative along e_a, e.g.,

$$\hat{X}_{a..b}{}^{c..d} \equiv e^f D_f X_{a..b}{}^{c..d}; \quad (7)$$

and δ_a represents the covariant derivative projected with N_{ab}, e.g.,

$$\delta_l X_{a..b}{}^{c..d} \equiv N_a{}^f \ldots N_b{}^g N_h{}^c \ldots N_i{}^d N_l{}^j D_j X_{f..g}{}^{h..i}. \quad (8)$$

At this point, the kinematics and dynamics of any spacetime can be described via the definition of some specific quantities constructed with the derivatives of u_a, e_a, and N_{ab}.

If one considers a spacetime endowed with a local rotational symmetry (LRS)—i.e., a spacetime in which a multiply transitive isometry group acting on the spacetime manifold—the $1+1+2$ formalism allows one to write the equations in terms only of scalar quantities. In our case, only the quantities

$$\mathcal{A} = e_a u_b \nabla^b u^a = e_a \dot{u}^a,$$
$$\phi = N_{ab} \nabla^b e^a = \delta_a e^a,$$
$$A_b = N_{ab} \dot{u}^a,$$
$$a_b = \hat{e}_b,$$
$$\zeta_{ab} = \left(N^c{}_{(a} N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd}\right) \nabla_c e_d \quad (9)$$

will be necessary.

Notice that our treatment will not involve vorticity, as vortical spacetimes are inherently stationary and we are interested here only in static spacetimes. In the following, for the sake of simplicity, we will call such general irrotational spacetimes "curved."

It is important to clarify the limits of the approach that we will follow to extend Derrick's idea. We first assume that our curved spacetime is such that, at any point, the quantities u_a, e_a, and N_{ab} can be defined as $C(1)$ tensor fields. In other words, the spacetime must be regular enough to be consistent with those fields.

III. COVARIANT DERRICK THEOREM IN FLAT SPACETIME

The equation of motion for a real scalar field φ minimally coupled to gravity is a Klein-Gordon equation of the form

$$\Box\varphi - V_{,\varphi} = 0, \quad (10)$$

where $\Box = \nabla^a \nabla_a$ is the d'Alembert operator, ∇_a is the covariant derivative with respect to the metric g_{ab}, $V = V(\varphi)$ is the scalar field potential, and V_φ denotes a derivative with respect to the scalar field φ.

In Ref. [1], to explore stability, one expresses the variation of the action deforming the spatial coordinates with a constant parameter λ in the Klein-Gordon action. We will use here the properties of the covariant approach to perform an equivalent operation. For simplicity, let us consider first the spherically symmetric case in a Minkowski spacetime. In the covariant language, a deformation like the one used by Derrick in Ref. [1] can be represented by the *quasiconformal* transformation

$$u_a \Rightarrow \bar{u}_a = u_a,$$
$$e_a \Rightarrow \bar{e}_a = \frac{1}{\lambda} e_a,$$
$$N_{ab} \Rightarrow \bar{N}_{ab} = \frac{1}{\lambda^2} N_{ab}, \tag{11}$$

where λ is assumed to be a generic positive function. Under Eq. (11), the d'Alembertian of φ

$$\Box\varphi = \frac{1}{\sqrt{-g}} \partial_a(\sqrt{-g} g^{ab} \varphi_{,b})_{,a} \tag{12}$$

transforms as

$$\Box\varphi \Rightarrow \lambda^2 \Box\varphi - \lambda\lambda^{,a}\varphi_{,a}, \tag{13}$$

which in static flat spacetime can be written as

$$\Box\varphi \Rightarrow \lambda^2 \varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q}, \tag{14}$$

where q is a parameter associated with the congruence e_a. Using the relation above, Eq. (10) becomes

$$\lambda^2 \varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} - V_\varphi = 0. \tag{15}$$

Equations of this type do not satisfy, in general, the Helmholtz conditions [19], and therefore they cannot be directly obtained as the Euler-Lagrange equations of any Lagrangian. However, Darboux showed [20–22] that, in one dimension, there is an equivalent second-order equation for which a variational principle can be found, namely,

$$e^\Phi(\lambda^2 \varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} - V_\varphi) = 0, \tag{16}$$

where e^Φ is known as the *integrator multiplier*. The form of the integrator multiplier in the case of an equation with the structure of Eq. (16) can be found via the relation

$$\frac{d}{d\varphi_{,q}} \mathcal{Q} - \frac{d}{dq}\left[\frac{d}{d\varphi_{,qq}} \mathcal{Q}\right] = 0, \tag{17}$$

where $\mathcal{Q}$ represents Eq. (16) and we have assumed that Φ does not depend on the derivatives of φ. In our case, it turns out that $\Phi = -3\ln\lambda + \Phi_0$, where Φ_0 is a constant. We will choose here $\Phi_0 = 0$ so that $\Phi = 0$ for $\lambda = 1$, and we recover the original action. With this choice, the action for Eq. (15) is given by

$$S(\lambda) = -\frac{1}{2}\int \frac{1}{\lambda^3}[\lambda^2\varphi_{,q}^2 + 2V(\varphi)]dq. \tag{18}$$

If the solution of Eq. (17) is localized, this integral will be well defined and finite. Derrick's deformation is given by

$$\lambda = \text{const}, \tag{19}$$

which implies that Eq. (18) can be written as

$$S(\lambda) = -\frac{1}{2}\left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{20}$$

where

$$I_1 = \int \varphi_{,q}^2 dq,$$
$$I_2 = 2\int V[\varphi(q)]dq. \tag{21}$$

Following Ref. [1], we now impose that Eq. (10) corresponds to an extremum of the action requiring that

$$\frac{\partial S(\lambda)}{\partial\lambda} = 0 \to \frac{I_1}{\lambda^2} + 3\frac{I_2}{\lambda^4} = 0. \tag{22}$$

Setting $\lambda = 1$, we obtain that

$$I_2 = -\frac{I_1}{3}, \tag{23}$$

i.e., the Pohozaev identity. This relation tells us that the Klein-Gordon equation can be an extremum of action (20) only if the integral of the potential is negative. This implies that, for example, a mass potential, which is defined positive, would never lead to an equilibrium.

We can determine the character of the extremum by considering the second-order derivative of $S(\lambda)$:

$$\frac{\partial^2 S(\lambda)}{\partial\lambda^2} = -\frac{I_1}{\lambda^3} - 6\frac{I_2}{\lambda^5}. \tag{24}$$

Substituting in Eq. (23) and setting $\lambda = 1$, we obtain

$$\frac{\partial^2 S(\lambda)}{\partial\lambda^2} = I_1 > 0. \tag{25}$$

Hence Eq. (10) is a minimum for the action provided that the integral of the potential V is negative.

Now, in the static case, the energy function (as defined in Ref. [23]) of φ can be related to the action via the relation[1]

$$E = -2S, \tag{26}$$

which implies

$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = -2I_1 < 0. \tag{27}$$

[1]This relation can be easily verified by calculating directly the (0,0) of the stress energy density for the scalar field which corresponds to the Hamiltonian or, more precisely, to the Lagrangian energy.

Therefore a minimum of the action corresponds to a maximum of the energy, and a localized solution $\varphi(q)$ of Eq. (10) must be unstable.

We can generalize this reasoning to the nonspherically symmetric case in which δ derivatives also appear. From Eq. (12), using the parameters w_2 and w_3 to map the two surface Υ, the d'Alembertian can be written as

$$\Box\varphi \Rightarrow \lambda^2\varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} + \sum_{i=2}^{3}(\lambda^2\varphi_{,w_iw_i} - \lambda\lambda_{,w_i}\varphi_{,w_i}). \tag{28}$$

Hence the Klein-Gordon equation is

$$\lambda^2\varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} + \sum_{i=2}^{3}(\lambda^2\varphi_{,w_iw_i} - \lambda\lambda_{,w_i}\varphi_{,w_i}) - V_\varphi = 0. \tag{29}$$

Considering the equivalent equation

$$e^{\Phi}\left[\lambda^2\varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} + \sum_{i=2}^{3}(\lambda^2\varphi_{,w_iw_i} - \lambda\lambda_{,w_i}\varphi_{,w_i}) - V_\varphi\right] = 0, \tag{30}$$

the integrator multiplier can be calculated using a condition similar to Eq. (17)[2]:

$$\sum_{i=1}^{3}\left\{\frac{d}{d\varphi_{,p_i}}\mathcal{Q} - \frac{d}{dp_i}\left[\frac{d}{d\varphi_{,p_ip_i}}\mathcal{Q}\right]\right\} = 0, \tag{31}$$

where $p_i = (0, q, w_2, w_3)$, $\mathcal{Q}$ is Eq. (30), and we have assumed again that Φ does not depend on the derivatives of φ. This relation amounts to the partial differential equation

$$\sum_{i=1}^{3}\left[\Phi_{,p_i} + 3\frac{\lambda_{,p_i}}{\lambda}\right] = 0. \tag{32}$$

Using the method of the characteristics, we can find the solutions

$$\Phi = -3\ln\lambda + C(w_2 - q, w_3 - q). \tag{33}$$

Since we want to return to the standard action $e^{\Phi} = 1$ for $\lambda = 1$, we can set $C = 0$. Thus the action can be written as

$$S = -\frac{1}{2}\int\frac{1}{\lambda^3}\left[\lambda^2\varphi_{,q}^2 + \lambda^2\sum_{i=2}^{3}\varphi_{,w_i}^2 + 2V(\varphi)\right]d\Omega, \tag{34}$$

where the $d\Omega = \prod_{a=1}^{3} dp_i$. Repeating the procedure above, we obtain

$$S(\lambda) = -\frac{1}{2}\left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{35}$$

where

$$I_1 = \int\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right]d\Omega,$$
$$I_2 = 2\int V[\varphi(q)]d\Omega, \tag{36}$$

which implies

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \rightarrow I_1 = -3I_2,$$
$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = -2I_1 < 0, \tag{37}$$

and shows that the solution φ is unstable.

This result is called in the literature Derrick's theorem, and it is the main reason why localized solutions of real scalar fields are generally considered unphysical. In the next sections, we will give a generalization of this result in the case of irrotational LRS spacetimes and explore its validity in more general spacetimes and in the context of modified gravity.

IV. COVARIANT DERRICK THEOREM IN CURVED SPACETIMES

Let us now prove Derrick's theorem in curved spacetimes. As before, for simplicity, we will start with the spherically symmetric case and then we will consider more complex cases.

A. Spherically symmetric spacetimes

Decomposing Eq. (10) in the $1+1+2$ variables and considering spherically symmetric LRSII spacetimes, the transformed Klein-Gordon equation reads

$$\lambda^2\varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} + [\mathcal{A}(\lambda) + \phi(\lambda)]\lambda\varphi_{,q} - V_\varphi = 0. \tag{38}$$

The above equation can be generated by the action

$$S(\lambda) = -\frac{1}{2}\int e^{\Phi(\lambda)}[\lambda^2\varphi_{,q}^2 + 2V(\varphi)]dq, \tag{39}$$

where

[2]Here we appear to force the original approach by Darboux, which works only for one-dimensional actions. However, we will show in the Appendix and in the following sections that the integrator multiplier can be associated with the volume form for the scalar field action, and therefore it can be determined with the Darboux procedure also in multidimensional actions (at least in our specific case). Indeed it will become clear that the form of the integrator multiplier is actually irrelevant for our purposes because its transformation properties can be deduced in general.

$$\Phi(\lambda) = \int \left\{\frac{1}{\lambda}[\mathcal{A}(\lambda) + \phi(\lambda)] - 3\frac{\lambda_{,q}}{\lambda}\right\} dq. \tag{40}$$

The integral (40) can be simplified by remembering that, under transformation (14), we have

$$\mathcal{A}(\lambda) = \lambda e_a \dot{u}^a = \lambda\mathcal{A}, \qquad \phi(\lambda) = \lambda\delta_a e^a = \lambda\phi. \tag{41}$$

This means that

$$\Phi(\lambda) - \Phi_0 = \int [\mathcal{A} + \phi] dq - 3\ln\lambda, \tag{42}$$

which yields

$$e^{\Phi(\lambda)} \Rightarrow \frac{e^\Phi}{\lambda^3}, \tag{43}$$

where we have chosen $\Phi_0 = 0$ so that $\lambda = 1$ implies $e^{\Phi(\lambda)} = e^\Phi$. In this way, Eq. (38) can be derived from the action

$$S = -\frac{1}{2}\int \frac{e^\Phi}{\lambda^3}[\lambda^2\varphi_{,q}^2 + 2V(\varphi)] dq. \tag{44}$$

The above expression is consistent with the interpretation of e^Φ as the volume form for action (39) of Eq. (38) (see the Appendix for details). In this perspective, the choice that we made for Φ_0 corresponds to a choice of the asymptotic properties of the metric. This fact can be understood by bearing in mind that, by definition, $\mathcal{A}$ and ϕ are identically zero when the spacetime is Minkowskian [24]. As we consider localized solutions for φ, it is only natural to choose an "asymptotically flat" Φ by choosing $\Phi_0 = 0$.

Setting $\lambda = \text{const}$, we can write

$$S(\lambda) = -\frac{1}{2}\left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{45}$$

where this time

$$I_1 = \int e^\Phi \varphi_{,q}^2 dq,$$
$$I_2 = 2\int e^\Phi V[\varphi(q)] dq. \tag{46}$$

Action (45) is the same as Eq. (18) and leads to the same conditions. This implies that Derrick's theorem is valid also in the curved spherically symmetric case.

It is important to stress that the Darboux procedure we have used so far to deduce the action is valid only if the integrator multiplier is other than zero. One can prove [24] that this condition implies that the spacetime we are considering does not contain a perfect or Killing horizon. Such a constraint excludes the case of spacetimes describing black holes, trapped surfaces, etc.

B. General irrotational spacetimes

What about more complex spacetimes? If vorticity is zero, upon the transformations (14), the Klein-Gordon equation reads

$$\lambda^2\varphi_{,qq} - \lambda\lambda_{,q}\varphi_{,q} + [\mathcal{A}(\lambda) + \phi(\lambda)]\lambda\varphi_{,q} + \sum_{b=2}^{3}[\lambda^2\varphi_{,w_iw_i} + \lambda\lambda_{,w_i}\varphi_{,w_i} + \mathcal{A}_b(\lambda)\lambda^2\varphi_{,w_i} + a_b(\lambda)\lambda^2\varphi_{,w_i}] - V_\varphi = 0. \tag{47}$$

It is clear that in nonspherical irrotational LRS spacetimes where $\mathcal{A}_b$ and a_b are identically zero (which still belong to the LRSII class), Derrick's theorem holds. We have

$$S(\lambda) = -\frac{1}{2}\left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{48}$$

where

$$I_1 = \int e^\Phi\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right] d\Omega,$$
$$I_2 = 2\int e^\Phi V[\varphi] d\Omega, \tag{49}$$

and Φ is given by Eq. (40) for $\lambda = \text{const}$.

If we consider more general spacetimes ($\mathcal{A}_b \neq 0$ and $a_b \neq 0$), we have to explore the transformation of the acceleration vectors under Eq. (11). We have

$$\mathcal{A}_b(\lambda) = N_b{}^c\dot{u}_c = \mathcal{A}_b,$$
$$a_b(\lambda) = N_b{}^c\hat{e}_c = N_b{}^c\lambda\widehat{\left(\frac{e_c}{\lambda}\right)} = a_b. \tag{50}$$

Defining the four-vector

$$V_a = (\mathcal{A} + \phi)e_a + (\mathcal{A}_c + a_c)N_b^c, \tag{51}$$

condition (31) for this case takes the form of the partial differential equation involving the components of V_a,

$$\sum_{i=1}^{3}\left[\Phi_{,p_i} + 3\frac{\lambda_{,p_i}}{\lambda} + V_i\right] = 0. \tag{52}$$

We can use the method of characteristics to solve the above equation and, as in all partial differential equations, the existence and properties of the solutions will depend critically on the boundary conditions. As we have seen in the spherically symmetric case (see also the Appendix), the boundary conditions are strictly related to the asymptotic properties of the specific metric which one is considering. Since we have assumed that the scalar field is localized, it is natural to assume asymptotic flatness. However, as far as it generates the correct field equation, the exact form of Φ is irrelevant for our purposes. We only need to determine the transformation of the quantity (53) under Eq. (11). From Eq. (52), it is evident that Φ will transform such that

$$\Phi(\lambda) = \Phi - 3\ln\lambda. \tag{53}$$

Using the above result, we obtain the action

$$S(\lambda) = -\frac{1}{2}\left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{54}$$

where

$$I_1 = \int e^{\Phi}\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right]d\Omega, \quad I_2 = 2\int e^{\Phi}V[\varphi]d\Omega. \tag{55}$$

This is the same result obtained in the flat case.

V. INTRODUCING BACKREACTION

In the previous section, we have made the tacit assumption that the mass of the confined scalar field solution would not perturb the assigned metric of the spacetime. In other words, we have neglected *backreaction*.

Is it possible to generalize the strategy above to the case in which the localized scalar field solution also contributes to the spacetime metric? In this case, one should add the Hilbert-Einstein term to the action for the scalar field

$$S = \frac{1}{2}\int dx^4\sqrt{-g}[R - \nabla_a\varphi\nabla^a\varphi - 2V(\varphi)] \tag{56}$$

and derive its transformation under Eq. (11).

It should be pointed out that, at present, there is no general consensus on the definition of the energy of the gravitational field. One should ask, then, if it makes sense to extend Derrick's results also to the backreaction case. A positive answer can be provided thinking that we are considering a very special case. First of all, in order to keep finite the action/energy integral, we have to assume an asymptotically flat background. In addition, since our choice of the vector field u_a corresponds to a timelike Killing field for the spacetimes we consider, the class of observers we consider is static.

From the results of Ref. [25], we have that, in stationary spacetimes, the energy of the gravitation field can be written as the scalar

$$E_G = \int\sqrt{-g}(t^m{}_n u^n + \sigma^{[mn]}{}_p\partial_n u^p)u_m d\Omega, \tag{57}$$

where $t^m{}_n$ is the Einstein pseudotensor and $\sigma^{[mn]}{}_p$ is Freud's complex [26] given by

$$\sigma^{[mn]}{}_p = \frac{1}{g}g_{pr}(gg^{r[m}g^{n]s})_{,s}. \tag{58}$$

Equation (57) can be written, in our assumptions, as

$$E_G = -\frac{1}{2}\int\sqrt{-g}\mathcal{L}^{(3)}_{\bar\Gamma\bar\Gamma}d\Omega, \tag{59}$$

where

$$\mathcal{L}^{(3)}_{\bar\Gamma\bar\Gamma} = h^{ab}(\bar\Gamma_{ad}{}^c\bar\Gamma_{cb}{}^d - \bar\Gamma_{ab}{}^c\bar\Gamma_{cd}{}^d), \tag{60}$$

the $\bar\Gamma$ being the Christoffel symbols of the three surface W. Now, starting from the Hilbert-Einstein action in the static case, we can write

$$S_G = \frac{1}{2}\int\sqrt{-g}R\,dt\,d\Omega = \frac{T_0}{2}\int\sqrt{-g}R\,d\Omega, \tag{61}$$

where T_0 is a constant, which we can set to 1 without loss of generality. Using the contracted Gauss-Codazzi equation, we have

$$S_G = \frac{1}{2}\int\sqrt{-g}(R^{(3)} + K^2 - K_a{}^bK^a{}_b + \nabla_a[\dot u^a + u^aK])d\Omega, \tag{62}$$

where $R^{(3)}$ is the Ricci scalar for submanifold W,

$$K_{ab} = h_a{}^ch_b{}^d\nabla_c u_d, \tag{63}$$

is the second fundamental form of W and $K = K^a{}_a$. Using the Gauss theorem, we can integrate out the last factor in Eq. (62). In addition, in static irrotational spacetimes the terms associated with the extrinsic curvature are identically zero. Now, $R^{(3)}$ can be decomposed in a similar way as R in Eq. (62), employing the extrinsic curvature of Υ. Integrating out the second (projected) derivatives and rewriting the expression in terms of the Christoffel symbols, we arrive at

$$S_G = S_0 + \frac{1}{2}\int\sqrt{-g}\mathcal{L}^{(3)}_{\bar\Gamma\bar\Gamma}d\Omega = -\frac{E_0}{2} - E_G, \tag{64}$$

where S_0 is a constant and we have defined $E_0 = -2S_0$. Thus in our case (and only in this case), modulus an irrelevant constant, the energy of the gravitational field can be linked to the Hilbert-Einstein action

$$E_G = -\frac{E_0}{2} - \frac{1}{2}\int e^{\Phi} R d\Omega. \quad (65)$$

Here, using the derivation of the Appendix, we have connected the volume form to e^{Φ}.

The next task is to evaluate how E_G transforms under Eq. (11). Using the Gauss-Codazzi equation also on $R^{(3)}$ gives, in static irrotational spacetimes,

$$R = R^{(2)} - 2\hat{\phi} - \frac{3}{2}\phi^2 - 2\zeta^2 - 2a_b a^b + 2\delta_b a^b. \quad (66)$$

Now, since

$$R^{(2)} = 2K_G, \quad (67)$$

where K_G is the Gaussian curvature, the Brioschi formula implies that, under Eq. (11),

$$R^{(2)} \Rightarrow R^{(2)}(\lambda) = \lambda^2 R^{(2)} + \ldots, \quad (68)$$

where the dots represent terms which contain derivatives of λ. In addition, from definition (9), one finds

$$\zeta_{ab} \Rightarrow \zeta_{ab}(\lambda) = \frac{\zeta_{ab}}{\lambda}, \qquad \zeta \Rightarrow \zeta(\lambda) = \lambda^2\zeta. \quad (69)$$

Using also Eqs. (41) and (50), we arrive at

$$R(\lambda) = \lambda^2 R + \ldots, \quad (70)$$

where, again, the dots represent additional terms which contain derivatives of λ. As we will eventually set up $\lambda = 1$, these terms are irrelevant and can be neglected.

The total action derived from Eq. (56) transforms as

$$S(\lambda) = \frac{S_G}{\lambda} + -\frac{1}{2}\int \frac{e^{\Phi}}{\lambda^3}\left[\lambda^2\varphi_{,q}^2 + \lambda^2\sum_{i=2}^{3}\varphi_{,w_i}^2 + 2V(\varphi)\right] d\Omega. \quad (71)$$

In the case $\lambda = \text{const}$, defining

$$I_1 = \int e^{\Phi}\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right] d\Omega, \qquad I_2 = 2\int e^{\Phi} V(\varphi) d\Omega, \qquad I_3 = -E_0 - 2E_G, \quad (72)$$

one can write

$$E(\lambda) = -2S(\lambda) = \left(\frac{I_1}{\lambda} - \frac{I_3}{\lambda} + \frac{I_2}{\lambda^3}\right). \quad (73)$$

Since, as we have seen, relation (26) between the energy and the action is still valid, we can examine the stability of the backreacting solution with the same strategy as the previous section. We have

$$\left.\frac{\partial E(\lambda)}{\partial \lambda}\right|_{\lambda=1} = 0 \rightarrow I_3 = I_1 + 3I_2, \qquad \left.\frac{\partial^2 E(\lambda)}{\partial \lambda^2}\right|_{\lambda=1} = 6I_2. \quad (74)$$

Now the trace of the field equations provides another relation that should be taken into consideration. We have

$$R = \nabla_a\varphi\nabla^a\varphi + 4V, \quad (75)$$

i.e., upon integration,

$$I_3 = I_1 + 2I_2. \quad (76)$$

Combining the above results with the first line of Eq. (74) gives $I_2 = 0$. This yields

$$\left.\frac{\partial^2 E(\lambda)}{\partial \lambda^2}\right|_{\lambda=1} = 8I_2 = 0. \quad (77)$$

Since the above quantity has opposite signs if we consider $I_2 \rightarrow I_2 \pm \epsilon$, where ϵ is a small constant, we have an inflection. Hence the presence of gravity has weakened the instability but cannot eliminate it completely.

The weakest point of the reasoning given above is, undoubtedly, the definition of the gravitational energy of the system. One might object that, even with our specific assumptions, the definition of energy we have used might miss some crucial aspect of the physics of these systems. We can argue here that this is not the case going around the problem of the definition of E_G by eliminating the Hilbert-Einstein term from the action using the field equations, i.e., considering the *on shell* action.

For example, using relation (75), we have

$$E^{\text{tot}}(\lambda) = -2\int \frac{e^{\Phi}}{\lambda^3} V(\varphi) d\Omega, \quad (78)$$

which immediately implies the result (77). This result shows that our previous argument is correct and, at the same time, suggests an easy shortcut to prove Derrick's theorem with backreaction. In the following, we will make ample use of this shortcut, especially in dealing with more complex settings.

We can use the on shell action to probe further in the validity of Derrick's theorem, by considering, for example, the case in which the scalar field backreacts with a spacetime with nonzero cosmological constant Λ. Equation (73) now reads

$$E(\lambda) = \left(\frac{I_1}{\lambda} - \frac{I_3}{\lambda} + \frac{I_2}{\lambda^3} + \frac{2I_4}{\lambda^3}\right), \tag{79}$$

where

$$I_4 = \Lambda \int e^{\Phi} d\Omega. \tag{80}$$

We obtain, on shell,

$$\left.\frac{\partial E(\lambda)}{\partial \lambda}\right|_{\lambda=1} = 0 \to I_3 = I_1 - I_2,$$
$$\left.\frac{\partial^2 E(\lambda)}{\partial \lambda^2}\right|_{\lambda=1} = 3(I_1 + 2I_2). \tag{81}$$

Hence in this case, stability is possible if

$$I_2 > -\frac{1}{2} I_1. \tag{82}$$

Therefore the presence of a cosmological constant can lead to stable solutions. However, these solutions make sense physically only at scales in which Λ is relevant, and they therefore exclude microscopic or astrophysical objects. Yet, the picture that emerges is that Derrick's instability cannot be avoided by minimal modifications of the model. In the following, we will explore further the validity of Derrick's theorem looking at the effect of scalar field coupling, noncanonical scalar field, and modified gravity.

VI. THE ROLE OF SCALAR FIELD COUPLINGS

Derrick's instability is very robust. No additional standard coupling of the scalar field with matter or other fields can prevent its appearance. A coupling with another scalar field of the type $f(\varphi)g(\psi)$ would just make more complicated the definition of the integral I_2. In fact, starting from the corresponding Klein-Gordon equations

$$\Box\varphi - V_{,\varphi} - f_{,\varphi} g(\psi) = 0, \tag{83}$$

and, proceeding as in the previous section, we obtain

$$S(\lambda) = -\frac{1}{2}\left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{84}$$

where

$$I_1 = \int e^{\Phi}\left[\varphi_{,q}^2 + \sum_{i=2}^{3} \varphi_{,w_i}^2\right] d\Omega,$$
$$I_2 = 2\int e^{\Phi}\{V[\varphi] + f[\varphi]g[\psi]\} d\Omega. \tag{85}$$

It follows that we can prove Derrick's theorem also in this case. This conclusion is independent from the sign of the terms appearing in the above integral. The presence of the coupling, however, changes the physical significance of the Pohozaev equilibrium condition. The same happens when we introduce backreaction.

What about other types of coupling? The strategy of the proof we have presented shows that, whatever the coupling, the key point in the determination of the stability of localized scalar field configurations relies on the λ dependence of the transformation of the integrator multiplier. If the transformation of $\exp(\Phi)$ is such that the action can be written as a combination of λ terms and λ-independent integrals, like, e.g., in Eq. (43), there will be a chance to prove (in)stability. In other cases, Derrick's approach does not lead to a definite answer.

A simple example is the case of derivative coupling of the type $a\hat{\varphi}g(\psi)$. For this coupling, the transformation of the integrator multiplier is given by

$$\Phi_g = \Phi + a\int \varphi_{,q} \frac{g(\psi)}{\lambda^2} dq, \tag{86}$$

and λ is not factorizable. This fact makes it impossible to find a form of the action similar to Eq. (48).

Instead, considering a coupling of the type $a\hat{\varphi}\hat{\psi}^2$ will yield

$$e^{\Phi_{gd}(\lambda)} \Rightarrow \frac{e^{\Phi_{gd}}}{\lambda}, \qquad \Phi_{gd} = \Phi + a\int \varphi_{,q}(\psi_{,q})^2 dq, \tag{87}$$

which leads to an action similar to Eq. (84) and thus implies instability.

VII. NONCANONICAL SCALAR FIELDS

In the context of cosmology and in particular when dealing with the problem of dark energy, a number of noncanonical scalar fields have been introduced. Using the strategy above, we can extend Derrick's theorem to these cases. In the following, we will consider the cases of phantom fields [27], quintom fields [28], and k-essence [29]. In these models, as in the ones of the next section, for the sake of brevity, we will make Derrick's deformation directly in the action rather than proving that the transformed action comes from the modified Klein-Gordon equation. This connection is, however, always valid. We will also consider only the case of LRSII spacetimes, as the generalization to more complicated geometries can be derived easily from the considerations above.

Phantom fields are scalar fields whose action contains a kinetic term with opposite sign with respect to the canonical one:

$$S = \frac{1}{2}\int dx^4 \sqrt{-g}[R + \nabla_a\varphi\nabla^a\varphi - 2V(\varphi)]. \tag{88}$$

Excluding backreaction, we have

$$E(\lambda) = -\left(\frac{I_1}{\lambda} - \frac{I_2}{\lambda^3}\right), \tag{89}$$

where

$$I_1 = \int e^{\Phi}\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right]d\Omega,$$
$$I_2 = 2\int e^{\Phi}V[\varphi]d\Omega. \tag{90}$$

Equation (89) yields

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \to I_1 = 3I_2,$$
$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = 2I_1 > 0, \tag{91}$$

which implies that a localized solution of phantom fields is actually stable. This result reveals that a key element of Derrick's instability is the sign of the scalar field kinetic terms.

The inclusion of backreaction, however, introduces instability. On shell, the energy can be written as

$$E(\lambda) = \frac{I_2}{\lambda^3}, \tag{92}$$

and we obtain, as in the standard case,

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \to -3I_2 = 0,$$
$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = 12I_2 = 0, \tag{93}$$

which is again an inflection.

In the case of quintom fields, we have two interacting fields: one canonical and the other noncanonical. The action reads

$$S = \frac{1}{2}\int dx^4\sqrt{-g}[R - \nabla_a\varphi\nabla^a\varphi + \nabla_a\psi\nabla^a\psi - 2V(\varphi,\psi)]. \tag{94}$$

Excluding backreaction, the energy function associated with Eq. (94), this action transforms under Eq. (11) as

$$E(\lambda) = \left(\frac{I_1}{\lambda} - \frac{I_4}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{95}$$

where

$$I_1 = \int e^{\Phi}\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right]d\Omega,$$
$$I_2 = 2\int e^{\Phi}V[\varphi]d\Omega,$$
$$I_4 = \int e^{\Phi}\left[\psi_{,q}^2 + \sum_{i=2}^{3}\psi_{,w_i}^2\right]d\Omega. \tag{96}$$

This leads to

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \to I_4 = 3I_2 + I_1,$$
$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = 6I_2 > 0, \tag{97}$$

which, as in the case of the phantom field, can be stable if I_2 is positive. Again, using the trace of the gravitational field equations to include backreaction, we can write the action above on shell,

$$S = \int dx^4\sqrt{-g}V(\varphi,\psi), \tag{98}$$

which leads to instability. This was an expected result, which confirms the general conclusions that we have drawn in Sec. VI: a multifield system becomes unstable if one of its components presents instability.

In the case of k-essence, the action is generalized as

$$S = \frac{1}{2}\int dx^4\sqrt{-g}[R + P(\varphi, X)], \tag{99}$$

where $X = \nabla_a\varphi\nabla^a\varphi$. Using the fact that, for $\lambda = 1$, $P_{,\lambda} = 2XP_{,X}$, under Eq. (11) and without backreaction, one has

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \to 2X\partial_X P - 3P = 0, \tag{100}$$

which means that

$$P = P_0(\varphi)X^{3/2}. \tag{101}$$

With this result, one obtains

$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = 0. \tag{102}$$

In other words, we always have instability.

Considering backreaction, we have

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \to X\partial_X P - P = 0, \tag{103}$$

which implies another inflection,

$$\frac{\partial^2 E(\lambda)}{\partial\lambda^2} = 0, \tag{104}$$

and therefore again instability.

On top of its intrinsic value, this result shows that conditions of Derrick's theorem can be used to constrain modifications of general relativity in which undetermined functions are present. In the next section, we will look at some interesting examples of such constraints.

VIII. SCALAR TENSOR GRAVITY

Using the results from the previous sections, we can proceed to the generalization of Derrick's theorem to nonminimal couplings. Let us consider, for example, the case of scalar tensor theories. This class of theories of gravity is characterized by

$$S = \frac{1}{2}\int dx^4\sqrt{-g}[F(\varphi)R - \nabla_a\varphi\nabla^a\varphi - 2V(\varphi)], \tag{105}$$

whose variation gives the field equations

$$\begin{aligned} FG_{ab} = \nabla_a\varphi\nabla_b\varphi - g_{ab}\left[\frac{1}{2}\nabla_a\varphi\nabla^a\varphi + V(\varphi)\right] \\ + \nabla_a\nabla_b F - g_{ab}\Box F, \end{aligned} \tag{106}$$

and the Klein-Gordon equation

$$\Box\varphi + \frac{1}{2}RF_{,\varphi} - V_{,\varphi} = 0, \tag{107}$$

where F represents the nonminimal coupling of the geometry (the Ricci scalar) with the field φ.

Notice that, since the Ricci scalar naturally enters in Eq. (107), there is no need to add by hand backreaction. We will then treat the full case by writing the action on shell.

Using the trace of the gravitational field equations and the Klein-Gordon equation, Eq. (105) can be written as

$$S = \frac{1}{2}\int dx^4\sqrt{-g}[K(\varphi)\nabla_a\varphi\nabla^a\varphi + W(\varphi)], \tag{108}$$

where

$$\begin{aligned} K(\varphi) &= -\frac{3(F_{,\varphi}^2 - 2FF_{,\varphi\varphi})}{2F + 3F_{,\varphi}^2}, \\ W(\varphi) &= \frac{VF_{,\varphi}^2 - FF_{,\varphi}V_{\varphi} - 2FV}{2F + 3F_{,\varphi}^2}. \end{aligned} \tag{109}$$

At this point, defining the integrals

$$\begin{aligned} I_1 &= \int e^{\Phi}K(\varphi)\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right]d\Omega, \\ I_2 &= 2\int e^{\Phi}W(\varphi)d\Omega, \end{aligned} \tag{110}$$

we can write

$$E(\lambda) = \left(\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{111}$$

which leads to

$$\begin{aligned} \left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} &= 0 \to I_1 + 3I_2 = 0, \\ \left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} &= 2I_1; \end{aligned} \tag{112}$$

therefore stability with a nonminimal coupling is possible if $I_1 > 0$ and $I_2 < 0$. This result allows us to give some limits on the functions F and V. In particular,

$$-\frac{1}{6}\varphi^2 < F \le \frac{\beta^2}{4\alpha}\varphi^2 + \beta\varphi + \alpha \tag{113}$$

and

$$V_0\varphi^4 < V \le V_0\left(\varphi - \frac{2\alpha}{\beta}\right)^{2-\frac{4\alpha}{3\beta^2}}, \tag{114}$$

where $V_0 < 0$, $\alpha > 0$, and β can be chosen freely.

Notice that Eq. (108) can be used only if we exclude the coupling

$$F_C = -\frac{1}{6}\varphi^2. \tag{115}$$

In order to also obtain a result in this case, we can construct an action on shell using the Klein-Gordon equation only. We obtain

$$S = -\frac{1}{2}\int dx^4\sqrt{-g}[\varphi\Box\varphi + \nabla_a\varphi\nabla^a\varphi + 2W_C(\varphi)], \tag{116}$$

where

$$W_C(\varphi) = V - \varphi V_{,\varphi}. \tag{117}$$

The above is, in principle, a noncanonical action. However, it can be converted into a canonical one by integrating by parts the higher-order term. Indeed

$$\varphi\Box\varphi = \nabla_a(\varphi\nabla^a\varphi) - \nabla_a\varphi\nabla^a\varphi. \tag{118}$$

Thus the action on shell can be written as

$$S = -\frac{1}{2}\int dx^4\sqrt{-g}[2\nabla_a\varphi\nabla^a\varphi + W_C(\varphi)]. \tag{119}$$

In this way, we can employ the usual procedure to explore this case. We have

$$E(\lambda) = \left(2\frac{I_1}{\lambda} + \frac{I_2}{\lambda^3}\right), \tag{120}$$

with

$$I_1 = \int e^{\Phi}\left[\varphi_{,q}^2 + \sum_{i=2}^{3}\varphi_{,w_i}^2\right]d\Omega,$$
$$I_2 = 2\int e^{\Phi}W_C(\varphi)d\Omega, \tag{121}$$

which leads to

$$\left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0 \to 2I_1 + 3I_2 = 0,$$
$$\left.\frac{\partial^2 E(\lambda)}{\partial\lambda^2}\right|_{\lambda=1} = -4I_1 < 0 \tag{122}$$

and implies instability. All in all, therefore only very specific combination of coupling and potential can lead to stable scalar field configurations.

It is widely believed today that the most general model with a single additional scalar degree of freedom (d.o.f.) and second-order field equations is given by the so-called Horndeski theory [30]. This theory has been proved to be equivalent to the curved spacetime generalization of a scalar field theory with Galilean shift symmetry in flat spacetime [31,32]. We will now apply the equilibrium and stability conditions we have derived above to this class of theories.

It is useful for our purposes to write their action in the form

$$S = \sum_{i=1}^{4}\int d^4x\sqrt{-g}\mathcal{L}_i, \tag{123}$$

where

$$\mathcal{L}_1 = G_1, \tag{124}$$

$$\mathcal{L}_2 = -G_2\Box\varphi, \tag{125}$$

$$\mathcal{L}_3 = G_3R + G_{3,X}[(\Box\varphi)^2 - \varphi_{;mn}\varphi^{;mn}], \tag{126}$$

$$\mathcal{L}_4 = G_4\mathbb{G}_{mn}\varphi^{;mn} - \frac{G_{4,X}}{6}[(\Box\varphi)^3 \tag{127}$$

$$+2\varphi_{;mn}\varphi^{;m}{}_{;a}\varphi^{;an} - 3\varphi_{;mn}\varphi^{;mn}\Box\varphi]. \tag{128}$$

Here $\mathbb{G}_{mn}$ is the Einstein tensor, and G_i are functions of X and φ. Under Eq. (11), we have, together with Eq. (70),

$$\Box\varphi \Rightarrow \lambda^2\Box\varphi + \ldots,$$
$$\varphi_{;mn} \Rightarrow \varphi_{;mn} + \ldots,$$
$$\mathbb{G}_{mn}\varphi^{;mn} \Rightarrow \lambda^6\mathbb{G}_{mn}\varphi^{;mn} + \ldots, \tag{129}$$

where, as before, the dots represent additional terms which contain derivatives of λ and therefore are irrelevant in our case.

Before proceeding, we should point out that this theory has a noncanonical Lagrangian, and therefore the energy function is not the standard one. However, the Ostrogradski approach [33] can be used to show that in this case one can also define a function that has the characteristics of the energy of the system (i.e., it is conserved and generates a time evolution).

The transformed action, in the static case and assuming, for brevity, $\lambda = \text{const}$, can be written as

$$S(\lambda) = \int d^4x\frac{\sqrt{-g}}{\lambda^3}[\mathcal{L}_1 + \lambda^2\mathcal{L}_2 + \lambda^4\mathcal{L}_3 + \lambda^6\mathcal{L}_4], \tag{130}$$

where $\mathcal{L}_i = \mathcal{L}_i(\lambda)$.

The Pohozaev identity for $\lambda = 1$ reads

$$2X\mathcal{L}_{1,X} - 3\mathcal{L}_1 + 2X\mathcal{L}_{2,X} - \mathcal{L}_2$$
$$+ 2X\mathcal{L}_{3,X} + \mathcal{L}_3 + 2X\mathcal{L}_{4,X} + 3\mathcal{L}_4 = 0. \tag{131}$$

Equation (131) contains derivatives of the scalar field, and therefore, without further assumptions, it cannot be used to obtain general constraints of the functions G_i, as we did in the previous section.

When such assumptions are provided, one can find a number of different combinations of these functions which can lead to stability. The trivial ones are the ones corresponding to general relativity ($G_1 = -\frac{1}{2}X - V$, $G_2 = 0$, $G_{3,X} = 0$, and $G_4 = 0$) and scalar tensor gravity [$G_1 = -\frac{1}{2}C_1(\varphi)X - V$, $G_2 = 0$, $G_3 = C_3(\varphi)$, and $G_4 = 0$]. A more general analytical condition can be obtained, for example, assuming only $G_4 = 0$. In this case, stability is possible for

$$G_2 = C_{2,1}(\varphi)X^{1/2} + C_{2,2}(\varphi)X,$$
$$G_3 = C_{3,1}(\varphi)X^{3/2} + C_{3,2}(\varphi), \tag{132}$$

and G_1 within the functions

$$G_1^* = C_{1,1}(\varphi)X^{5/2} + C_{1,2}(\varphi)X^{3/2} + C_{1,3}(\varphi)X,$$
$$G_2^* = C_{1,5}X^{3/2} + C_{1,6}(\varphi)X + C_{1,4}(\varphi)\ln X,$$
$$+ C_{1,7}(\varphi). \tag{133}$$

Note that this is only a particular solution for the G_i. Obtaining a general solution would require the resolution of a nonlinear third-order differential equation for G_3, which cannot be achieved in general.

As we said, there are a number of other combinations of forms of the function G_i which can lead to stability. The fact that such a big number of different conditions is possible is to be ascribed to the high level of generality of this class of theories. Members of the Horndeski group of theories can have a wildly different physical behavior and, this is reflected also in the stability of the localized solution of their scalar d.o.f.

IX. CONCLUSION

In this paper, we have presented a complete proof of Derrick's theorem in curved spacetime. The proof follows the same strategy of the original paper by Derrick, but it has the advantage of being completely covariant and not requiring any assumption on the potential for the scalar field or on the underlying geometry (other than the absence of vorticity). This is made possible by combining the $1+1+2$ covariant formalism and a technique first developed by Darboux which allows one to write down Lagrangians for dissipative systems.

For scalar fields in a fixed background, i.e., nonbackreacting, we have been able to prove that no stable localized solution of the Klein-Gordon equation is possible. In addition, we found that the coupling of the scalar field with other types of matter can change only the conditions necessary to achieve equilibrium.

The results we have obtained can be understood by recognizing that a scalar field can be represented macroscopically as an effective fluid with negative pressure (tension). Such fluids will tend naturally to collapse in flat spacetime. In the nonflat case, if the energy of the scalar field is not enough to appreciably influence the curvature of spacetime (i.e., without backreaction), a localized solution will again be unstable. When couplings are considered, the interaction influences the tension of the scalar field, changing its magnitude.

In the case of real scalar field stars, as the metric of these compact objects is determined by its matter distribution, backreaction cannot be neglected. Our generalization of Derrick's theorem, therefore, implies that, in curved spacetimes, stable relativistic stars made of real scalar fields cannot exist, even when considering couplings. Indeed the introduction of gravity "mitigates" the instability in the sense that the maximum of the energy of the scalar field solution is turned into an inflection point.

In terms of the fluid interpretation, this is also clear: if the scalar field is able to influence the spacetime in which it is embedded, its tension will induce a repulsion, much in the same way as dark energy. As a result, the configuration is less unstable and, in principle, with enough tension stability could be possible. Our result suggests, however, that no sufficient level of tension can be achieved to support a localized solution.

It should be remarked, however, that full stability is not a necessary condition for physical validity. In order to be able to consider scalar field stars as possible astrophysical objects, one can require the weaker conditions of "long-term" stability (e.g., longer/comparable to the age of the Universe). It is not possible with solely the tools of Derrick's proof to evaluate this aspect of the stability of scalar field stars. Its study will be the topic of future works.

Unlike full stability, the equilibrium condition still constitutes a strong physical constraint. Its application in the context of a noncanonical scalar field and some classes of modified gravity reveals that the extended Pohozaev identity can be used to select potentials and classes of theories which present an equilibrium. This is particularly relevant in the case of modified gravity, as for these theories all compact stars are also real scalar field stars in which the scalar field coincides with the gravitational scalar d.o.f. We found some very stringent criteria for scalar tensor gravity and Horndeski-type theories. Also, in this case, coupling with other fields as well as standard matter might be able to modify these constraints.

We conclude by pointing out that our analysis was performed in a purely classical setting: we have neglected completely the quantum nature of the scalar field and the corresponding modification to its action. These corrections, which are necessary to build more realistic models of scalar field stars, could lead to modifications of the equilibrium conditions we have derived, and even to stability. Future studies will be dedicated to the analysis of these cases.

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APPENDIX: CONNECTING $\sqrt{-g}$ AND e^{Φ}

Let us start with the covariant divergence of the vector u_a. It is well known that

$$\nabla_a u^a = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}u^a). \tag{A1}$$

On the other hand, by definition, $\nabla_a u^a = \Theta$, i.e., the left-hand side is the expansion. In this way, one can write

$$u^a\partial_a \ln|g| = 2(\Theta - \partial_a u^a). \tag{A2}$$

The same procedure can be applied to the quantity $\nabla_a e^a$ to obtain

$$e^a \partial_a \ln|g| = 2(\mathcal{A} + \phi - \partial_a e^a), \tag{A3}$$

Instead, for $N_{ca}\nabla_b N^{ab}$, since

$$\nabla_b N^{ab} = \frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}N^{ab}) + \Gamma^a_{bc}N^{bc}, \tag{A4}$$

we have

$$N_c{}^a \partial_a \ln|g| = 2(\mathcal{A}_c + a_c) + N_c{}^b \partial_b \ln N, \tag{A5}$$

where we call N the determinant of the nonzero minor of N_{ab}. In this way, we have

$$\partial_a \ln|g| = 2\{(\partial_c u^c - \Theta)u_a + (\mathcal{A} + \phi - \partial_c e^c)e_a + (\mathcal{A}_c + a_c)N^c_a\} + N_a{}^b \partial_b \ln N. \tag{A6}$$

Let us now define

$$V_a = -\Theta u_a + (\mathcal{A} + \phi)e_a + (\mathcal{A}_c + a_c)N^c_a,$$
$$W_a = -(\partial_b u^b)u_a + (\partial_b e^b)e_a + N_a{}^b \partial_b \ln\sqrt{N} \tag{A7}$$

so that

$$\partial_a \ln|g| = 2(V_a - W_a). \tag{A8}$$

The partial differential equation (A6) is equivalent to the one we have encountered in the main text to determine Φ [see Eq. (42) for $\lambda = 1$].

In the case of a static and spherically symmetric spacetime and choosing e_a normalized to 1, we have $\Theta = 0$, $\mathcal{A}_c = 0$, $a_c = 0$, and $\partial_a u^a = 0$. Inserting this result into Eq. (A8) and integrating out the total divergences in W_a, we obtain

$$\sqrt{|g|} = g_0 \exp\left[\int(\mathcal{A} + \phi)dq\right], \tag{A9}$$

i.e., modulus an irrelevant constant, Eq. (40). This suggests that we can write, in a $(-,+,+,+)$ signature,

$$e^\Phi = \sqrt{-g}. \tag{A10}$$

In more general spacetimes, Eq. (A8) is a system of partial differential equations which one must solve in order to find the expression of the volume form. As in the case of Eq. (52), we can use the method of characteristics to obtain some solutions, but we need an accurate description of the boundary conditions to determine a solution. However, as discussed for Φ, the exact form of the metric tensor is irrelevant to our discussion, the only important thing is the transformation of $\sqrt{-g}$.

Let us then look at the transformation of g. It is easy to see that, under a conformal transformation $g_{ab}(\lambda) = \lambda g_{ab}$, one has

$$V_a(\lambda) = V_a,$$
$$W_a(\lambda) = W_a + \partial_a \ln\lambda, \tag{A11}$$

and this implies

$$\Phi(\lambda) = \Phi - 4\ln\lambda, \tag{A12}$$

$$g(\lambda) = \frac{g}{\lambda^4}, \tag{A13}$$

which is consistent with the known conformal transformation of a tensor density.

Under Eq. (11), one has, again,

$$V_a(\lambda) = V_a,$$
$$W_a(\lambda) = W_a + \partial_a \ln\lambda. \tag{A14}$$

However, the above results imply

$$\Phi(\lambda) = \Phi - 3\ln\lambda,$$
$$g(\lambda) = \frac{g}{\lambda^3}, \tag{A15}$$

i.e., the same transformation for Φ obtained in Eq. (42).

[1] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, J. Math. Phys. (N.Y.) **5**, 1252 (1964).

[2] U. Enz, Discrete mass, elementary length, and a topological invariant as a consequence of a relativistic invariant variational principle, Phys. Rev. **131**, 1392 (1963).

[3] J. A. Wheeler, Geons, Phys. Rev. **97**, 511 (1955).

[4] D. R. Brill and J. B. Hartle, Method of the self-consistent field in general relativity and its application to the gravitational geon, Phys. Rev. **135**, B271 (1964); P. R. Anderson and D. R. Brill, Gravitational geons revisited, Phys. Rev. D **56**, 4824 (1997).

[5] N. S. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004).

[6] S. Coleman, Q-balls, Nucl. Phys. **B262**, 263 (1985).
[7] S. I. Pohozaev, On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Sov. Math. Dokl. **6**, 1408 (1965).
[8] H. Berestycki and P. L. Lions, Nonlinear scalar field equations, I—Existence of a ground state, Arch. Ration. Mech. Anal. **82**, 313 (1983).
[9] T. N. Palmer, Derrick's theorem in curved space, J. Phys. A **12**, L17 (1979).
[10] P. M. Radmore and G. Stephenson, Nonlinear wave equations in a curved background space, J. Phys. A **11**, L149 (1978).
[11] M. Heusler and N. Straumann, Scaling arguments for the existence of static, spherically symmetric solutions of self-gravitating systems, Classical Quantum Gravity **9**, 2177 (1992).
[12] C. A. R. Herdeiro and E. Radu, Asymptotically flat black holes with scalar hair: A review, Int. J. Mod. Phys. D **24**, 1542014 (2015).
[13] S. Hod, No-go theorem for static boson stars, Phys. Lett. B **778**, 239 (2018).
[14] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual, and J. A. Font, Spontaneous Scalarization of Charged Black Holes, Phys. Rev. Lett. **121**, 101102 (2018).
[15] G. Alestas and L. Perivolaropoulos, Evading Derrick's theorem in curved space: Static metastable spherical domain wall, Phys. Rev. D **99**, 064026 (2019).
[16] C. A. Clarkson and R. K. Barrett, Covariant perturbations of Schwarzschild black holes, Classical Quantum Gravity **20**, 3855 (2003).
[17] G. Betschart and C. A. Clarkson, Scalar and electromagnetic perturbations on LRS class II space-times, Classical Quantum Gravity **21**, 5587 (2004).
[18] C. Clarkson, Covariant approach for perturbations of rotationally symmetric spacetimes, Phys. Rev. D **76**, 104034 (2007).
[19] W. Starlet, The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics, J. Phys. A **15**, 1503 (1982).
[20] G. Darboux, *Leçons sur la Théorie Générale des Surfaces*, Vol. 3 (Gauthier-Villars, Paris, 1894).
[21] M. C. Baldiotti, R. Fresneda, and D. M. Gitman, Quantization of the damped harmonic oscillator revisited, Phys. Lett. A **375**, 1630 (2011).
[22] H. von Helmholtz, On the physical meaning of the principle of least action, *J. Reine Angew. Math.* 100, 137 (1887); *Selected Writings of Hermann von Helmholtz*, edited by R. Kahl (Wesleyan University Press, Middletown, CT, 1971).
[23] H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics: Pearson New International Edition* (Pearson Higher Education, London, 2014).
[24] S. Carloni and P. K. S. Dunsby, The $1+1+2$ formalism for scalar-tensor gravity, Gen. Relativ. Gravit. **48**, 136 (2016).
[25] M. Cantor, A necessary and sufficient condition for York data to specify an asymptotically flat spacetime, J. Math. Phys. (N.Y.) **20**, 1741 (1979); J. Katz, D. Lynden-Bell, and J. Bicak, Gravitational energy in stationary spacetimes, Classical Quantum Gravity **23**, 7111 (2006); J. Katz, J. Bicak, and D. Lynden-Bell, Relativistic conservation laws and integral constraints for large cosmological perturbations, Phys. Rev. D **55**, 5957 (1997).
[26] P. Freud, On the expression of the total energy-momentum of a retrial system in general relativity, Ann. Math. J. **40**, 417 (1939).
[27] R. Caldwell, A phantom menace, Phys. Lett. B **545**, 23 (2002); R. R. Caldwell and M. Kamionkowski, and N. N. Weinberg, Phantom Energy and Cosmic Doomsday, Phys. Rev. Lett. **91**, 071301 (2003).
[28] Y.-F. Cai, E. N. Saridakis, M. R. Setare, and J.-Q. Xia, Quintom cosmology: Theoretical implications and observations, Phys. Rep. **493**, 1 (2010).
[29] T. Chiba, T. Okabe, and M. Yamaguchi, Kinetically driven quintessence, Phys. Rev. D **62**, 023511 (2000).
[30] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. **10**, 363 (1974).
[31] C. Deffayet, S. Deser, and G. Esposito-Farese, Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors, Phys. Rev. D **80**, 064015 (2009).
[32] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Generalized G-inflation: Inflation with the most general second-order field equations, Prog. Theor. Phys. **126**, 511 (2011).
[33] M. Ostrogradsky, An essay on differential equations related to the isoperimetric problem, Mem. Acad. St. Petersbourg **6**, 385 (1850).