Galileon-like vector fields

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(Received 25 January 2019; published 16 July 2019)

We construct simple Lagrangians of vector fields which involve second derivatives but nevertheless lead to second-order field equations. These vector fields are, therefore, analogs of generalized Galileons. Our construction is given first in Minkowski space and then generalized to include dynamical gravity. We show that the speed of gravitational waves about homogeneous and isotropic backgrounds is equal to the speed of light. We present examples of backgrounds that are stable and ghost free despite the absence of gauge invariance. Some of these backgrounds violate the null energy condition.

DOI: 10.1103/PhysRevD.100.025006

I. INTRODUCTION

Scalar theories with Lagrangians involving second derivatives, which nevertheless lead to second-order field equations, attract considerable interest. These are theories of generalized Galileons [1] of which the versions with dynamical gravity are Horndeski theories [2,3]. From the cosmological viewpoint, these theories are particularly interesting because they are capable of violating the null energy condition (NEC) in a healthy way (for a review, see Ref. [4]). It is natural to try to generalize these theories to fields other than scalar. If one insists on gauge invariance, then no generalization is possible in four dimensions[1] [6], while in higher dimensions, one arrives at a theory of p-form Galileons [7,8]. Giving up gauge invariance is dangerous but may not be fatal. Indeed, there are vector theories (with Lagrangians involving first derivatives only) which are not gauge invariant but, nevertheless, stable. One class of such theories is the generalized Proca theories, or vector Galileons [9,10]. Theories of another class [11] are stable in nontrivial backgrounds. An interesting property of the latter is that they also may violate the NEC in a healthy way [12].

In this paper, we also consider vector field and give up gauge invariance. Our purpose is to construct the simplest vector-field Lagrangians involving second derivatives and yet giving rise to second-order field equations. We do this first in Minkowski space and find that there are at least three fairly large classes of theories that have the desired property. We then switch on the dynamical gravity and observe that all field equations remain second order for theories belonging to two of these classes. A possible area of application of the classes of theories we consider is the early Universe and/or dark energy. Concerning the dark energy application, a prerequisite is that the gravitational waves travel at the speed of light, cf. Ref. [13]; we discuss this point in our paper. A particularly interesting possibility is NEC violation by homogeneous and isotopic background. In this paper, we take the first step in this direction. Namely, we consider one class of theories and give an example of vector background in Minkowski space that violates the NEC. Then, we derive the conditions for stability (absence of ghosts and gradient instabilities) about this background in Minkowski space and find the range of parameters where the NEC-violating background is stable. Thus, theories we consider may be viewed as vector analogs of the generalized Galileons. In a straightforward application, we describe a model of early genesis stage based on our NEC-violating vectorlike Galileons. It is worth noting that our class of models is substantially different from vector Galileons (generalized Proca fields) of Refs. [9,10]. The defining feature of the latter is the existence of 3 degrees of freedom (d.o.f.) in an arbitrary background. We do not impose this requirement, so our vector field genuinely has four propagating d.o.f. As a result, these d.o.f. are not pathological in nontrivial backgrounds only, while there exist ghosts in trivial background $A_\mu = 0$. It remains to be understood whether or not the absence of the stable Lorentz-invariant vacuum is a drawback of our class of models.

This paper is organized as follows. In Sec. II, we construct non–gauge invariant second-derivative Lagrangians with second-order field equations for vector fields in

[1]This not the case in curved space-time [5].

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Minkowski space. In Sec. III, we turn on dynamical gravity and show that all equations of motion remain second order in theories belonging to two of the classes found in Sec. II. In Sec. IV, we prove that Lagrangians, belonging to the two classes of Sec. III, do not modify the propagation speed of gravitational waves; this speed is equal to the speed of light. In Sec. V, we give an example of nontrivial homogeneous vector field background in Minkowski space that violates the NEC and derive the conditions for stability and for the absence of superluminal propagation of perturbations. Then, we determine the range of parameters, in which the background is stable and violates the NEC in Minkowski space. Finally, we describe a model of early genesis stage based on our NEC-violating vectorlike Galileons. We conclude in Sec. VI.

II. SECOND-DERIVATIVE LAGRANGIANS IN MINKOWSKI SPACE

Let us construct a nongauge-invariant simplest theory for vector field in Minkowski space which has the Lagrangian satisfying the following requirements:

(1) The Lagrangian $\mathcal{L}$ has second derivatives, along with first derivatives and the field itself.
(2) Field equations obtained from this Lagrangian have derivatives of at most second order.
(3) The Lagrangian cannot be reduced by integration by parts to the Lagrangian involving first derivatives only.
(4) The Lagrangian is linear in the second derivatives:

$$\mathcal{L} = S^{\mu\nu\rho}(A_\lambda; A_{\tau;\xi})A_{\rho;\mu\nu} + L(A_\tau, A_{\lambda;\xi}) \tag{2.1}$$

It is convenient to think of $S^{\mu\nu\rho}$ as a sum

$$S^{\mu\nu\rho} = \frac{1}{2}(K^{\mu\nu\rho} + K^{\nu\mu\rho}), \tag{2.2}$$

where $K^{\mu\nu\rho}$ does not have to be symmetric in μ, ν.
(5) The function $K^{\mu\nu\rho}$ in (2.2) is a monomial in variables A_μ, $A_{\nu;\tau}$ which does not involve the totally antisymmetric tenzor:

$$K^{\mu_\alpha\mu_\beta\mu_\gamma} = \text{const}\cdot\eta^{\mu_{\sigma(1)}\mu_{\sigma(2)}}\ldots\eta^{\mu_{\sigma(n+2m+2)}\mu_{\sigma(n+2m+3)}} \times A_{\mu_1}\ldots A_{\mu_n}A_{\mu_{n+1};\mu_{n+2}}\ldots A_{\mu_{n+2m-1};\mu_{n+2m}}, \tag{2.3}$$

where n is odd, σ denotes a permutation of $(n+2m+3)$ indices, and μ_α, μ_β, μ_γ are non-convoluted indeces, $(\mu_\alpha,\mu_\beta,\mu_\gamma)=(\mu_{\sigma^{-1}(n+2m+1)}, \mu_{\sigma^{-1}(n+2m+2)},\mu_{\sigma^{-1}(n+2m+3)})$.

The Euler-Lagrange equations for a theory with this Lagrangian have the following form:

$$\frac{\partial\mathcal{L}}{\partial A_\rho} - \partial_\mu\frac{\partial\mathcal{L}}{\partial A_{\rho;\mu}} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial A_{\rho;\mu\nu}} = 0, \tag{2.4}$$

where $A_{\rho;\mu} = \partial_\mu A_\rho$, $A_{\rho;\mu\nu} = \partial_\mu\partial_\nu A_\rho$. The third order terms in Eq. (2.4) for the Lagrangian (2.1) read:

$$\left(\frac{\partial S^{\mu\nu\rho}}{\partial A_{\tau;\lambda}} - \frac{\partial S^{\mu\nu\tau}}{\partial A_{\rho;\lambda}}\right)A_{\tau;\lambda\mu\nu}.$$

Thus, to have second-order field equations, we require that

$$\frac{\partial S^{\mu\nu\rho}}{\partial A_{\tau;\lambda}} - \frac{\partial S^{\mu\nu\tau}}{\partial A_{\rho;\lambda}} = 0. \tag{2.5}$$

In accordance with (2.3), the indices μ, ν, ρ in the function $K^{\mu\nu\rho}$ come from the metrics or vector field or derivative of vector field. The last index in $K^{\mu\nu\rho}$ plays a different role in Eq. (2.5) than the other indices, and so it is convenient to classify functions $K^{\mu\nu\rho}$ according to the "origin" of the index ρ. In this way we arrive at four possibilities (other options give the same $S^{\mu\nu\rho}$ in (2.2):

I. $K^{\mu\nu\rho} = L^{\mu\nu}_{\varkappa}(A_\sigma, A_{\tau;\lambda})A^{\varkappa;\rho}$
II. $K^{\mu\nu\rho} = f^\mu(A_\sigma, A_{\tau;\lambda})\eta^{\nu\rho}$
III. $K^{\mu\nu\rho} = B^{\mu\nu}(A_\sigma, A_{\tau;\lambda})A^\rho$
IV. $K^{\mu\nu\rho} = \tilde{L}^{\mu\nu}{}_{\varkappa}(A_\sigma, A_{\tau;\lambda})A^{\rho;\varkappa}$,

where functions $L^{\mu\nu}{}_\alpha$, $\tilde{L}^{\mu\nu}{}_\varkappa$, $B^{\mu\nu}$ and f^μ are again monomials in two variables A_σ, $A_{\tau;\lambda}$ that do not involve totally antisymmetric tensor. Furthermore, it is convenient to classify the functions $\tilde{L}^{\mu\nu}{}_\varkappa$ according to the "origin" of index $\varkappa$:

IV.1. $\tilde{L}^{\mu\nu}{}_\varkappa = T^\mu(A_\sigma, A_{\tau;\lambda})\delta^\nu_\varkappa$; $K^{\mu\nu\rho} = T^\mu(A_\sigma, A_{\tau;\lambda})A^{\rho;\nu}$
IV.2. $\tilde{L}^{\mu\nu}{}_\varkappa = X^{\mu\nu}{}_\alpha(A_\sigma, A_{\tau;\lambda})A^\alpha{}_\varkappa$; $K^{\mu\nu\rho} = X^{\mu\nu}{}_\alpha(A_\sigma, A_{\tau;\lambda})\times A^{\alpha;}{}_\varkappa A^{\rho;\varkappa}$
IV.3. $\tilde{L}^{\mu\nu}{}_\varkappa = Z^{\mu\nu}{}_\alpha(A_\sigma, A_{\tau;\lambda})A_{\varkappa;}{}^\alpha$; $K^{\mu\nu\rho} = Z^{\mu\nu}{}_\alpha(A_\sigma, A_{\tau;\lambda})\times A_{\varkappa;}{}^\alpha A^{\rho;\varkappa}$
IV.4. $\tilde{L}^{\mu\nu}{}_\varkappa = V^{\mu\nu}(A_\sigma, A_{\tau;\lambda})A_\varkappa$; $K^{\mu\nu\rho} = V^{\mu\nu}(A_\sigma, A_{\tau;\lambda})A_\varkappa A^{\rho;\varkappa}$.

Making use of this classification, we analyse Eq. (2.5) in Appendix A. We find that there are three independent Lagrangians which satisfy above requirements 1–5, namely

$$\mathcal{L}_1 = (F)^{l_1}(D)^{n_1}(B)^{k_1}\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu}, \tag{2.6}$$

$$\mathcal{L}_2 = (F)^{l_2}(D)^{n_2}(B)^{k_2}A_\sigma A_\tau A^{\sigma;\mu}A^{\tau;\nu}A^\rho A_{\rho;\mu\nu}, \tag{2.7}$$

$$\mathcal{L}_3 = (F)^{l_3}(C)^{n_3}\eta^{\mu\nu}A^{\rho;\sigma}A_\sigma A_{\rho;\mu\nu}, \tag{2.8}$$

where k_i, l_i, n_i are nonnegative integers, and

$$F = A_\mu A^\mu, \tag{2.9}$$

$$D = A^\nu A^\lambda A_{\nu;\lambda}, \tag{2.10}$$

$$B = A_\mu A^\nu A^{\mu;\lambda}A_{\nu;\lambda}, \tag{2.11}$$

$$C = A^{\mu;\tau}A_\tau A^\rho A_{\mu;\rho}. \tag{2.12}$$

The Lagrangians (2.6) and (2.7) have the structure corresponding to the case III above for function $K^{\mu\nu\rho}$, while the Lagrangian (2.8) corresponds to IV.4.

The Lagrangians (2.6)–(2.8) contain second derivatives, provided that

$$k_1 \neq 0 \quad \text{and/or} \quad n_1 > 1, \tag{2.13}$$

$$k_2 \neq 0 \quad \text{and/or} \quad n_2 \neq 0, \tag{2.14}$$

$$n_3 \neq 0, \tag{2.15}$$

respectively. Lagrangians (2.6) and (2.7) are independent when

$$n_1 > 1. \tag{2.16}$$

Straightforward generalizations of (2.6)–(2.8) are

$$\mathcal{L}_1 = f^{(1)}(B, D, F)\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu},$$
$$f_B^{(1)} \neq 0 \quad \text{and/or} \quad f_{DD}^{(1)} \neq 0, \tag{2.17}$$

$$\mathcal{L}_2 = f^{(2)}(B, D, F)A_\varkappa A_\tau A^{\varkappa;\mu}A^{\tau;\nu}A^\rho A_{\rho;\mu\nu},$$
$$f_B^{(2)} \neq 0 \quad \text{and/or} \quad f_D^{(2)} \neq 0, \tag{2.18}$$

$$\mathcal{L}_3 = f^{(3)}(C, F)\eta^{\mu\nu}A^{\rho;\lambda}A_\lambda A_{\rho;\mu\nu}, \qquad f_C^{(3)} \neq 0, \tag{2.19}$$

where $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$ are arbitary functions of their arguments, and $f_B = \frac{\partial f}{\partial B}$, $f_{DD} = \frac{\partial^2 f}{\partial D^2}$, etc.

It is worth pointing out that there may exist linear combinations of Lagrangians whose structure is different from (2.17)–(2.19), but which nevertheless lead to second order field equations due to cancellations between different terms. One of the examples is

$$\mathcal{L} = \left(\frac{1}{2}A^\rho A^{\mu;\nu}A_\nu A_{\mu;\lambda}A^\lambda + A^{\rho;\tau}A_\tau A_{\mu;\nu}A^\mu A^\nu\right)\Box A_\rho.$$

We do not consider this fairly contrived possibility in this paper.

Coming back to the Lagrangians (2.17)–(2.19), we point out that since they lead to the second order field equations, the number of propagation degrees of freedom is generally speaking, equal to four. We will see this explicitly in Sec. V B.

III. TURNING ON GRAVITY

In the previous section, we constructed three non–gauge invariant vector-field Lagrangians involving second derivatives and yet giving rise to second-order and/or lower field equations in Minkowski space, Eqs. (2.17)–(2.19). Our purpose here is to figure out which of these Lagrangians lead to the second-order or lower equations of motion and energy-momentum tensor.

Let us consider Lagrangian (2.17). One assumes minimal coupling to gravity; then, $-\sqrt{-g}T^{\rho\sigma}\delta g_{\rho\sigma}$ for this theory reads

$$\begin{aligned} &-\sqrt{-g}T^{\rho\sigma}\delta g_{\rho\sigma} = 2\delta(\sqrt{-g}\mathcal{L}_{(1)}) \\ &\quad = \delta(\sqrt{-g}f^{(1)}(B, D, F)\Box F) + \cdots \\ &\quad \Rightarrow \sqrt{-g}f_B^{(1)}g^{\mu\nu}((\partial_\mu\partial_\nu F)\delta B + (\partial_\mu\partial_\nu B)\delta F) \\ &\qquad + \sqrt{-g}f_D^{(1)}g^{\mu\nu}((\partial_\mu\partial_\nu F)\delta D + (\partial_\mu\partial_\nu D)\delta F) + \cdots, \end{aligned} \tag{3.1}$$

where omitted terms do not contain third derivatives and the arrow denotes integration by parts and $\delta B = \frac{\delta B}{\delta g_{\rho\sigma}}\delta g_{\rho\sigma}$, etc. It is convenient to represent Eq. (3.1) in the following form,

$$-\sqrt{-g}T^{\rho\sigma}\delta g_{\rho\sigma} \Rightarrow I_1 + I_2 + \cdots,$$

where

$$I_1 = \sqrt{-g}f_B^{(1)}g^{\mu\nu}((\partial_\mu\partial_\nu F)\delta B + (\partial_\mu\partial_\nu B)\delta F),$$
$$I_2 = \sqrt{-g}f_D^{(1)}g^{\mu\nu}((\partial_\mu\partial_\nu F)\delta D + (\partial_\mu\partial_\nu D)\delta F).$$

We see that $T^{\mu\nu}$ does not contain third-order derivatives of the vector field and/or metric. Indeed, using the fact that $B = \frac{1}{4}F_{;\mu}F^{;\mu}$, we obtain that I_1 is second order or lower,

$$\begin{aligned} I_1 \Rightarrow \frac{\sqrt{-g}}{2}f_B^{(1)}(&-(\partial^\tau F)(\partial_\tau\partial_\mu\partial_\nu F)\delta F \\ &+ (\partial^\tau F)(\partial_\mu\partial_\nu\partial_\tau F)\delta F) + \cdots = 0 + \cdots \end{aligned}$$

I_2 does not contain third-order derivatives, either:

$$\begin{aligned} I_2 &= f_D^{(1)}\sqrt{-g}A^\lambda A_\varkappa A^\nu A^\rho A^\mu g^{\sigma\alpha}((\partial_\sigma\partial_\alpha g_{\mu\nu})\delta\Gamma^\varkappa_{\rho\lambda} + (\partial_\sigma\partial_\alpha\Gamma^\varkappa_{\rho\lambda})\delta g_{\mu\nu}) \\ &\quad - f_D^{(1)}\sqrt{-g}A^\lambda A^\rho g^{\sigma\alpha}(2A^\mu(\partial_\sigma\partial_\alpha A_\mu)A_\varkappa\delta\Gamma^\varkappa_{\rho\lambda} + (\partial_\sigma\partial_\alpha\partial_\lambda A_\rho)A^\mu A^\nu\delta g_{\mu\nu}) + \cdots \\ &\Rightarrow \frac{1}{2}f_D^{(1)}\sqrt{-g}A^\lambda A^\varkappa A^\nu A^\rho A^\mu g^{\sigma\alpha}((\partial_\sigma\partial_\alpha\partial_\varkappa g_{\mu\nu})\delta g_{\rho\lambda} - (\partial_\sigma\partial_\alpha\partial\rho g_{\mu\nu})\delta g_{\lambda\varkappa}) \\ &\quad - f_D^{(1)}\sqrt{-g}A^\lambda A^\rho A^\mu A^\nu g^{\sigma\alpha}((\partial_\sigma\partial_\alpha\partial_\lambda A_\mu)\delta g_{\rho\nu} - (\partial_\sigma\partial_\alpha\partial_\lambda A_\rho)\delta g_{\mu\nu}) + \cdots \\ &= 0 + \cdots \end{aligned}$$

Now, $\frac{\delta(\sqrt{-g}\mathcal{L}_{(1)})}{\sqrt{-g}\delta A_\sigma}\delta A_\sigma$ also does not contain third-order derivatives. Indeed,

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_1) &= \frac{1}{2}\sqrt{-g}\left[f_B\left(\frac{\delta(F_{;\tau}F^{;\tau})}{4}\Box F + \frac{\Box(F_{;\tau}F^{;\tau})}{4}\delta F\right)\right)\\ &\quad + f_D((\Box F)\delta D + (\Box D)\delta F)\Bigg] + \cdots\\ &\Rightarrow \frac{1}{2}\sqrt{-g}[0.5f_B((F_{;\tau} - F_{;\tau})\Box F^{;\tau})\delta F\\ &\quad + f_D(A^\lambda A_\rho A^\mu A_\nu \partial_\tau\partial_\mu\partial^\tau g^{\rho\nu} - A^\lambda A^\rho A_\mu A_\nu \partial_\tau\partial_\rho\partial^\tau g^{\mu\nu})\delta A_\lambda]\\ &\quad + \cdots = 0 + \cdots.\end{aligned}$$

Thus, the equation of motion has derivatives of second order and/or lower. Summarizing, we see that Lagrangian (2.17) leads to the second-order and/or lower field equation and energy-momentum tensor.

Let us consider Lagrangian (2.18). Using the fact that $B = \frac{1}{4}F_{;\mu}F^{;\mu}$ and $D = \frac{1}{2}F^{;\mu}A_\mu$, we find that

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_2) &= \frac{\sqrt{-g}}{4}\delta(f^{(2)}(B,D,F)F^{;\nu}B_{;\nu}) + \cdots \Rightarrow -\frac{\sqrt{-g}}{4}\delta(f^{(2)}(\Box F)B)\\ &\quad - \frac{\sqrt{-g}}{4}\delta(f^{(2)}_{;\nu}F^{;\nu}B) + \cdots = -\frac{\sqrt{-g}}{4}\delta(f^{(2)}_{;\nu}F^{;\nu}B) + \cdots \Rightarrow -\frac{\sqrt{-g}f^{(2)}_B}{4}(-(BF^{;\nu})_{;\nu}\delta B\\ &\quad - (\Box B)B\delta F + B_{;\nu}F^{;\nu}\delta B) - \frac{\sqrt{-g}f^{(2)}_D}{4}(-(F^{;\nu}B)_{;\nu}\delta D - B(\Box D)\delta F + D_{;\nu}F^{;\nu}\delta B) + \cdots\\ &\Rightarrow \frac{\sqrt{-g}f^{(2)}_B B}{8}(-F^{;\tau}\Box(F_{;\tau}) + (\Box F)_{;\tau}F^{;\tau})\delta F - \frac{\sqrt{-g}f^{(2)}_D A^\lambda F^{;\nu}}{16}(F^{;\tau}F_{;\tau\nu\lambda} - F^{;\tau}F_{;\lambda\nu\tau})\delta F = 0 + \cdots,\end{aligned}$$

where omitted terms do not contain third derivatives and $F_{;\nu} = \nabla_\nu F$, $F_{;\nu\mu} = \nabla_\mu\nabla_\nu F$, etc., $\delta B = \frac{\delta B}{\delta g_{\rho\sigma}}\delta g_{\rho\sigma} + \frac{\delta B}{\delta A_\mu}\delta A_\mu$, etc. Thus, all field equations have derivatives of second order and/or lower.

The minimal extension of Lagrangian (2.19) leads to the third-order field equations. We were unable to find additional terms involving the Riemann tensor that would give rise to the cancellation of the third derivatives in the field equations. Thus, we conjecture that Lagrangian (2.19) cannot be generalized to the theory with dynamical gravity in such a way that the equations of motion remain second order. We do not consider Lagrangian (2.19) in what follows.

To summarize, in the case when we switch on the dynamical gravity, all field equations remain second order for two Lagrangians (2.17) and (2.18).

IV. SPEED OF GRAVITATIONAL WAVES

Our purpose here is to figure out whether the second-derivative terms in Galileon-like Lagrangians (2.17) and (2.18) can modify the propagation speed of tensors modes. Even though we consider minimal coupling to gravity in the sense that we do not have any direct couplings between the vector field and curvature tensors, the second derivatives acting on the vector field do induce explicit couplings between the vector field and the metric and its derivatives, which could result in a modified propagation speed of gravitational waves. Let us consider the following background and gauge fixing for metric perturbation,

$$ds^2 = dt^2 - a^2(t)(\delta_{ij} + h^{TT}_{ij})dx^i dx^j, \tag{4.1}$$

with $\partial_i h^{TT}_{ij} = 0$, $h^{TT}_{ii} = 0$, and for vector field

$$A_\mu = (A(t), 0, 0, 0) + \delta A_\mu. \tag{4.2}$$

The term with second derivatives in Lagrangian (2.17) reads

$$\frac{1}{2}f^{(1)}(B,D,F)\Box F, \tag{4.3}$$

while for Lagrangian (2.18), we have

$$\frac{1}{16}f^{(2)}(B,D,F)F^{,\lambda}(F_{,\rho}F^{,\rho})_{,\lambda}. \tag{4.4}$$

The terms (4.3) and (4.4) may give rise to the combination like $\mathcal{L} \supset \mathcal{F}(t)\dot{h}^{TT}_{ij}\partial_i\delta A_j$. However, these terms in the action vanish upon integration by parts. In other words, helicity-2 field h_{ij}^{TT} does not mix with the field δA_μ, which has helicity 1 or 0.

Now, let us see that the terms (4.3) and (4.4) do not lead to the term $(\dot{h}_{ij}^{TT})^2$ or $(\partial_k h_{ij}^{TT})^2$. To this end, we set $\delta A_\mu = 0$.

Functions $f^{(1)}(B,D,F)$ and $f^{(2)}(B,D,F)$ do not contain h_{ij}^{TT}. This follows from the facts that $B = \frac{1}{4}F_{,\mu}F^{,\mu}$, $D = \frac{1}{2}F^{,\nu}A_\nu$, and $F = A^\nu A_\nu = \dot{A}^2$. We also notice that $(F_{,\rho}F^{,\rho})_{,\lambda}$ does not contain h_{ij}^{TT} either. Therefore, the term (4.4) does not contain h_{ij}^{TT} at all.

Turning to the term (4.3), we recall that $\Box F$ can be written in the following form:

$$\Box F = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu F).$$

Thus, $\Box F$, and hence the whole term (4.3), is at most linear in $\partial_\mu h_{ij}^{TT}$. This completes the argument.

Thus, terms (4.3) and (4.4) do not modify the propagation speed of gravitational waves in homogeneous and isotropic background.

V. STABLE NEC-VIOLATING SOLUTION IN MINKOWSKI SPACE

A. Solution

Our purpose here is to figure out if there are Lagrangians in the set (2.17) and (2.18), which lead to stable NEC-violation solutions. In this section, we give such an example in the Minkowski background. Let us consider the Lagrangian (2.17) with additional first-order terms

$$\mathcal{L}_1 = qD^2A^\rho\Box A_\rho + kB^2 + lC^2 + vF^6, \tag{5.1}$$

where q, k, l, and v are free parameters and B, C, D, and F are given by Eqs. (2.9)–(2.12), respectively. The specific choice of the Lagrangian functions here is such that all terms have the same transformation property under rescaling $x^\mu \Rightarrow \lambda x^\mu$, $A_\mu \Rightarrow \lambda^{-1}A_\mu$, namely, $\mathcal{L}_1 \Rightarrow \lambda^{-12}\mathcal{L}_1$. Then, there exists a nontrivial homogeneous solution of the field equation

$$A_\mu^{bg} = (\beta t^{-1}, 0, 0, 0), \qquad t > 0. \tag{5.2}$$

For this solution, the field equation gives

$$\beta = \left(\frac{3k + 3l - 5q}{v}\right)^{1/4}. \tag{5.3}$$

This solution exists when

$$\begin{aligned} &3k + 3l - 5q > 0 \quad \text{and} \quad v > 0 \quad \text{or} \\ &3k + 3l - 5q < 0 \quad \text{and} \quad v < 0. \end{aligned} \tag{5.4}$$

We will need the expression for the energy-momentum tensor for this solution:

$$T_{\mu\nu}|_{g_{\rho\sigma}=\eta_{\rho\sigma};A_\tau=A_\tau^{bg}} = \frac{2\delta(\sqrt{-g}\mathcal{L})}{\sqrt{-g}\delta g^{\mu\nu}}\bigg|_{g_{\rho\sigma}=\eta_{\rho\sigma};A_\tau=A_\tau^{bg}}.$$

To this end, we again consider minimal coupling to the metric, i.e., set $\Box A_\rho = \nabla^\mu\nabla_\mu A_\rho$ and $D = A_{\mu;\nu}A_\tau A_\lambda g^{\mu\tau}g^{\nu\lambda}$, etc., in curved space-time. The Lagrangian (5.1) can be written in the following form,

$$\mathcal{L}_1 = \frac{1}{2}f^{(1)}(D)\Box F - f^{(1)}(D)A_{\tau;\sigma}A^{\tau;\sigma} + L(B,C,D,F),$$

where

$$\begin{aligned} f^{(1)}(D) &= qD^2, \\ L(B,C,D,F) &= kB^2 + lC^2 + vF^6. \end{aligned}$$

Using the fact that $\partial_0 T^{0\rho}|_{A_\mu=A_\mu^{bg}} = 0$, we find that

$$\begin{aligned} &T_{00} = 0, \\ &T_{ij} = p\delta_{ij}, \\ &p = \left(-\frac{1}{2}\partial_\tau f\partial^\tau F + L - fA_{\tau;\sigma}A^{\tau;\sigma}\right)\bigg|_{g_{\mu\nu}=\eta_{\mu\nu};A_\mu=A_\mu^{bg}}, \end{aligned} \tag{5.5}$$

where i, $j = 1, 2, 3$. This gives

$$p = \beta^8 t^{-12}(v\beta^4 + k + l - 9q) = \beta^8 t^{-12}(4(k+l) - 14q). \tag{5.6}$$

Thus, the background (5.2) violates the NEC, provided that

$$l + k < \frac{7q}{2}. \tag{5.7}$$

This is possible in both cases listed in (5.4).

B. Stability conditions in Minkowski space

Let us consider the stability of the solution (5.2). Having in mind Refs. [14,15], we also require subluminality of the perturbations about it. To this end, we study the somewhat more general Lagrangian

$$\mathcal{L}_1 = f^{(1)}(B,D,F)A^\rho\Box A_\rho + L(B,D,F,C), \tag{5.8}$$

where $L(B,D,F)$ and $f^{(1)}(B,D,F,C)$ are arbitrary functions of their arguments. We consider homogeneous background $A_\mu^{bg} = (A_0^{bg}(t), 0, 0, 0)$ and expand Lagrangian (5.8) up to the second order. In the expansion, we are only interested in coefficients of $(\delta A^{0,i}\delta A_0^{,i})$, $(\delta A^{0,0}\delta A_{0,0})$ and $(\delta A^{i,0}\delta A^i_{,0})$, $(\delta A^{i,j}\delta A^{i,j})$ because here we consider the high momentum regime, meaning that the variation of δA_μ in space and time occurs at scales much shorter than the timescale characteristic of the background $A_\mu^{bg}(t)$; the terms $\delta A_{0,i}\delta A_{i,0}$ are not present. We find

$$\delta\mathcal{L}_1 = \mathcal{L}_1(A_\mu^{bg} + \delta A_\mu) - \mathcal{L}_1(A_\mu^{bg}) = K_{01}(\delta A^{0,i}\delta A^{,i}_{0}) + K_{00}(\delta A^{0,0}\delta A_{0,0}) + K_{10}(\delta A^{i,0}\delta A^{i}_{,0}) + K_{11}(\delta A^{i,j}\delta A^{i,j}) + (\ldots)(\delta A^0\delta A_0) + (\ldots)(\delta A^i\delta A^i) + \cdots, \tag{5.9}$$

where dots denote terms with fewer than two derivatives and $A \equiv A_0^{bg}$. Here,

$$K_{00} = 2\dot{A}^2A(L_{BB}+L_{CC}) + 4L_{BC}A^4\dot{A} + \frac{1}{2}A^4L_{DD} + 2A^4\dot{A}(L_{BD}+L_{CD}) + 2A^5\dot{A}^2\ddot{A}(f^{(1)}_{BB}) + \frac{1}{2}A^5\ddot{A}f^{(1)}_{DD} + 2A^5\dot{A}\ddot{A}(f^{(1)}_{BD}) + A^3\ddot{A}(f^{(1)}_{B}) - \frac{1}{2}\frac{d}{dt}(2A^3\dot{A}(f^{(1)}_{B}) + A^3f^{(1)}_{D}) - f^{(1)} - 2A^2\dot{A}^2(f^{(1)}_{B}) - 2A^2\dot{A}f^{(1)}_{D} - 2A^2f^{(1)}_{F} + A^2(L_C+L_B),$$
$$K_{01} = -L_BA^2 - A^3\ddot{A}f^{(1)}_{B} + f^{(1)} - \frac{1}{2}\frac{d}{dt}(2A^3\dot{A}(f^{(1)}_{B}) + A^3f^{(1)}_{D}) + 2A^2\dot{A}^2(f^{(1)}_{B}) + 2A^2\dot{A}f^{(1)}_{D} + 2A^2f^{(1)}_{F},$$
$$K_{10} = f^{(1)} - L_CA^2,$$
$$K_{11} = -f^{(1)}.$$

So, the conditions of stability are

$$K_{00} > 0, \quad K_{01} < 0, \quad K_{10} > 0, \quad K_{11} < 0, \tag{5.10}$$

and the condition of the absence of superluminal perturbations is

$$|K_{00}| > |K_{01}|, \qquad |K_{10}| > |K_{11}|. \tag{5.11}$$

The conditions (5.10) and (5.11) for Lagrangian (5.2) read

$$l > 3q - k, \quad l < 0, \quad l > \frac{36q - 12k}{13}, \quad v > 0. \tag{5.12}$$

We see that Lagrangian (5.2) gives rise to the stable homogeneous NEC-violating solution (5.3) when the parameters satisfy the relations (5.12), (5.7), and (5.4). In fact, all these conditions are satisfied, provided that

$$v > 0, \qquad q > 0, \qquad l < 0, \qquad 3q < k < \frac{19q}{2}, \qquad \frac{36q}{13} - \frac{12k}{13} < l < \frac{7q}{2} - k.$$

Thus, our example shows that there are stable homogeneous solutions in our vector theories that violate the NEC.

It is worth noting that Lagrangian (5.9) describes, generally speaking, four propagating d.o.f. None of them is pathological, however, in the background (5.2), whereas trivial background $A_\mu = 0$ may well be unstable because of ghost perturbations.

C. Early genesis stage

For a straightforward application of the NEC-violating solution (5.2), we construct an initial stage of the cosmological genesis scenario, similar to Ref. [16]. In the asymptotic past, space-time is assumed to be Minkowskian, and in accordance with (5.5) and (5.6), energy-momentum tensor vanishes as $t \to -\infty$. At large but finite $|t|$, gravitational effects on the vector-field evolution are negligible, so to the leading order in $M_{\rm Pl}^{-1}$, the energy density and pressure are given by (5.5) and (5.6). Then, the Hubble parameter is obtained from

$$\dot{H} = -4\pi G(\rho + p).$$

We find

$$H = \frac{4\pi G\beta^8|(4(k+l) - 14q)|}{11|t|^{11}}.$$

Thus, the Universe undergoes an accelerated expansion characteristic of the early genesis epoch. At this stage, perturbations about the background are stable and subluminal. It remains to be seen whether this genesis scenario with vector field can be made complete. In particular, it would be interesting to see whether or not the model suffers from instabilities at later times, analogous to those in Horndeski theories [17,18]. We leave this analysis for future work.

VI. CONCLUSION

In this paper, we have constructed simplest vector-field Lagrangians involving second derivatives and leading to second-order field equations. We found that there exist three large classes of such theories in Minkowski space. However, we observed that in the case when the dynamical gravity is switched on only theories belonging to two of these classes have all field equations remaining second order. These two Lagrangians are

$$\mathcal{L}_1 = f^{(1)}(B,D,F)\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu},$$
$$\mathcal{L}_2 = f^{(2)}(B,D,F)A_\varkappa A_\tau A^{\varkappa;\mu}A^{\tau;\nu}A^\rho A_{\rho;\mu\nu}, \tag{6.1}$$

where B, D, and F are defined in (2.9)–(2.11). Furthermore, we have seen that the gravitational waves travel with the speed of light in homogeneous and isotropic backgrounds. This opens up the possibility of using our vector field as a model of dark energy.

We have shown that the theory (6.1) can violate the NEC in a healthy way in Minkowski space. Then, for a straightforward application, we described a model of early genesis stage based on our NEC-violating vectorlike Galileons. Other applications may have to do with dark energy or a bouncing Universe.

As we already pointed out, our vector field generically has four propagating d.o.f. It would be interesting to find a subclass of our models (if any) in which there are only 3 d.o.f., like in the vector Galileons of Refs. [9,10]. We leave this issue for the future.

APPENDIX: SIMPLEST LAGRANGIANS

As we discussed in Sec. II, we have seven possibilities for the structure of the function $K^{\mu\nu\rho}$:

(I) $K^{\mu\nu\rho} = L^{\mu\nu}{}_{\varkappa}(A_\sigma, A_{\tau;\lambda})A^{\varkappa;\rho}$
(II) $K^{\mu\nu\rho} = f^{\mu}(A_\sigma, A_{\tau;\lambda})\eta^{\nu\rho}$
(III) $K^{\mu\nu\rho} = B^{\mu\nu}(A_\sigma, A_{\tau;\lambda})A^{\rho}$
(IV.1) $K^{\mu\nu\rho} = T^{\mu}(A_\sigma, A_{\tau;\lambda})A^{\rho;\nu}$
(IV.2) $K^{\mu\nu\rho} = X^{\mu\nu}{}_{\alpha}(A_\sigma, A_{\tau;\lambda})A^{\alpha;}{}_{\varkappa}A^{\rho;\varkappa}$
(IV.3) $K^{\mu\nu\rho} = Z^{\mu\nu}{}_{\alpha}(A_\sigma, A_{\tau;\lambda})A_{\varkappa;}{}^{\alpha}A^{\rho;\varkappa}$
(IV.4) $K^{\mu\nu\rho} = V^{\mu\nu}(A_\sigma, A_{\tau;\lambda})A_{\varkappa}A^{\rho;\varkappa}$.

1. Case I

Considering option I, we find that the requirement (2.5) is equivalent to

$$A^{\varkappa;\rho}\frac{\partial L^{(\mu\nu)}{}_{\varkappa}}{\partial A_{\sigma;\lambda}} - A^{\varkappa;\sigma}\frac{\partial L^{(\mu\nu)}{}_{\varkappa}}{\partial A_{\rho;\lambda}} + L^{(\mu\nu)\sigma}\eta^{\rho\lambda} - L^{(\mu\nu)\rho}\eta^{\sigma\lambda} = 0, \tag{A1}$$

where parentheses denote symmetrization. $L^{\mu\nu\tau}$ is a monomial, so $L^{(\mu\nu)\tau}$ can be represented in the following form,

$$L^{(\mu\nu)\rho} = (A^{\tau;}{}_{\tau})^{n}\tilde{L}^{(\mu\nu)\rho},$$

where n is non-negative integer and $\tilde{L}$ does not contain $A^{\tau;}{}_{\tau}$. So, Eq. (A1) reads

$$\eta^{\rho\lambda}(-A^{\varkappa;\sigma}n(A^{\tau;}{}_{\tau})^{n-1}\tilde{L}^{(\mu\nu)}{}_{\varkappa} + (A^{\tau;}{}_{\tau})^{n}\tilde{L}^{(\mu\nu)\sigma}) + \cdots = 0, \tag{A2}$$

where omitted terms do not contain the structures proportional to $\eta^{\rho\lambda}$. We see that (A2) cannot be satisfied because the two terms in parentheses have different powers of $A^{\tau;}{}_{\tau}$. Thus, option I does not work.

2. Case II

Considering option II, we find that the requirement (2.5) is equivalent to

$$\frac{1}{2}\left(\eta^{\rho\nu}\frac{\partial f^{\mu}}{\partial A_{\tau;\lambda}} + \eta^{\rho\mu}\frac{\partial f^{\nu}}{\partial A_{\tau;\lambda}} - \eta^{\tau\nu}\frac{\partial f^{\mu}}{\partial A_{\rho;\lambda}} - \eta^{\tau\mu}\frac{\partial f^{\nu}}{\partial A_{\rho;\lambda}}\right) = 0. \tag{A3}$$

We have three possibilities for function f^{μ}:

(IIa) $f^{\mu} = A^{\mu}h(A_\sigma, A_{\nu;\lambda})$
(IIb) $f^{\mu} = A^{\mu;\varkappa}v_{\varkappa}(A_\sigma, A_{\nu;\lambda})$
(IIc) $f^{\mu} = A^{\varkappa;\mu}v_{\varkappa}(A_\sigma, A_{\nu;\lambda})$

In case IIa, we obtain that (A3) is equivalent to

$$\eta^{\nu\rho}\frac{\partial h}{\partial A_{\tau;\lambda}} = \eta^{\nu\tau}\frac{\partial h}{\partial A_{\rho;\lambda}},$$

which is satisfied only in the case $h = h(A_\sigma)$, so that

$$f^{\mu} = A^{\mu}h(A_\sigma).$$

However, the corresponding Lagrangian $\mathcal{L} = h(A_\sigma) \times A^{(\mu}\eta^{\nu)\rho}A_{\rho;\mu\nu}$ does not contain second-order derivatives after integration by parts.

In case IIb, we find that (A3) is equivalent to

$$A^{\mu;\varkappa}\eta^{\nu\rho}\frac{\partial v_{\varkappa}}{\partial A_{\tau;\lambda}} = A^{\mu;\varkappa}\eta^{\nu\tau}\frac{\partial v_{\varkappa}}{\partial A_{\rho;\lambda}}.$$

This is possible only if $v_{\varkappa} = v_{\varkappa}(A_\sigma)$. This leads to the following Lagrangian:

$$\mathcal{L} = A_{\varkappa}(A^{\tau}A_{\tau})^{n}\eta^{\rho(\nu}A^{\mu);\varkappa}A_{\rho;\mu\nu}.$$

It can be reduced by integration by parts to a Lagrangian involving first derivatives only,

$$\begin{aligned} &A_{\varkappa}(A^{\tau}A_{\tau})^{n}\eta^{\rho(\nu}A^{\mu);\varkappa}A_{\rho;\mu\nu} \\ &\quad = A_{\varkappa}(A^{\tau}A_{\tau})^{n}\frac{1}{2}(\eta^{\rho\nu}A^{\mu;\varkappa} + \eta^{\rho\mu}A^{\nu;\varkappa})A_{\rho;\mu\nu} \\ &\quad \Rightarrow -(A^{\tau}A_{\tau})^{n}A^{\varkappa}A^{\mu;}{}_{\mu\varkappa}A^{\rho;}{}_{\rho} + \cdots \\ &\quad \Rightarrow \frac{1}{2}((A^{\tau}A_{\tau})^{n}A_{\varkappa})_{;\varkappa}(A^{\rho;}{}_{\rho})^{2} + \cdots = 0 + \cdots, \end{aligned}$$

where omitted terms do not contain second derivatives and arrows denote integration by parts.

Finally, in case IIc, Eq. (A3) is equivalent to

$$v^{\rho}\eta^{\mu\tau}\eta^{\nu\lambda} = -\eta^{\mu\tau}A^{\varkappa;\nu}\frac{\partial v_{\varkappa}}{\partial A_{\rho;\lambda}}.$$

This equation cannot be satisfied.

Summarizing, we see that option II does not lead to the desired Lagrangians.

3. Option III

Let us consider option III. It is convenient to classify the functions $B^{\mu\nu}$ according to the origin of the indices μ and ν. In this way, we arrive at nine possibilities [other options give the same $S^{\mu\nu\rho}$ in (2.2)]:

(IIIa) $B^{\mu\nu} = h(A_\theta, A_{\tau;\lambda})A^\mu A^\nu$
(IIIb) $B^{\mu\nu} = h(A_\theta, A_{\tau;\lambda})\eta^{\mu\nu}$
(IIIc) $B^{\mu\nu} = h(A_\theta, A_{\tau;\lambda})A^{\mu;\nu}$
(IIId) $B^{\mu\nu} = v_\xi(A_\theta, A_{\tau;\lambda})A^{\mu;\xi}A^\nu$
(IIIe) $B^{\mu\nu} = v_\xi(A_\theta, A_{\tau;\lambda})A^{\xi;\mu}A^\nu$
(IIIf) $B^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\mu;\xi}A^{\nu;\phi}$
(IIIg) $B^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\xi;\mu}A^{\phi;\nu}$
(IIIh) $B^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\xi;\mu}A^{\nu;\phi}$
(IIIi) $B^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\mu;\xi}A^{\phi;\nu}$

a. Cases IIIa and IIIb

In cases IIIa and IIIb, we obtain that the requirement (2.5) is equivalent to

$$A^\rho \frac{\partial h}{\partial A_{\tau;\lambda}} - A^\tau \frac{\partial h}{\partial A_{\rho;\lambda}} = 0. \tag{A4}$$

This is possible only if

$$h = (F)^l (D)^n (B)^k,$$

where n, l, and k are non-negative integers and

$$F = A^\rho A_\rho,$$
$$D = A^\nu A^\lambda A_{\nu;\lambda},$$
$$B = A^\nu A_\mu A^{\mu;\lambda} A_{\nu;\lambda}.$$

Thus, this option leads to the following Lagrangians,

$$\mathcal{L}_1 = (F)^{l_1}(D)^{n_1}(B)^{k_1} A^\mu A^\nu A^\rho A_{\rho;\mu\nu}, \tag{A5}$$

$$\mathcal{L}_2 = (F)^{l_2}(D)^{n_2}(B)^{k_2} \eta^{\mu\nu} A^\rho A_{\rho;\mu\nu}, \tag{A6}$$

where $l_{1,2}$, $k_{1,2}$, and $n_{1,2}$ are non-negative integers. We consider these Lagrangians, along with other cases, in the end of this Appendix to figure out which of them are independent.

b. Case IIIc

In case IIIc we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{h}{2}(A^{\mu;\nu} + A^{\nu;\mu})A^\rho, \tag{A7}$$

Using (A7), we find that the requirement (2.5) is equivalent to

$$\eta^{\tau(\mu}\eta^{\nu)\lambda} h A^\rho - \eta^{\rho(\mu}\eta^{\mu)\lambda} h A^\tau + A^{(\mu;\nu)}\left(\frac{\partial(hA^\rho)}{\partial A_{\tau;\lambda}} - \frac{\partial(hA^\tau)}{\partial A_{\rho;\lambda}}\right) = 0. \tag{A8}$$

We see that (A8) cannot be satisfied because the first term in (A8) cannot be canceled out by other terms. Thus, option IIIc does not work.

c. Case IIId

In case IIId, we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{v_\varkappa}{2}(A^{\mu;\varkappa}A^\nu + A^{\nu;\varkappa}A^\mu)A^\rho. \tag{A9}$$

Using (A9), we observe that the requirement (2.5) is equivalent to

$$\left(\frac{\partial(A^\rho v_\varkappa)}{\partial A_{\tau;\lambda}} - \frac{\partial(A^\tau v_\varkappa)}{\partial A_{\rho;\lambda}}\right) A^{(\mu;\nu)} + A^\rho v^\lambda A^{(\nu}\eta^{\mu)\tau}$$
$$- A^\tau v^\lambda A^{(\nu}\eta^{\mu)\rho} = 0.$$

We see that (A9) cannot be satisfied because the third term in (A9) cannot be canceled out by other terms. Thus, case IIId does not lead to the desired Lagrangians.

d. Case IIIe

In case IIIe, we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{v_\varkappa}{2}(A^{\varkappa;\mu}A^\nu + A^{\varkappa;\nu}A^\mu)A^\rho.$$

Then, the requirement (2.5) is equivalent to

$$\frac{\partial(A^\rho v_\varkappa)}{\partial A_{\tau;\lambda}} - \frac{\partial(A^\tau v_\varkappa)}{\partial A_{\rho;\lambda}} = 0,$$
$$A^\rho v_\varkappa = A_\varkappa v^\rho. \tag{A10}$$

Using the second equation in (A10), we find that $v^\varkappa$ must have the following form:

$$v_\varkappa = A_\varkappa h(A_\sigma, A_{\mu;\nu}).$$

From this, we obtain that h must obey Eq. (A4), so we get the Lagrangian

$$\mathcal{L}_3 = (F)^{l_3}(D)^{n_3}(B)^{k_3} A_\varkappa A^{\varkappa;(\nu}A^{\mu)} A^\rho A_{\rho;\mu\nu}, \tag{A11}$$

where l_3, k_3, and n_3 are non-negative integers. We consider this Lagrangian in the end of the Appendix.

e. Case IIIf

In case IIIf, we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = L_{(\varkappa\tau)} A^{\mu;\varkappa} A^{\nu;\sigma} A^{\rho}. \tag{A12}$$

Using (A12), we find that the requirement (2.5) is equivalent to

$$\begin{aligned}&\left(\frac{(\partial A^{\rho} L_{(\varkappa\tau)})}{\partial A_{\sigma;\lambda}} - \frac{\partial (A^{\sigma} L_{(\varkappa\tau)})}{\partial A_{\rho;\lambda}}\right) A^{\mu;\varkappa} A^{\nu;\tau} \\ &\quad + A^{\rho}(L^{(\lambda\varkappa)} \eta^{\sigma\nu} A^{\mu;}_{\xi} + L^{(\lambda\varkappa)} \eta^{\sigma\mu} A^{\nu;}{}_{\varkappa}) \\ &\quad - A^{\sigma}(L^{(\lambda\varkappa)} \eta^{\rho\nu} A^{\mu;}{}_{\varkappa} + L^{(\lambda\varkappa)} \eta^{\rho\mu} A^{\nu;}{}_{\varkappa}) = 0. \end{aligned} \tag{A13}$$

We see that (A13) cannot be satisfied because the third term in (A13) cannot be canceled out by other terms. Thus, case IIIf does not lead to the desired Lagrangians.

f. Case IIIg

In case IIIg, we find the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = L_{(\varkappa\tau)} A^{\varkappa;\mu} A^{\tau;\nu} A^{\rho}. \tag{A14}$$

Using (A14), we obtain that the requirement (2.5) is equivalent to

$$\begin{aligned}&A^{\rho} L^{(\varkappa\tau)} = A^{\tau} L^{(\varkappa\rho)}, \\ &\frac{\partial (A^{\rho} L^{(\varkappa\tau)})}{A_{\sigma;\lambda}} - \frac{\partial (A^{\sigma} L^{(\varkappa\tau)})}{A_{\rho;\lambda}} = 0. \end{aligned} \tag{A15}$$

Using the first equation in (A15), we find that $L^{\mu\nu}$ must have the following form:

$$L^{\mu\nu} = A^{\mu} A^{\nu} h(A_{\theta}, A_{\xi;\tau}).$$

From this, we obtain that h must satisfy Eq. (A4), and the Lagrangian is

$$\mathcal{L}_4 = (F)^{l_4} (D)^{n_4} (B)^{k_4} A_{\varkappa} A_{\lambda} A^{\varkappa;\mu} A^{\lambda;\nu} A^{\rho} A_{\rho;\mu\nu}, \tag{A16}$$

where l_4, k_4, and n_4 are numbers.

g. Case IIIh

In case IIIh we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{1}{2} L_{\varkappa\tau} (A^{\varkappa;\mu} A^{\nu;\tau} + A^{\varkappa;\nu} A^{\mu;\tau}) A^{\rho}. \tag{A17}$$

Using the (A17), we find that the requirement (2.5) is equivalent to

$$\begin{aligned}&\frac{1}{2}(A^{\varkappa;\mu} A^{\nu;\tau} + A^{\varkappa;\nu} A^{\mu;\tau}) \left(A^{\rho} \frac{\partial (L_{\varkappa\tau})}{\partial A_{\sigma;\lambda}} - A^{\theta} \frac{\partial (L_{\varkappa\tau})}{\partial A_{\rho;\lambda}} \right) \\ &\quad + A^{\rho} L_{\varkappa\tau} (\eta^{\lambda(\mu} A^{\nu);\tau} \eta^{\varkappa\sigma} + A^{\varkappa;(\mu} \eta^{\nu)\sigma} \eta^{\tau\lambda}) - A^{\sigma} L_{\varkappa\tau} (\eta^{\lambda(\mu} A^{\nu);\tau} \eta^{\varkappa\rho} \\ &\qquad + A^{\varkappa;(\mu} \eta^{\nu)\rho} \eta^{\tau\lambda}) = 0. \end{aligned} \tag{A18}$$

This equation cannot be satisfied because the term $A^{\rho} L_{\varkappa\tau} A^{\varkappa;(\mu} \eta^{\nu)\sigma} \eta^{\tau\lambda}$ in (A18) cannot be canceled out by other terms. Thus, option IIIh does not work.

h. Case IIIi

This case is similar to the previous one IIIh, and so it does not lead to the desired Lagrangians.

Summarizing, we see that option III leads to the four Lagrangians (A5), (A6), (A11), and (A16).

4. Case IV.1

Considering option IV.1, we find that the requirement (2.5) is equivalent to

$$A^{\rho;\mu} \frac{\partial T^{\nu}}{\partial A_{\tau;\lambda}} = A^{\tau;\nu} \frac{\partial T^{\mu}}{\partial A_{\rho;\lambda}}. \tag{A19}$$

We have three possibilities for function T^{μ}:
(IV.1a) $T^{\mu} = A^{\mu} h(A_{\sigma}, A_{\nu;\lambda})$
(IV.1b) $T^{\mu} = A^{\mu;\varkappa} v_{\varkappa}(A_{\sigma}, A_{\nu;\lambda})$
(IV.1c) $T^{\mu} = A^{\varkappa;\mu} v_{\varkappa}(A_{\sigma}, A_{\nu;\lambda})$
In case IV.1a, we obtain that (A19) is equivalent to

$$A^{\rho;\mu} A^{\nu} \frac{\partial h}{A_{\tau;\lambda}} = A^{\tau;\nu} A^{\mu} \frac{\partial h}{A_{\rho;\lambda}},$$

which is satisfied only in the case $h = h(A_{\sigma})$, so that

$$T^{\mu} = A^{\mu} h(A_{\sigma}).$$

However, the corresponding Lagrangian $\mathcal{L} = h(A_{\sigma}) \times A^{\rho;(\nu} A^{\mu)} A_{\rho;\mu\nu}$ does not contain second-order derivatives after integration by parts.

In case IV.1b, we find that (A19) is equivalent to

$$A^{\rho;\mu} \left(\eta^{\nu\tau} v^{\lambda} + A^{\nu;\varkappa} \frac{\partial v_{\varkappa}}{\partial A_{\tau;\lambda}} \right) = A^{\tau;\nu} \left(\eta^{\mu\rho} v^{\lambda} + A^{\mu;\varkappa} \frac{\partial v_{\varkappa}}{\partial A_{\rho;\lambda}} \right). \tag{A20}$$

We see that (A20) cannot be satisfied because the first term $A^{\rho;\mu} \eta^{\nu\tau} v^{\lambda}$ in (A20) cannot be canceled out by other terms.

Finally, in case IV.1c, we obtain that (A19) is equivalent to

$$A^{\rho;\mu} \left(\eta^{\nu\lambda} v^{\tau} + A^{\varkappa;\nu} \frac{\partial v_{\varkappa}}{\partial A_{\tau;\lambda}} \right) = A^{\tau;\nu} \left(\eta^{\mu\lambda} v^{\rho} + A^{\varkappa;\mu} \frac{\partial v_{\varkappa}}{\partial A_{\rho;\lambda}} \right). \tag{A21}$$

We see that (A21) cannot be satisfied because the first term $A^{\rho;\mu}\eta^{\nu\lambda}v^{\tau}$ in (A21) cannot be canceled out by other terms.

Summarizing, we see that option IV.1 does not work.

5. Case IV.2

Considering option IV.2, we find that the requirement (2.5) is equivalent to

$$L^{(\mu\nu)\sigma}A^{\rho;\lambda} + A^{\tau;}{}_{\varkappa}A^{\rho;\varkappa}\frac{\partial L^{(\mu\nu)}{}_{\tau}}{\partial A_{\sigma;\lambda}} - L^{(\mu\nu)\rho}A^{\sigma;\lambda} - A^{\tau;}{}_{\varkappa}A^{\sigma;\varkappa}\frac{\partial L^{(\mu\nu)}{}_{\tau}}{\partial A_{\rho;\lambda}} = 0. \quad \text{(A22)}$$

$L^{\mu\nu\sigma}$ is a monomial, so $L^{(\mu\nu)\sigma}$ can be represented in the following form,

$$L^{(\mu\nu)\sigma} = (A^{\varkappa;\tau}A_{\varkappa;\tau})^n\tilde{L}^{(\mu\nu)\sigma},$$

where n is a natural number and $\tilde{L}^{(\mu\nu)\sigma}$ does not contain $(A^{\rho;\tau}A_{\rho;\tau})$. So, Eq. (A22) reads

$$((A^{\varkappa;\tau}A_{\varkappa;\tau})^n\tilde{L}^{(\mu\nu)\sigma} - 2nA^{\tau;}{}_{\varkappa}A^{\sigma;\varkappa}\tilde{L}^{(\mu\nu)}{}_{\tau}(A^{\varkappa;\tau}A_{\varkappa;\tau})^{n-1})A^{\rho;\lambda} + \cdots = 0, \quad \text{(A23)}$$

where omitted terms do not contain the structures proportional to $A^{\rho;\lambda}$. We see that (A23) cannot be satisfied because the two terms in parentheses have different powers of $(A^{\varkappa;\tau}A_{\varkappa;\tau})$. Thus, option IV.2 does not work.

6. Case IV.3

Considering option IV.3, we find that the requirement (2.5) is equivalent to

$$A^{\rho;\tau}L^{(\mu\nu)\lambda} + A^{\rho;\varkappa}A_{\varkappa;\sigma}\frac{\partial L^{(\mu\nu)\sigma}}{\partial A_{\tau;\lambda}} - A^{\tau;\rho}L^{(\mu\nu)\lambda} - A^{\tau;\varkappa}A_{\varkappa;\sigma}\frac{\partial L^{(\mu\nu)\sigma}}{\partial A_{\rho;\lambda}} = 0. \quad \text{(A24)}$$

We see that (A24) cannot be satisfied because the first term in (A24) cannot be canceled out by others terms in (A24). Thus, option IV.3 does not lead to the desired Lagrangians.

7. Option IV.4

We now consider option IV.4. It is convenient to classify the functions $Z^{\mu\nu}$ according to the origin of the indices μ, ν. In this way, we arrive at nine possibilities [other options give the same $S^{\mu\nu\rho}$ in (2.2)]:

(IV.4a) $Z^{\mu\nu} = h(A_\theta, A_{\tau;\lambda})A^\mu A^\nu$
(IV.4b) $Z^{\mu\nu} = h(A_\theta, A_{\tau;\lambda})\eta^{\mu\nu}$
(IV.4c) $Z^{\mu\nu} = h(A_\theta, A_{\tau;\lambda})A^{\mu;\nu}$
(IV.4d) $Z^{\mu\nu} = v_\xi(A_\theta, A_{\tau;\lambda})A^{\mu;\xi}A^\nu$
(IV.4e) $Z^{\mu\nu} = v_\xi(A_\theta, A_{\tau;\lambda})A^{\xi;\mu}A^\nu$
(IV.4f) $Z^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\mu;\xi}A^{\nu;\phi}$
(IV.4g) $Z^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\xi;\mu}A^{\phi;\nu}$
(IV.4h) $Z^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\xi;\mu}A^{\nu;\phi}$
(IV.4i) $Z^{\mu\nu} = L_{\xi\phi}(A_\theta, A_{\tau;\lambda})A^{\mu;\xi}A^{\phi;\nu}$.

a. Cases IV.4a and IV.4b

In cases IV.4a and IV.4b, we find that the requirement (2.5) is equivalent to

$$A^{\rho;\varkappa}A_\varkappa\frac{\partial h}{\partial A_{\tau;\lambda}} - A^{\tau;\varkappa}A_\varkappa\frac{\partial h}{\partial A_{\rho;\lambda}} = 0,$$

which can be satisfied only in the case $h = (F)^l \times (A^{\mu;\tau}A_\tau A^\rho A_{\mu;\rho})^n$, so that we have the following Lagrangians,

$$\mathcal{L}_5 = (F)^{l_5}(C)^{n_5}A^\mu A^\nu A^{\rho;\varkappa}A_\varkappa A_{\rho;\mu\nu}, \quad \text{(A25)}$$

$$\mathcal{L}_6 = (F)^{l_6}(C)^{n_6}\eta^{\mu\nu}A^{\rho;\varkappa}A_\varkappa A_{\rho;\mu\nu}, \quad \text{(A26)}$$

where $l_{5,6}$ and $n_{5,6}$ are non-negative integers and

$$C = A^{\mu;\tau}A_\tau A^\rho A_{\mu;\rho}.$$

We discuss Lagrangians (A25) and (A26) in the end of this Appendix.

b. Case IV.4c

In case IV.4c, we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{h}{2}(A^{\mu;\nu} + A^{\nu;\mu})A^\rho. \quad \text{(A27)}$$

Using (A27), we find that the requirement (2.5) is equivalent to

$$\eta^{\tau(\mu}\eta^{\nu)\lambda}hA^{\rho;\varkappa}A_\varkappa - \eta^{\rho(\mu}\eta^{\mu)\lambda}hA^{\tau;\varkappa}A_\varkappa + A^{(\mu;\nu)}\left(\frac{\partial(hA^{\rho;\varkappa}A_\varkappa)}{\partial A_{\tau;\lambda}} - \frac{\partial(hA^{\tau;\varkappa}A_\varkappa)}{\partial A_{\rho;\lambda}}\right) = 0. \quad \text{(A28)}$$

We see that (A28) cannot be satisfied because the first term in (A28) cannot be canceled out by other terms. Thus, option IV.4c does not work.

c. Case IV.4d

In case IV.4d, we obtain the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{v_\varkappa}{2}(A^{\mu;\varkappa}A^\nu + A^{\nu;\varkappa}A^\mu)A^\rho. \quad \text{(A29)}$$

Using (A29), we find that the requirement (2.5) is equivalent to

$$\left(\frac{\partial(A^{\rho;\sigma}A_\sigma v_\varkappa)}{\partial A_{\tau;\lambda}} - \frac{\partial(A^{\tau;\sigma}A_\sigma v_\varkappa)}{\partial A_{\rho;\lambda}}\right)A^{(\mu;\nu)} + A^{\rho;\sigma}A_\sigma v^\lambda A^{(\nu}\eta^{\mu)\tau} - A^{\tau;\sigma}A_\sigma v^\lambda A^{(\nu}\eta^{\mu)\rho} = 0. \tag{A30}$$

We see that (A30) cannot be satisfied because the third term in (A30) cannot be canceled out by other terms. Thus, option IV.4d does not lead to the desired Lagrangians.

d. Case IV.4e

In case IV.4e, we find the following function $S^{\mu\nu\rho}$:

$$S^{\mu\nu\rho} = \frac{v_\varkappa}{2}(A^{\varkappa;\mu}A^\nu + A^{\varkappa;\nu}A^\mu)A^{\rho;\sigma}A_\sigma.$$

Then, the requirement (2.5) is equivalent to

$$\frac{\partial(A^{\rho;\sigma}A_\sigma v_\varkappa)}{\partial A_{\tau;\lambda}} - \frac{\partial(A^{\tau;\sigma}A_\sigma v_\varkappa)}{\partial A_{\rho;\lambda}} = 0, \quad A^{\rho;\sigma}A_\sigma v_\varkappa = A_{\varkappa;\sigma}A_\sigma v^\rho. \tag{A31}$$

Using the second equation in (A31), we find that $v_\varkappa$ must have the following form:

$$v_\varkappa = A_{\varkappa;\sigma}A^\sigma h(A_\tau, A_{\mu;\nu}).$$

So, Eq. (A31) reads

$$A_\varkappa{}^{;(\nu}A^{\mu)}\left(A^{\rho;\tau}A_\tau A^{\varkappa;\tau}A_\tau\frac{\partial h}{\partial A_{\sigma;\lambda}} + A^{\rho;\tau}A^\lambda h A_\tau \eta^{\varkappa\sigma} - A^{\sigma;\tau}A_\tau A^{\varkappa;\tau}A_\tau\frac{\partial h}{\partial A_{\rho;\lambda}} - A^{\sigma;\tau}A^\lambda h A_\tau\eta^{\varkappa\rho}\right) = 0. \tag{A32}$$

This equation cannot be satisfied because the second term in (A32) cannot be canceled out by other terms. Thus, option IV.4e does not work.

e. Case IV.4f

In case IV.4f, we find that (2.5) is equivalent to

$$\left(\frac{\partial(A^{\rho;\sigma}A_\sigma L_{(\varkappa\tau)})}{\partial A_{\alpha;\lambda}} - \frac{\partial(A^{\alpha;\sigma}A_\sigma L_{(\varkappa\tau)})}{\partial A_{\rho;\lambda}}\right)A^{\mu;\varkappa}A^{\nu;\tau} + A^{\rho;\sigma}A_\sigma(L^{(\lambda\varkappa)}\eta^{\alpha\nu}A^{\mu;}{}_\varkappa + L^{(\lambda\varkappa)}\eta^{\alpha\mu}A^{\nu;}{}_\varkappa) - A^{\alpha;\sigma}A_\sigma(L^{(\lambda\varkappa)}\eta^{\rho\nu}A^{\mu;}{}_\varkappa + L^{(\lambda\varkappa)}\eta^{\rho\mu}A^{\nu;}{}_\varkappa) = 0. \tag{A33}$$

We see that (A33) cannot be satisfied because the third term in Eq. (A33) cannot be canceled out by other terms. Thus, this option does not lead to the desired Lagrangians.

f. Case IV.4g

In case IV.4g, we obtain that (2.5) is equivalent to

$$f^\rho L^{(\varkappa\tau)} = f^\tau L^{(\varkappa\rho)}, \quad \frac{\partial(f^\rho L^{(\varkappa\tau)})}{A_{\sigma;\lambda}} - \frac{\partial(f^\sigma L^{(\varkappa\tau)})}{A_{\rho;\lambda}} = 0, \tag{A34}$$

where $f^\rho = A^{\rho;\mu}A_\mu$. This is possible only if $L^{\mu\nu} = f^\mu f^\nu h(A_\sigma, A_{\varkappa;\tau})$. From this, we find that (A34) is equivalent to

$$A^{\rho;\tau}A_\tau A^{\varkappa;\sigma}A_\sigma A^{\alpha;\mu}A_\mu\left(\frac{\partial h}{A_{\nu;\lambda}}\right) - A^{\nu;\tau}A_\tau A^{\varkappa;\sigma}A_\sigma A^{\alpha;\mu}A_\mu\left(\frac{\partial h}{A_{\rho;\lambda}}\right) + h(A^{\alpha;\mu}A_\mu A^{\rho;\tau}A_\tau\eta^{\varkappa\nu}A^\lambda - A^{\alpha;\mu}A_\mu A^{\nu;\tau}A_\tau\eta^{\varkappa\rho}A^\lambda + A^{\varkappa;\mu}A_\mu A^{\rho;\tau}A_\tau\eta^{\alpha\nu}A^\lambda - A^{\varkappa;\mu}A_\mu A^{\nu;\tau}A_\tau\eta^{\alpha\rho}A^\lambda) = 0. \tag{A35}$$

We see that (A35) cannot be satisfied because the third term in Eq. (A35) cannot be canceled out by other terms. Thus, option IV.4g does not work.

g. Case IV.4h

In case IV.4h, we obtain that (2.5) is equivalent to

$$\frac{1}{2}(A^{\varkappa;\mu}A^{\nu;\tau} + A^{\varkappa;\nu}A^{\mu;\tau})\left(f^\rho\frac{\partial(L_{\varkappa\tau})}{\partial A_{\alpha;\lambda}} - f^\alpha\frac{\partial(L_{\varkappa\tau})}{\partial A_{\rho;\lambda}}\right) + f^\rho L_{\varkappa\tau}(\eta^{\lambda(\mu}A^{\nu);\tau}\eta^{\varkappa\alpha} + A^{\varkappa;(\mu}\eta^{\nu)\alpha}\eta^{\tau\lambda}) - f^\alpha L_{\varkappa\tau}(\eta^{\lambda(\mu}A^{\nu);\tau}\eta^{\varkappa\rho} + A^{\varkappa;(\mu}\eta^{\nu)\rho}\eta^{\tau\lambda}) = 0, \tag{A36}$$

where $f^\rho = A^{\rho;\sigma}A_\sigma$. We see that (A36) cannot be satisfied because the term $A^{\varkappa;(\mu}\eta^{\nu)\alpha}\eta^{\tau\lambda}$ in (A36) cannot be canceled out by other terms. Thus, this option does not lead to the desired Lagrangians.

h. Case IV.4i

This case is similar to the previous one, IV.4i, and so it does not lead to the desired Lagrangians.

8. Independent Lagrangians

To summarize, we have arrived at the six Lagrangians (A5), (A6), (A11), (A16), (A25), and (A26). We write them again for reference:

$$\mathcal{L}_1 = (F)^{l_1}(D)^{n_1}(B)^{k_1}A^\mu A^\nu A^\rho A_{\rho;\mu\nu}, \tag{A37}$$

$$\mathcal{L}_2 = (F)^{l_2}(D)^{n_2}(B)^{k_2}\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu}, \tag{A38}$$

$$\mathcal{L}_3 = (F)^{l_3}(D)^{n_3}(B)^{k_3}A_\varkappa A^{\varkappa;(\nu}A^{\mu)}A^\rho A_{\rho;\mu\nu}, \tag{A39}$$

$$\mathcal{L}_4 = (F)^{l_4}(D)^{n_4}(B)^{k_4}A_\varkappa A_\sigma A^{\varkappa;\mu}A^{\sigma;\nu}A^\rho A_{\rho;\mu\nu}, \tag{A40}$$

$$\mathcal{L}_5 = (F)^{l_5}(C)^{n_5}A^\mu A^\nu A^{\rho;\varkappa}A_\varkappa A_{\rho;\mu\nu}, \tag{A41}$$

$$\mathcal{L}_6 = (F)^{l_6}(C)^{n_6}\eta^{\mu\nu}A^{\rho;\varkappa}A_\varkappa A_{\rho;\mu\nu}. \tag{A42}$$

Upon integration by parts. some of these Lagrangians are reduced to the Lagrangian containing the first derivatives only. Our purpose here is to figure out which of these Lagrangians are independent modulo first-order Lagrangians.

The Lagrangian (A41) can be reduced by integration by parts to a Lagrangian involving first derivatives only,

$$\begin{aligned}\mathcal{L}_5 &= (F)^{l_5}(C)^{n_5}A^\mu A^\nu A^{\rho;\varkappa}A_\varkappa A_{\rho;\mu\nu}\\ &= \frac{1}{2}(F)^{l_5}(C)^{n_5}A^\nu C_{;\nu} + \cdots\\ &= \frac{1}{2(n_5+1)}(F)^{l_5}((C)^{n_5+1})_{;\nu}A^\nu + \cdots\\ &\Rightarrow -\frac{1}{2(n_5+1)}(F)^{l_5}(C)^{n_5+1}A^{\nu;}_{\nu} + \cdots = 0 + \cdots,\end{aligned}$$

where, as before, omitted terms do not contain second derivatives and the arrow denotes integration by parts.

Upon integration by parts and adding terms containing first derivatives only, the remaining Lagrangians (A37)–(A40) and (A42) can be expressed through three Lagrangians (A38), (A40), and (A42). Indeed, Lagrangian (A37) can be expressed through Lagrangian (A39):

$$\begin{aligned}\mathcal{L}_1 &= (F)^{l_1}(D)^{n_1}(B)^{k_1}A^\mu A^\nu A^\rho A_{\rho;\mu\nu} + \cdots\\ &= \frac{1}{2}(F)^{l_1}(D)^{n_1}(B)^{k_1}A^\nu D_{;\nu} + \cdots\\ &\Rightarrow -\frac{k_1}{2(n_1+1)}((F)^{l_1}(D)^{n_1+1}(B)^{k_1-1})F^{;\nu}D_{;\nu} + \cdots\\ &= -\frac{k_1}{(n_1+1)}(F)^{l_1}(D)^{n_1+1}(B)^{k_1-1}A_\varkappa A^{\varkappa;(\nu}A^{\mu)}A^\rho A_{\rho;\mu\nu}\\ &\quad + \cdots\end{aligned}$$

Lagrangian (A39) can in turn be expressed through two Lagrangians (A38) and (A40):

$$\begin{aligned}\mathcal{L}_3 &= (F)^{l_3}(D)^{n_3}(B)^{k_3}A_\varkappa A^{\varkappa;(\nu}A^{\mu)}A^\rho A_{\rho;\mu\nu} + \cdots = \frac{1}{2}(F)^{l_3}(D)^{n_3}(B)^{k_3}D_{;\mu}F^{;\mu} + \cdots\\ &\Rightarrow -\frac{1}{2(n_3+1)}(F)^{l_3}(D)^{n_3+1}(B)^{k_3}\Box F - \frac{k_3}{2(n_3+1)}(F)^{l_3}(D)^{n_3+1}(B)^{k_3-1}B_{;\nu}F^{;\nu} + \cdots\\ &= -\frac{1}{(n_3+1)}(F)^{l_3}(D)^{n_3+1}(B)^{k_3}\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu}\\ &\quad -\frac{2k_3}{(n_3+1)}(F)^{l_3}(D)^{n_3+1}(B)^{k_3-1}A_\varkappa A_\tau A^{\varkappa;\mu}A^{\tau;\nu}A^\rho A_{\rho;\mu\nu} + \cdots\end{aligned}$$

There are four special cases in which the remaining Lagrangians (A38), (A40), and (A42) are, in fact, first order or are not independent. One is Lagrangian (A38) with $n_2 = 0$, 1 and $k_2 = 0$:

$$\begin{aligned}F^{l_2}D\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu} + \cdots &= \frac{1}{2}F^{l_2}D\Box F + \cdots \Rightarrow -\frac{1}{2}F^{l_2}D_{;\nu}F^{;\nu} + \cdots\\ &= -\frac{1}{4}F^{l_2}A^\theta F_{;\theta\nu}F^{;\nu} + \cdots = -\frac{1}{8}F^{l_2}A^\lambda(F_{;\nu}F^{;\nu})_{;\lambda} + \cdots\\ &\Rightarrow \frac{1}{8}(F^{l_2}A^\lambda)_{;\lambda}F_{;\nu}F^{;\nu} + \cdots = 0 + \cdots\end{aligned}$$

Another is Lagrangian (A40) with $n_4 = 0$ and $k_{4=0}$, which is effectively first order. The third special case is Lagrangian (A38) with $n_2 = 1$ and arbitrary k_2, which can be expressed through Lagrangian (A40):

$$\begin{aligned}\mathcal{L}_2 &= (F)^{l_2}(B)^{k_2}D\eta^{\mu\nu}A^\rho A_{\rho;\mu\nu} = \frac{1}{2}(F)^{l_2}(B)^{k_2}D\Box F + \cdots\\ &\Rightarrow -\frac{1}{2}(F)^{l_2}((B)^{k_2}D)_{;\nu}F^{;\nu} + \cdots = -\frac{1}{2}(F)^{l_2}(B)^{k_2}D_{;\nu}F^{;\nu} - \frac{k_2}{2}(F)^{l_2}(B)^{k_2-1}DB_{;\nu}F^{;\nu} + \cdots\\ &= -\frac{1}{2}(F)^{l_2}(B)^{k_2}B_{;\nu}A^\nu - 2k_2(F)^{l_2}(B)^{k_2-1}DA_\varkappa A_\tau A^{\varkappa;\mu}A^{\tau;\nu}A^\rho A_{\rho;\mu\nu} + \cdots\\ &\Rightarrow -2k_2(F)^{l_2}(B)^{k_2-1}DA_\varkappa A_\tau A^{\varkappa;\mu}A^{\tau;\nu}A^\rho A_{\rho;\mu\nu} + \cdots\end{aligned}$$

Finally, Lagrangian (A42) is effectively first order for $n_6 = 0$. This completes the analysis leading to the result quoted in the end of Sec. II, Eqs. (2.6)–(2.16).

We point out that in the cases when the above Lagrangians are first order in derivatives (upon integration by parts) their structure does not coincide with any of the Lagrangian for vector Galileons [9,10].

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