Charged gravastars in modified gravity

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In this paper, we investigate the effects of electromagnetic field on the isotropic spherical gravastar models in metric $f(R,T)$ gravity. For this purpose, we have explored singularity-free exact models of relativistic spheres with a specific equation of state. After considering Reissner-Nordström spacetime as an exterior region, the interior charged manifold is matched at the junction interface. Several viable realistic characteristics of the spherical gravastar model are studied in the presence of electromagnetic field through graphical representations. It is concluded that the electric charge has a substantial role in the modeling of the proper length, energy contents, entropy, and equation of state parameter of the stellar system. We have also explored the stable regions of the charged gravastar structures.

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I. INTRODUCTION

Recent interesting outcomes from a few observational experiments like cosmic microwave background radiation, type Ia supernovae, etc., [1], claimed that our cosmos is in the phase of accelerating expansion. The observational ingredients from the BICEP2 experiment [2–4], the Planck satellite [5–7], and the Wilkinson Microwave Anisotropy Probe [8,9] state that our cosmos is composed of only 5% baryonic matter and the rest is composed of dark matter and dark energy (DE). Their percentages are observed to be 27% and 68%, respectively.

A popular approach to understanding the structure formation and evolution of the Universe is modified gravity theories. Such theories are obtained by generalizing the usual Einstein-Hilbert action. The importance as well as the need of such theories have been discussed in detail by Nojiri and Odintsov [10]. The $f(R)$ (R is the Ricci scalar) [11], $f(\mathcal{T})$ ($\mathcal{T}$ is the torsion scalar) [12], $f(R,\Box R,T)$ ($\Box$ is the de Alembert's operator, and T is the trace of energy-momentum tensor) [13], $f(G)$ (G is the Gauss-Bonnet term) [14], $f(G,T)$ [15], etc., are among the most attractive models of modified theories (for further reviews on such models, see, for instance, Ref. [16]). Harko *et al.* [17] introduced a notion of $f(R,T)$ theory by introducing T in the well-known theory of $f(R)$ gravity. This quantity T may be considered by exotic imperfect fluids or quantum effects. They presented dynamical equations and the associated equations of motion for a test particle. Houndjo [18] did reconstruction in order to solve cosmological issues in this gravity and found some models that could be useful to understand matter-dominated eras of our Universe. Jamil *et al.* [19] did the same process and found some quite consistent outcomes with low-redshifts surveys.

Adhav [20] studied the homogeneous and anisotropic cosmic model with the help of a constant deceleration parameter and presented a few analytical $f(R,T)$ solutions under some conditions. Shabani and Farhoudi [21] examined the cosmological solutions of $f(R,T)$ gravity for a perfect fluid using a spatially Friedmann-Lemaître-Robertson-Walker universe. To simplify equations, they presented some dimensionless parameters and variables. Baffou *et al.* [22] analyzed the stability of power-law models with the help of de Sitter cosmic models against linear perturbations. They concluded that these models could be taken as an efficient candidate for DE. Das and Ali [23] elaborated the anisotropic and homogeneous axially symmetric Bianchi type-I bulk viscous cosmological models using the time-varying cosmological and gravitational constant. By using the Hubble parameter, they solved the field equations and discussed the kinematical and physical properties of the models. Kiran and Reddy [24] investigated the Bianchi type III in the presence of viscous fluid and concluded that this model does not exist in $f(R,T)$ gravity. Momeni *et al.* [25] analyzed the Noether symmetry problem for two types of modified theories. First is mimetic $f(R)$, and second is a nonminimally coupled model, which is known as $f(R,T)$. Pankaj and Singh [26] discussed the viscous cosmology with matter creation under $f(R,T)$ gravity. Sun and Huang [27] studied the issues of an isotropic and homogeneous universe under modified $f(R,T)$ gravity with nonminimal coupling. Moraes *et al.* [28] studied the hydrostatic equilibrium conditions of compact objects by relating their pressure and density through an equation of state.

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A gravastar (gravitational vacuum star) is an astronomical substance hypothesized as a substitute to the black hole. The conception of a gravastar was established from the theory of Mazur and Mottola [29,30]. By increasing the idea of Bose-Einstein, this new form of the solution was introduced as a consequence of gravitational collapse. Such a kind of model is hypothesized to contain no event horizon. Though the gravastar may look identical to a black hole, there are certain experiments, such as x-ray radiations by infalling matter, that may allow one to discriminate them. As such, gravastars are of interest from a purely theoretical perspective. Such a kind of stellar structures could be used to explain the role of DE in the accelerating expansion of the Universe. These could be helpful to explain why some galaxies have high or low DM concentrations. The gravastars could be described with the help of three different zones, in which I is the interior region ($0 \leq r,\ r < r_1$); II is the intermediate thin shell, with $r_1 < r,\ r < r_2$; while III is an exterior region ($r_2 < r$). It so happens that in the region I the isotropic pressure produces a force of repulsion over the intermediate thin shell, which is equal to -ρ (where ρ is the energy density). This intermediate thin shell is supposed to be supported by a fluid pressure and ultrarelativistic plasma. However, region III can be represented by the vacuum solution of the field equations. The pressure has zero value at this zone. It contains a thermodynamically stable solution and maximum entropy under small fluctuations [29,30].

Visser [31] developed a simple mathematical model for the description of Mazur-Mottola scenario and described the stability of gravastars after exploring some realistic values of the equation of state (EoS) parameter. Cattoen *et al.* [32] extended their results for the case of anisotropic gravastar structures. They calculated the anisotropic factor Δ from the equations of motion for the spherically symmetric spacetime and analyzed the inclusion of pressure anisotropy could be useful to support relatively high compact gravastars. Based upon the ranges of the parameters involved, Carter [33] studied the stability of the gravastar and checked the existence of thin shell. He used the Israel junction condition in order to join de Sitter spacetime (interior) with a Reissner-Nordström exterior metric. He also analyzed the role of the EoS parameter in the modeling of gravastar structures.

Horvat *et al.* [34] presented two different theoretical models of the gravastars in the presence of an electromagnetic field. After joining the interior metric with an appropriate exterior vacuum geometry, they obtained viability constraints for the stability of the gravastar though dominant energy conditions. They also studied the effects of electromagnetic field on the formulations as well as graphical representations of the EoS, the speed of sound and the surface redshift. Rahaman *et al.* [35] discussed the existence of a charged gravastar in an environment of $(2+1)$-dimensional spacetime. They studied various physical properties like length and energy within the thin shell and entropy of the charged gravastars and claimed that their solutions are nonsingular and could present a viable alternative to the black hole.

De Felice *et al.* [36] found exact solutions of the spherically symmetric spacetimes in the presence of electric charge and also compared their results with the existing black holes models. It can be concluded that one can mollify the phenomenon of gravitational collapse to a great extent in the presence of a electric charge. Yousaf and Bhatti [37] investigated the modeling of relativistic structures in the presence of electromagnetic field. They concluded that the electric charge has greatly weakened the influence of modified gravity, leading to the production of a repulsive field. Turimov *et al.* [38] studied the slowly rotating magnetized gravastars in the presence of electromagnetic field.

Rahaman *et al.* [39] studied the three-dimensional neutral spherically symmetric model of a gravitational vacuum star of which the exterior region is elaborated by the *Bañados-Teitelboim-Zanelli metric*. He presented a nonsingular and stable model and discussed various physical features, for example, length, energy conditions, entropy, and junction conditions of the spherical distribution by using static spherically symmetric matter distribution as an interior spacetime. Lobo and Garattini [40] performed linearized stability analysis with noncommutative geometry of gravastars and investigated a few exact solutions of the gravastar after exploring their physical features and characteristics.

Usmani *et al.* [41] studied a charged gravastar experiencing conformal motion and elaborated the dynamics of the formation of the thin shell and the entropy of the system. Herrera and de León [42] discussed the role of anisotropic/isotropic pressure on charged spheres and found some exact solutions of the nonlinear field equations by assuming spherical symmetry spacetime. The same authors [43] also analyzed the dynamics of anisotropic spheres by introducing the one-parameter group of conformal motions. They inferred that on the boundary of matter all of their calculated solutions can exist on the Schwarzschild exterior metric. Esculpi and Aloma [44] studied the conformal motion of the charged fluid sphere with a linear EoS. They also discussed the dynamical stability analysis of the relativistic structures. The process of dynamical instability [45] and the regularity of certain physical quantities [46] on the surface of evolving matter distribution are also discussed in the literature. Ray and Das [47] discussed the electromagnetic mass model with the help of the conformal killing vector. By considering the existence of a one-parameter group, the corresponding conformal motion have been described for the charged strange quark star model.

This paper is devoted to understanding the existence of a gravastar under spherically symmetric spacetime in the realm of Maxwell-$f(R,T)$ gravity. The paper is arranged as follows. In Sec. II, we shall describe the basic frame work of $f(R,T)$ theory and their conservation equation. Section III is based on the formulation of field/conservation equations and gravitational mass of the static spherically symmetric manifold. In Sec. III, we shall calculate the mass of the thin shell by using certain matching conditions. The Sec. V is aimed at discussing the effects of charge on the various physical features of the gravastar. Finally, we summarize our findings.

II. $f(R,T)$ GRAVITY

The action of the $f(R,T)$ theory is given as

$$\mathbf{S} = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}d^4x + \int \text{Ł}_m\sqrt{-g}d^4x, \quad (1)$$

where $f(R,T)$ is the arbitrary function of R and T with R as the Ricci scalar and T as the trace of the energy-momentum tensor, Ł_m is the Lagrangian matter density, and g is the determinant of metric tensor $g_{\zeta\eta}$. In this paper, we have consider $G = c = 1$. By varying the action of $f(R,T)$ with respect to the metric $g_{\zeta\eta}$, we can deduce the field equations of $f(R,T)$ of gravity as

$$f_R(R,T)R_{\zeta\eta} - \frac{1}{2}f(R,T)g_{\zeta\eta} - (\nabla_\zeta\nabla_\eta - g_{\zeta\eta}\Box)f_R(R,T) + f_T(R,T)(T_{\zeta\eta} + \Theta_{\zeta\eta}) = 8\pi T g_{\zeta\eta}, \quad (2)$$

where $f_R(R,T)$ is the derivative of generic function f with respect to the Ricci scalar R, $f_T(R,T)$ is the derivative of generic function with respect to trace of the energy-momentum tensor T, and $\Box$ is the product of the contravariant and covariant derivative,

$$\Theta_{\zeta\eta} = g_{\zeta\eta}\frac{\partial T_{\zeta\eta}}{\partial g_{\zeta\eta}}, \quad (3)$$

where the stress-energy tensor is defined below:

$$T_{\zeta\eta} = g_{\zeta\eta}\text{Ł}_m - 2\frac{\partial \text{Ł}_m}{\partial g^{\zeta\eta}}. \quad (4)$$

We assume perfect fluid as the energy-momentum tensor

$$T_{\zeta\eta} = (\rho + p)U_\zeta U_\eta - p g_{\zeta\eta}, \quad (5)$$

where ρ is the density, p is the pressure, and U_ζ is the 4-velocity vector. It is interesting to mention that here the equations of motion are dependant on the role of fluid distribution. Therefore, one can take specific equations of motion by choosing Ł_m. In the literature, researchers have chosen $\text{Ł}_m = p$ and $\text{Ł}_m = \rho$ [17,18,22,48,49]. Here, we are interested in studying the charged isotropic fluid; therefore, we shall take $\text{Ł}_m = -(p + \mathcal{F})$, where $\mathcal{F}$ shows the contribution of electromagnetic field and is defined through Maxwell tensor $(F_{\alpha\beta})$ as $\mathcal{F} = \frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\sigma}g^{\alpha\gamma}g^{\beta\sigma}$. The Maxwell field tensor is defined through the 4-potential ϕ_α as $F_{\alpha\beta} = \phi_{\beta,\alpha} - \phi_{\alpha,\beta}$. The Maxwell equations of motion are

$$F^{\xi\beta}_{;\beta} = K_0 J^\xi, \qquad F_{[\xi\beta;\gamma]} = 0, \quad (6)$$

in which J^ξ is the 4-current and K_0 indicates the permeability of the magnetic field. In view of this scenario, Eq. (3) reduces to

$$\Theta_{\zeta\eta} = -2T_{\zeta\eta} - p g_{\zeta\eta} - \mathcal{F} g_{\zeta\eta}.$$

In this work, we use the functional form of $f(R,T) = 2\chi T + R$. Substituting this relation in Eq. (2), we get

$$G_{\zeta\eta} = 8\pi(T_{\zeta\eta} + E_{\zeta\eta}) + \chi T g_{\zeta\eta} + 2\chi(T_{\zeta\eta} + E_{\zeta\eta} + p g_{\zeta\eta} + \mathcal{F} g_{\zeta\eta}), \quad (7)$$

where $G_{\zeta\eta}$ is the Einstein tensor and $E_{\zeta\eta}$ is the energy-momentum tensor for an electromagnetic field and is given by [50]

$$E_{\xi\beta} = \frac{1}{4\pi}\left(F^\gamma_\xi F_{\beta\gamma} - \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g_{\alpha\xi}\right). \quad (8)$$

In this paper, we consider a scenario in which the system is evolving by keeping the charged particles at the state of rest. This will give zero contribution to the magnetic field. Therefore,

$$\phi_\xi = \Phi(t,r)\delta^0_\xi, \qquad J^\xi = K_1(t,r)V^\xi,$$

where Φ represents the corresponding scalar potential and K_1 indicates the charge density.

To understand $f(R,T)$ theory as an appropriate gravitational theory, one must consider a viable and effective distribution of the $f(R,T)$ function. Besides its physical consistency with the observations of current cosmic acceleration, it should pass the stability tests and should meet the viability requirements from solar and terrestrial static/nonstatic systems. Usually, the $f(R,T)$ models are presented in the following different ways:

(1) $f(R,T) = R + 2g(T)$. Such a kind of selection in the geometric part of the Lagrangian describes cosmological constant Λ as a time-dependent entity and hence represents the ΛCDM model.

(2) $f(R,T) = f_1(R) + f_2(T)$. This type of choice corresponds to the minimal knowledge to understand modified relativistic dynamics. This could be regarded as the corrections to the notable $f(R)$ theory. By considering any linear combination of f_2,

various distinct results can be obtained from the choices of the $f(R)$ function.

(3) $f(R,T) = f_1(R) + f_2(T)f_3(R)$. This explicitly describes the nonminimal coupled matter-geometry theory of gravity. The comparison of results found from this selection may be different from the minimal interacting choices.

The criteria for understanding the viability of $f_1(R)$ model are as follows:

(i) For a positive value of $f_{1R}(R)$ with $R > \mathbf{R}$, where $\mathbf{R}$ represents today's choice of the Ricci scalar. This condition is necessary to prevent the appearances of a ghost state. Ghosts that notify that DE is responsible for cosmic acceleration under modified gravity theories often appear. This state could be induced by a mysterious force that creates a repulsion between the supermassive or massive stellar object. For retaining the attractive feature of gravity, the constraint should maintain a positive sign with a consistent gravitational constant, $G_{\mathrm{eff}} = G/f_{1R}$.

(ii) The positive value of $f_{1RR}(R)$ with $R > \mathbf{R}$. This requirement is introduced for making the evolving system not to conceive situations in which tachyons appear. A hypothetical object that could move faster than the speed of light is known as a tachyon.

If a model of $f_1(R)$ does not fulfill those conditions, it would not be considered viable. Haghani *et al.* [51] as well as Odintsov and D. Sáez-Gómez [52] proposed that Dolgov-Kawasaki instability in $f(R,T)$ gravity requires a similar sort of limitations as in $f(R)$ gravity, and one needs to satisfy $1 + f_T > 0$ with $G_{\mathrm{eff}} > 0$. Thus, in the realm of $f(R,T)$ models, the following conditions should be fulfilled:

$$f_R > 0, \qquad 1 + f_T > 0, \qquad f_{RR} > 0, \qquad R > \mathbf{R}.$$

Thus, throughout in our paper, we assume that $1 + 2\chi > 0$. One thing that must be taken into account is that the divergence of the energy-momentum tensor is not zero in $f(R,T)$ gravity and is defined as

$$\nabla^\zeta T_{\zeta\eta} = \frac{f_T}{8\pi - f_T}\left[(T_{\zeta\eta} + \Theta_{\zeta\eta})\nabla^\zeta \ln f_T + \nabla^\zeta \Theta_{\zeta\eta} - \frac{1}{2} g_{\zeta\eta}\nabla^\zeta T\right]. \tag{9}$$

The non-zero value of divergence of energy momentum tensor causes the breaking of all equivalence principle in $f(R,T)$ gravity. According to the weak equivalence principle, "All test particles in a given gravitational field will undergo the same acceleration, independent of their properties, including their rest mass [53]." In this modified theory, the equation of motion is based on those features of the particle that are thermodynamic in nature, e.g., pressure, energy density, etc. Further, the strong equivalence principle states that "The gravitational motion of a small test body depends only on its initial position and velocity, and not on its configuration." This principle also does not hold in $f(R,T)$ theory, thus causing the particles to experience nongeodesic motion along the world lines. In the background of quantum theory, one can relate the nonzero divergence of the effective energy-momentum tensor with the violation of energy conservation in the scattering phenomenon. In this theory, the energy nonconservation can cause an energy flow between the four-dimensional spacetime and a compact extra-dimensional metric [54]. It is worth noting that the constraint $f(T) = 0$ in Eq. (9) would reduce our dynamics to that of $f(R)$ gravity. One can write Eq. (9) as follows:

$$\nabla^\zeta T_{\zeta\eta} = \frac{-2\chi}{8\pi + 2\chi}\left[\nabla^\zeta(p g_{\zeta\eta}) + \nabla^\zeta(\mathcal{F} g_{\zeta\eta}) + \frac{1}{2} g_{\zeta\eta}\nabla^\zeta T\right]. \tag{10}$$

It is worth stressing that Eq. (9) corresponds to a general $f(T)$ part, while Eq. (10) describes the covariant divergence of stress-energy tensor with a linear contribution on the $f(T)$ model.

III. SPHERICALLY SYMMETRIC SPACETIME MODELS

This section is devoted to exploring modified equations of motion, including field and conservation laws. By simultaneously solving these equations, we evaluate the evolution equation. After using a specific combination of EoS, we shall evaluate the value of the corresponding scale factors that would eventually lead to gravitational mass of the relativistic structure. We consider an irrotational static form of the spherically spacetime as follows:

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{11}$$

The nonzero components of the Einstein tensor for above equation are

$$G^{00} = \frac{\lambda' r + e^\lambda - 1}{r^2 e^\nu e^\lambda}, \tag{12}$$

$$G^{11} = \frac{\nu' r - e^\lambda + 1}{r^2 e^{\lambda^2}}, \tag{13}$$

$$G^{22} = \frac{-2\lambda' - 2\nu' + (2\nu'' + \nu'^2 - \nu'\lambda')r}{4r^3 e^\lambda}. \tag{14}$$

The nonzero components of Eq. (8) are given as

$$E_{00} = 2\pi E^2 e^\nu, \quad E_{11} = 2\pi E^2 e^\lambda, \quad E_{22} = 2\pi E^2 r^2, \tag{15}$$

where E is the electric intensity, which is defined via electric charge (q) as

$$E = \frac{q}{4\pi r^2}.$$

After using Eqs. (12)–(15) in Eq. (6), we get

$$\lambda' r + e^{\lambda} - 1 = r^2 e^{\lambda}\left[8\pi\rho + \chi(3\rho - p) + \frac{q^2}{r^4} + \frac{\chi}{r^4}\left(\frac{1}{4\pi} + 1\right)q^2\right], \tag{16}$$

$$-\nu' r + e^{\lambda} - 1 = r^2 e^{\lambda}\left[-8\pi p + \chi(\rho - 3p) + \frac{q^2}{r^4} + \frac{\chi}{r^4}\left(\frac{1}{4\pi} + 1\right)q^2\right], \tag{17}$$

$$-\frac{r}{2}(\nu' - \lambda') + \frac{r^2}{4}(\nu'\lambda' - 2\nu'' - \nu'^2) = r^2 e^{\lambda}\left[-8\pi p + \chi(\rho - 3p) - \frac{q^2}{r^4} + \frac{\chi}{r^4}\left(-\frac{1}{4\pi} + 1\right)q^2\right]. \tag{18}$$

The hydrostatic equilibrium condition can be evaluated with the help of the conservation law as

$$\frac{\nu'}{2}(\rho + p) + \frac{dp}{dr} + \frac{3q^2}{8\pi r^5} + \frac{\chi}{4\pi + \chi}\left[\frac{2q^2}{r^5} + \frac{1}{2}(\rho' - p')\right] = 0. \tag{19}$$

With the help of the Misner-Sharp formula [55] and G_{00} component of the Einstein tensor, the corresponding component of the line element g_{00} becomes

$$e^{-\lambda} = 1 - \frac{2m}{r} - \chi\left(\rho - \frac{p}{3}\right)r^2 + \frac{2q^2}{r^4} + \frac{\chi}{r^4}\left(\frac{1}{4\pi} + 1\right)q^2. \tag{20}$$

Using Eq. (17) in Eq. (19), we get

$$\frac{dp}{dr} = \frac{-\frac{2q^2\chi}{r^5(4\pi+\chi)} - \frac{3q^2}{8\pi r^5} - \frac{\nu'}{2}(\rho + p)}{[1 + \frac{\chi}{2(4\pi+\chi)}(1 - \frac{d\rho}{dp})]}, \tag{21}$$

where

$$\nu' = \frac{r[8\pi p - \chi(\rho - 3p) - \frac{q^2}{r^4} - \frac{\chi}{r^4}(\frac{1}{4\pi} + 1)q^2] + \frac{1}{r}[\frac{2m}{r} + \chi(\rho - \frac{p}{3}) - \frac{2q^2}{r^4} - \frac{2\chi}{r^4}(\frac{1}{4\pi} + 1)q^2]}{[1 + \frac{2q^2}{r^4} - \frac{2m}{r} - \chi(\rho - \frac{p}{3})r^2 + \frac{2\chi}{r^4}(\frac{1}{4\pi} + 1)q^2]}.$$

Gravastars [29,30] consist of three regions characterized by an EoS $p = \omega\rho$, where ω is constant. Here, we assume that the interior region is filled with an enigmatic gravitational source. The corresponding EoS for the dark energy model is given as

$$p = -\rho, \quad \text{with} \quad \omega = -1. \tag{22}$$

Using $\rho = \rho_0$ (constant) in Eq. (22), we get

$$p = -\rho_0. \tag{23}$$

After using Eq. (22) in Eq. (16), it follows that

$$e^{-\lambda} = 1 - \frac{4r^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{r^2} + \frac{\chi}{r^2}\left(\frac{1}{4\pi} + 1\right)q^2 + \frac{H}{r}, \tag{24}$$

where H is an integration constant, the value of which, after applying the regularity condition, is found to be zero. Therefore, Eq. (25) becomes

$$e^{-\lambda} = 1 - \frac{4r^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{r^2} + \frac{\chi}{r^2}\left(\frac{1}{4\pi} + 1\right)q^2. \tag{25}$$

Substituting an EoS in Eqs. (16) and (17), we have

$$e^{-\lambda} = Ie^{\nu}, \tag{26}$$

where I is an integration constant. The gravitational mass $M(D)$ can be found as

$$M(D) = \int_0^{r=D} 4\pi\left(\rho_0 + \frac{q^2}{2r^2}\right)r^2 dr = 2\pi D\left(\frac{2}{3}D^2 + q^2\right), \tag{27}$$

where ρ_0 is the constant density. Equation (27) describes that the interior gravitational mass and radius of the stellar system are directly proportional to each other. This is the characteristic feature of the stellar compact object. Furthermore, the above equation also states the substantial dependence of M on the specific value $r = D$ in the presence of electric charge. This integral becomes improper on substituting $r = \infty$. However, this choice is not realistic as one cannot consider the infinite radius of the stellar body.

IV. INTERMEDIATE SHELL OF THE CHARGED GRAVASTAR

In this section, we tend to discuss the effect of electromagnetic charge on the formulation of the intermediate shell of the corresponding gravastars. We shall also explore the smooth matching conditions for the joining of interior and exterior manifolds of the gravastar structures by using

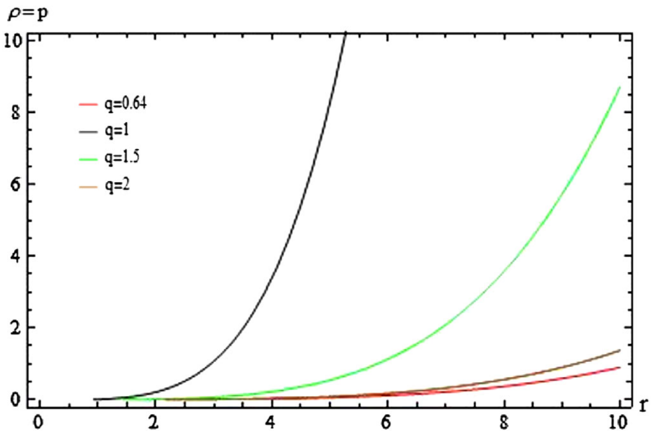

FIG. 1. Plot of pressure p within the shell with respect to radius r (km) with different charges.

Darmois-Israel formalism. For this purpose, we assume that the intermediate shell is formed by an ultrarelativistic fluid with nonvacuum background with an equation of state $p = -\rho$. It is hard to calculate the solution of the cumbersome set of field equations in the nonvacuum region. To avoid this query, we will use some approximation and find the analytical solution, i.e., $0 < e^{-\lambda} \ll 1$. By solving field Eqs. (16)–(18) under the EoS, we end up with the following two equations:

$$\frac{de^{-\lambda}}{dr} = -\frac{2q^2}{r^3} - \frac{2\chi}{r^3}\left(\frac{1}{4\pi}+1\right)q^2 + \frac{2}{r}, \tag{28}$$

$$\left(\frac{3}{2r}+\frac{\nu'}{4}\right)\frac{de^{-\lambda}}{dr} = -\frac{2\chi q^2}{r^4} + \frac{1}{r^2}. \tag{29}$$

Integrating Eq. (28), we get

$$e^{-\lambda} = \frac{q^2}{r^2} + \frac{2\chi}{r^2}\left(\frac{1}{4\pi}+1\right)q^2 + 2lnr + B, \tag{30}$$

where B is an integration constant and r is the radius belonging to $D \ll r \ll D+\epsilon$, under $\epsilon \ll 1$. To get analytical values of the pressure and radius of the thin shell, we use Eqs. (28) and (29) in Eq. (19), and we get the behavior of the pressure with respect to radius, which are shown in Fig. 1.

To discuss the structure of the gravastar, we take static Schwarzschild spacetime as an exterior geometry given as follows:

$$ds^2 = \left(1-\frac{2M}{r}+\frac{q^2}{D^2}\right)dt^2 - \left(1-\frac{2M}{r}+\frac{q^2}{D^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{31}$$

Darmois [56] and Israel [57] introduced conditions for the matching of interior and exterior geometries over the surface. The metric coefficients are continuous at the junction surface (Σ); i.e., their derivatives might not be continuous at interior surfaces. The surface tension and surface stress energy of the joining surface S may be resolved from the discontinuity of the extrinsic curvature of S at $r = D$. The field equation of the intrinsic surface is defined by the Lanczos equation as

$$S_\alpha^\beta = -\frac{1}{8\pi}(\Upsilon_\alpha^\beta - \delta_\beta^\alpha \Upsilon_\kappa^\kappa), \tag{32}$$

where S_{ji} is the stress-energy tensor for the surface, $\Upsilon_{\alpha\beta} = \eta^+_{\alpha\beta} - \eta^-_{\alpha\beta}$ tells the extrinsic curvatures or second fundamental forms, and the (+) sign indicates the interior surface while the (−) sign indicates the exterior surface. The second fundamental forms connect the interior and exterior surfaces of the thin shell and are defined as

$$\eta^\pm_{\mu\nu} = -n_i^\pm\left[\frac{\partial^2 x_i}{\partial\xi^\mu\partial\xi^\nu} + \Gamma^i_{\gamma\delta}\frac{\partial x^\gamma}{\partial\xi^\mu}\frac{\partial x^\delta}{\partial\xi^\nu}\right]_\Sigma, \tag{33}$$

where ξ^μ represents the coordinate of intrinsic metric and $n_i^\pm$ describes the unit normals on the surface of gravastar.

$$n_i^\pm = \pm\left|g^{\alpha\beta}\frac{\partial f(r)}{\partial x^\alpha}\frac{\partial f(r)}{\partial x^\beta}\right|^{-\frac{1}{2}}\frac{\partial f(r)}{\partial x^i}, \qquad n_j n^j = 1, \tag{34}$$

where $f(r)$ illustrates the coordinate of the exterior metric. Using the Lanczos equations, we can get the surface energy density (φ) and surface pressure ψ as

$$\varphi = -\frac{1}{4\pi D}[\sqrt{f(r)}]^\pm_-, \tag{35}$$

$$\psi = -\frac{\varphi}{2} + \frac{1}{16\pi}\left[\frac{f(r)'}{\sqrt{f(r)}}\right]^+_-. \tag{36}$$

Making use of Eqs. (35) and (36), we get

$$\varphi = -\frac{1}{4\pi D}\left[\sqrt{1-\frac{2M}{D}+\frac{q^2}{D^2}} - \sqrt{1-\frac{4D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{D^2}+\frac{\chi}{D^2}\left(\frac{1}{4\pi}+1\right)q^2}\right], \tag{37}$$

$$\psi = \frac{1}{8\pi D}\left[\frac{(1-\frac{M}{D})}{\sqrt{1-\frac{2M}{D}+\frac{q^2}{D^2}}} - \frac{[1-\frac{8D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{2D^2}+\frac{\chi}{2D^2}(\frac{1}{4\pi}+1)q^2]}{\sqrt{1-\frac{4D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{D^2}+\frac{\chi}{D^2}(\frac{1}{4\pi}+1)q^2}}\right]. \tag{38}$$

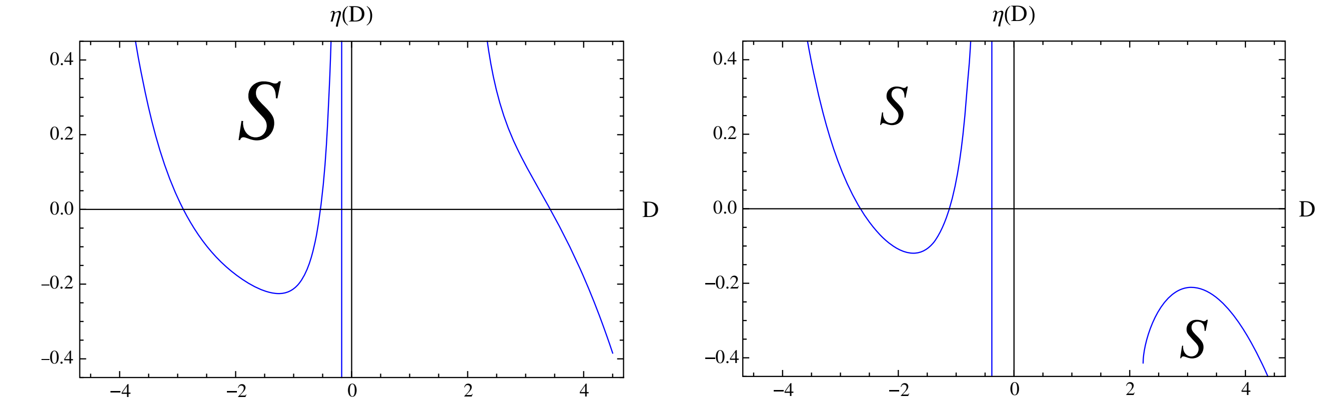

FIG. 2. Stability regions of the charged gravastar in terms of $\eta = \frac{\psi'}{\varphi'}$. We have chosen $\chi = 0.2$, $M = 1.2$, and $\rho = 0.002$ at $q = 1$ and $q = 1.5$.

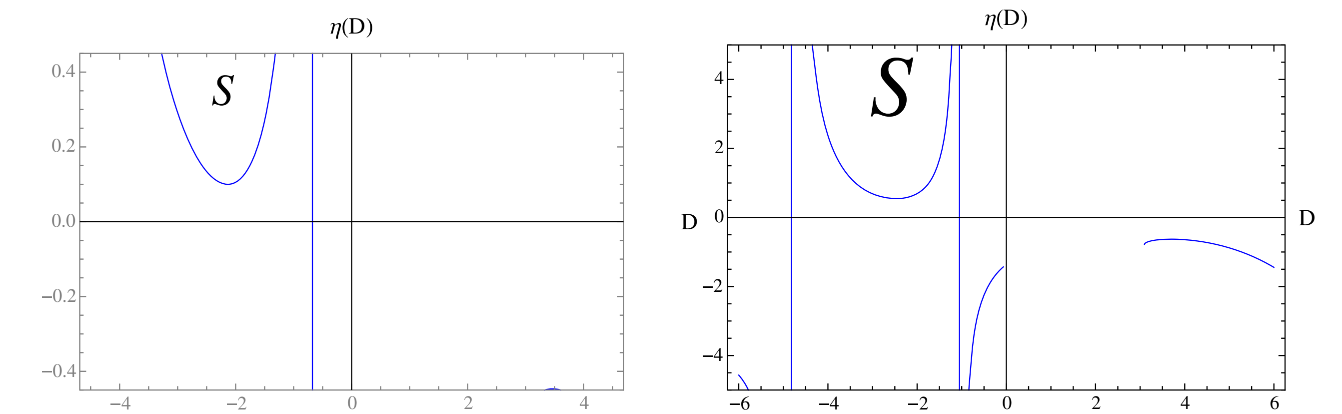

FIG. 3. Stability regions of the charged gravastar in terms of $\eta = \frac{\psi'}{\varphi'}$. We have chosen $\chi = 0.2$, $M = 1.2$, and $\rho = 0.002$ at $q = 2$ and $q = 2.5$.

One can find the mass of the intermediate thin shell by using the areal density as

$$m_s = 4\pi D^2 \varphi = -D\left[\sqrt{1-\frac{2M}{D}+\frac{q^2}{D^2}} - \sqrt{1-\frac{4D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{D^2}+\frac{\chi}{D^2}\left(\frac{1}{4\pi}+1\right)q^2}\right], \tag{39}$$

where

$$M = \frac{m_s}{D}\sqrt{1-\frac{4D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{D^2}+\frac{\chi}{D^2}\left(\frac{1}{4\pi}+1\right)q^2} - \frac{m_s^2}{2D} + \frac{2\rho_o D^3}{3}(2\pi+\chi) - \frac{\chi}{2D}\left(\frac{1}{4\pi}+1\right)q^2$$

represents the total mass of the gravitational vacuum star with $m_s = m$. It can be noticed that one can calculate the value of M, once the mass of the intermediate thin shell (m_s), the radial distance (D), and the value of the $f(R,T)$ correction term (χ) or electric charge (q) are known. It also indicates that the physical quantities m_s, D, and q have dominated their influence over the $f(R,T)$ dark source terms. This is because one can diminish the role of χ by substituting zero to m_s, D, and q. This situation could be different if one considers the Palatini $f(R,T)$ gravity [58] instead of the metric $f(R,T)$ gravity [17].

It will be very useful to understand the stability of gravastars by defining a parameter (η) as the ratio of the derivatives of ψ and φ as follows:

$$\eta(a) = \frac{\psi'(a)}{\varphi'(a)}\bigg|_{r=a_0}.$$

The stability regions can be explored by analyzing the behavior of η as a function of $r = a_0$. Övgün *et al.* [59] considered the static form of the spherically symmetric spacetime and analyzed the stability of a charged thin-shell gravastar with the help of a similar parameter as defined above. We have investigated the stable regimes of gravastars with specific choices of parameters involved. The letter S in Figs. 2 and 3 describes the stable epochs of spherically symmetric gravastar structures.

V. SOME FEATURES OF GRAVASTARS

This section is devoted to examining the impact of electromagnetic field on different physical features of the gravastar. In this context, we shall calculate the proper length of the thin shell as well as the energy of relativistic structure. After examining the entropy of gravastars, the role of the EoS parameter will be analyzed on the dynamical formulation of gravastars. We shall also describe our results by drawing various diagrams and graphs.

A. Proper length of the thin shell

In this subsection, we shall consider $r = D$ for describing the radius of an interior region, while $r = D + \epsilon$ (with $\epsilon \ll 1$) and ϵ represent the radius of the exterior region and the thickness of the intermediate thin shell, respectively. The proper thickness between two surfaces can be described mathematically as

$$\ell = \int_D^{D+\epsilon} \sqrt{e^{\lambda}} dr = \int_D^{D+\epsilon} \frac{1}{\sqrt{\frac{q^2}{r^2} + \frac{2\chi}{r^2}(\frac{1}{4\pi} + 1)q^2 + 2lnr + C}} dr. \quad (40)$$

The analytic solution of the above expression with Maxwell-$f(R,T)$ gravity corrections is not possible. We shall solve it by the numerical method and examine the behavior of the charge. The behavior of the length of the shell vs its thickness has been shown in Fig. 4.

B. Energy of the charged gravastar

The energy content within the shell is given as

$$\varepsilon = \int_D^{D+\epsilon} 4\pi\rho r^2 dr,$$

which is found after using the corresponding values from the equation of motion as follows:

$$\varepsilon = \frac{4\pi H}{7}[(D+\epsilon)^7 - D^7] + 2q^2\epsilon. \quad (41)$$

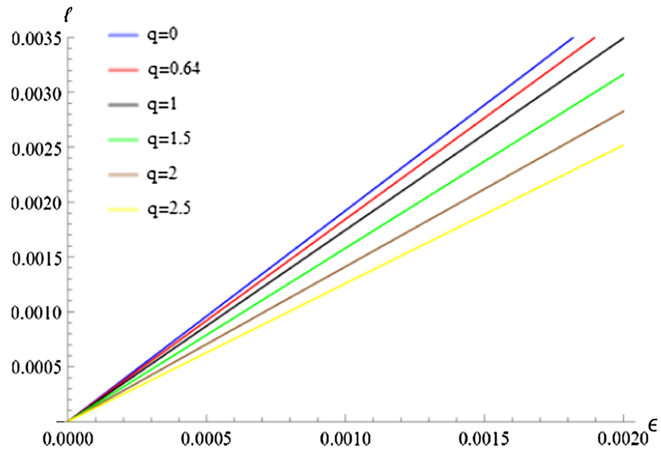

FIG. 4. Plot of length ℓ(km) of the shell with thickness of the shell ϵ(km). Fixing $\chi = 1$, $C = 0.00006$, $D = 1$.

The graphical representation about the role of energy and thickness of the intermediate shell is being explored from the above equation and is shown in Fig. 5. The graph 5 shows the linear relationship between the energy and thickness of the shell, while energy of the system tends to increase by increasing the corresponding charge values.

C. Entropy of the charged gravastars

Entropy is the disorderliness within the body of a gravastar. It is found in the literature that the entropy density of the interior region of the charged gravastar is zero. The entropy relation for the shell can be calculated through the formula

$$S = \int_D^{D+\epsilon} 4\pi r^2 S(r)\sqrt{e^{\lambda}} dr, \quad (42)$$

where

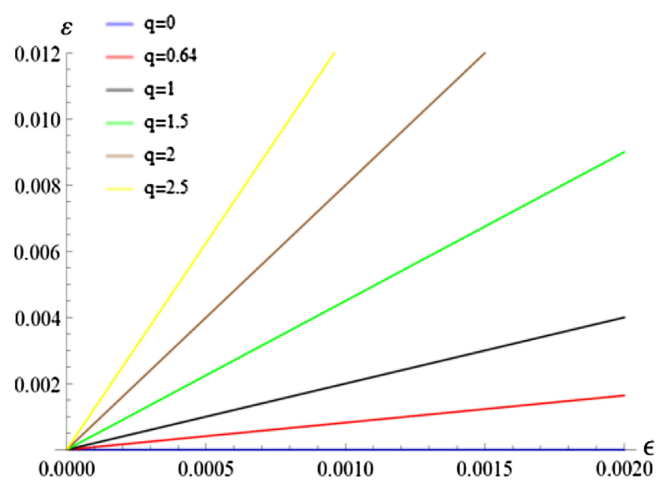

FIG. 5. Plot of energy ε with thickness of the shell ϵ(km). Fixing $H = 0.00001$, $D = 1$.

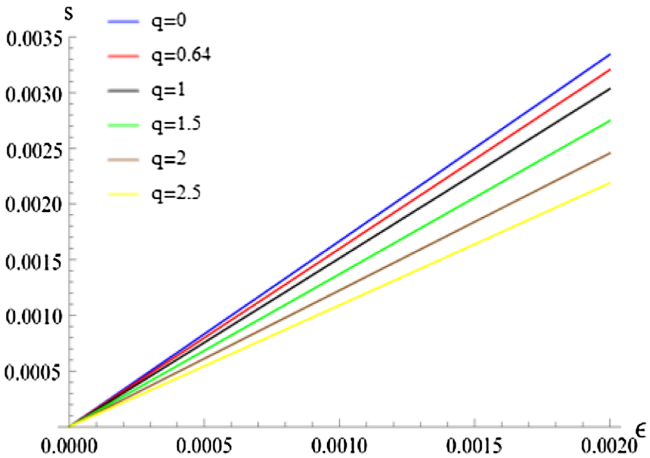

FIG. 6. Plot of entropy S with thickness of the shell ϵ(km). Fixing $\chi = 1$, $C = 0.00006$, $H = 0.00001$, $D = 1$, and $\alpha = 1$.

$$S(r) = \frac{\alpha^2 K_B^2 T(r)}{4\pi h^2} = \alpha\left(\frac{K_B}{h}\right)\sqrt{\frac{P}{2\pi}} \tag{43}$$

describes the entropy density corresponding to a specific temperature $T(r)$. In the above expression, α is a constant term that has no dimension. It is noteworthy that we are using geometrical ($G = C = 1$) as well as Planck units ($K_B = \hbar = 1$) in our computation; therefore, $S(r)$ becomes

$$S(r) = \alpha\sqrt{\frac{P}{2\pi}}. \tag{44}$$

Then, Eq. (42) turns out to be

$$S = (8\pi H)^{\frac{1}{2}}\alpha\int_D^{D+\epsilon}\frac{r^4}{\sqrt{\frac{q^2}{r^2} + \frac{2\chi}{r^2}(\frac{1}{4\pi} + 1)q^2 + 2lnr + C}}dr. \tag{45}$$

The above equation contains the contribution of charge as well as corrections from $f(R, T)$ gravity. The analytical solutions of the above are not possible. After using the numerical method, we have drawn graphs to examine the behavior of an electric charge, which are given in Fig. 6.

D. EoS parameter

At a particular radius $r = D$, the EoS can written as

$$\omega(D) = \frac{\nu}{\sigma}. \tag{46}$$

Substituting the values of ν and σ in Eq. (46), we get

$$\omega(D) = \frac{\left[\frac{(1-\frac{M}{D})}{\sqrt{1-\frac{2M}{D}+\frac{q^2}{D^2}}} - \frac{\left[1-\frac{8D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{2D^2}+\frac{\chi}{2D^2}(\frac{1}{4\pi}+1)q^2\right]}{\sqrt{1-\frac{4D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{D^2}+\frac{\chi}{D^2}(\frac{1}{4\pi}+1)q^2}}\right]}{-2\left[\sqrt{1-\frac{2M}{D}+\frac{q^2}{D^2}} - \sqrt{1-\frac{4D^2\rho_0}{3}(2\pi+\chi)+\frac{q^2}{D^2}+\frac{\chi}{D^2}(\frac{1}{4\pi}+1)q^2}\right]}. \tag{47}$$

To get real solutions of the above equation, we have a tendency to use some approximations, i.e., $\frac{2M}{D} < 1$ and $\frac{4(2\pi+\chi)\rho_0 D^2}{3} < 1$. We use binomial expansion for avoiding square root terms, which is responsible for producing increments in the sensitivity of the equation and some approximations, i.e., $\frac{M}{D} < 1$, $\frac{4(2\pi+\chi)\rho_0 D^2}{3} < 1$, and $\frac{q^2}{2D^2} \ll 1$; then, we get

$$\omega(D) \approx \frac{-3Dq^2 + 3Mq^2 + 12(2\pi+\chi)\rho_o D^5 - 2(2\pi+\chi)q^2\rho_o D^2[5 + 6(\frac{1}{4\pi}+1)\chi]}{2D[-4(2\pi+\chi)\rho_o D^4 + 3\chi(\frac{1}{4\pi}+1)q^2 + 6M]}, \tag{48}$$

which can be rewritten as

$$\omega(D) \approx \frac{\phi_1 - \phi_2}{8D^5(2\pi+\chi)\rho_o[\phi_3 - 1]}, \tag{49}$$

where

$$\phi_1 = 3Mq^2 + 12(2\pi+\chi)\rho_o D^5,$$
$$\phi_2 = 2(2\pi+\chi)q^2\rho_o D^2\left[5 + 6\left(\frac{1}{4\pi}+1\right)\chi\right],$$
$$\phi_3 = \frac{3\chi(\frac{1}{4\pi}+1)q^2 + 6M}{4(2\pi+\chi)\rho_o D^4}.$$

The sign of the EoS parameter is being controlled by the signs of the numerator and denominator. The EoS parameter becomes positive, if $\phi_1 > \phi_2$ along with $\phi_3 > 1$ or $\phi_1 < \phi_2$ with $\phi_3 < 1$. However, if during evolution, the stellar system satisfies the constraints $\phi_1 > \phi_2$, $\phi_3 < 1$ or $\phi_1 < \phi_2$, $\phi_3 > 1$, then the ω will enter into a negative phase. For instance, the choice $\omega = -1$ incorporates the DE effects of the cosmological constant Λ. This scenario could be helpful in understanding the theoretical modeling of gravastars.

VI. CONCLUSION

In this work, we have investigated the role of the electromagnetic field on an isotropic stellar model with extra degrees of freedom coming from $f(R, T)$ gravity. The gravastar is the short form of gravitationally vacuum stars, which takes up a new idea in the gravitational system. Such a kind of stellar model can be considered as an alternative to black holes. The gravastar can be described through three different regions: first is the interior region with radius r, second is the intermediate thin shell with thickness ϵ, and third is the exterior region with radius $r+$. The evolution of fluid is dealt with by a specific EoS. We have worked out a set of singularity-free solutions of the gravastar that represents different features of the isotropic relativistic system. Some of the discussed properties of our systems are described below.

(1) *Pressure-density profile.*—The relationship between pressure and density of the ultrarelativistic fluid within the intermediate thin shell is shown in Fig. 1 against the radial coordinate r. We can see the effect of electromagnetic charge on pressure and density.
(2) *Proper length of the thin shell.*—Figure 4 is plotted between the proper length of the shell and the thickness of the shell. We can conclude from the graph that if charge within the gravastar is increasing then the length of thin shell is decreasing, and if charge within the gravastar is decreasing, then length of thin shell is increasing. The electromagnetic field and thickness of the shell are having an inverse relation.
(3) *Energy content.*—Energy within the shell and thickness of the shell are directly proportional to each other. It can be seen from Fig. 5 that the increase of the charge energy would directly increase the thickness of the shell. Furthermore, the thickness can be enhanced by including a huge amount of charge in gravastars.
(4) *Entropy.*—To see the role of entropy, thickness, and electric charge, we have drawn a graph mentioned in Fig. 6. This graph shows the linear relationship between the entropy and thickness of the shell. By studying the effect of electromagnetic charge, we can conclude that if the charge in the gravastar is increasing then the shell's entropy is decreasing and vice versa.
(5) *Equation of state.*—We use some approximation on binomial expressions to require a real solution of $\omega(D)$. The constraints depend upon the electric charge, $f(R, T)$ corrections, mass, and radius of the metric.

We have an overall observation regarding the contribution of $f(R, T)$ gravity is that unlike GR the involvement of extra degrees of freedom coming from χ has made our analysis quite different in both mathematical and graphical point of view. Assigning a zero value to this coupling constant would eventually provide the limiting case of GR.

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