Multicharged black lens

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We construct an asymptotically flat, stationary, and biaxisymmetric supersymmetric black lens solution in five-dimensional $U(1)^3$ supergravity. It is shown that the spatial cross section of the horizon is topologically the lens space of $L(n,1)$, and the spacetime is regular on/outside the event horizon. The black lens carries $(3n+2)$ physical quantities, three electric charges, two angular momenta, and $3(n-1)$ magnetic fluxes.

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I. INTRODUCTION

It was shown in Refs. [1–4] that for an asymptotically flat, stationary, and biaxisymmetric five-dimensional black hole spacetime, the spatial cross section of each connected component of the event horizon must be topologically either a sphere S^3, a ring $S^1 \times S^2$, or lens spaces $L(p,q)$ (p,q: coprime integers) under the dominant energy condition. Concerned with the first two cases, one has already known the exact solutions in five-dimensional Einstein theory [5–7]. Nevertheless, as for the third case, a few authors [8–10] attempted to construct an exact solution to the five-dimensional vacuum Einstein equation by using the inverse scattering method, but it turned out that the resultant solution always suffered from naked singularities.

The supersymmetric solutions provide us various useful information on the corresponding vacuum solutions. In particular, as for asymptotically flat supersymmetric black objects in the five-dimensional minimal supergravity (the constructions of these solutions are based on the scheme developed by Gauntlett *et al.* [11]), the properties have been so far studied by many authors. For instance, the possible topologies of the supersymmetric black holes must be either a sphere S^3, a ring $S^1 \times S^2$, a torus T^3, or quotient thereof [12]. For the spherical topology, Breckenridge *et al.* first found a supersymmetric black hole with equal angular momenta [13]. The supersymmetric black ring with the topology $S^1 \times S^2$ was found in [14]. Furthermore, recently, the asymptotically flat supersymmetric black hole solution with the special horizon topology of $L(2,1) = S^3/\mathbb{Z}_2$ has been constructed by Kunduri and Lucietti [15] (see [16] for the extension to $\mathrm{U}(1)^3$ supergravity). Subsequently, the black lens solution with more general lens space topologies $L(n,1) = S^3/\mathbb{Z}_n (n \geq 3)$ was constructed in [17] and it was also immediately generalized to the solution with multiple horizons in [18]. Furthermore, it was extended to de Sitter space in [19].

In this paper, we generalize the asymptotically flat supersymmetric black lens solution [17] with the horizon topology of $L(n,1)$ in the five-dimensional minimal supergravity to the supersymmetric solution in the five-dimensional $U(1)^3$ supergravity, whereas this is also the extension of the black lens solution [16] with the horizon topology $L(2,1)$ to the more general lens spaces $L(n,1)$ ($n \geq 3$). Our strategy to construct such a solution is to consider the Gibbons-Hawking space as a hyper-Kähler base space and allow the harmonic functions to have n point sources with appropriate coefficients. By imposing suitable boundary conditions on the solution, we consider the configuration of the point sources in which the cross section of the horizon becomes $L(n,1)$ and the spatial infinity becomes S^3. The black lens spacetime possesses a nontrivial domain of outer communication by the existence of $(n-1)$ nontrivial 2-cycles supported by magnetic fluxes outside the horizon. One of these 2-cycles has disk topology D^2, whereas the others have the topology of S^2.

We organize the remaining sections of this paper as follows. In Sec. II, we present the supersymmetric solution of black lenses admitting stationarity and biaxisymmetry in the five-dimensional $U(1)^3$ supergravity. In Sec. III, we impose the boundary conditions on the solution in order that the spacetime is asymptotically flat, has no curvature/conical singularities in the domain of outer communication and no orbifold/Dirac-Misner string singularities on the axis. In Sec. IV, we compute the conserved charges and discuss the physical aspects of such a black lens. In Sec. V,

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we devote ourselves to the summary and discussion on our results.

II. BLACK LENS SOLUTION

In the bosonic sector of the five-dimensional $U(1)^3$ supergravity, the supersymmetric solution can be written as a timelike fiber bundle over a hyper Kähler space [20]. In this work, we choose such a space as the Gibbons-Hawking space, whose metric can be written in spherical polar coordinates (r,θ,ϕ) on $\mathbb{E}^3$ as

$$ds_M^2 = H^{-1}(d\psi+\chi)^2 + H ds_3^2, \qquad d\chi = *dH, \tag{1}$$

$$ds_3^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{2}$$

where H is a harmonic function on $\mathbb{E}^3$ and $*$ denotes the Hodge duarity on $\mathbb{E}^3$. When we assume that $\partial/\partial\psi$ is also a Killing vector for the five-dimensional metric, the metric, gauge potentials A^I of Maxwell fields and scalar fields $X^I (I = 1,2,3)$ with the constraint $X^1X^2X^3 = 1$ for the supersymmetric solutions are expressed as

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}ds_M^2, \tag{3}$$

$$A^I = \frac{1}{3}H_I^{-1}(dt+\omega) + \frac{1}{2}[K^I H^{-1}(d\psi+\chi)+\xi^I], \tag{4}$$

$$X^I = H_I^{-1}(H_1H_2H_3)^{\frac{1}{3}}, \tag{5}$$

where the functions f^{-1}, $H_I (I = 1,2,3)$ and 1-forms ω, ξ^I are given by

$$f^{-1} = 3(H_1H_2H_3)^{\frac{1}{3}}, \tag{6}$$

$$H_I = L_I + \frac{1}{24}H^{-1}|\epsilon_{IJK}|K^J K^K, \tag{7}$$

$$\omega = \omega_\psi(d\psi+\chi) + \hat\omega, \tag{8}$$

$$d\xi^I = -*dK^I \tag{9}$$

with

$$\omega_\psi = -\frac{K^1K^2K^3}{8H^2} - \frac{3L_IK^I}{4H} + M, \tag{10}$$

$$*d\hat\omega = HdM - MdH - \frac{3}{4}(K^I dL_I - L_I dK^I). \tag{11}$$

K^I, L_I, and M are the harmonic functions on $\mathbb{E}^3$. We assume that the harmonic functions H, K^I, L_I $(I = 1,2,3)$, and M have n point sources at $\boldsymbol{r} = (0,0,z_i)$ $(z_i < z_j$ for $i < j$, $i,j = 1,\ldots,n)$ on $\mathbb{E}^3$ as

$$H = \frac{h_i}{r_i} \coloneqq \frac{n}{r_1} - \sum_{i=2}^{n}\frac{1}{r_i}, \tag{12}$$

$$K^I = \sum_i \frac{k^I{}_i}{r_i}, \qquad L_I = l_{I0} + \sum_i \frac{l_{Ii}}{r_i},$$
$$M = m_0 + \sum_i \frac{m_i}{r_i}, \tag{13}$$

where $r_i = \sqrt{r^2 - 2rz_i\cos\theta + z_i^2}$. It can be shown from Eqs. (1), (11), and (9) that the 1-forms $(\chi, \hat\omega, \xi^I)$ are explicitly expressed as

$$\chi = \sum_i h_i \frac{r\cos\theta - z_i}{r_i}d\phi, \tag{14}$$

$$\hat\omega = \sum_{i,j}\left(h_i m_j - \frac{3}{4}\sum_I k^I{}_i l_{Ij}\right)\frac{r^2-(z_i+z_j)r\cos\theta+z_iz_j}{z_{ji}r_ir_j}d\phi$$
$$-\sum_i\left(m_0h_i - \frac{3}{4}\sum_I k^I{}_i l_{I0}\right)\frac{r\cos\theta - z_i}{r_i}d\phi + c\,d\phi, \tag{15}$$

$$\xi^I = -\sum_i k^I{}_i \frac{r\cos\theta - z_i}{r_i}d\phi, \tag{16}$$

where c is a constant and $z_{ji} \coloneqq z_j - z_i$.

III. BOUNDARY CONDITIONS

We require the following boundary conditions:

(1) The spacetime is asymptotically flat at infinity $r\to\infty$.

(2) The point, $\boldsymbol{r} = \boldsymbol{r}_1$, at which the harmonic functions H, K^I, L_I, M diverge, corresponds to a smooth degenerate Killing horizon whose spatial topology is the lens space of $L(n,1)$.

(3) The remaining $(n-1)$ points $\boldsymbol{r} = \boldsymbol{r}_i (i = 2,\ldots,n)$ of the harmonic functions behave as regular points. We also demand the regularity conditions that there exist no curvature singularities, no conical singularities, and no Dirac-Misner string singularities.

In addition, we impose the absence of closed timelike curves (CTCs) in the domain of outer communication. In order that the $(8n+5)$ parameters $(k^I{}_i, l_{I0}, l_{Ii}, m_0, m_i, z_i, c)$ satisfy all of these boundary conditions, we will derive the constraint equations in what follows.

A. Infinity

For $r\to\infty$, the function f^{-1} and 1-form ω behave, respectively, as

$$f^{-1} \simeq 3(l_{10}l_{20}l_{30})^{\frac{1}{3}}, \tag{17}$$

$$\omega \simeq \left(m_0 - \frac{3}{4}\sum_{i,I} k^I{}_i l_{I0}\right)(d\psi + \cos\theta d\phi) - \sum_i \left(m_0 h_i - \frac{3}{4}\sum_I k^I{}_i l_{I0}\right)\cos\theta d\phi + \left(\sum_{i,j(i\neq j)} \frac{h_i m_j - \frac{3}{4}\sum_I k^I{}_i l_{Ij}}{z_{ji}} + c\right)d\phi. \quad (18)$$

The asymptotic flatness requires that f^{-1} approaches to 1 and ω vanishes at infinity in the given coordinate system. Therefore, we impose the following conditions on the parameters (l_{I0}, c, m_0):

$$l_{I0} = \frac{1}{3}, \quad (19)$$

$$c = -\sum_{i,j(i\neq j)} \frac{h_i m_j - \frac{3}{4}\sum_I k^I{}_i l_{Ij}}{z_{ji}}, \quad (20)$$

$$m_0 = \frac{3}{4}\sum_{i,I} k^I{}_i l_{I0}. \quad (21)$$

In terms of the five-dimensional radial coordinate $\rho \coloneqq 2\sqrt{r}$, for $\rho \to \infty (r \to \infty)$ the metric is asymptotically approximated as

$$ds^2 \simeq -dt^2 + d\rho^2 + \frac{\rho^2}{4}[(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2], \quad (22)$$

which coincides with the metric of Minkowski spacetime written in terms of the Euler angles. The regularity of the metric at infinity demands the range of the coordinates $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $0 \leq \psi < 4\pi$ and the identification of $\phi \sim \phi + 2\pi$ and $\psi \sim \psi + 4\pi$.

B. Horizon $r = r_1$

To see that the point $\boldsymbol{r} = \boldsymbol{r}_1$ denotes a Killing horizon with the topology of the lens space of $L(n,1) = S^3/\mathbb{Z}_n$, let us introduce new spherical polar coordinates such that $\boldsymbol{r} = \boldsymbol{r}_1$ becomes an origin in $\mathbb{E}^3$ of the Gibbons-Hawking space. Near the origin $r \coloneqq |\boldsymbol{r} - \boldsymbol{r}_1| = 0$, the functions f^{-1} and ω_ψ are approximated as

$$f^{-1} \simeq \frac{P}{4h_1 r} + c'_1, \quad (23)$$

$$\omega_\psi \simeq -\frac{k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1 \sum_I k^I{}_1 l_{I1} - 8h_1^2 m_1}{8h_1^2 r} + c'_2, \quad (24)$$

where we have defined the constants c'_1 and c'_2, respectively, by

$$c'_1 \coloneqq \frac{1}{3P^2}\left[k^1{}_1 k^2{}_1 k^3{}_1 \sum_I k^I{}_1 + 72h_1^2 \sum_{I,J,K} |\epsilon_{IJK}| l_{I1} l_{J1} + 6h_1 \sum_{I,J,K} |\epsilon_{IJK}| k^I{}_1 k^J{}_1 (l_{I1} + l_{J1})\right] + \frac{1}{12P^2 h_1^2}\sum_{i\neq 1}\frac{1}{|z_{i1}|}\left[-3\left(k^1{}_1 k^2{}_1 k^3{}_1\right)^2 + 8h_1 k^1{}_1 k^2{}_1 k^3{}_1 \sum_I k^I{}_1 l_{I1} + 24h_1^2 \sum_{I,J,K}|\epsilon_{IJK}| k^I{}_1 k^J{}_1 l_{I1} l_{J1}\right) h_i + h_1 \sum_I \left(\sum_{J,K} |\epsilon_{IJK}| k^J{}_1 k^K{}_1 + 24h_1 l_{I1}\right) \times \left(k^1{}_1 k^2{}_1 k^3{}_1 + 3h_1 \sum_{J,K} |\epsilon_{IJK}| (k^J{}_1 l_{J1} + k^K{}_1 l_{K1})\right) k^I{}_i + 6h_1^2 \sum_{I,J,K} |\epsilon_{IJK}| (k^I{}_1 k^K{}_1 + 12h_1 l_{J1})(k^I{}_1 k^J{}_1 + 12h_1 l_{K1}) l_{Ii}\Bigg], \quad (25)$$

$$c'_2 \coloneqq m_0 - \frac{1}{4h_1}\sum_I k^I{}_1 + \sum_{i\neq 1}\frac{1}{8h_1^3 |z_{i1}|}\left[\left(2k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1 \sum_I k^{I1} l_{I1}\right) h_i - \frac{h_1}{2}\sum_I\left(\sum_{J,K} |\epsilon_{IJK}| k^J{}_1 k^K{}_1 + 12h_1 l_{I1}\right) k^I{}_i - 6h_1^2 \sum_I k^I{}_1 l_{Ii} + 8h_1^3 m_i\right], \quad (26)$$

with

$$P \coloneqq \{(12h_1 l_{11} + k^2{}_1 k^3{}_1)(12h_1 l_{21} + k^3{}_1 k^1{}_1)(12h_1 l_{31} + k^1{}_1 k^2{}_1)\}^{\frac{1}{3}}. \quad (27)$$

Moreover, to eliminate the apparent divergence of the metric at $r=0$, let us use new coordinates (v,ψ') defined by

$$dv = dt - \left(\frac{A_0}{r^2}+\frac{A_1}{r}\right)dr, \qquad d\psi' = d\psi + \sum_{i\neq 1}\frac{z_{i1}}{|z_{i1}|}d\phi - \frac{B_0}{r}dr \tag{28}$$

where

$$\begin{aligned}4A_0 = \Bigg[&-9(k^1{}_1 l_{11}-k^2{}_1 l_{21})^2 + 18 l_{31}(k^1{}_1 k^3{}_1 l_{11} + k^2{}_1 k^3{}_1 l_{21} + 24 h_1 l_{11} l_{21}) - 9(k^3{}_1 l_{31})^2\\ &+4\left(k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1\sum_I k^I{}_1 l_{I1}\right)m_1 - 16 h_1^2 m_1^2\Bigg]^{\frac{1}{2}},\end{aligned} \tag{29}$$

$$-8A_0B_0 = k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1\sum_I k^I{}_1 l_{I1} - 8h_1^2 m_1, \tag{30}$$

$$\begin{aligned}16A_0A_1 = &\frac{3}{2}\sum_{I,J,K}|\epsilon_{IJK}|(-k^I{}_1 + k^J{}_1 + k^K{}_1)k^I{}_1 l_{I1} + 36h_1\sum_{I,J,K}|\epsilon_{IJK}| l_{J1} l_{K1} + 2\left(k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1\sum_I k^I{}_1 l_{I1}\right)m_0\\ &+4h_1\left(\sum_I k^I{}_1 - 4h_1 m_0\right)m_1 + \sum_{j\geq 2}\frac{1}{|z_{j1}|}\Bigg[4\left(54 l_{11} l_{21} l_{31} + 3\sum_I k^I{}_1 l_{I1} m_1 - 4h_1 m_1^2\right)h_j\\ &+\sum_I\left[\frac{9}{2}\sum_{J,K}(-k^I{}_1 l_{I1} + k^J{}_1 l_{J1} + k^K{}_1 l_{K1}) l_{I1} + \left(\sum_{J,K}|\epsilon_{IJK}| k^K{}_1 k^J{}_1 + 12 h_1 l_{I1}\right)m_1\right]k^I{}_j\\ &+\sum_I\left[\frac{9}{2}\sum_{J,K}(-k^I{}_1 l_{I1} + k^J{}_1 l_{J1} + k^K{}_1 l_{K1}) k^I{}_1 + 12h_1\sum_{J,K}|\epsilon_{IJK}| k^K{}_1 k^J{}_1 + 12 h_1 k^I{}_1 m_1\right] l_{Ij}\\ &+2\left(k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1\sum_I k^I{}_1 l_{I1} - 8h_1^2 m_1\right)m_j\Bigg].\end{aligned} \tag{31}$$

It is shown from $g_{vv}=\mathcal{O}(r)$ that $r=0$ is a null Killing horizon. We can obtain the near-horizon geometry by putting $(v,r)=(v/\epsilon,\epsilon r)$ and taking the limit $\epsilon\to 0$ as

$$\begin{aligned}ds^2_{NH} = &\frac{R_2^2}{4}\left[d\psi' + n\cos\theta d\phi - \frac{16A_0B_0}{R_1^2R_2^2}r dv\right]^2\\ &+R_1^2(d\theta^2+\sin^2\theta d\phi^2) - \frac{4r^2}{R_1^2R_2^2}dv^2 - \frac{4}{R_2}dvdr,\end{aligned} \tag{32}$$

where

$$R_1^2 := \frac{P}{4}, \tag{33}$$

$$R_2^2 := \frac{P^3 - [k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1\sum_I k^I{}_1 l_{I1} - 8h_1^2 m_1]^2}{16h_1^2 R_1^4}. \tag{34}$$

As expected, this is locally isometric to the near-horizon geometry of the Breckenridge-Myers-Peet-Vafa black hole.

The metric of the cross section of the event horizon with $v,r=$ const. surfaces gives

$$ds_H^2 = \frac{R_2^2}{4}[d\psi' + n\cos\theta d\phi]^2 + R_1^2(d\theta^2+\sin^2\theta d\phi^2), \tag{35}$$

which is the metric of the squashed lens space $L(n,1)=S^3/\mathbb{Z}_n$. To remove CTCs near the horizon, we must impose

$$R_1^2>0, \qquad R_2^2>0. \tag{36}$$

Note that it is sufficient to impose the second inequality only, namely,

$$P > \left[k^1{}_1 k^2{}_1 k^3{}_1 + 6h_1\sum\nolimits_I k^I{}_1 l_{I1} - 8h_1^2 m_1\right]^{\frac{2}{3}}. \tag{37}$$

C. $r=r_i$ $(n=2,\ldots,n)$

Next, we introduce new spherical polar coordinates such that $\boldsymbol{r}=\boldsymbol{r}_i(n=2,\ldots,n)$ becomes an origin in $\mathbb{E}_3$ of the Gibbons-Hawking space. Near the points $r=|\boldsymbol{r}-\boldsymbol{r}_i|$

($n = 2, \ldots, n$), the functions f^{-1} and ω_ψ behaves, respectively, as

$$f^{-1} \simeq \frac{\{3(12l_{1i} - k^2{}_i k^3{}_i)(12l_{2i} - k^3{}_i k^1{}_i)(12l_{3i} - k^1{}_i k^2{}_i)\}^{\frac{1}{3}}}{12r} + c_{1(i)}, \tag{38}$$

$$\omega_\psi \simeq \frac{-k^1{}_i k^2{}_i k^3{}_i + 6(k^1{}_i l_{1i} + k^2{}_i l_{2i} + k^3{}_i l_{3i}) + 8m_i}{8r} + c_{2(i)}, \tag{39}$$

where the constants $c_{1(i)}$ and $c_{2(i)}$ $(i = 2, \ldots, n)$ are given by

$$c_{1(i)} \coloneqq -\frac{1}{4} \prod_{I,J(I<J)} \left(-4 + \sum\nolimits_{j(\neq i)} \frac{k^I{}_i k^J{}_i h_j + k^J{}_i k^I{}_j + k^I{}_i k^J{}_j - 12 \sum_K \epsilon_{IJK} l_{Kj}}{|z_{ji}|} \right)^{\frac{1}{3}}, \tag{40}$$

$$c_{2(i)} \coloneqq m_0 + \frac{1}{4} \sum_I k^I{}_i + \sum_{j \neq i} \frac{-k^1{}_i k^2{}_i k^3{}_i h_j - \frac{1}{2} \sum_{I,J,K} |\epsilon_{IJK}| k^I{}_i k^J{}_i k^K{}_j + 12 \sum_I k^I{}_i l_{Ij} + 16 m_j}{16 |z_{ji}|}. \tag{41}$$

The 1-forms χ and $\hat{\omega}$ are approximated, respectively, as

$$\chi \simeq (-\cos\theta + \chi_{(0)}) d\phi, \tag{42}$$

$$\hat{\omega} \simeq (\hat{\omega}_{(1)} \cos\theta + \hat{\omega}_{(0)}) d\phi, \tag{43}$$

where

$$\chi_{(0)} = -\sum_{j(\neq i)} \frac{h_j z_{ji}}{|z_{ji}|}, \tag{44}$$

$$\hat{\omega}_{(0)} \coloneqq \sum_{k,j(k\neq j,k,j\neq i)} \left(h_k m_j - \frac{3}{4} \sum_I k^I{}_k l_{Ij} \right) \frac{z_{ji} z_{ki}}{z_{jk} |z_{ji}| |z_{ki}|} - \sum_{j(\neq i)} \left(m_0 h_j - \frac{3}{4} \sum_I k^I{}_j l_{I0} \right) \frac{-z_{ji}}{|z_{ji}|} + c, \tag{45}$$

$$\hat{\omega}_{(1)} \coloneqq \sum_{j(j\neq i)} \left[h_i m_j - h_j m_i - \frac{3}{4} \sum_I (k^I{}_i l_{Ij} - k^I{}_j l_{Ii}) \right] \frac{-1}{|z_{ji}|} - \left(m_0 h_i - \frac{3}{4} \sum_I k^I{}_i l_{I0} \right). \tag{46}$$

To eliminate the divergence at the points $\boldsymbol{r} = \boldsymbol{r}_i$ ($n = 2, \ldots, n$) of the function f^{-1} and ω_ψ, we need to impose the following conditions on the parameters (l_{Ii}, m_i) $(i = 2, \ldots, n, I = 1, 2, 3)$:

$$l_{1i} = \frac{k^2{}_i k^3{}_i}{12}, \qquad l_{2i} = \frac{k^3{}_i k^1{}_i}{12}, \qquad l_{3i} = \frac{k^1{}_i k^2{}_i}{12}, \tag{47}$$

$$m_i = -\frac{k^1{}_i k^2{}_i k^3{}_i}{16}. \tag{48}$$

This immediately leads to the important constraint

$$\hat{\omega}_{(1)} = c_{2(i)}, \qquad (i = 2, \ldots, n). \tag{49}$$

Using these conditions and new coordinates defined by

$$\rho = \sqrt{-c_{1(i)} r}, \qquad \psi' = \psi + \chi_{(0)} \phi, \qquad \phi' = \phi, \tag{50}$$

we find that near the points $\boldsymbol{r} = \boldsymbol{r}_i (i = 2, \ldots, n)$ the metric is locally isometric to the Minkowski metric

$$ds^2 \simeq -c_{1(i)}^{-2} d(t + c_{2(i)} \psi' + \hat{\omega}_{(0)} \phi')^2 + d\rho^2 + \frac{\rho^2}{4} \{ (d\psi' - \cos\theta d\phi')^2 + d\theta^2 + \sin^2\theta d\phi'^2 \}. \tag{51}$$

However, the existence of $c_{2(i)}$ and $\hat{\omega}_{(0)}$ in the metric yields CTCs around the origin $\rho = 0$ $(\boldsymbol{r} = \boldsymbol{r}_i (i = 2, \ldots, n))$. Therefore, we must require

$$c_{2(i)} = 0, \qquad \hat{\omega}_{(0)} = 0, \tag{52}$$

where it should be noted that the former automatically guarantees the latter. Moreover, from Eq. (50), we must require $c_{1(i)} < 0$ $(i = 2, \ldots, n)$, which can be written as

$$\prod_{I,J(I<J)} \left(-4 + \sum_{j(\neq i)} \frac{k^I{}_i k^J{}_i h_j + k^J{}_i k^I{}_j + k^I{}_i k^J{}_j - 12 \sum_K \epsilon_{IJK} l_{Kj}}{|z_{ji}|} \right) > 0. \tag{53}$$

D. Axis

On $\mathbb{E}^3$ in the Gibbons-Hawking space [Eq. (2)], let us introduce the Cartesian coordinates (x, y, z), which are defined by $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.

First, we show that there exist no Dirac-Misner string singularities on the z axis of $\mathbb{E}^3$ (i.e., $x = y = 0$) in the Gibbons-Hawking space. To do so, it is sufficient to show $\hat{\omega} = 0$ on the z axis. The z axis is split into the $(n+1)$ intervals as $I_- = \{(x, y, z)|x = y = 0, z < z_1\}$, $I_i = \{(x, y, z)|x = y = 0, z_i < z < z_{i+1}\}$ $(i = 1, \ldots, n-1)$, and $I_+ = \{(x, y, z)|x = y = 0, z > z_n\}$. On the z axis, the 1-form $\hat{\omega}$ can be written in the form

$$\hat{\omega} = \sum_{k,j(k\neq j)}\left(h_k m_j - \frac{3}{4}\sum_I k^I{}_k l_{Ij}\right)\frac{(z - z_k)(z - z_j)}{z_{jk}|z - z_k||z - z_j|}d\phi - \sum_j\left(m_0 h_j - \frac{3}{4}\sum_I k^I{}_j l_{I0}\right)\frac{z - z_j}{|z - z_j|}d\phi - \sum_{k,j(k\neq j)}\left(h_k m_j - \frac{3}{4}\sum_I k^I{}_k l_{Ij}\right)\frac{d\phi}{z_{jk}}, \tag{54}$$

where we have used Eq. (20). On $I_\pm$, $\hat{\omega}$ vanishes since

$$\hat{\omega} = \mp\sum_j\left(m_0 h_j - \frac{3}{4}\sum_I k^I{}_j l_{I0}\right)d\phi = 0, \tag{55}$$

where we have used Eq. (21). For $z \in I_i$ $(i = 1, 2, \ldots, n-1)$, we find

$$\hat{\omega}_\phi|_{I_i} = \hat{\omega}_\phi(x = y = 0, z_i < z < z_{i+1}) \tag{56}$$

$$= \hat{\omega}_\phi(x = y = 0, z = z_i + 0) \tag{57}$$

$$= \hat{\omega}_\phi(r \coloneqq |\boldsymbol{r} - \boldsymbol{r}_i| = 0, \theta = 0) \tag{58}$$

$$= \hat{\omega}_{(1)} + \hat{\omega}_{(0)} \tag{59}$$

$$= 0 \tag{60}$$

where we have used the fact that $\hat{\omega}$ is constant on I_i in the second equality and Eq. (43) in the fourth equality, respectively. Furthermore, we have used Eq. (49) and Eq. (52) in the last equality. Thus, it has been shown that $\hat{\omega} = 0$ holds at each interval, which proves that no Dirac-Misner string pathologies exist throughout the spacetime.

Next, we show that there exist no orbifold singularities on the z axis. On $I_\pm$, χ can be written as

$$\chi = \pm d\phi, \tag{61}$$

whereas on I_i we have

$$\chi = \left(n\frac{z - z_1}{|z - z_1|} - \sum_{2\leq j\leq i}\frac{z - z_j}{|z - z_j|} - \sum_{i+1\leq j\leq n-1}\frac{z - z_j}{|z - z_j|}\right)d\phi = (2n - 2i + 1)d\phi. \tag{62}$$

Let us use the coordinate basis vectors $(\partial_{\phi_1}, \partial_{\phi_2})$ of 2π periodicity, which are defined by $\phi_1 \coloneqq (\psi + \phi)/2$ and $\phi_2 \coloneqq (\psi - \phi)/2$. We can show that the Killing vector $v \coloneqq \partial_\phi - \chi_\phi\partial_\psi$ vanishes on each interval. Therefore, the rod structure is given by

(i) on I_+, the Killing vector $v_+ \coloneqq \partial_\phi - \partial_\psi = (0, -1)$ vanishes,

(ii) on each I_i $(i = 1, \ldots, n-1)$, the Killing vector $v_i \coloneqq \partial_\phi - (2n - 2i + 1)\partial_\psi = (i - n, i - n - 1)$ vanishes,

(iii) on I_-, the Killing vector $v_- \coloneqq \partial_\phi + \partial_\psi = (1, 0)$ vanishes.

From these, we can observe that the Killing vectors $v_\pm$, v_i on the intervals satisfy

$$\det(v_+^T, v_{n-1}^T) = -1, \qquad \det(v_i^T, v_{i-1}^T) = -1, \tag{63}$$

with

$$\det(v_1^T, v_-^T) = n. \tag{64}$$

Equation (63) means that there exist no orbifold singularities at adjacent intervals $z = z_i$ $(1 \leq i \leq n)$, and Eq. (64) shows that the spatial topology of the horizon is the lens space $L(n, 1) = S^3/\mathbb{Z}_n$.

IV. PHYSICAL PROPERTIES

Let us count the number of physical degree of freedom. First, note that there exists a gauge freedom of redefining harmonic functions

$$K^I \to K^I - a^I H,$$
$$L_I \to L_I - \frac{1}{12}|\epsilon_{IJK}|a^J K^K - \frac{1}{24}|\epsilon_{IJK}|a^J a^K H,$$
$$M \to M - \frac{3}{4}a^I L_I + \frac{1}{96}|\epsilon_{IJK}|(a^I a^J a^K H - 3a^I a^J K^K), \tag{65}$$

where a^I are three arbitrary constants. Under these transformations, $f, H_I, X^I, \omega, \chi$ remain invariant. The 1-forms ξ^I transforms as $\xi \to \xi + a^I\chi$, which means merely the gauge transformation since $A^I \to A^I - a^I d\psi$. Therefore, the transformations (65) makes the solution invariant. This gauge invariance enables us to eliminate one term which appears in K^I and to simplify the form of the solution. We can use this gauge invariance to put, for instance,

$$k^I{}_1 = 0. \tag{66}$$

The regularity of the metric at each boundary has required the boundary conditions (19)–(21), (47)–(48), (52). From the gauge freedom of $z \to z + z_0$ (z_0: constant) and (66), these conditions reduce the number of the independent parameters that appear in the solution from $(8n+5)$ to $(3n+1)$, Furthermore, the absence of CTCs demands that these should satisfy the inequalities (37) and (53).

We compute the conserved charges of the black lens. The Arnowitt-Deser-Misner (ADM) mass, the electric charges, and two ADM angular momenta are written as

$$M = \sum_I Q_I = \sum_I \frac{3\pi}{G}\left(\sum_i l_{Ii} + \frac{1}{24}\sum_{\substack{J,K\\ i,j}} |\epsilon_{IJK}| k^J{}_i k^K{}_j\right), \tag{67}$$

$$J_\phi = \frac{\pi}{2G}\sum_{\substack{I\\ i,j}} (h_i k^I{}_j - h_j k^I{}_i) z_i, \tag{68}$$

$$J_\psi = \frac{\pi}{G}\left[\frac{1}{24}\sum_{\substack{I,J,K\\ i,j,k}} |\epsilon_{IJK}| k^I{}_i k^J{}_j k^K{}_k + \frac{3}{2}\sum_{\substack{I\\ i,j}} k^I{}_i l_{Ij} - 2\sum_i m_i\right]. \tag{69}$$

As expected, the mass saturates the Bogomolnyi-Prasad-Sommerfield (BPS) bound $M \geq \sum_I Q_I$. The magnetic fluxes through $I_i (i = 1, \ldots, n-1)$ are defined as $q[I_i] \coloneqq \frac{1}{4\pi}\int_{I_i} F$, which are computed as

$$q^I[I_1] = \frac{k^1{}_1 k^2{}_1 k^3{}_1 + 6nk^I{}_1 l_{I1} - 8n^2 m_1}{2n(24nl_{I1} + k^I{}_1 k^K{}_1 |\epsilon_{IJK}|)} + \frac{1}{2}\left(\frac{k^I{}_1}{n} + k^I{}_2\right), \tag{70}$$

$$q^I[I_i] = \frac{1}{2}(k^I{}_i - k^I{}_{i+1}). \tag{71}$$

V. SUMMARY

In this paper, we have generalized the asymptotically flat supersymmetric black lens solution with the horizon topology of $L(n,1)$ in the five-dimensional minimal supergravity [17] to the five-dimensional $U(1)^3$ supergravity, which also corresponds to the extension of the black lens solution with the lens space $L(2,1)$ to the black lens solution with the more genera lens spaces $L(n,1)$ $(n \geq 3)$. We have shown that the black lens with the horizon topology of $L(n,1)$ includes $3n+2$ parameters, which must obey n inequalities. We have computed $(3n+1)$ physical charges, three electric charges $Q_I (I = 1,2,3)$ (the sum is equal to the mass, and hence it follows that the Bogomolny bound is saturated), two angular momenta J_ϕ, J_ψ, and $(n-1)$ magnetic fluxes $q^I{}_i (i = 1, \ldots, n-1)$, which are subject to a constraint.

In this work, we have imposed the boundary conditions such that the point source $\boldsymbol{r} = \boldsymbol{r}_1$ in the harmonic functions corresponds to the Killing horizon and the other $\boldsymbol{r} = \boldsymbol{r}_i$ $(i = 2, \ldots, n)$ are simply regular points like an origin of Minkowski spacetime. It may be also interesting to consider the different boundary conditions, such that $\boldsymbol{r} = \boldsymbol{r}_i (2 \leq i \leq [\frac{n+1}{2}])$ is a Killing horizon and the other are regular points, since such a solution is expected to have physically different properties (see Ref. [15] for the corresponding examples in five-dimensional minimal supergravity). Furthermore, it may be straightforward to generalize the present solution to five-dimensional $U(1)^N$ supergravity. These issues deserve further study.

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