Minimal length effect on the broadening of free wave packets and its physical implications

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We study the generalized uncertainty principle (GUP) modified time evolution for the width of wave packets for a scalar potential. The free particle case is solved exactly where the wave-packet broadening is modified by a coupling between the GUP parameter and higher-order moments in the probability distribution in momentum space. We consider two popular forms of deformations widely used in the literature—one of which modifies the commutator with a quadratic term in momentum, while the other modifies it with terms both linear and quadratic in momentum. Unlike the standard case, satisfying Heisenberg uncertainty, here the GUP modified broadening rates, for both deformations, not only depend on the initial size (in both position and momentum space) of the wave packet but also on the initial probability distribution and momentum of the particle. The new rates of wave-packet broadening, for both situations, are modified by a handful of new terms—such as the *skewness* and *kurtosis* coefficients, as well as the (constant) momentum of the particle. Comparisons with the standard Heisenberg uncertainty principle–based results show potentially measurable differences in the rates of free wave-packet broadening for physical systems such as the C_{60} and C_{176} molecules, and more so for large organic molecular wave packets. In doing so, we open a path to scan the GUP parameter space by several orders of magnitude *inside* the best existing upper bounds for both forms of GUP.

I. INTRODUCTION

One of the key features of quantum mechanics (QM) is the fact that it sets, by means of the Heisenberg uncertainty principle (HUP), a fundamental limit on the precise and simultaneous knowledge of two canonically conjugate dynamical variables for any quantum system. This, along with other fundamental principles, when put together, ensures the dispersion of free wave packets through space in a manner that the width of the packet tends to always increase over time [1]. These insights are important to understand classical-quantum correspondence in general. For example, one can easily compute that the wave packet corresponding to a free electron will disperse in space very rapidly, and therefore the likelihood of pointing down a free electron to be present at a specific point in space is negligible. For a classical particle the wave packet does not have a detectable dispersion in space over the age of the universe.

The above features are, of course, very well in agreement with our experiences which, nonetheless, are also verified in certain cases. On the other hand, there is a growing consensus that, inspired by certain quantum gravity theories, is advocating about the existence of a fundamental

minimal length scale (at the Planck length). Among them, studies in string theory [2–8], doubly special relativity [9–12], black hole physics [13–15], loop quantum gravity (LQG) [16,17], noncommutative *quantum* geometries [18–20], and more general approaches concerning QM and general relativity [21–30] manifest this existence of a minimal length by replacing the HUP by a generalized uncertainty principle (GUP) whose exact form, however, often disagrees among various proposals (for a broad overview see [7,13,31–35] and references therein). The GUP-based approaches have a motivation to provide a shorthand exercise in the search for quantum gravity effects, hypothesized to be realized in the form of a minimal length, in low-energy physics, and, if it is indeed found, then ask for an appropriate fundamental theory, from first principles, to explain this effective description of physical reality [36] (which may well be one of the existing theories—LQG or string theory or an entirely new theory).

One main focus of GUP-based studies is to calculate the modified spectrum of different observables that can be useful to test the validity of the theory, and in case no measurable differences are found it may still give bounds on the GUP parameters. Some of the studies in this line are reported in several works [37], and, in fact, a number of new experiments have been proposed [38] to measure these GUP contributions.

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We, on the other hand, are opening a new avenue in this quest for understanding the fundamental insights that are brought in by the GUP modification (or the minimal length scale) on the *wave function itself* and thereby giving some new information on a distributional level. To do this we consider the wave packet corresponding to a free particle which can give an account of the *bare effect of the minimal length scale* on the otherwise very well understood situation. There exist preliminary works on the GUP effect on free particle wave packets [39], but they were not developed enough to highlight the theoretical and experimental impact discussed here. Particularly, in this article, we present a detailed account of the basic setting of wave-packet evolution both within the standard HUP setting (which is well known) and within the GUP framework (which is a new study). The GUP modification will be shown to imply a nontrivial distributional ramification on the rate and fundamental properties of the broadening of the free wave packets. We shall also compare two situations, i.e., GUP vs HUP explicitly to clarify various outcomes, both mathematically and physically. There will be a considerable effort to estimate the time difference between the GUP and HUP broadening of the free wave packets and the likelihood of experimentally detecting this departure. Interestingly, while doing so, we can also put some bounds on the GUP parameter, and that will be an improvement of several orders of magnitude to the current best upper bounds that come from studying the spectrum of a number of observables. All these will be done using the two most popular forms of the GUP given by the Ali-Das-Vagenas (ADV) form and the Kempf-Mann-Mangano (KMM) form.

This paper is organized in the following manner: in Sec. II we provide a review of the basic setup to derive the evolution law for the width of the free wave packet. In Sec. III we shall take the first step to include the GUP effect by generalizing Ehrenfest's equations. Section IV is used for the derivation of the governing equation for the spreading of free wave packets in the GUP scenario. Here we consider two popular forms of GUP, given by the ADV and KMM forms. The Sec. V is dedicated to solving these equations, exactly, for the free particle case. Section VI physically explains the new results. Moving on, in Sec. VII we shall elaborate on the possibility of testing our results within the present technology, for both forms of GUP. Finally, in Sec. VIII we conclude.

II. THE MOTION AND SPREADING OF WAVE PACKETS

In this section we review the standard picture of wave-packet broadening in quantum mechanics. This is a standard textbook exercise (see, e.g, [1]); however, it is important to review it here for the sake of clarity and completeness of the paper.

A. Ehrenfest's theorem: A classical analogy

In quantum mechanics, the fundamental principle that sets a limit in the precision to which one can simultaneously measure two given physical quantities is the HUP

$$[q_i, p_j] = i\hbar\delta_{ij}, \qquad i, j = 1, 2, ..., N, \qquad (1)$$

where N is the number of spatial dimensions under consideration. This is equivalent to the uncertainty relationship between the position and momentum of a particle satisfying

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \qquad (2)$$

We also have the identity applied to the time derivative of the average (expectation) value of the observable

$$i\hbar \frac{d}{dt}\langle A \rangle = \langle [A, H] \rangle + i\hbar \left\langle \frac{\partial A}{\partial t} \right\rangle, \qquad (3)$$

where the observable A is understood as a self-adjoint operator and $H = H(q_1, ..., q_N; p_1, ..., p_N)$ is the system's Hamiltonian. Using this identity on coordinates of position and momentum, we obtain *Ehrenfest's equations*

$$\frac{d}{dt}\langle q_i \rangle = \frac{1}{i\hbar}\langle [q_i, H] \rangle = \left\langle \frac{\partial H}{\partial p_i} \right\rangle, \quad i = 1, 2, ..., N, \qquad (4)$$

and

$$\frac{d}{dt}\langle p_j \rangle = \frac{1}{i\hbar}\langle [p_j, H] \rangle = -\left\langle \frac{\partial H}{\partial q_j} \right\rangle, \quad j = 1, 2, ..., N, \qquad (5)$$

which are deduced from Ehrenfest's theorem. Notice that these equations are formally identical to Hamilton's equations in classical mechanics, although this formal analogy can be rigorously made only when the conditions

$$\left\langle \frac{\partial}{\partial p_i} H(q_1, ..., q_N; p_1, ..., p_N) \right\rangle$$
$$= \frac{\partial}{\partial p_i} H(\langle q_1 \rangle, ..., \langle q_N \rangle; \langle p_1 \rangle, ..., \langle p_N \rangle)$$

and

$$\left\langle \frac{\partial}{\partial q_j} H(q_1, ..., q_N; p_1, ..., p_N) \right\rangle$$
$$= \frac{\partial}{\partial q_j} H(\langle q_1 \rangle, ..., \langle q_N \rangle; \langle p_1 \rangle, ..., \langle p_N \rangle)$$

are fulfilled. The above equations need not hold for an arbitrary potential; however, both of them hold up perfectly

to the quadratic potential that then includes the cases such as the free particle and the harmonic oscillator.

Now, let us consider a one-dimensional wave packet $\Psi(q,t)$ with Hamiltonian

$$H = \frac{p^2}{2m} + V(q). \tag{6}$$

In order to study the time evolution of the expectation values $\langle q \rangle$ and $\langle p \rangle$ let us first define their mean-square deviations,

$$\xi = (\Delta q)^2 = \langle q^2 \rangle - \langle q \rangle^2, \qquad \eta = (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2. \tag{7}$$

Note that in the classical approximation $\Psi(q,t)$ represents a particle with position, momentum, and energy given by

$$q_{cl} = \langle q \rangle, \qquad p_{cl} = \langle p \rangle, \quad \text{and} \quad E_{cl} = \frac{\langle p \rangle^2}{2m} + V(\langle q \rangle). \tag{8}$$

Now, let us define the quantity that tracks the difference

$$\varepsilon = \langle H \rangle - E_{cl} = \frac{1}{2m}\eta + \langle V \rangle - V_{cl}, \tag{9}$$

where $V_{cl} = V(\langle q \rangle)$.

For the classical approximation to hold, we require the extension Δq of the wave packet to remain small as compared to the characteristic distances of the problem under consideration, so that we can make the following Taylor expansions around $\langle q \rangle$:

$$V(q) = V_{cl} + (q - \langle q \rangle)V'_{cl} + \frac{1}{2}(q - \langle q \rangle)^2 V''_{cl} + \cdots,$$

$$V'(q) = V'_{cl} + (q - \langle q \rangle)V''_{cl} + \frac{1}{2}(q - \langle q \rangle)^2 V'''_{cl} + \cdots, \tag{10}$$

where $V'_{cl} = \frac{dV}{dq}\big|_{q=\langle q \rangle}$. Using this expansion will guarantee the results are entirely general, i.e., valid for any V. Taking the expectation values of (10), we obtain

$$\langle V \rangle = V_{cl} + \frac{1}{2}\xi V''_{cl} + \cdots,$$

$$\langle V' \rangle = V'_{cl} + \frac{1}{2}\xi V'''_{cl} + \cdots. \tag{11}$$

By (4), (5), and (6), we have

$$\frac{d}{dt}\langle q \rangle = \frac{\langle p \rangle}{m}, \qquad \frac{d}{dt}\langle p \rangle = -\langle V' \rangle. \tag{12}$$

Notice that, if we use $\langle V' \rangle = V'_{cl}$ [(11) up to first order], then Eqs. (12) reduce to "classical" equations of motion

for the mean values $\langle q \rangle$ and $\langle p \rangle$. This result holds if $V(q)$ varies slowly over a distance $\sim\sqrt{\xi}$, so that the effect of V''' and higher derivatives in (11) is negligible. This condition holds trivially for the cases $V(q) = cq^2$ (harmonic oscillator) and $V(q) = 0$ (free particle), and for every $V(q)$ of at most order 2 in q. Assuming these conditions hold [i.e., series (11) are rapidly converging], we have [see (9)]

$$\varepsilon \simeq \frac{1}{2m}(\eta + mV''_{cl}\xi) = \text{const.} \tag{13}$$

B. Deriving the master equation

We have described the motion of wave packets, by means of $\langle q \rangle$ and $\langle p \rangle$; now, in order to study the spreading of wave packets over time, we want to obtain functions $\xi(t)$ and $\eta(t)$ (i.e., *spread* in configuration and momentum space) explicitly. Notice that $\xi = \langle u \rangle$, where $u = q^2 - \langle q \rangle^2$ and $\langle q \rangle = f(t)$, so applying identity (3) to this operator yields

$$\frac{d}{dt}\xi = \frac{1}{m}(\langle pq + qp \rangle - 2\langle p \rangle\langle q \rangle). \tag{14}$$

Analogously, for the operator $d\xi/dt$, using again (3) and (12) we obtain

$$\frac{d^2\xi}{dt^2} = \frac{2\eta}{m^2} - \frac{1}{m}(\langle V'q + qV' \rangle - 2\langle q \rangle\langle V' \rangle). \tag{15}$$

By using (10) in (15), we get the approximate equation

$$\frac{d^2\xi}{dt^2} \simeq \frac{2}{m^2}(\eta - mV''_{cl}\xi), \tag{16}$$

and finally, taking (13) into account, we can rewrite it as

$$\frac{d^2\xi}{dt^2} \approx \frac{4}{m}(\varepsilon - V''_{cl}\xi), \tag{17}$$

which we refer to here as the *master equation*. Upon solving it, and knowing the deviations ξ_0, η_0, and $\dot{\xi}_0 \equiv d\xi_0/dt$ at $t = t_0$, we obtain $\xi(t)$, the spread of the wave function over time in configuration space; $\eta(t)$ can then be found with (13), using the fact that ε is constant.

Two interesting cases arise: the free particle and harmonic oscillator potential, in which the motion of the center of the packet is rigorously identical to that of a classical particle [1]. In the case of the free particle, $V = 0$, and thus from (13) we have $\eta = 2m\varepsilon = \eta_0$; that is, $\eta = (\Delta p)^2$ remains constant. However, we have rigorously $d^2\xi/dt^2 = 2\eta_0/m^2$ and thus

$$\xi(t) = \xi_0 + \dot{\xi}_0 t + \frac{\eta_0}{m^2}t^2. \tag{18}$$

This result tells us that the free wave packet spreads indefinitely, as is well known, *so this sets a limit for the time interval during which the classical-particle analogy holds*. If we have $\dot{\xi}_0 = 0$ (e.g., the packet has the *minimum width at t_0*, so that $\xi_0 \eta_0 = \frac{1}{2}\hbar^2$), then (18) is simplified to $\xi = \xi_0 + \eta_0 t^2/m^2$ or, equivalently,

$$\Delta q(t) = \sqrt{\xi(t)} = \left[(\Delta q_0)^2 + \left(\frac{\Delta p_0 t}{m}\right)^2\right]^{1/2}, \quad (19)$$

where Δq_0 and Δp_0 are the initial uncertainties in position and momentum space corresponding to the minimum wave packet. This is a truly remarkable equation and fundamental to our physical understanding of quantum theory, which explains why we cannot see an electron as a localized object and why classical objects are seem to be localized forever. Take, for instance, the case of free electron—the second term in (19) increases with time as t^2 and matches the initial width in time $t = \frac{2\pi(\Delta q_0)^2}{c\lambda_e}$ (by using the minimum wave-packet uncertainty relation $\Delta q_0 \Delta p_0 = \hbar/2$ and the definition for the Compton wavelength for the electron). Using $\lambda_e = 2.4 \times 10^{-12}$ m and initial width $\Delta q_0 \simeq 10^{-10}$ m we get the time it takes for the second term in (19) to equate the first term is $t \sim 10^{-16}$ s. This is why it is hard to detect the electron as a localized object confined to a small space—the wave packet gets quickly delocalized. On the other hand, for most of the classical objects this time is more than the age of the universe.

III. THE GENERALIZED EHRENFEST EQUATIONS

The general form of the GUP commutator we are considering here is given by [40,41]

$$[q_i, p_j] = i\hbar \left\{\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \beta^2(p^2\delta_{ij} + 3p_i p_j)\right\}. \quad (20)$$

This relationship is valid for the nonrelativistic quantum mechanical context where the coordinates are usually identified as the Cartesian coordinates (which is true even with a standard commutator without GUP modifications [1]). This leads to the generalized Ehrenfest equations

$$\frac{d}{dt}\langle q_i\rangle = \sum_{j=1}^{N}\left\{\delta_{ij}\left\langle\frac{\partial H}{\partial p_j}\right\rangle - \alpha\left(\delta_{ij}\left\langle p\frac{\partial H}{\partial p_j}\right\rangle + \left\langle\frac{p_i p_j}{p}\frac{\partial H}{\partial p_j}\right\rangle\right)\right.$$
$$\left. + \beta^2\left(\delta_{ij}\left\langle p^2\frac{\partial H}{\partial p_j}\right\rangle + 3\left\langle p_i p_j\frac{\partial H}{\partial p_j}\right\rangle\right)\right\}, \quad (21)$$

$$\frac{d}{dt}\langle p_i\rangle = \sum_{j=1}^{N}\left\{-\delta_{ij}\left\langle\frac{\partial H}{\partial q_j}\right\rangle + \alpha\left(\delta_{ij}\left\langle p\frac{\partial H}{\partial q_j}\right\rangle + \left\langle\frac{p_i p_j}{p}\frac{\partial H}{\partial q_j}\right\rangle\right)\right.$$
$$\left. - \beta^2\left(\delta_{ij}\left\langle p^2\frac{\partial H}{\partial q_j}\right\rangle + 3\left\langle p_i p_j\frac{\partial H}{\partial q_j}\right\rangle\right)\right\}. \quad (22)$$

Putting these equations into the forms

$$\frac{d}{dt}\langle q_i\rangle = \sum_{j=1}^{N}\left\{\delta_{ij}\left(\left\langle\frac{\partial H}{\partial p_j}\right\rangle - \alpha\left\langle p\frac{\partial H}{\partial p_j}\right\rangle + \beta^2\left\langle p^2\frac{\partial H}{\partial p_j}\right\rangle\right)\right.$$
$$\left. - \alpha\left\langle\frac{p_i p_j}{p}\frac{\partial H}{\partial p_j}\right\rangle + 3\beta^2\left\langle p_i p_j\frac{\partial H}{\partial p_j}\right\rangle\right\} \quad (23)$$

and

$$\frac{d}{dt}\langle p_i\rangle = \sum_{j=1}^{N}\left\{-\delta_{ij}\left(\left\langle\frac{\partial H}{\partial q_j}\right\rangle - \alpha\left\langle p\frac{\partial H}{\partial q_j}\right\rangle + \beta^2\left\langle p^2\frac{\partial H}{\partial q_j}\right\rangle\right)\right.$$
$$\left. + \alpha\left\langle\frac{p_i p_j}{p}\frac{\partial H}{\partial q_j}\right\rangle - 3\beta^2\left\langle p_i p_j\frac{\partial H}{\partial q_j}\right\rangle\right\}, \quad (24)$$

we recognize, in the δij term, the pattern $(1 - \alpha\sqrt{p_i p_i} + \beta^2 p_i p_i)$ that arises in various results of the GUP modified angular momentum algebra [40].

Since we shall be interested in the one-dimensional problem of a free wave-packet expansion, we express the above-mentioned equations for one dimension, starting from the commutator,

$$[q, p]_{\text{GUP}} = i\hbar(1 - 2\alpha p + 4\beta^2 p^2), \quad (25)$$

which leads to the other commutation relations

$$[q, H]_{\text{GUP}} = i\hbar\tilde{\gamma}\frac{p}{m}; \qquad [p, H]_{\text{GUP}} = -i\hbar\tilde{\gamma}V',$$

where $\tilde{\gamma} \equiv 1 - 2\alpha p + 4\beta^2 p^2$. If we consider a free particle, then, of course, $[p, H] = 0$. Using the above results and (3) we find

$$\frac{d}{dt}\langle q\rangle = \frac{1}{m}\langle\tilde{\gamma}p\rangle$$

and

$$\frac{d}{dt}\langle p^n\rangle = \frac{1}{i\hbar}\langle[p^n, H]\rangle = \frac{1}{i\hbar}\left\langle\sum_{j=0}^{n-1}p^j[p, H]p^{n-(j+1)}\right\rangle = 0$$

for the free particle in one dimension. Now we move to the next section to derive the modified master equation with GUP.

IV. ONE-DIMENSIONAL WAVE PACKETS AND GENERALIZED MASTER EQUATION

Using the results of Sec. III and Eq. (3), for a free particle we find

$$\ddot{\xi}_{\text{GUP}} = \frac{2}{m^2} \{ \langle \tilde{\gamma} p^2 \rangle - 2\alpha \langle \tilde{\gamma} p^3 \rangle + 4\beta^2 \langle \tilde{\gamma} p^4 \rangle - \langle \tilde{\gamma} p \rangle^2 \}. \quad (26)$$

As a quick consistency check, let $\alpha = \beta = 0$, i.e., $\tilde{\gamma} = 1$; the result is $\ddot{\xi} = \frac{2}{m^2} \eta_0$, which is just the standard result for a free particle using the HUP. For future convenience we want to rewrite (26) in terms of some new variables, in the form

$$\ddot{\xi}_{\text{GUP}} = \frac{2}{m^2} (\eta_0 - 4\alpha \tilde{C}_1 + 4\alpha^2 \tilde{C}_2 + 8\beta^2 \tilde{C}_3), \quad (27)$$

which is correct up to second order in GUP parameters. This consideration is in line with the very definition of the GUP commutator (20), which is an approximate expression up to quadratic powers. The new variables in (27) are defined as

$$\tilde{C}_1 = \eta(2p_{cl} + \Gamma_1 \eta^{1/2}), \quad (28)$$

$$\tilde{C}_2 = \eta^2(\Gamma_2 - 1) + 4\eta p_{cl}(p_{cl} + \eta^{1/2}\Gamma_1), \quad (29)$$

$$\tilde{C}_3 = \eta(3p_{cl}^2 + 3\eta^{1/2} p_{cl}\Gamma_1 + \eta\Gamma_2), \quad (30)$$

where $\eta = \eta_0$ is the square of the (constant) standard deviation in momentum, which also appeared with HUP.

Notice that the new variables $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ involve *the higher-order moments*, which introduce a novel statistical interpretation to our discussion, regarding the shape of the probability distribution for free wave packets. To understand this meaningfully, we have introduced Pearson's *skewness coefficient* (Γ_1), which represents the third order moment, as

$$\Gamma_1 = \frac{\langle (p - \langle p \rangle)^3 \rangle}{\sigma^3} = \frac{1}{\eta^{3/2}} \langle (p - \langle p \rangle)^3 \rangle. \quad (31)$$

Further, we have also introduced the fourth order moment given by the *kurtosis coefficient* Γ_2 as

$$\Gamma_2 = \frac{\langle (p - \langle p \rangle)^4 \rangle}{\sigma^4} = \frac{1}{\eta^2} \langle (p - \langle p \rangle)^4 \rangle. \quad (32)$$

The term $\sigma \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \eta^{1/2}$ is the *standard deviation* of the momentum distribution. It is important to recall that both Γ_1 and Γ_2 measure the departure of the probability distribution from the *normal distribution*. While Γ_1 measures the *asymmetry* about its mean $\langle p \rangle$, Γ_2 measures its *tailedness*. While the skewness can take either

positive or negative values, kurtosis is positive definite. For a normal (true Gaussian) distribution $\Gamma_1 = 0$ and $\Gamma_2 = 3$. This is a remarkable result since every new correction coming from the GUP has a distributional interpretation and, therefore, can be explained physically. Just for completeness, let us note the expanded form of Γ's, given by

$$\Gamma_1 = \frac{1}{\eta^{3/2}} (\langle p^3 \rangle + 2\langle p \rangle^3 - 3\langle p \rangle \langle p^2 \rangle)$$

and

$$\Gamma_2 = \frac{1}{\eta^2} (\langle p^4 \rangle - 4\langle p \rangle \langle p^3 \rangle + 6\langle p^2 \rangle \langle p \rangle^2 - 3\langle p \rangle^4).$$

V. GUP MODIFIED BROADENING OF WAVE PACKETS: THE FREE PARTICLE

It is straightforward, once again, to write down a solution of (27), under the assumption that the initial wave packet is minimal $[\dot{\xi}(t_0) = 0]$. The resulting equation is

$$\Delta q_{\text{free}}(t) = \left[(\Delta q_0)^2 + \frac{1}{m^2} \{ (\Delta p_0)^2 - 4\alpha \tilde{C}_1 + 4\alpha^2 \tilde{C}_2 + 8\beta^2 \tilde{C}_3 \} t^2 \right]^{1/2}, \quad (33)$$

which depends on $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ (which carry the information on the standard deviation, skewness, and kurtosis of the probability distribution in momentum space), as well as GUP parameters α and β. From here on, we shall branch our discussion in two directions, with two special cases of the GUP (i) with $\alpha = \beta$, which is the ADV form of GUP, and (ii) $\alpha = 0$, which is the KMM form.

A. Ali-Das-Vagenas GUP

We can arrive at the form of GUP prescribed by Ali, Das, and Vagenas in [35,40,41] just by setting $\alpha = \beta$ in the commutator (25), giving us

$$[q, p]_{\text{GUP}} = i\hbar(1 - 2\alpha p + 4\alpha^2 p^2). \quad (34)$$

The solution (33) dictating the spread over time for the free wave packet for this case is given by

$$\Delta q_{\text{free}}(t) = \sqrt{\xi(t)}$$

$$= \sqrt{\Delta q_0^2 + \frac{1}{m^2}(\Delta p_0^2 - 4\alpha C_1 + 4\alpha^2 C_2)t^2}, \quad (35)$$

where the coefficients $C_1 = \tilde{C}_1$ and $C_2 = 3\langle p^4 \rangle - \langle p^2 \rangle^2 - 2p_{cl}\langle p^3 \rangle$. Now, before going on to the analysis of the

GUP-modified spread of free wave packets, we need to find an expression for η as a function of the initial *size* of the wave packet $\xi_0 = (\Delta q_0)^2$. To do this, note that (20) leads to the minimum uncertainty relation, which for the ADV form is

$$\Delta q_0 \Delta p_0 = \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2\right)\Delta p_0^2 \right.$$
$$\left. + 4\alpha^2 p_{cl}^2 - 2\alpha\sqrt{\langle p^2 \rangle}\right]. \quad (36)$$

Using this and the fact that $\langle p^2 \rangle = \eta_0 + p_{cl}^2$, we find that

$$\frac{2}{\hbar}(\Delta q_0 \sqrt{\eta_0}) - [1 + 4\alpha^2(\eta_0 + p_{cl}^2)] + \alpha\left[\frac{\eta_0 + 2p_{cl}^2}{\sqrt{\eta_0 + p_{cl}^2}}\right] = 0. \quad (37)$$

Upon solving this equation for η_0 we find the expression $\eta_0 = \eta_0(\Delta q_0, \alpha, \beta, p_{cl})$ that we were looking for. Notice that, since both Δq_0 and p_{cl} are constant parameters that depend on the particle (or molecule) under consideration, and β is the GUP parameter, solving (37) will yield a numerical value for η_0 that will be different for the different systems that one is considering. We shall take advantage of this in the following section.

B. Kempf-Mann-Mangano GUP

The KMM form of GUP (proposed in [24] and further discussed in [30]) does not include any linear term in the momentum and is given by

$$[q, p]_{\text{GUP}} = i\hbar(1 + \tilde{\beta}^2 p^2), \quad (38)$$

which is identical to (25) with the identification $\tilde{\beta} = 2\beta$. To get the solution for the free wave-packet expansion in this case we can just set $\alpha = 0$ in (33), and this gives

$$\Delta q_{\text{free}}(t) = \left[(\Delta q_0)^2 + \frac{1}{m^2}\{(\Delta p_0)^2 + 2\tilde{\beta}^2 \tilde{C}_3\}t^2\right]^{1/2} \quad (39)$$

and it does not include $\tilde{C}_1$ and $\tilde{C}_2$. However, statistically speaking it has the same interpretation in terms of Γ_1, Γ_2, η since all of them are included in the definition of $\tilde{C}_3$. The corresponding minimum uncertainty relation and the relationship to find η are now given by

$$\Delta q_0 \Delta p_0 = \frac{\hbar}{2}(1 + 4\beta^2(\Delta p_0^2 + p_{cl}^2)) \quad (40)$$

and

$$\frac{2}{\hbar}(\Delta q_0 \sqrt{\eta_0}) - [1 + 4\beta^2(\eta_0 + p_{cl}^2)] = 0, \quad (41)$$

respectively.

VI. RESULTS AND PHYSICAL INTERPRETATION

Now, let us elaborate on the results obtained in the previous sections.

The standard discussion based on the HUP provides a universal time-evolution law (19) for the wave packet's width, irrespective of the initial probability distribution at time t_0. The only requirement for (19) is that the wave packet's width was minimal at t_0. This will apply for a normal distribution (which is quite ideal) and also for all other situations where the initial probability distribution is not normal. For all cases, the evolution law is the same and is given by (19). On the other hand, as evident from our analysis, that is not true if we have to believe a GUP-based calculation, irrespective of the particular form one may choose (such as the ADV or KMM form). The modified time evolution laws (35) and (39) are, indeed, dependent on the type of initial probability distribution. That is to say, for two wave packets of the same initial width but different forms (different values of skewness or kurtosis) the dispersion rate will be different for both (35) and (39). With that said, the distributions do not need to be skewed or have excess kurtosis in order to exhibit GUP-induced effects (the evolution of normal Gaussian wave packets is modified as well). Furthermore, these rates are dependent on *both* the initial momentum and the uncertainty in momentum, as opposed to the standard case (18) where it does not depend on the initial momentum.

One may now ask the question: Why do we have to consider different initial probability distributions, at all, for a free particle? To answer this question we may think about a stream of particles that were under some sort of applied force fields for some time and then those force fields were switched off at time t_0, and from that instant on (or a little while after, depending on the relaxation time) these particles start behaving as free wave packets. Then the initial configuration of the wave packet at time t_0, when all the force fields are switched off, depends on the details of the interaction between the particles and said force fields, which can, of course, be arbitrary and, therefore, the initial configuration of the stream of free particles at t_0 need not be a normal distribution. In fact, it is likely to have any other distribution including the possibility to have a nonzero skewness and excess kurtosis.

Therefore, from our discussion it follows that, while an HUP-based calculation is blind to the moments higher than second order of the initial probability distribution in momentum space, GUP-based approaches do differentiate between two different initial templates; it shows *an*

enhanced memory of the initial probability distribution (such as skewness and kurtosis) at any later instant of time. Note that, however, since all of the physical parameters, such as the skewness and kurtosis in momentum space, and the average momentum are constants in time for a free particle, their initial values will be unchanged during the course of time. Further, $\eta = \langle p^2 \rangle - \langle p \rangle^2$ is also constant in time for a free particle so that the initial uncertainty in momentum space remains unchanged over the course of time—there is no spreading in momentum space.

To start analyzing these GUP-induced effects, let us first consider a *skewed* probability distribution (with vanishing excess kurtosis) of the initial wave packet. A template of such a wave packet can be expressed in terms of the following function (we are only considering the positive amplitude):

$$f(\Gamma_1, t) := \frac{\exp\left(-\frac{q^2}{2\xi(t)}\right)\left(\mathrm{erf}\left(\frac{q\Gamma_1}{2\sqrt{\xi(t)}}\right) + 1\right)}{(\pi\xi(t))^{1/4}}. \quad (42)$$

It is easy to check that the probability distribution associated with this wave packet [that is, the square of (42)] is normalized over the configuration space and, therefore, satisfies the probability conservation condition at all times. This function corresponds to a skewed distribution with normal kurtosis $\Gamma_2 = 3$; its width satisfies, depending on the form of GUP, Eqs. (39) or (35), and for a given instant of time t the shape of the wave packet will change for a given value of the skewness Γ_1. In Fig. 1 we plot this behavior for both C_{60} and C_{176} "buckyball" molecules. The details of parameter values are given in the figure. We chose C_{60} and C_{176}, also known as the buckyballs (scientific name *Buckminsterfullerene*), molecules for this analysis because they are one of the, commercially available, bigger-sized molecules that behave as a single wave packet; thus they can be used for experimental studies on our proposal. This point will be further clarified in the next section where we discuss a possible test of our results. Just for a reminder, plots with positive skewness have a higher probability that the particle will be found on the right-hand side than the left-hand side of the mean value and vice versa. Here we have assumed characteristic values for several parameters including the mass and the initial *size* (taken to be the van der Waals diameter [42]) of the molecule. Note that the GUP coupling constants α and β are taken to be order 1. These plots are therefore more for a qualitative understanding. Accurate quantitative analysis for testing our result will be carried out in the next section.

Notice that, even though Γ_1 and Γ_2 are defined as the skewness and kurtosis coefficients in *momentum space* [see (31) and (32)], this does, of course, introduce skewness and kurtosis in position space as well, so that the shape of the wave packet in position space will also be affected, as shown in the figures. The difference is that the skewness

and kurtosis coefficients in *position space* will change over time; we can readily see this from the fact that, generally speaking, $\langle q^n \rangle = \langle q^n \rangle(t)$ for the free particle. With that said, notice that the GUP-modified spread evolution laws for free wave packets (39) and (35) do *not* depend explicitly on these coefficients in position space, but rather in momentum space, so we do not need to compute the former for our present analysis.

Now, let us plot the time evolutions of this wave packet governed by (42), in Fig. 2, for both C_{60} and C_{176} parameters for both KMM and ADV time evolutions (39) and (35). Again, the shape and the rate at which it spreads depends on the value of Γ_1 appearing in (39) and (35) [through $\tilde{C}_3$ in (39) and through both C_1 and C_2 in (35)]. Clearly, the initial distribution has an important role to play in the time evolution of the wave packet, and this is a new insight coming from the GUP-based analysis, again irrespective of the KMM or ADV forms that one may consider—both modifications have the same statistical interpretation.

In Fig. 3, we compare the wave-packet evolution with and without the GUP modifications. The sample distribution is again given by (42) with either C_{60} or C_{176} parameters, and we consider the normal (Gaussian) part of it by setting $\Gamma_1 = 0$. We find some important insights by looking at these plots: first, for the KMM GUP (38) the minimum uncertainty wave packet, defined at the initial time, has a larger width for the GUP-based calculation than the HUP-based standard result, whereas for the ADV GUP (34) it is opposite—the minimal wave packet has a smaller width than the HUP-based minimal width. It is therefore consistent to say that for the ADV form of GUP, for a physical quantum system, such as the one given by these buckyballs, the existence of a minimal length scale in the form of (34) minimizes the uncertainty in the probability distribution in position space for the same momentum distribution. Interestingly, this ADV result of further squeezing the free, minimal wave packet may be related, of course with certain differences, with an expectation that gravity might have a natural tendency to localize the wave function, as first pointed out by Penrose and Díosi [43]. This localization process could be evident in the ADV form. However, the KMM form predicts the opposite behavior where the minimal wave packet increases its width as compared to the standard HUP case, due to their specific form of GUP. The interpretation of this behavior is not well known.

Furthermore, as we shall see in the next section, the broadening rate for the KMM form will be quicker than the HUP broadening rate, whereas the ADV form will predict a slower broadening rate for the minimal width wave packet, for a vast range of parameter values.

So far our discussion did not include a distribution with an excess kurtosis. In order to study this let us assume the probability density function of the logistic distribution, given by
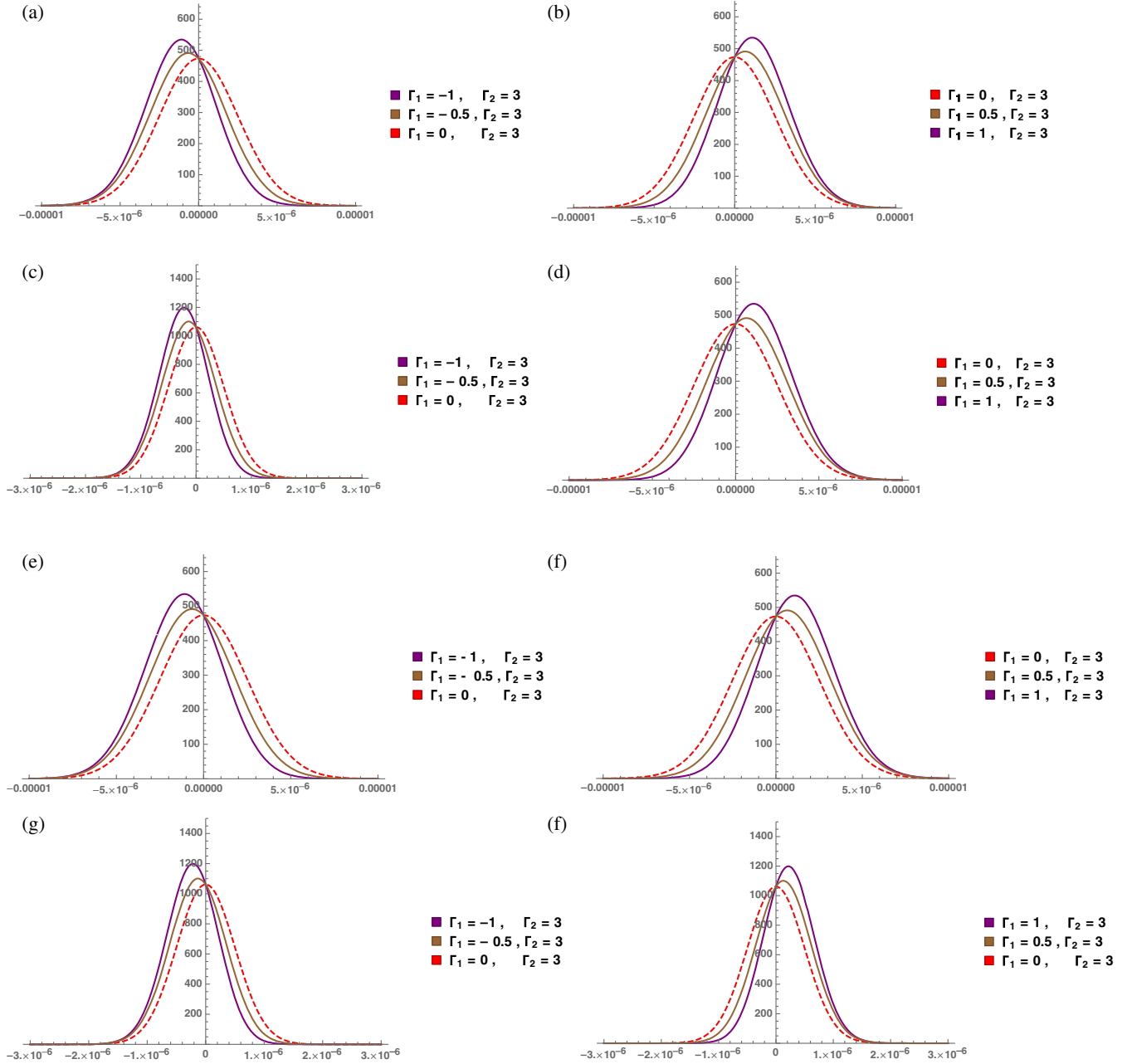
FIG. 1. Comparison of GUP wave packets with various skewness and normal kurtosis. The spatial coordinate is along the X axis and the wave function is along the Y axis. The left column of parts (a), (c), (e), and (g) corresponds to the normal vs negative skewness, whereas the right column of parts (b), (d), (f), and (h) corresponds to the normal vs positive skewness. The KMM form is used in (a)–(d) and the ADV form is used in (e)–(h). The first and third rows (a), (b), (e), and (f) correspond to C_{60}, while the second and fourth rows (c), (d), (g), and (h) correspond to C_{176}.

$$G(q, \xi(t)) := \frac{\exp\left(-\frac{q}{\xi(t)}\right)}{\xi(t)\left(\exp\left(-\frac{q}{\xi(t)}\right) + 1\right)^2}. \qquad (43)$$

$$g(q, \xi(t)) := \frac{\exp\left(-\frac{q}{2\xi(t)}\right)}{\sqrt{\xi(t)}\left(\exp\left(-\frac{q}{\xi(t)}\right) + 1\right)}. \qquad (44)$$

This function has skewness $\Gamma_1 = 0$ and excess kurtosis with $\Gamma_2 = 4.2$. The wave packet associated with this probability distribution is (again considering the positive amplitude)

The rate at which the spreading takes place with the GUP modifications, for both the KMM (39) and the ADV (35) forms for (44) include the kurtosis Γ_2 as opposed to the standard prediction from the HUP, where the rate of
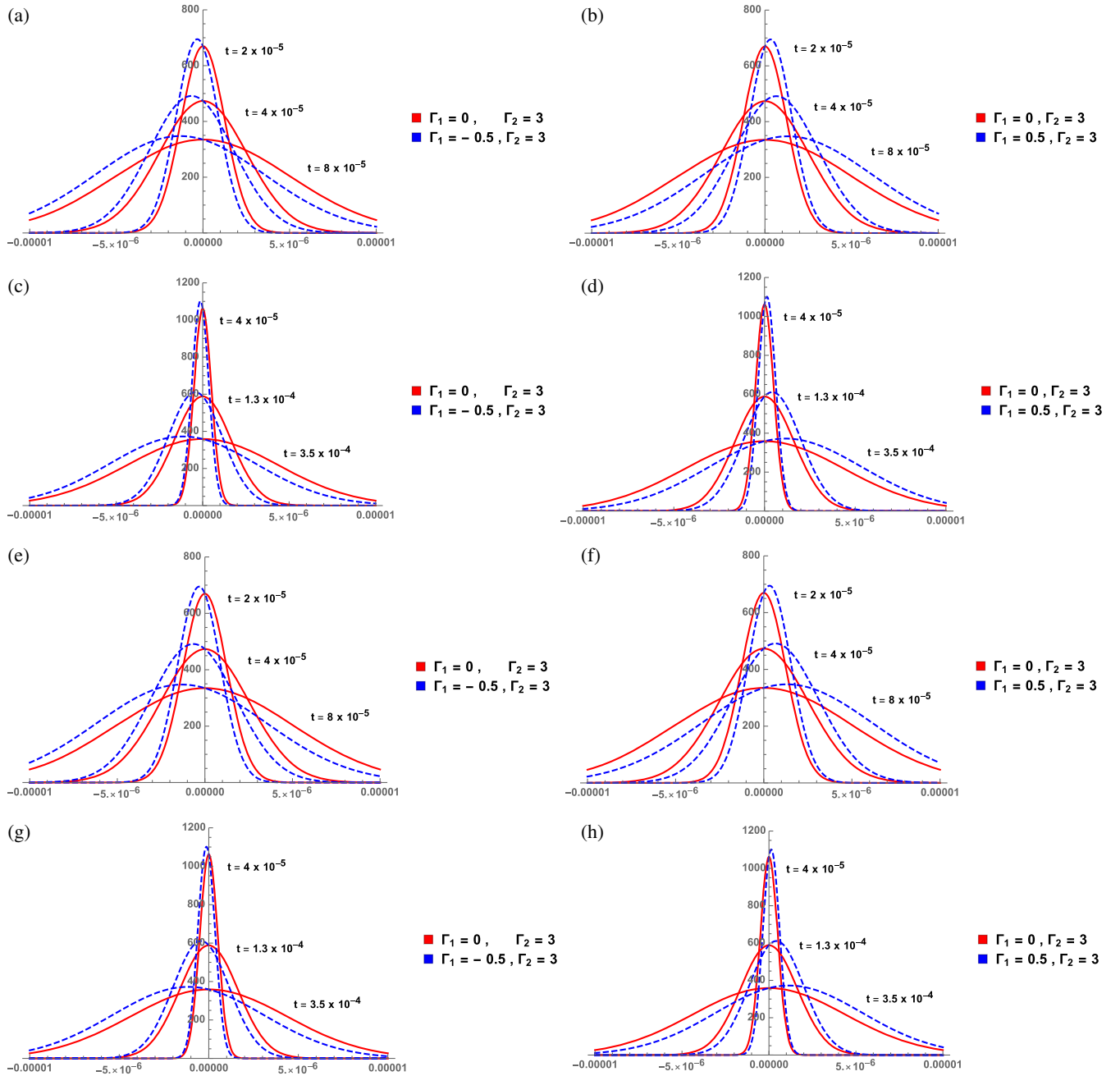
FIG. 2.   Comparison of (all) GUP time evolution between the normal vs skewed wave packets for both KMM and ADV forms. The spatial coordinate is along the X axis and the wave function is along the Y axis. The left column of parts (a), (c), (e), and (g) corresponds to the negative skewness, whereas the right column of (b), (d), (f), and (h) corresponds to the positive skewness. The KMM form is used in (a)–(d) and the ADV form is used in (e)–(h). The first and third rows (a), (b), (e), and (f) correspond to C_{60}, while the second and fourth rows (c), (d), (g), and (h) correspond to C_{176}.

expansion of $\xi(t)$ (18) is independent of the value of kurtosis. In Fig. 4 we plot the GUP time evolution of (44) starting from the minimal width wave packet for KMM and ADV forms of GUP. Figures 4(a) and 4(b) belong to the KMM form and correspond to (a) C_{60} and (b) C_{176} molecules, whereas Figs. 4(c) and 4(d) belong to the ADV form and correspond to (c) C_{60} and (d) C_{176} molecules. Note that, when considering any initial distribution (be it

normal, skewed, or with excess kurtosis), if one takes the GUP parameters $\alpha, \tilde{\beta} \sim 1$, the time evolution is practically identical to the HUP-based calculation, and it is hard to differentiate between the two in the plots irrespective of the KMM or ADV forms. However, given that the allowed parameter space for $\tilde{\beta}, \alpha$ is quite wide [37], for larger values of these parameters these plots do show a significant difference between the width of the wave packet with or
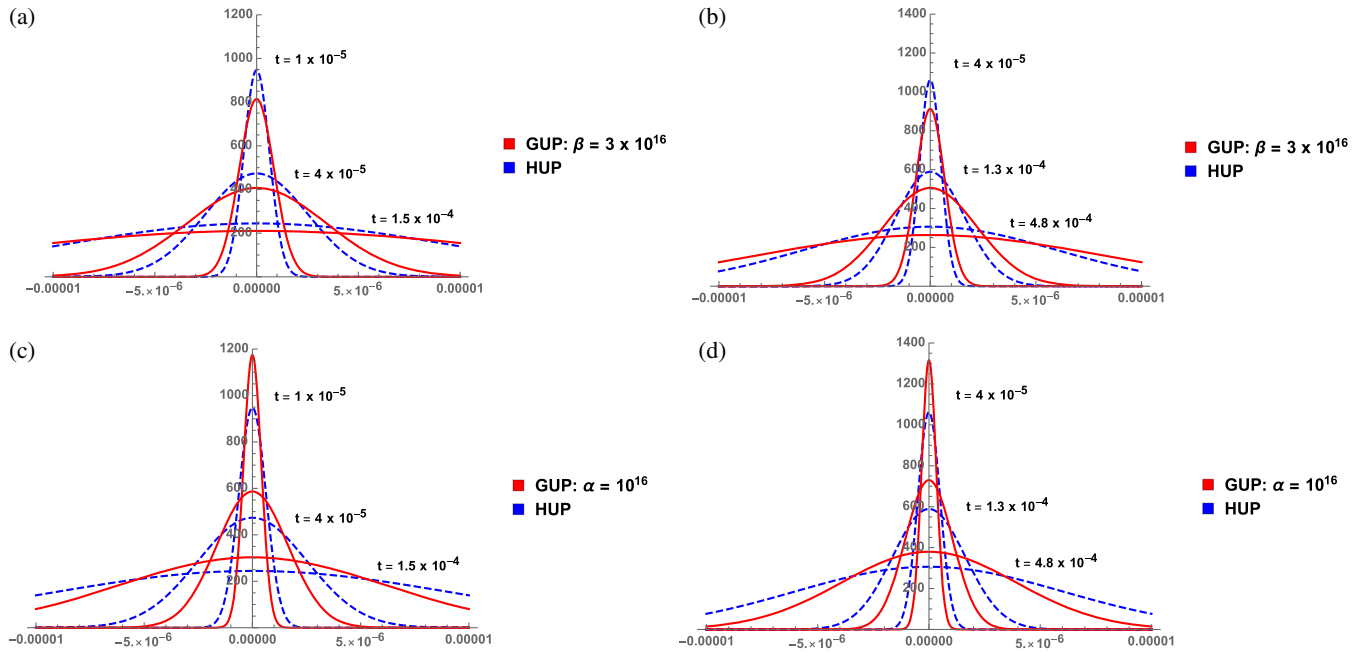
FIG. 3. GUP vs HUP time evolution (broadening) of free wave packets. The spatial coordinate is along the X axis and the wave function is along the Y axis. The left column of parts (a) and (c) is for C_{60} and the right column of parts (b) and (d) is for the C_{176} molecule. The first row is for KMM and the second row is for the ADV form. For more discussion see text.
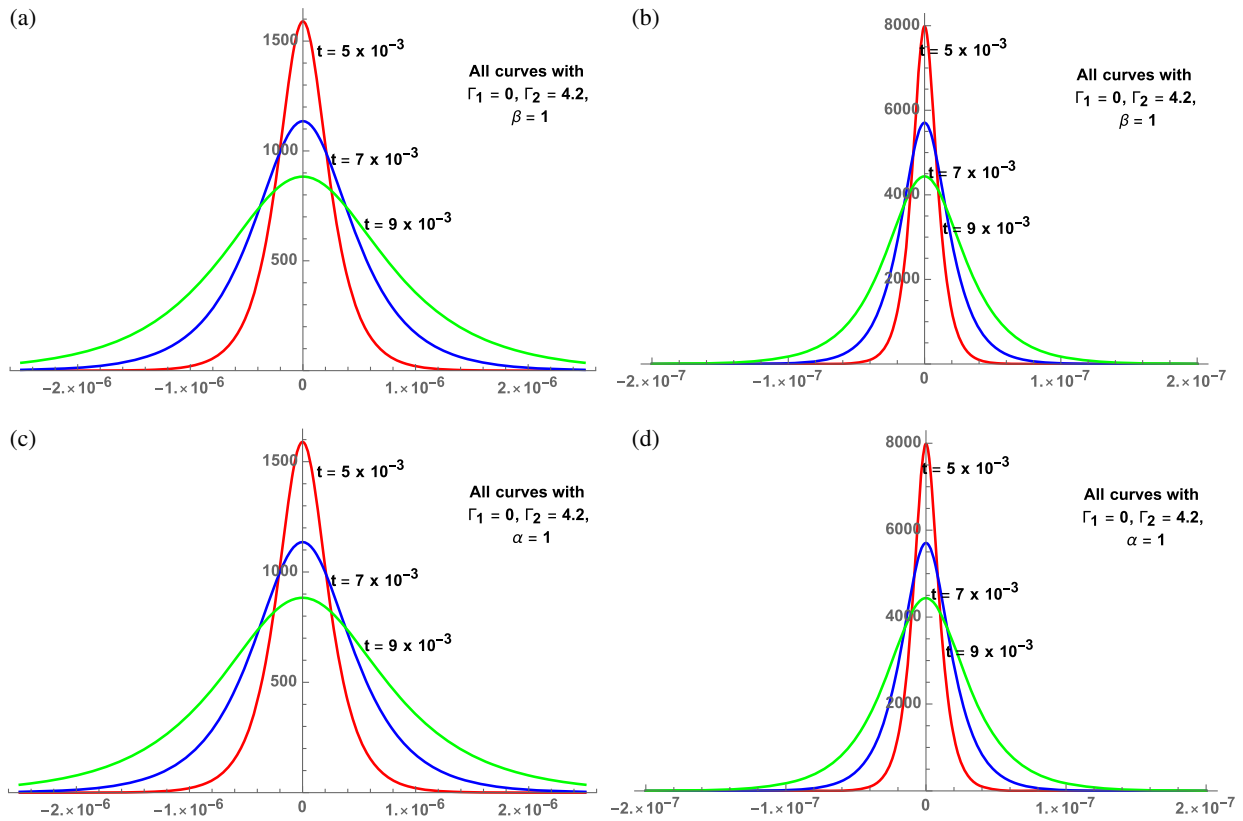

FIG. 4. GUP modified free wave packets with fixed excess kurtosis. The spatial coordinate is along the X axis and the wave function is along the Y axis. The left column of parts (a) and (c) is for C_{60} and the right column of parts (b) and (d) is for the C_{176} molecule. The first row is for KMM and the second row is for the ADV form.
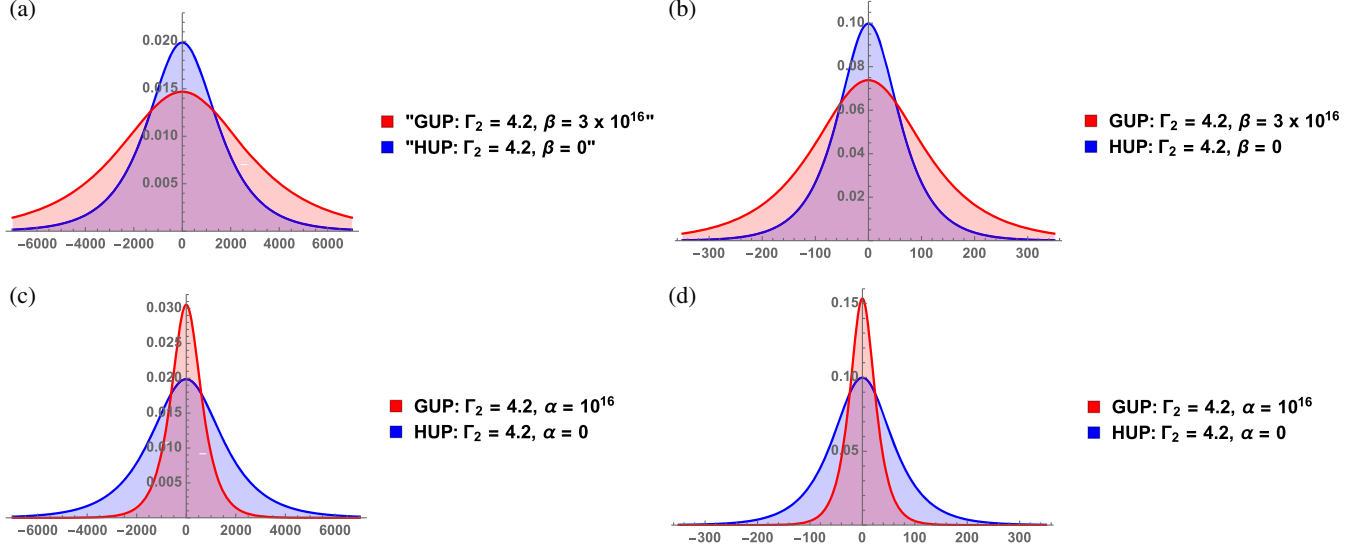
(a)

(b)

(c)

(d)

FIG. 5.   Comparison between wave packets predicted by GUP and HUP for a probability distribution with excess kurtosis. The spatial coordinate is along the X axis and the wave function is along the Y axis. The left column of parts (a) and (c) is for C_{60} and the right column of parts (b) and (d) is for the C_{176} molecule. The first row is for KMM and the second row is for the ADV form.

without GUP corrections. This can be visualized in Fig. 5 where we have considered values $\alpha, \tilde{\beta} \sim \mathcal{O}(10^{16})$ for both KMM and ADV GUP parameters. Figures 5(a) and 5(b) belong to the KMM form while 5(c) and 5(d) belong to the ADV form. This characteristic of time evolution and its difference with or without GUP modifications is just similar as that we considered before in Fig. 3—just that here we plot one snapshot while in Fig. 3 we have three of them. In the next section we shall speak more about the numbers and the likelihood of measuring them in realistic experiments.

VII. POSSIBLE TESTS

In this section we study the possibility of experimental verification of the minimal length effect on the dispersion of the free wave packets. The scheme that we propose here is quite simple—*one needs to measure the timescale in which the wave packet (describing a particle or a system of particles behaving as a single wave packet) doubles its initial width.* In fact, one can choose any final size that is permissible, but our calculation here will be done considering that the wave packet is doubling its size. HUP-based calculation gives a precise estimate for that which we already discussed for the case of electrons in Sec. II.

Let us redo the analysis, now in presence of the GUP modifications. Clearly the doubling time will be different depending on the choice of modifying the commutator such as KMM or ADV definitions. For the KMM case, when using (39) this doubling time is found to be

$$t_{\text{double}}^{\text{KMM}} = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 2\tilde{\beta}^2 \tilde{C}_3}}, \tag{45}$$

whereas for the ADV case using (35) we can easily calculate this time to be

$$t_{\text{double}}^{\text{ADV}} = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 - 4\alpha C_1 + 4\alpha^2 C_2}}, \tag{46}$$

where the minimum uncertainty wave packet now satisfies either (40) or (36), depending on the form of GUP under consideration. Plugging in our expressions for C_1, C_2, and $\tilde{C}_3$, we get

$$t_{\text{double}}^{\text{KMM}} = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 2\tilde{\beta}^2 \eta[3p_{cl}(p_{cl} + \eta^{1/2}\Gamma_1) + \eta\Gamma_2]}}, \tag{47}$$

$$t_{\text{double}}^{\text{ADV}} = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 4\eta[\alpha^2((3\Gamma_2 - 1)\eta + 10p_{cl}(\Gamma_1\eta^{1/2} + p_{cl})) - \alpha(2p_{cl} + \Gamma_1\eta^{1/2})]}}. \tag{48}$$

If we, for the sake of simplicity, consider a Gaussian wave packet, then we can set the skewness and kurtosis coefficients to $\Gamma_1 = 0$ and $\Gamma_2 = 3$, respectively. With this the above expressions get simplified, giving

$$t_{\text{double}}^{\text{KMM}}(\Gamma_1 = 0, \Gamma_2 = 3) = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 6\tilde{\beta}^2\eta(p_{cl}^2 + \eta)}} \quad (49)$$

and

$$t_{\text{double}}^{\text{ADV}}(\Gamma_1 = 0, \Gamma_2 = 3)$$

$$= \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 8\eta(\alpha^2(4\eta + 5p_{cl}^2) - \alpha p_{cl})}}. \quad (50)$$

With expressions (49) and (50) at hand, we can use the relations (41) and (37) to replace $\eta = \Delta p_0^2$ in terms of Δq_0 and other parameters. Therefore, we now have everything we need for doing a numerical calculation with realistic molecular wave packets.

First, let us go back to the case of the free electron (where the initial wave packet had a width of 10^{-10} m) so we can use (49) and (50) to estimate the magnitude of the GUP modification for both the KMM and ADV forms, respectively.

For the KMM form, a simple numerical check for the free electron case shows that for the values $1 \leq \tilde{\beta} \leq 10^{16}$ the difference between the HUP and KMM-GUP predictions is negligible. If we go to values such as $\tilde{\beta} = 10^{17}$, we get a difference between both predictions $\mathcal{O}(10^{-30})s$— not anywhere near a potentially detectable value. Going to larger values such as $\tilde{\beta} = 10^{22}$ we get a difference $\mathcal{O}(10^{-20})$ s, and even larger values such as $\tilde{\beta} = 10^{30}$ give a time difference of 4.06218×10^{-16} s, which is somewhat close to a potentially detectable value.

The numbers are better for the ADV form for the free electron case, but the results are still effectively the same with or without GUP in the parameter range $1 \leq \alpha \leq 10^{21}$. The numerical calculations show that the difference between HUP and ADV-GUP is at most $\mathcal{O}(10^{-30})$ s for $\alpha \leq 10^{10}$, and for higher values such as $\alpha = 10^{17}$ we start getting differences $\mathcal{O}(10^{-23})$ s, but practically undetectable still.

It is therefore clear that for both KMM and ADV forms a free electron wave-packet expansion is almost identical to the original HUP results—the difference being unlikely to be detected even with the utmost precision of atomic clocks available today. Furthermore, if we have to believe an upper bound for α or $\tilde{\beta}$, we can infer that the GUP modification does not give a major difference in the doubling time for the case of free electrons, at least in the initial stage where it is more likely to be detectable by a laboratory-based experiment.

In order for these effects to be detectable in a laboratory, we must magnify the GUP modifications somehow. To do this, we must consider probes whose wave packets have initial size and associated mass *bigger* than that of an electron wave packet. One obvious way to achieve this is to

consider atoms instead of electrons or, even better, use bigger molecules that can behave as a single wave packet. This brings us to the so-called buckyball systems and large organic molecules (LOM). Buckyballs or *Buckminsterfullerene* molecules are basically a bunch of carbon atoms behaving as a single quantum wave packet [44]. We shall consider again C_{60} and C_{176} molecules— which we already considered in various plots in the previous section. On the other hand, LOMs are probably the most exciting candidates since they are the largest molecules (in terms of the combination of size and mass scale) found so far that behave as a single wave packet [45].

Below we do an analysis for these three objects, using both forms, where we shall keep the relevant GUP parameter α or $\tilde{\beta}$ as a free parameter from the beginning, and see how the wide range of values for these parameters affect the time difference between the HUP and GUP results for the minimal width wave packet to double its initial width.

A. Ali-Das-Vagenas GUP

In the case of a C_{60} buckyball molecule, with a mass of 1.19668×10^{-24} kg (720 u) and an initial width Δq_0 equal to its van der Waals diameter (7 Å) [42], the HUP prediction for the *doubling* time is $t_{\text{double}}(C_{60}, \text{HUP}) = 1.92719 \times 10^{-8}$ s. If we start considering ADV type GUP modifications, first with $\alpha = 1$ as the value of the GUP parameter, then we get practically the same value t_{double}, the difference between them being

$$t_{\text{double}}^{\text{ADV}}(C_{60}, \alpha = 1) - t_{\text{double}}(C_{60}, \text{HUP})$$
$$= -6.61744 \times 10^{-24} \text{ s}.$$

However, if we take large values such as $\alpha = 10^{10}$, then we find that

$$t_{\text{double}}^{\text{ADV}}(C_{60}, \alpha = 10^{10}) - t_{\text{double}}(C_{60}, \text{HUP})$$
$$= 1.15631 \times 10^{-14} \text{ s},$$

and if we go even further, such as $\alpha = 10^{16}$, we find $t_{\text{double}}(C_{60}, \alpha = 10^{16}) = 2.96189 \times 10^{-8}$ s and

$$t_{\text{double}}^{\text{ADV}}(C_{60}, \alpha = 10^{16}) - t_{\text{double}}(C_{60}, \text{HUP})$$
$$= 1.0347 \times 10^{-8} \text{ s}.$$

That is, the difference between both predictions is of the order of the original HUP prediction ($\sim 10^{-8}$ s) while taking $\alpha \sim 10^{16}$ as the GUP parameter.

This analysis shows that depending on the wide range of values for α, the difference between the HUP and GUP predictions for t_{double} for C_{60} buckyballs stays in an interval where the lower end is undetectable even with the most

precise clocks currently available, but the upper end stays well within the available range of precision.

Furthermore, since we want to amplify the GUP-induced effects (and thus make them easier to detect at laboratory-based experiments), let us now consider a C_{176} buckyball. Using this molecule's parameters [$m = 3.50706 \times 10^{-24}$ kg (2112 u) and $\Delta q_0 = 1.2$ nm [42]], we find that the HUP prediction for the doubling time is $t_{\text{double}}(C_{176}, \text{HUP}) = 1.6598 \times 10^{-7}$ s and, again, taking small values of α (as order unity) yields an effectively undetectable difference between the HUP and GUP predictions. However, if we again set $\alpha = 10^{10}$, we get

$$t_{\text{double}}^{\text{ADV}}(C_{176}, \alpha = 10^{10}) - t_{\text{double}}(C_{176}, \text{HUP})$$
$$= 9.9588 \times 10^{-14} \text{ s},$$

which is better by a factor of more than 8 as compared with C_{60}, and going to higher values such as $\alpha = 10^{16}$ yields $t_{\text{double}}(C_{176}, \alpha = 10^{16}) = 2.55094 \times 10^{-7}$ s and

$$t_{\text{double}}^{\text{ADV}}(C_{176}, \alpha = 10^{16}) - t_{\text{double}}(C_{176}, \text{HUP})$$
$$= 8.9114 \times 10^{-8} \text{ s}.$$

This is again an improvement by a factor of almost 9 over the time difference (1.0347×10^{-8} s) that we got for the C_{60} molecule. Therefore, we see that bigger (larger van der Waals diameter) and more massive molecules tend to show stronger deviations from the HUP behavior when considering GUP-modified t_{double} calculations.

Now let us consider the case of recently discovered LOM wave packets [45], such as a TPPF152 or *tetraphenylpor-phyrin* molecule (which consists of 430 atoms and is formally known as $C_{168}H_{94}F_{152}O_8N_4S_4$), with a mass of 5,310 u (~8.81746×10^{-24} kg) and an initial *size* of 60 Å. Taking $\alpha = 1$, once again, does not bring the time difference in a detectable range. However, if we go to larger values of α such as 10^{10}, we find

$$t_{\text{double}}^{\text{ADV}}(\text{TPPF152}, \alpha = 10^{10}) - t_{\text{double}}(\text{TPPF152}, \text{HUP})$$
$$= 6.25961 \times 10^{-12} \text{ s},$$

which improves the result of the C_{176} molecule by a factor of 63, and this number is 500 times better than for the C_{60} molecule. Further, moving to $\alpha \sim 10^{16}$ the difference becomes

$$t_{\text{double}}^{\text{ADV}}(\text{TPPF152}, \alpha = 10^{16}) - t_{\text{double}}(\text{TPPF152}, \text{HUP})$$
$$= 5.60129 \times 10^{-6} \text{ s},$$

which is again better by a factor of 63 from C_{176} and 560 from C_{60}.

In Fig. 6, we have plotted the difference between the doubling times for various values of α (difference between
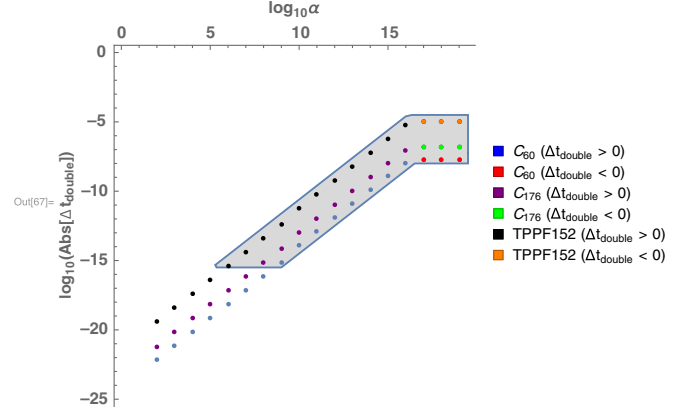

FIG. 6.   The log-log plots between the GUP parameter α and the doubling time difference $\Delta t_{\text{double}}^{\text{ADV}} = t_{\text{double}}^{\text{ADV}}(\text{GUP}) - t_{\text{double}}(\text{HUP})$ between the GUP and HUP time evolutions for (a) the C_{60} molecule (the lower plot), (b) the C_{176} molecule (the middle plot), and (c) the large organic molecule TPPF152 (the upper plot). The X axis represents the GUP parameter and the Y axis represents the difference between GUP and HUP doubling times. The shaded region indicates the region of parameter space that can be probed by the above-mentioned molecular wave packets with an atomic clock of maximum precision 10^{-15} s.

the ADV type GUP-based and HUP-based calculations). This is a log-log plot where values of $\Delta t_{\text{double}}^{\text{ADV}}$ are shown for the parameter space $1 \le \alpha \le 10^{19}$. Note that for the larger values of $\alpha \ge 10^{16}$ we get a doubling time difference $\mathcal{O}(10^{-8}$ s) for the C_{60} molecule, which can easily be detected by today's atomic clocks. This result is even better (10^{-7} s) for C_{176} and in μs range for TPPF152. On the other side, we can scan the complete parameter space of α (up to order unity), if we can measure a time difference of the order of 10^{-21} s to 10^{-23} s, just by considering these molecules. However, if we have to believe that highly precise atomic clocks can differentiate the time measurement by at most 10^{-15} s, the use of C_{60} molecules can scan the parameter space $\alpha \ge 10^9$, and it is again better for C_{176}, for which we can scan $\alpha \ge 10^8$. The best of the three, however, stands for TPPF152, which can scan, on the lower side, down to $\alpha \sim 10^6$. Therefore, if the experiments with TPPF152 do not show the deviation in doubling time within femtoseconds, we automatically get an improvement by 4 orders of magnitude on the best existing bound found coming from the Lamb shift [46], which is $\alpha \le 10^{10}$. In addition, if we are lucky and nature behaves in such a manner, we might be able to verify (34) with these molecular wave packets. If not, we can put a new bound and move on to redo the experiments with even bigger and more massive wave packets, which could scan the whole parameter space. This is a totally new avenue that has not been proposed before. *In fact, any departure from HUP, irrespective of the manner it differs, will be a pathbreaking discovery since it will challenge the standard quantum*

mechanical prediction anyway. We expect that, perhaps, colleagues from the experimental side will find this result interesting.

B. Kempf-Mann-Mangano GUP

Performing a similar analysis for the KMM form (38) we find that, for the case of the spreading wave packet of a C_{60} molecule, for values of the GUP parameter $1 \leq \tilde{\beta} \leq 10^9$ the difference between the HUP-predicted and GUP-predicted *doubling* times is practically negligible and far from being in a detectable range. Even going to values $\tilde{\beta} \sim 10^{10}$ still gives a very small value for this difference,

$$t_{\text{double}}^{\text{KMM}}(C_{60}, \tilde{\beta} = 10^{10}) - t_{\text{double}}(C_{60}, \text{HUP})$$
$$= -7.74241 \times 10^{-22} \text{ s},$$

which is unlikely to be detected even with today's best atomic clocks.

Now, going up to values $\sim \mathcal{O}(10^{16})$ of $\tilde{\beta}$ yields a much more optimistic result,

$$t_{\text{double}}^{\text{KMM}}(C_{60}, \tilde{\beta} = 10^{16}) - t_{\text{double}}(C_{60}, \text{HUP})$$
$$= -7.38707 \times 10^{-10} \text{ s},$$

which might be well within the range of precision of the current technology.

If we start to consider C_{176} molecules, again $1 \leq \tilde{\beta} \leq 10^9$ yields practically negligible results, and $\tilde{\beta} \sim \mathcal{O}(10^{10})$ gives

$$t_{\text{double}}^{\text{KMM}}(C_{176}, \tilde{\beta} = 10^{10}) - t_{\text{double}}(C_{176}, \text{HUP})$$
$$= -6.64391 \times 10^{-21} \text{ s},$$

an improvement of almost 1 order of magnitude over the C_{60} result. On the other hand, $\tilde{\beta} \sim \mathcal{O}(10^{16})$ gives

$$t_{\text{double}}^{\text{KMM}}(C_{176}, \tilde{\beta} = 10^{16}) - t_{\text{double}}(C_{176}, \text{HUP})$$
$$= -6.36215 \times 10^{-9} \text{ s},$$

which is, again, a much better result.

Notice that, up to this point, all values of $t_{\text{double}}(\text{GUP}) - t_{\text{double}}(\text{HUP})$ have been negative for the KMM form— meaning that the KMM-predicted spreading is faster than predicted by the HUP, as opposed to the ADV-based predictions (at least for a majority range of values of the GUP parameter).

Finally, taking our analysis again to the TPPF152 large organic molecule gives much better values of the *doubling* time difference, where taking $\tilde{\beta} \sim \mathcal{O}(10^{10})$ gives

$$t_{\text{double}}^{\text{KMM}}(\text{TPPF152}, \tilde{\beta} = 10^{10}) - t_{\text{double}}(\text{TPPF152}, \text{HUP})$$
$$= -4.1674 \times 10^{-19} \text{ s}.$$

Going even further to $\tilde{\beta} \sim \mathcal{O}(10^{16})$ yields

$$t_{\text{double}}^{\text{KMM}}(\text{TPPF152}, \tilde{\beta} = 10^{16}) - t_{\text{double}}(\text{TPPF152}, \text{HUP})$$
$$= -3.99893 \times 10^{-7} \text{ s},$$

which, as is the case for large organic molecules in both types of GUP, is well within the range of possible detection in experiments and once again gives results ~63 times better than the C_{176} case and ~500 times better than for C_{60} molecules—albeit the time differences predicted by a KMM form of GUP are considerably smaller than those predicted by an ADV-type modification of the HUP. This is because the deformation of the commutator only includes a term of quadratic order in momentum and not a term of linear order in momentum in the KMM case.

In Fig. 7 we present a plot for the doubling time differences predicted by the KMM GUP, analogous to that of Fig. 6 for the ADV form. The values of $\Delta t_{\text{double}}^{\text{KMM}}$ are shown for the parameter space $10^{10} \leq \tilde{\beta} \leq 10^{19}$ since these are the values for which the time differences have significant values (that may be possible to measure in a laboratory). The shaded region represents the portion of parameter space that could be possible to probe, given the assumption that today's best atomic clocks can resolve time differences $\sim \mathcal{O}(10^{-15})$ s. Note, however, that it is possible for the currently achievable precision to be even better than this conservative estimate. Assuming this time
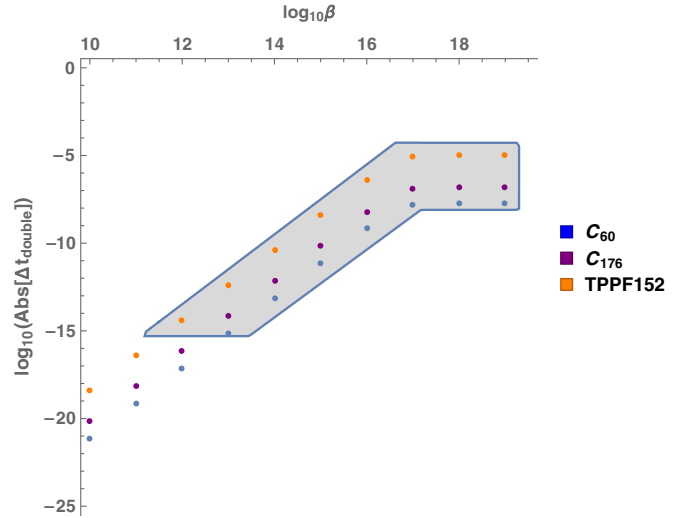

FIG. 7. The log-log plots between the GUP parameter $\tilde{\beta}$ and the doubling time difference $\Delta t_{\text{double}}^{\text{KMM}} = t_{\text{double}}^{\text{KMM}}(\text{GUP}) - t_{\text{double}}(\text{HUP})$ between the GUP and HUP time evolutions for (a) the C_{60} molecule (the lower plot), (b) the C_{176} molecule (the middle plot), and (c) the large organic molecule TPPF152 (the upper plot). The X axis represents the GUP parameter and the Y axis represents the difference between GUP and HUP doubling times. The shaded region indicates the region of parameter space that can be probed by the above molecular wave packets with an atomic clock of maximum precision 10^{-15} s.

resolution, C_{60} and C_{176} molecules can probe, on the lower end, down to $\tilde{\beta} \sim \mathcal{O}(10^{13})$, while using TPPF152 molecules would let us probe down to $\tilde{\beta} \sim \mathcal{O}(10^{12})$. These values of $\tilde{\beta}$ will constrain the coupling (the quadratic term) of KMM form $\tilde{\beta}^2 \leq 10^{24}$. This is again 15 orders of magnitude better than the bound claimed $\tilde{\beta}^2 \leq 10^{39}$ (coming from the study of the cold atom recoil experiment as reported in [47]).

VIII. CONCLUSIONS AND DISCUSSIONS

We have introduced a novel approach and, to some extent, established the fact that studying the dispersion of free wave packets might lead to indirect evidence for the long anticipated minimal length scale in nature. Our result here is based on the possibility that HUP should be replaced by a GUP in the presence of the minimal length. Nonetheless, it is very important to stress that our approach is quite general and independent of the specific manner in which the commutator bracket has to be modified. This specific study, based on two popular choices given by the ADV form (34) and the KMM form (38), has several interesting outcomes, which we list below.

(i) For both forms of GUP the deformations have brought a rich distributional consequence on the expansion rate of free wave packets. The rate of dispersion depends not only on the initial uncertainty and standard deviation (in position and momentum) but also on the higher-order moments in momentum space (such as skewness and kurtosis). In addition, it also depends on the initial momentum of the wave packet.

(ii) The minimal width of a free wave packet is modified—for the ADV form it is normally squeezed in position space while for the KMM form it is further widened.

(iii) We have shown that by measuring the "doubling time," that is, the time in which a free, minimal width wave packet doubles its size, we may get important clues on the minimal length scale. The difference between the doubling times of HUP- and GUP-based predictions may well be in the detectable range if we use highly precise atomic clocks and measure the broadening rates of molecular wave packets.

(iv) This difference in broadening time is more for massive molecular wave packets in comparison with the wave packets representing small objects such as electrons. Large organic molecules (such as TPPF152) and buckyball (such as C_{60}, C_{176}) wave packets may be useful for verifying or falsifying the GUP proposals.

(v) For the ADV form, in the absence of detecting any difference for doubling time with an atomic clock of precision level 10^{-15} s, with C_{60}, we can better the best existing upper bound for $\alpha (< 10^{10})$ by 1 order of magnitude, for C_{176} by 2 orders of magnitude, and

for TPPF152 by 4 orders of magnitudes ($\alpha < 10^6$). This bound can be further sharpened by using atomic clocks more precise than femtoseconds.

(vi) For the KMM form, the use of TPPF152 and a clock of femtosecond precision can provide an upper limit of $\tilde{\beta}^2 \leq 10^{24}$. This is an improvement by 15 orders of magnitude of the coupling $\tilde{\beta}^2$ obtained from the cold atom recoil experiments [47].

(vii) There are two ways to improve the numbers presented here and to reach even closer to testing the GUP theory. One of them is to consider larger and heavier molecular wave packets, and the other is to come up with new atomic clocks that can measure the time difference even beyond a femtosecond.

We want to stress that coming up with an experiment to test our results might not be impossible in the near future, especially because of the remarkable progress that has been achieved to test the superposition principle with increasingly massive molecular wave packets [45]. Perhaps an experiment in our context will be easier to conduct since the wave packet does not pass through the double slit; rather, it only needs to be set free until it doubles its size.

Finally, we want to add some general remarks on the relativistic extension of the GUP setting. It should be noted that our work in this paper is based on the nonrelativistic quantum mechanics as the molecular wave packets are highly nonrelativistic objects. If someone wants to study an ultra-high-energy particle with relativistic velocity, this approach may not be as useful. One would need to reformulate the question in terms of fundamental quantum fields, their particle excitation, etc. One has to also rethink the physical variables that must be built from the quantum field and its derivatives that might be considered as observables. The width of the wave packet would not have a meaningful usage there. Therefore, one naturally asks the question to oneself that, should, if at all, GUP be used in such a context with relativistic particles? A satisfactory answer to this question can be found in an important work by Magueijo and Smolin [11], where a concise picture of the GUP modification in a relativistic setting was outlined. It was shown that the two forms of GUP (which we used here) can be derived by satisfying following five criteria: (i) the validity of relativity in inertial frames, (ii) an invariant energy/length scale at Planck value, (iii) a varying speed of light at higher energies, (iv) a modified dispersion relation at higher energy (inspired by the ultra-high-energy-cosmic-ray anomaly), and (iv) the theory should have a maximum momentum. The idea was to keep the principle of the relativity of inertial frames by modifying the laws by which energy/momenta measured by various inertial observers are related to each other. The only possibility to achieve all of these conditions was argued to be achieved by a nonlinear action of the ordinary Lorentz group (in momentum space) on the states of the theory.

The appearance of the Lorentz group is quite interesting and renders the theory as Lorentz invariant. In fact, a naive judgment that the minimal length scale breaks the Lorentz invariance is false, and recent works on "modular spaces" further hint at this possibility [48]. However, the issue of a fully relativistic approach is beyond the scope of this paper since our aim is to use low-energy atomic-molecular experiments in the search of a fundamental length scale.

ACKNOWLEDGMENTS

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