Discrete fuzzy de Sitter cosmology

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We analyze the spectrum of time observable in the noncommutative cosmological model introduced by Buric and Madore [Eur. Phys. J. C **75**, 502 (2015)], defined by the $(\rho, s = \frac{1}{2})$ representation of the de Sitter group. We find that time has a peculiar property: it is not self-adjoint, but appropriate restrictions to the space of physical states give self-adjoint extensions. Extensions have a discrete spectrum with a logarithmic distribution of eigenvalues, $t_n \sim \ell \log n + \mathrm{const}$, where ℓ characterizes noncommutativity and the usual assumption is $\ell = \ell_{\mathrm{Planck}}$. When calculated on physical states, the radius of the universe is bounded below by $\ell\sqrt{\frac{3}{4}(\frac{1}{4}+\rho^2)}$, which resolves the big bang singularity. An immediate consequence of the model is a specific breaking of the original symmetry at the Planck scale.

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I. INTRODUCTION

The expression "quantum space" was introduced in the early days of quantum mechanics by Heisenberg, along with "quantum derivative" introduced by Dirac who observed that a commutator is a derivation; "points" of the quantum space are "q numbers" or operators. Today the idea that spacetime, as seen by quantum particles, is described by operators gives strong heuristic and physical motivation for noncommutative geometry.

There is a surprisingly simple covariantization of the usual flat space of quantum mechanics to curved noncommutative spaces. If we identify flat quantum space with the Heisenberg algebra,

$$[ip_i, x^j] = \partial_i x^j = \delta_i^j \tag{1.1}$$

($\hbar = 1$, p_i, x^j Hermitian), curved quantum space can be defined by a moving frame e_α^μ,

$$[ip_\alpha, x^\mu] = e_\alpha x^\mu = e_\alpha^\mu(x), \tag{1.2}$$

as in general relativity [1]. Adding to the last relation property (which one expects in the quantum-gravity regime) that spacetime at the Planck scale is discrete or has a minimal quantum of length, i.e., that coordinates may be noncommuting,

$$[x^\mu, x^\nu] = i k\!\!\!{}^{-} J^{\mu\nu}(x), \tag{1.3}$$

we have a general situation, a noncommutative algebra of coordinates and momenta, $\mathcal{A}$. In principle, $\mathcal{A}$ may not have a Schrödinger-type representation of momenta through the partial derivatives; in fact, some representations might be finite dimensional. In this picture, position algebra (1.3) determines the structure of the points of noncommutative space, i.e., the algebraic properties of coordinates, while (1.2) and the related commutators between momenta define the differential-geometric structure and enable us to introduce connection and curvature. Algebraic and geometric structures are intertwined by the assumption that one deals with operators, i.e., by associativity [2].

This is the general framework that we use. Its algebraic part is, in various descriptions of noncommutative spaces, more or less invariant, while the differential-geometric part is specific in every approach: we use the noncommutative frame formalism of Madore. The frame formalism has proven in many aspects to be successful, in particular in describing spaces of Euclidean signature with finite-dimensional representations such as the fuzzy sphere and a number of other models in two and three dimensions [1,3,4]. For further development of this concept it is crucial to provide realistic cosmological and astrophysical configurations in four dimensions: this is the main motivation for our work.

Noncommutative geometry is but one of the approaches to quantum gravity. Other approaches are, perhaps, in light of the description in terms of Lagrangian and quantization procedures, more fundamental. String theory introduces an elementary substructure that after quantization, macroscopically, gives spacetime geometry and classical gravity. In loop quantum gravity, vielbein and connection fields

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are basic variables that are quantized in a background-space independent way. In these approaches quantum space with its properties is a derived quantity or notion. But in most cases, being effective or not, coordinates are operators in the Hilbert space of states: therefore models, algebras with physically plausible features, are common to many theories. We thus hope that properties of fuzzy de Sitter space and its physical interpretation discussed here will be of wider interest.

The plan of the paper is as follows. In Sec. II we introduce fuzzy de Sitter space as a unitary irreducible representation of the de Sitter group, i.e., identify its coordinates and differential structure. In Sec. III we give Hilbert space representation for a specific de Sitter space defined by $(\rho, \frac{1}{2})$ representation of the principal continuous series of $SO(1,4)$ and solve the eigenvalue equation for the observable of cosmic time τ. In Sec. IV we examine the obtained solutions and show how to redefine time to render it self-adjoint. Finally, in the last section we discuss physical properties and some cosmological implications of the given fuzzy geometry.

II. FUZZY DE SITTER SPACE

Our task is to study the observable of time in the cosmological model introduced in [5,6]. In commutative geometry, four-dimensional de Sitter space can be defined as an embedding in five-dimensional flat space [7],

$$v^2 - w^2 - x^2 - y^2 - z^2 = -\mathsf{L}^2,$$
$$ds^2 = dv^2 - dw^2 - dx^2 - dy^2 - dz^2, \tag{2.1}$$

where $v \in (-\infty, \infty)$ is the embedding time. Introducing

$$\frac{\mathsf{t}}{\mathsf{L}} = \log\frac{v+w}{\mathsf{L}}, \qquad \frac{\mathsf{x}}{\mathsf{L}} = \frac{x}{v+w}, \qquad \frac{\mathsf{y}}{\mathsf{L}} = \frac{y}{v+w},$$
$$\frac{\mathsf{z}}{\mathsf{L}} = \frac{z}{v+w}, \tag{2.2}$$

one obtains the line element in the Friedmann-Robertson-Walker form, the "steady state universe,"

$$ds^2 = d\mathsf{t}^2 - e^{\frac{2\mathsf{t}}{\mathsf{L}}}(d\mathsf{x}^2 + d\mathsf{y}^2 + d\mathsf{z}^2). \tag{2.3}$$

Time $\mathsf{t} \in (-\infty, \infty)$ is defined only for $v + w > 0$: coordinates (2.2) cover only half of the de Sitter space and the steady state space is incomplete and extendible.

Fuzzy de Sitter space can be defined in an analogous manner. The general idea, realized in all details for the fuzzy sphere [3], is to identify spacetime with the algebra of a Lie group, realizing the embedding through the Casimir relations: then, in fact, fuzzy spacetime is given by an irreducible representation of a Lie group. We start with the group $SO(1,4)$ with generators $M_{\alpha\beta}$ $(\alpha, \beta = 0, 1, 2, 3, 4)$,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}), \tag{2.4}$$

the signature is $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1,-1)$. Noncommutative extensions of v, w, x, y, z are embedding coordinates x^α: they are proportional to the "Pauli-Lubanski vector" W^α,

$$W^\alpha = \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta\eta}M_{\beta\gamma}M_{\delta\eta}, \qquad x^\alpha = \ell W^\alpha. \tag{2.5}$$

Dimensional constant ℓ fixes the length scale of noncommutativity: depending on the physical interpretation, it can lie between the grand unification scale and the Planck length [6,8], and usually one assumes $\ell \sim \ell_{\mathrm{Planck}}$. One of the two Casimirs of $SO(1,4)$,

$$\eta_{\alpha\beta}W^\alpha W^\beta = -\mathcal{W}, \tag{2.6}$$

defines the embedding equivalent to (2.1). We will for simplicity assume that the other Casimir operator

$$\mathcal{Q} = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} \tag{2.7}$$

is also fixed, i.e., that the fuzzy de Sitter space is given by a unitary irreducible representation (UIR) of the de Sitter group.

All UIR's of the $SO(1,4)$ are infinite dimensional, labeled by two quantum numbers: conformal weight ρ and spin s [9],

$$\mathcal{W} = s(s+1)\left(\frac{1}{4} + \rho^2\right), \qquad \mathcal{Q} = -s(s+1) + \frac{9}{4} + \rho^2. \tag{2.8}$$

In the following we will use UIR's of the principal continuous series, $\rho \geq 0$, $s = 0, 1/2, 1, 3/2$, and the Hilbert space representations; in fact, in this concrete calculation we use only the simplest nontrivial of them $(\rho, s = \frac{1}{2})$.

Various choices of differential calculi on fuzzy de Sitter space were discussed in [5]. The simplest one, which has the de Sitter metric as the commutative (macroscopic) limit, is the calculus generated by four momenta, translations $ip_i = M_{i4} + M_{0i}$, $i = 1, 2, 3$, and dilatation $ip_0 = M_{04}$. When calculated, expression (1.2) for vielbein suggests to choose comoving coordinates proportional to W^i and cosmic time τ proportional to $\log(W^0 + W^4)$ [5],

$$\frac{x^i}{\ell} = W^i, \qquad \frac{\tau}{\ell} = \log\frac{x^0 + x^4}{\ell} = \log(W_0 - W_4). \tag{2.9}$$

It is clear that correct identification of coordinates and momenta is very important for understanding various properties and limits of a given fuzzy space, as well as for its physical interpretation. One way to see if noncommutativity improves the singularity structure of spacetime is to determine the spectra of coordinates, in this case τ and x^i, or $\sum(x^i)^2$. As found in [6], spectra of x^i are continuous in (ρ, s) representations; embedding time W^0/l has a discrete spectrum. Here we wish to find eigenvalues of the cosmic time.[1]

Properties of the spectrum can often be inferred directly from the algebra. In this case we have relation

$$[iM_{04}, W_0 - W_4] = W_0 - W_4, \tag{2.10}$$

which implies that the group action of dilatation M_{04} is given by

$$e^{i\alpha M_{04}}(W_0 - W_4)e^{-i\alpha M_{04}} = e^{\alpha}(W_0 - W_4). \tag{2.11}$$

The last formula means, apparently, that the spectrum of $W_0 - W_4$ is continuous. Namely, it is easy to check formally that, if there is a nonzero eigenvalue $\lambda > 0$ of $W_0 - W_4$ and the corresponding eigenvector ψ_λ,

$$(W_0 - W_4)\psi_\lambda = \lambda\psi_\lambda, \tag{2.12}$$

and then for every real α, $e^{-i\alpha M_{04}}\psi_\lambda$ is the eigenvector for the eigenvalue $e^\alpha\lambda$. This would mean that the spectrum consists of all real $\lambda > 0$. We will show in the following that eigenvalues of $W_0 - W_4$, calculated in the Hilbert space representation $(\rho, \frac{1}{2})$, are, in fact, discrete. Namely, differential equation (3.9) corresponding to (2.12) has solutions of finite norm for all positive $\lambda \in \mathbb{R}$, which, due to appropriate functional-analysis theorems, means that $W_0 - W_4$ is not self-adjoint. The operator is only "formally symmetric" because the domains of $W_0 - W_4$ and $(W_0 - W_4)^\dagger$ are not equal. There are, however, self-adjoint extensions that we construct: each reduces the initial space of states to the "subspace of physical states," implying in consequence discreteness of time.

III. HILBERT SPACE REPRESENTATION

We work in the Hilbert space representation of the principal continuous series (ρ, s) [10]. It is constructed in the familiar Bargmann-Wigner representation space of the Poincaré group with mass $m > 0$ and spin s [11]. Generators of the Lorentz rotations are given by

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3, \tag{3.1}$$

where $S_{\mu\nu}$ are spin generators, $L_{ik} = i(p_i \frac{\partial}{\partial p^k} - p_k \frac{\partial}{\partial p^i})$, $L_{0k} = ip_0 \frac{\partial}{\partial p^k}$, $i, k = 1, 2, 3$, and $p_0 = \sqrt{m^2 + (p_i)^2}$. Generators of the Poincaré translations, multiplication operators p_μ, are used to define the remaining $M_{4\mu}$ by

$$M_{4\mu} = \frac{\rho}{m}p_\mu - \frac{1}{2m}(p^\rho M_{\rho\mu} + M_{\rho\mu}p^\rho). \tag{3.2}$$

This representation was used in [6]: we will introduce it here very briefly in order to fix the notation and stress a couple of technical details and simplifications.

Bargmann-Wigner space $\mathcal{H}$ for $s = \frac{1}{2}$ is the space of bispinors in momentum representation, $\psi(\vec{p})$, which are square-integrable solutions to the Dirac equation. Using Dirac representation of γ matrices,

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

$\psi(\vec{p})$ can be written as

$$\psi(\vec{p}) = \begin{pmatrix} \Phi(\vec{p}) \\ -\frac{\vec{p}\cdot\vec{\sigma}}{p_0+m}\Phi(\vec{p}) \end{pmatrix}, \tag{3.3}$$

where $\Phi(\vec{p})$ is an unconstrained spinor. The scalar product is given by

$$(\psi, \psi') = \int \frac{d^3p}{p_0}\psi^\dagger\gamma^0\psi' = \int \frac{d^3p}{p_0}\frac{2m}{p_0 + m}\Phi^\dagger\Phi'. \tag{3.4}$$

Written in blocks of 2×2 matrices, $M_{\alpha\beta}$ and W_α have the form

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

Matrix elements of such operators are[2]

$$\begin{aligned}(\psi, M\psi') &= \int d^3p\psi^\dagger\frac{\gamma^0}{p_0}M\psi' \\ &= \int \frac{d^3p}{p_0}\Phi^\dagger\Big(A - \frac{p_k\sigma^k}{p_0 + m}A\frac{p_i\sigma^i}{p_0 + m} \\ &\quad + \Big[B, \frac{p_k\sigma^k}{p_0 + m}\Big]\Big)\Phi'.\end{aligned}$$

Eigenvalue problem $M\psi = \lambda\psi$ can be written as a set of two spinor equations:

[1] An important observation is that components W^α are the Casimir operators of subgroups of $SO(1,4)$: W^0 of the $SO(4)$ and W^i of the $SO(1,3)$. Therefore, eigenvalues of W^α could in principle be determined group theoretically: by reduction of a given UIR of the $SO(1,4)$ to the sum of UIR's of the corresponding subgroup. A similar strategy is possible for τ, which is one of two Casimir operators of the $E(3)$ subgroup, generated by $M_{0i} + M_{i4}$ and M_{jk}: we have not succeeded in finding the appropriate reduction formula in the literature.

[2] At this point we fix the relative positions of γ^0, $1/p_0$, and M: this ordering is not essential and can be changed, but it implies appropriate changes in relations that follow.

$$\left(A - \frac{p_k\sigma^k}{p_0+m}A\frac{p_i\sigma^i}{p_0+m} + \left[B, \frac{p_k\sigma^k}{p_0+m}\right]\right)\Phi = \lambda\frac{2m}{p_0+m}\Phi, \tag{3.5}$$

$$\left(\left[A, \frac{p_k\sigma^k}{p_0+m}\right] + B - \frac{p_k\sigma^k}{p_0+m}B\frac{p_i\sigma^i}{p_0+m}\right)\Phi = 0. \tag{3.6}$$

One can easily check that the second equation is fulfilled for all solutions of the first, so essentially one has to solve (3.5).

In our problem $M = W_0 - W_4$, the blocks A and B are

$$A = -\frac{1}{2m}\left(\rho - \frac{i}{2}\right)p_i\sigma^i - \frac{i}{2m}p_0(p_0+m)\frac{\partial}{\partial p_i}\sigma_i, \tag{3.7}$$

$$B = -\frac{1}{2m}\epsilon^{ijk}(p_0+m)p_i\frac{\partial}{\partial p^j}\sigma_k - \frac{3i}{4m}(p_0+m). \tag{3.8}$$

The eigenvalue equation for $W_0 - W_4$ becomes

$$\left(-\frac{\rho}{2m}p_i\sigma^i - \frac{i}{2}(p_0+m)\frac{\partial}{\partial p_i}\sigma_i + \frac{i}{2m}p_i\frac{\partial}{\partial p_i}p_j\sigma^j\right)\Phi = \lambda\Phi. \tag{3.9}$$

As $W_0 - W_4$ commutes with 3-rotations M_{ij}, we can choose the eigenfunctions in the form

$$\Phi_{\lambda jm}(\vec{p}) = \frac{f(p)}{p}\phi_{jm}(\theta,\varphi) + \frac{h(p)}{p}\chi_{jm}(\theta,\varphi), \tag{3.10}$$

where p is the radial momentum, $p^2 = -p_ip^i = p_0^2 - m^2$, and ϕ_{jm} and χ_{jm} are the eigenfunctions of the angular momentum. Using (3.10) we obtain radial equations for f and h:

$$(p_0+1)\frac{df}{dp_0} + i\rho f - \frac{j+\frac{1}{2}}{p_0-1}f = 2i\lambda\frac{h}{p}, \tag{3.11}$$

$$(p_0+1)\frac{dh}{dp_0} + i\rho h + \frac{j+\frac{1}{2}}{p_0-1}h = 2i\lambda\frac{f}{p}. \tag{3.12}$$

Solutions to these equations are derived in Appendix A. They are expressed in terms of the Bessel functions using variable $z = \sqrt{\frac{p_0-m}{p_0+m}}$; this variable varies in a finite interval, $z \in (0,1)$. Of two linearly independent solutions for fixed λ and j one is regular,

$$f_{\lambda j} = C\left(\frac{2}{1-z^2}\right)^{-i\rho}\sqrt{z}J_j(2\lambda z),$$
$$h_{\lambda j} = iC\left(\frac{2}{1-z^2}\right)^{-i\rho}\sqrt{z}J_{j+1}(2\lambda z), \tag{3.13}$$

and therefore we conclude that the spectrum of $W_0 - W_4$ is the positive real axis, $\lambda \in (0,\infty)$. However, the given set of solutions is not orthonormal. The scalar product of two eigenfunctions is

$$\begin{aligned}(\psi_{\lambda jm}, \psi_{\lambda' j' m'}) = 2\delta_{jj'}\delta_{mm'}C^*C' &\times \int_0^1 z dz (J_j(2\lambda z)J_j(2\lambda' z) \\ &+ J_{j+1}(2\lambda z)J_{j+1}(2\lambda' z)).\end{aligned} \tag{3.14}$$

As Bessel functions $J_j(\zeta)$ are finite in any finite interval, integral (3.14) is bounded for $\lambda = \lambda'$; i.e., all solutions are normalizable, which is in contradiction with the statement that they belong to a continuous spectrum. Also they are not orthogonal for $\lambda \neq \lambda'$. Therefore, not all of formal solutions (3.13) can be the eigenfunctions of a self-adjoint operator, and self-adjointness is a property we would certainly like τ to have.

IV. SELF-ADJOINT EXTENSIONS

The obtained result requires additional analysis. We started with a unitary representation of the $SO(1,4)$, that is, with a set of self-adjoint (Hermitian) generators $M_{\alpha\beta}$. We defined W_α by (2.5), as a sum of products of operators that mutually commute. Therefore formally, $W_0 - W_4 = \tau/\ell$ is Hermitian and should have an orthonormal eigenbasis (discrete or continuous). But in concrete representation we obtained a continuous set of eigenfunctions of finite norm that are not mutually orthogonal. Hence τ is not self-adjoint: it can only be formally symmetric, with domain $\mathcal{D}(\tau)$ unequal to the domain of its adjoint, $\mathcal{D}(\tau^\dagger)$. To define self-adjoint extensions, if they exist, we need to resolve the issue of the domains.

The problem is obviously in the radial equation. Separation of angular variables gives a division of $\mathcal{H}$ into subspaces of fixed angular momentum j, in which τ reduces to operators T_j:

$$\begin{aligned}&(\psi_{jm}, (W_0 - W_4)\psi'_{j'm'}) \\ &\equiv \delta_{jj'}\delta_{mm'}\int_0^1 dz \Phi^\dagger T_j \Phi' \\ &= 2\delta_{jj'}\delta_{mm'}\int_0^1 dz (f^* \quad h^*) \\ &\times \begin{pmatrix} 0 & \rho\frac{2z}{1-z^2} - i(j+\frac{1}{2})\frac{1}{z} - i\frac{d}{dz} \\ \rho\frac{2z}{1-z^2} + i(j+\frac{1}{2})\frac{1}{z} - i\frac{d}{dz} & 0 \end{pmatrix}\begin{pmatrix} f' \\ h' \end{pmatrix} \\ &= \delta_{jj'}\delta_{mm'}\int_0^1 dz (F^* \quad H^*)\begin{pmatrix} 0 & -i\frac{d}{dz} \\ -i\frac{d}{dz} & 0 \end{pmatrix}\begin{pmatrix} F' \\ H' \end{pmatrix}.\end{aligned} \tag{4.1}$$

Functions F and H are defined by

$$F = \left(\frac{2}{1-z^2}\right)^{i\rho} z^{-j-\frac{1}{2}} f, \qquad H = \left(\frac{2}{1-z^2}\right)^{i\rho} z^{j+\frac{1}{2}} h, \tag{4.2}$$

and they are introduced in Appendix A to solve the radial equation; they simplify the matrix elements of T_j as well as the scalar product,

$$(\psi_{jm}, \psi'_{j'm'}) = 2\delta_{jj'}\delta_{mm'} \int_0^1 dz (F^* \quad H^*) \times \begin{pmatrix} z^{2j+1} & 0 \\ 0 & z^{-2j-1} \end{pmatrix} \begin{pmatrix} F' \\ H' \end{pmatrix}.$$

Let us examine properties of T_j. In order to find $T_j^\dagger$ we partially integrate (4.1),

$$\begin{aligned}(\psi_{jm}, (W_0 - W_4)\psi'_{j'm'}) &= -i\delta_{jj'}\delta_{mm'} \int_0^1 dz \left(F^* \frac{dH'}{dz} + H^* \frac{dF'}{dz} \right) \\ &= i\delta_{jj'}\delta_{mm'} \int_0^1 dz \left(\frac{dF^*}{dz} H' + \frac{dH^*}{dz} F' \right) \\ &\quad - i\delta_{jj'}\delta_{mm'} (F^* H' + H^* F') \Big|_0^1. \end{aligned} \tag{4.3}$$

We see that the "action" of T_j on functions, given by the first term in (4.3), is self-adjoint: but since the boundary term does not vanish, T_j and $T_j^\dagger$ are not equal. This is, in fact, a definition of being "formally symmetric" [12]. The other signatures of non-Hermiticity are nonzero deficiency indices, i.e., the existence of normalizable solutions to equations, $T_j \Phi = \pm i\Phi$. We show in Appendix B that the deficiency indices of T_j are $(n_+, n_-) = (1, 1)$.

There is a systematic way of extending formally symmetric operators to the self-adjoint [12,13]. The main idea is to find an appropriate subspace of $\mathcal{H}$ on which the boundary term vanishes: this subspace becomes the domain of both redefined or "extended" τ and $\tau^\dagger$. A necessary condition for the existence of self-adjoint extensions is that deficiency indices n_+ and n_- be equal. Analyzing (4.3) in Appendix B we find that T_j is self-adjoint if it is restricted to the subspace of functions (4.2), which satisfy

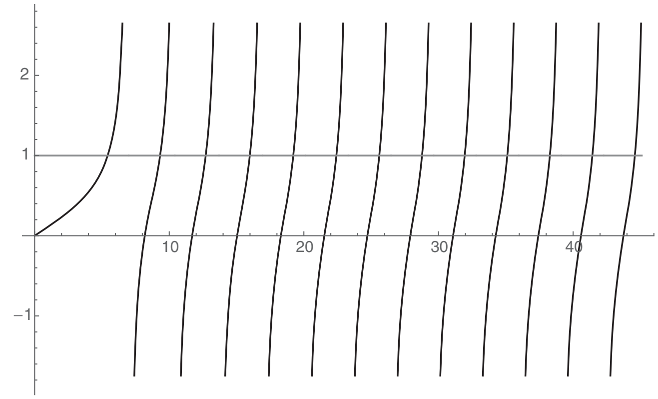

FIG. 1. Solutions to Eq. (4.5) for $j = \frac{7}{2}$, $c = 1$.

$$F(0) = H(0) = 0, \qquad H(1) = icF(1). \tag{4.4}$$

Let us check that eigenfunctions (3.13) can satisfy (4.4). The first relation is clearly true, and the second gives

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} = c = \text{const}, \tag{4.5}$$

that is, an equation for λ. This equation, as seen from Fig. 1, has infinitely many solutions for every real c; the set of solutions is discrete. The other way to see this is for large values of λ as, asymptotically,

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} \sim -\tan\left(2\lambda - \frac{(2j+1)\pi}{4}\right), \qquad \lambda \to \infty. \tag{4.6}$$

The eigenvalues can be labeled by a natural number n, and for large λ they become equidistant with period $\pi/2$. By a choice of c we can fix the value of one of the λ's; the other eigenvalues are determined by (4.5). This means that for every c we obtain a different self-adjoint extension $T_j^{(c)}$; i.e., we have a one-parameter family: we can take, for example, $c = 1$ as a preferred choice.

Let us check orthogonality. Using the recurrence relations between the Bessel functions we find

$$\begin{aligned}(\psi_{\lambda jm}, \psi_{\lambda' j'm'}) &= 2C^* C\delta_{jj'}\delta_{mm'} \int_0^1 z dz (J_j(2\lambda z) J_j(2\lambda' z) + J_{j+1}(2\lambda z) J_{j+1}(2\lambda' z)) \\ &= \delta_{jj'}\delta_{mm'} \frac{|C|^2}{\lambda^2 - \lambda'^2} (\lambda' J_j(2\lambda) J'_j(2\lambda') + \lambda' J_{j+1}(2\lambda) J'_{j+1}(2\lambda') \\ &\quad - \lambda J_j(2\lambda') J'_j(2\lambda) - \lambda J_{j+1}(2\lambda') J'_{j+1}(2\lambda)) \\ &= -\frac{\delta_{jj'}\delta_{mm'}}{\lambda - \lambda'} \frac{|C|^2}{J_{j+1}(2\lambda) J_{j+1}(2\lambda')} \left(\frac{J_j(2\lambda)}{J_{j+1}(2\lambda)} - \frac{J_j(2\lambda')}{J_{j+1}(2\lambda')} \right), \end{aligned} \tag{4.7}$$

where in the second line $J'_a(\zeta)$ denotes the derivative of $J_a(\zeta)$. The last expression is zero for $\lambda \neq \lambda'$ for a discrete set of eigenfunctions that satisfy (4.5), and we confirm that the given basis is orthogonal.

V. SINGULARITIES AND SYMMETRIES

Let us verify that fuzzy de Sitter space corresponds to an expanding cosmology and discuss the absence of the big bang singularity. The (squared) radius of the universe is given by

$$(x^i)^2 = -\ell^2 W_i W^i, \tag{5.1}$$

and its evolution can be traced by the expectation value $\langle (x^i)^2 \rangle$ in the eigenstates of time. Eigenvalue λ of $W_0 - W_4$ used in the previous calculation is related to the time eigenvalue t exponentially,

$$t = \langle \tau \rangle = (\psi_{\lambda jm}, \tau \psi_{\lambda jm}) = \ell \log \lambda. \tag{5.2}$$

Using Casimir relation (2.8),

$$-W_i W^i = \mathcal{W} + W_0^2 - W_4^2, \tag{5.3}$$

and taking normalized eigenstates $\psi_{\lambda jm}$,

$$(\psi_{\lambda jm}, \psi_{\lambda jm}) = 2C^* C \int_0^1 dzz(J_j^2(2\lambda z) + J_{j+1}^2(2\lambda z)) = 1, \tag{5.4}$$

we find

$$\begin{aligned} \langle -W_i W^i \rangle &= \mathcal{W} + \langle (W_0 + W_4)(W_0 - W_4) \rangle \\ &= \mathcal{W} + \lambda^2 + 2\lambda \langle W_4 \rangle. \end{aligned}$$

Expectation value $\langle W_4 \rangle$ can be estimated. We have

$$W_4 = -\frac{1}{2} \begin{pmatrix} p_0 \vec{r} \cdot \vec{\sigma} & i\vec{L} \cdot \vec{\sigma} + \frac{3i}{2} \\ i\vec{L} \cdot \vec{\sigma} + \frac{3i}{2} & p_0 \vec{r} \cdot \vec{\sigma} \end{pmatrix},$$

and therefore

$$\begin{aligned} &(\psi_{\lambda jm}, W_4 \psi_{\lambda jm}) \\ &= \int \frac{d^3 p}{p_0} \Phi^\dagger_{\lambda jm} \left(\frac{im \vec{p} \cdot \vec{\sigma}}{2(p_0 + m)^2} - \frac{m^2 \vec{r} \cdot \vec{\sigma}}{p_0 + m} \right. \\ &\quad \left. - \frac{im(\vec{p} \cdot \nabla)(\vec{p} \cdot \vec{\sigma})}{(p_0 + m)^2} \right) \Phi_{\lambda jm} \\ &= -\frac{i}{2} \int_0^1 dz(1 - z^2) \left(F^*_{\lambda j} \frac{dH_{\lambda j}}{dz} + H^*_{\lambda j} \frac{dF_{\lambda j}}{dz} \right) \\ &= \lambda C^* C \int_0^1 dzz(1 - z^2)(J_j^2(2\lambda z) + J_{j+1}^2(2\lambda z)). \end{aligned} \tag{5.5}$$

Comparing the last integral with (5.4),

$$\begin{aligned} 0 &\le \int_0^1 dzz(1 - z^2)(J_j^2(2\lambda z) + J_{j+1}^2(2\lambda z)) \\ &\le \int_0^1 dzz(J_j^2(2\lambda z) + J_{j+1}^2(2\lambda z)), \end{aligned} \tag{5.6}$$

we obtain that $0 \le (\psi_{\lambda jm}, W_4 \psi_{\lambda jm}) \le \frac{\lambda}{2}$; hence

$$\mathcal{W} + \lambda^2 \le (\psi_{\lambda jm}, -W_i W^i \psi_{\lambda jm}) \le \mathcal{W} + 2\lambda^2. \tag{5.7}$$

The expectation value of the radius of the universe is bounded below by $\ell\sqrt{\mathcal{W}}$: it does not vanish in physical states that lie in the domain of self-adjoint extensions $\tau^{(c)}$; i.e., it can be expanded in the corresponding eigenbases. The radius, on the other hand, grows with time exponentially: for late times we have $\sqrt{\langle -W_i W^i \rangle} \sim \lambda = e^{t/\ell}$.

Another important point is discreteness of time that, as explained, also comes through the self-adjointness of τ. Though Hermiticity is a usual condition in quantum mechanics, we rarely deal with operators that do not have unique self-adjoint extensions. This is related to the fact that quantum mechanics is defined on the flat unbounded space: one can expect boundary effects in curved spaces, spaces that are bounded or singular (geodesically incomplete, or with curvature singularities). In this context, formally symmetric Hamiltonians with a one-parameter family of self-adjoint extensions appear in various physical situations (and mathematical setups) in general relativity and cosmology [14–17]. The interpretation of nonuniqueness of the extension varies: from understanding that it is a further quantization ambiguity [16] to that it renders a definition of spacetimes that are singular for "quantum probes" (as in some cases, classically singular spacetimes can appear completely regular for quantized particles) [15]. Wald relates the necessity to choose one of the extensions with the fact that the initial-value problem is classically ill-defined at naked singularity, and regards the possibility of constructing a self-adjoint extension as a resolution to the singularity problem [14].

The last point of view is in some sense close to our example, though we are extending time and not the Hamiltonian. Discreteness of time becomes relevant in the "deep quantum region" $\lambda \to 0$, i.e., $t \to -\infty$, near the classical boundary through which the steady-state model can be extended to the complete de Sitter space. For values away from the Planck scale time is almost continuous: the difference between its consecutive eigenvalues is macroscopically negligible,

$$t_{n+1} - t_n \approx \ell \log\left(1 + \frac{1}{n}\right). \tag{5.8}$$

Discreteness obtained by requiring self-adjointness is known in other cases of quantum spaces. One example is the q-deformed Heisenberg algebra,

$$[p,x] = -i + (q-1)xp, \tag{5.9}$$

and its unitary representations [18,19]. The analysis shows that coordinate x is not self-adjoint, but the self-adjoint extensions exist; both x and p have discrete spectra. Another interesting case is the minimal-length Heisenberg algebra,

$$[p,x] = -i - i\beta p^2, \tag{5.10}$$

which is in [20] represented in the Schrödinger representation. Again it is found that x has a one-parameter family of self-adjoint extensions, which puts its spectrum on a lattice.

The q-deformed Heisenberg algebra (Manin plane) has, as symmetry, the quantum group $SU_q(2)$, so it is natural to ask whether in our model symmetry gets deformed as well. As shown in [5], our choice of frame, in fact, breaks the $SO(1,4)$ invariance, and *a priori* symmetries of fuzzy de Sitter space are rotations and time translations, $SO(3)\times U(1)$. Here $U(1)$ denotes the dilatation subgroup, $U(1) = \{e^{i\alpha M_{04}} | \alpha \in \mathbb{R}\}$, the dilatation generator plays the role of the Hamiltonian, $H = M_{04}$: it evolves the eigenstates of time, (2.11).

If we keep the standard notion that symmetry is defined by a group of transformations, the choice of a self-adjoint extension $\tau^{(c)}$ is spontaneous symmetry breaking. This can be seen easily: the elements of $U(1)$ do not preserve the space of physical states defined by (4.4) for arbitrary values of parameter α. However, there is a subgroup of dilatations, $U^{(c)}(1)$, determined by the allowed values of α that preserve condition (4.4): it is represented nonlinearly. For large eigenvalues, Eq. (4.4) becomes periodic and λ equally spaced: subgroup $U^{(c)}(1)$ becomes in this limit (in this region of physical parameters) the additive group of integers. In the continuum approximation $\ell \to 0$ that is valid on the macroscopic scale, the full symmetry is recovered. Another view is that, in the quantum regime, classical symmetries get deformed [21]: whether the corresponding transformations in our case have the structure of a quantum group is to be studied. In any case, what we find is that classical symmetries get broken or deformed on the Planck scale, due to the quantum structure of spacetime. To obtain other effects in cosmology that our model predicts we should introduce matter, for example, a scalar field. This is in principle a well-defined problem in noncommutative geometry, and we plan to address it in our future work.

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APPENDIX A: RADIAL EQUATION

In this appendix we solve the radial equations (3.9). In the signature that we use

$$\vec{p} = (p_i), \qquad \vec{L} = (L_i), \qquad \vec{\sigma} = (\sigma_i),$$
$$\vec{r} = (x^i) = i\frac{\partial}{\partial p_i}, \qquad \vec{p}\cdot\vec{\sigma} = -p_i\sigma^i,$$
$$\sigma_i\sigma_j = -\eta_{ij} - \epsilon_{ijk}\sigma^k,$$
$$(\vec{r}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma}) = i\left(3 + p\frac{\partial}{\partial p} + \vec{L}\cdot\vec{\sigma}\right).$$

The eigenvalue equation (3.9) is

$$\left(\frac{1}{2m}\rho(\vec{p}\cdot\vec{\sigma}) - \frac{1}{2}(p_0+m)(\vec{r}\cdot\vec{\sigma}) - \frac{1}{2m}(\vec{p}\cdot\vec{r})(\vec{p}\cdot\vec{\sigma})\right)\Phi = \lambda\Phi. \tag{A1}$$

We use the Ansatz that separates angular and radial variables,

$$\Phi_{\lambda jm}(\vec{p}) = \frac{f(p)}{p}\phi_{jm}(\theta,\varphi) + \frac{h(p)}{p}\chi_{jm}(\theta,\varphi), \tag{A2}$$

with $p^2 = -p_ip^i = p_0^2 - m^2$. The ϕ_{jm} and χ_{jm} are the spinor eigenfunctions of $M_{ij}M^{ij}$ and M_{12}; they are orthonormal and satisfy

$$\phi_{jm} = \frac{\vec{p}\cdot\vec{\sigma}}{p}\chi_{jm}, \qquad (\vec{L}\cdot\vec{\sigma})\phi_{jm} = \left(j-\frac{1}{2}\right)\phi_{jm},$$
$$\chi_{jm} = \frac{\vec{p}\cdot\vec{\sigma}}{p}\phi_{jm}, \qquad (\vec{L}\cdot\vec{\sigma})\chi_{jm} = -\left(j+\frac{3}{2}\right)\chi_{jm}. \tag{A3}$$

Introducing (A2) we obtain radial equations

$$(p_0+1)\frac{df}{dp_0} + i\rho f - \frac{j+\frac{1}{2}}{p_0-1}f = 2i\lambda\frac{h}{p}, \tag{A4}$$

$$(p_0+1)\frac{dh}{dp_0} + i\rho h + \frac{j+\frac{1}{2}}{p_0-1}h = 2i\lambda\frac{f}{p}. \tag{A5}$$

In order to simplify them we rescale momentum to be dimensionless, $p\to mp$, $p_0 \to mp_0$, $p\in(0,\infty)$, $p_0\in(1,\infty)$. Equations decouple when we introduce new functions F, H by

$$f = (p_0+1)^{-i\rho-\frac{2j+1}{4}}(p_0-1)^{\frac{2j+1}{4}}F,$$
$$h = (p_0+1)^{-i\rho+\frac{2j+1}{4}}(p_0-1)^{-\frac{2j+1}{4}}H. \tag{A6}$$

We then obtain

$$(p_0^2-1)\frac{d^2F}{dp_0^2}+2(p_0+j)\frac{dF}{dp_0}+\frac{4\lambda^2}{(p_0+1)^2}F=0, \tag{A7}$$

$$(p_0^2-1)\frac{d^2H}{dp_0^2}+2(p_0-j-1)\frac{dH}{dp_0}+\frac{4\lambda^2}{(p_0+1)^2}H=0, \tag{A8}$$

and additional relations

$$\begin{aligned}\frac{dF}{dp_0}&=2i\lambda(p_0+1)^{j-1}(p_0-1)^{-j-1}H,\\ \frac{dH}{dp_0}&=2i\lambda(p_0+1)^{-j-2}(p_0-1)^{j}F.\end{aligned} \tag{A9}$$

Equations (A7) and (A8) reduce to the Bessel equation

$$\zeta^2\frac{d^2Y}{d\zeta^2}+\zeta\frac{dY}{d\zeta}+(\zeta^2-a^2)Y=0 \tag{A10}$$

by compactification of the independent variable. Introducing z as

$$z=\sqrt{\frac{p_0-1}{p_0+1}}, \tag{A11}$$

both equations reduce to (A10) for $\zeta=2\lambda z\in(0,2\lambda)$. In Eq. (A7), $a=j$; in (A8), $a=-j-1$.

Linearly independent solutions to the Bessel equation are the Bessel functions $J_a(\zeta)$, $J_{-a}(\zeta)$ or $J_a(\zeta)$, $Y_a(\zeta)$ [a is half-integer, so $J_{-j-1}(\zeta)=(-1)^{j-\frac{1}{2}}Y_{j+1}(\zeta)$]. Therefore, $F\sim J_j, J_{-j}$ and $H\sim J_{j+1}, J_{-j-1}$. Taking into account additional relations (A9) that are satisfied through recurrence relation

$$\frac{1}{\zeta}\frac{d}{d\zeta}\zeta^a J_a(\zeta))=\zeta^{a-1}J_{a-1}(\zeta), \tag{A12}$$

we obtain two linearly independent solutions:

$$F_{\lambda j}=Cz^{-j}J_j(2\lambda z),\qquad H_{\lambda j}=iCz^{j+1}J_{j+1}(2\lambda z), \tag{A13}$$

$$\tilde F_{\lambda j}=\tilde Cz^{-j}J_{-j}(2\lambda z),\qquad \tilde H_{\lambda j}=-i\tilde Cz^{j+1}J_{-j-1}(2\lambda z). \tag{A14}$$

As the Bessel functions around $\zeta=0$ behave as

$$J_a(\zeta)\sim\frac{1}{\Gamma(a+1)}\left(\frac{\zeta}{2}\right)^a, \tag{A15}$$

the second solution diverges, $\tilde\psi_{\lambda jm}\sim\zeta^{-j-\frac{3}{2}}$, so we have one regular solution,

$$\begin{aligned}f_{\lambda j}&=C\left(\frac{2}{1-z^2}\right)^{-i\rho}\sqrt{z}J_j(2\lambda z),\\ h_{\lambda j}&=iC\left(\frac{2}{1-z^2}\right)^{-i\rho}\sqrt{z}J_{j+1}(2\lambda z).\end{aligned} \tag{A16}$$

It exists for every real λ. But $J_a(-\zeta)=(-1)^aJ_a(\zeta)$, so the spectrum can be restricted to the positive real axis, $\lambda>0$.

The scalar product of two eigenfunctions is given by

$$\begin{aligned}(\psi_{\lambda jm},\psi_{\lambda' j'm'})&=2\delta_{jj'}\delta_{mm'}\int_0^1 dz(f^*f'+h^*h')\\ &=2\delta_{jj'}\delta_{mm'}\int_0^1 dz(z^{2j+1}F^*F'+z^{-2j-1}H^*H')\\ &=2\delta_{jj'}\delta_{mm'}C^*C'\int_0^1 zdz(J_j(2\lambda z)J_j\\ &\quad\times(2\lambda' z)+J_{j+1}(2\lambda z)J_{j+1}(2\lambda' z)).\end{aligned} \tag{A17}$$

It is nonzero for $\lambda\neq\lambda'$ and finite for each λ, which as we discuss in the text, is a problem. Singular solutions do not have the right normalization to be eigenfunctions of the continuous spectrum: similar to (A17), we have

$$\begin{aligned}(\tilde\psi_{\lambda jm},\tilde\psi_{\lambda' j'm'})=2\delta_{jj'}\delta_{mm'}\tilde C^*\tilde C'\int_0^1 dz(J_{-j}(2\lambda z)J_{-j'}(2\lambda' z)\\ +J_{-j-1}(2\lambda z)J_{-j'-1}(2\lambda' z)).\end{aligned}$$

This integral is divergent in the lower limit, but the divergence depends on j and not on the difference $\lambda-\lambda'$; i.e. it does not have the required form $\delta(\lambda-\lambda')$.

APPENDIX B: DEFICIENCY INDICES AND SELF-ADJOINTNESS

We start with the deficiency indices of T_j. To determine them we need to solve equation

$$T_j\Phi=\pm i\Phi. \tag{B1}$$

This is, in fact, not difficult: solutions to these equations are the same as solutions to (A1) for $\lambda=\pm i$: the Bessel functions of the imaginary argument, i.e., the modified Bessel functions $I_a(\zeta)$ and $K_a(\zeta)$,

$$I_a(\zeta)=i^{-a}J_a(i\zeta),\qquad K_a(\zeta)=\frac{\pi}{2}i^{a+1}(J_a(i\zeta)+iY_a(i\zeta)). \tag{B2}$$

As before, $a=\pm j$, $\pm(j+1)$. The modified Bessel functions have similar behavior around zero as the Bessel functions: $K_a(\zeta)$ is divergent and the corresponding solution has an infinite norm. This implies that equation $T_j\Phi=i\Phi$ has just one regular solution,

$$F_+=Cz^{-j}I_j(2z),\qquad H_+=-Cz^{j+1}I_{j+1}(2z). \tag{B3}$$

Similarly there is one regular solution (F_-,H_-) to equation $T_j\Phi=-i\Phi$. This means that deficiency indices of T_j are $(n_+,n_-)=(1,1)$; hence T_j is not a self-adjoint operator but can be extended to one.

Next, let us briefly recall the procedure of constructing self-adjoint extensions of formally symmetric operators. We use the notation of [12], where proof of the main technical result that we use can also be found. We can write Eq. (4.3) abstractly as

$$(\Phi, T_j\Phi') = (T_j\Phi, \Phi') + B(\Phi, \Phi') = (T_j^\dagger\Phi, \Phi'), \tag{B4}$$

where the boundary term $B(\Phi, \Phi')$ is a bilinear form, which in our case reads

$$B(\Phi, \Phi') = (F^* H' + H^* F')|_0^1. \tag{B5}$$

Apparently, the domain of T_j is given by all normalizable functions Φ and Φ' that satisfy $B(\Phi, \Phi') = 0$, or in our case $F(0) = H(0) = 0$ and $F(1) = H(1) = 0$. Then $\mathcal{D}(T_j^\dagger) = \mathcal{H}$ and obviously the two domains are not equal, $\mathcal{D}(T_j) \subset \mathcal{H}$. To achieve self-adjointness, one should relax the condition that determines $\mathcal{D}(T_j)$ and restrict $\mathcal{D}(T_j^\dagger)$. This is done effectively by finding n_+ linearly independent functions Φ_k (more precisely, n_+ linearly independent vectors corresponding to their boundary values)—in our case one, Φ_1—that satisfy

$$B(\Phi_k, \Phi_l) = 0. \quad \forall\ k, l. \tag{B6}$$

The domain of a self-adjoint extension of T_j is then defined as a set of functions Φ,

$$\mathcal{D}(T_j) = \mathcal{D}(T_j^\dagger) = \{\Phi | B(\Phi, \Phi_k) = 0, \quad \forall\ k\}. \tag{B7}$$

In principle, boundary term (4.3) is a combination of values at both boundary points but often the constraints can be imposed separately. It is possible to do it in our case as well: we can choose $F(0) = H(0) = 0$, in accordance with the behavior of the eigenfunctions of τ that constitute a basis. If, at the other boundary, we denote the values of Φ_1 as

$$F_1(1) = \sigma e^{i\beta}, \qquad H_1(1) = i\sigma' e^{i\beta'}, \tag{B8}$$

we find $i\beta = i\beta' + n\pi$. Constants β, σ, and σ' are real numbers, so the domain of the self-adjoint extension $T_j^{(c)}$ is a set of functions that satisfies

$$F(0) = H(0) = 0, \qquad H(1) = \pm i \frac{\sigma}{\sigma'} F(1) = icF(1), \quad c \in \mathbb{R}. \tag{B9}$$

[1] J. Madore, An Introduction to Noncommutative Differential Geometry and its Physical Applications, London Mathematical Society Lecture Note Series Vol. **257**, 2nd ed. (Cambridge University Press, Cambridge, England, 2009), https://doi.org/10.1017/CBO9780511569357.

[2] M. Buric, T. Grammatikopoulos, J. Madore, and G. Zoupanos, J. High Energy Phys. 04 (2006) 054.

[3] J. Madore, Classical Quantum Gravity **9**, 69 (1992); J. Hoppe, Ph.D. Thesis, MIT, 1982.

[4] A. P. Balachandran, S. Kurkcuoglu, and S. Vaidya, *Lectures On Fuzzy And Fuzzy Susy Physics* (World Scientific, Singapore, 2007), p. 191.

[5] M. Buric and J. Madore, Eur. Phys. J. C **75**, 502 (2015).

[6] M. Buric, D. Latas, and L. Nenadovic, Eur. Phys. J. C **78**, 953 (2018).

[7] S. W. Hawking and G. F. R. Ellis, *Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1975).

[8] M. Buric, D. Latas, V. Radovanovic, and J. Trampetic, Phys. Rev. D **75**, 097701 (2007).

[9] J. Dixmier, Bull. Soc. Math. Fr. **89**, 9 (1961).

[10] P. Moylan, J. Math. Phys. (N.Y.) **24**, 2706 (1983); **26**, 29 (1985).

[11] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U.S.A. **34**, 211 (1948).

[12] V. Hutson, J. Pym, and M. J. Cloud, *Applications of Functional Analysis and Operator Theory* (Elsevier Science, Amsterdam, 2005).

[13] N. Dunford and J. T. Schwartz, *Linear Operators* (John Wiley & Sons, New York, 1957).

[14] R. M. Wald, J. Math. Phys. (N.Y.) **21**, 2802 (1980).

[15] G. T. Horowitz and D. Marolf, Phys. Rev. D **52**, 5670 (1995).

[16] A. A. Andrianov, C. Lan, O. O. Novikov, and Y. F. Wang, Eur. Phys. J. C **78**, 786 (2018).

[17] S. Gryb and K. P. Y. Thébault, Classical Quantum Gravity **36**, 035009 (2019).

[18] J. Schwenk and J. Wess, Phys. Lett. B **291**, 273 (1992).

[19] A. Hebecker, S. Schreckenberg, J. Schwenk, W. Weich, and J. Wess, Z. Phys. C **64**, 355 (1994).

[20] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995).

[21] M. Fichtmuller, A. Lorek, and J. Wess, Z. Phys. C **71**, 533 (1996).